Deviation from standard QED at large distances: Influence of transverse dimensions of colliding beams on bremsstrahlung

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The radiation at the collision of high-energy particles is formed over a rather long distance and therefore is sensitive to the environment. In particular, the smallness of the transverse dimensions of the colliding beams leads to suppression of the bremsstrahlung cross section for soft photons. This beam-size effect was discovered and investigated at INP, Novosibirsk. At that time an incomplete expression for the bremsstrahlung spectrum was calculated and used. This is because a subtraction associated with the extraction of the pure fluctuation process was not performed. Here this procedure is done. The complete expression for the spectral-angular distribution of incoherent bremsstrahlung probability is obtained. The case of the Gaussian colliding beams is investigated in detail. In the case of flat beams the expressions for the bremsstrahlung spectrum are essentially simplified. The comparison shows quite reasonable agreement between the theory and the VEPP-4 and DESY HERA data. The possible application to the tuning of beams in a linear e^+e^- collider is discussed.

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I. INTRODUCTION

The bremsstrahlung process at high energy involves a very small momentum transfer. In the space-time picture this means that the process occurs over a rather large (macroscopic) distance. The corresponding longitudinal length (with respect to the direction of the initial momentum) is known as the *coherence* (formation) length l_f. For the emission of a photon with the energy ω the coherence length is $l_f(\omega)\sim\varepsilon(\varepsilon-\omega)/m^2\omega$, where ε and m are the energy and the mass of the emitting particle (here the system $\hbar=c=1$ is used). If the particle experiences some action over this length, the radiation pattern changes (in the case when the action is the multiple scattering of the emitting particle one observes the famous Landau-Pomeranchuk effect [1]).

A different situation arises in the bremsstrahlung process in the electron-electron (-positron) collision. The point is that the external factors act differently on the radiating particle and on the recoil particle. For the radiating particle the criterion of influence of external factors is the same both at a electron scattering from a nucleus and at a collision of particles. For the recoil particle the effect turns out to be enhanced by the factor ε^2/m^2. This is due to the fact that the main contribution to the bremsstrahlung cross section gives the emission of a virtual photon with very low energy q_0 by the recoil particle

$$q_0\sim\frac{m^2\omega}{\varepsilon(\varepsilon-\omega)}, \tag{1.1}$$

so that the formation length of the virtual photon is

$$L_v(\omega)=l_f(q_0)=\frac{4\varepsilon^3(\varepsilon-\omega)}{m^4\omega}. \tag{1.2}$$

This means that the effect for the recoil particles appears much earlier than for the radiating particles. For example, the Landau-Pomeranchuk effect distorted the whole bremsstrahlung spectrum at the TeV range (for heavy elements) while it turns out that the action on the recoil particle can be important for contemporary colliding beam facilities at the GeV range [2].

There are a few factors which could act on the recoil electron. One of them is the presence of an external magnetic field in the region of particle collision [2–4]. If the formation length of the virtual photon L_v turns out to be larger than the formation length $l_H(\omega)$ of a photon with energy ω in a magnetic field H then the magnetic field will limit the region of minimal momentum transfers. This will lead to a decrease of the bremsstrahlung cross section and a change of its spectrum. Another effect can appear due to the smallness of the linear interval l where the collision occurs in comparison with $L_v(\omega)$ [see Eq. (1.2)]. This was pointed out in [5].

A special experimental study of bremsstrahlung was performed at the electron-positron colliding beam facility VEPP-4 of the Institute of Nuclear Physics, Novosibirsk [6]. The deviation of the bremsstrahlung spectrum from the standard QED spectrum was observed at the electron energy $\varepsilon=1.84$ GeV. The effect was attributed to the smallness of the transverse size of the colliding beams. In theory the problem was investigated in [7], where the bremsstrahlung spectrum at the collision of electron-electron (-positron) beams with small transverse size was calculated to within the power accuracy (the neglected terms are of the order $1/\gamma=m/\varepsilon$). Later the problem was analyzed in [8–10] where the bremsstrahlung spectra found coincide with those obtained in [7].

It should be noted that in [7] (as well as in all the other papers mentioned above) an incomplete expression for the bremsstrahlung spectrum was calculated. One has to perform the subtraction associated with the extraction of the pure fluctuation process. Let us discuss this item in some detail. The momentum transfer $\mathbf{q}$ at collision is an important characteristic of the radiation process (the cross section contains the factor $\mathbf{q}^2$ at $\mathbf{q}^2\ll m^2$). At the beam collision the momentum transfer may arise due to the interaction of the emitting particle with the opposite beam as a whole (due to the coherent interaction with the averaged field of the beam) and due to the interaction with an individual particle of the opposite beam. Here we are considering the *incoherent* process only (connected with the incoherent fluctuation of density) and so we have to subtract the coherent contribution. The expression for the bremsstrahlung spectrum found in [7] contains the mean value $\langle \mathbf{q}^2 \rangle$, while the coherent contribution contains $\langle \mathbf{q} \rangle^2$ and this term has to be subtracted. We encountered an analogous problem in the analysis of incoherent processes in oriented crystals [11] where it was pointed out (see p. 407) that the subtraction has to be done in the spectrum calculated in [7]. Without the subtraction the results for the incoherent processes in oriented crystals would be qualitatively erroneous.

In Sec. II a qualitative analysis of the incoherent radiation process is given. In Sec. III the general formulas for the spectral-angular distributions of incoherent bremsstrahlung are derived. The incoherent bremsstrahlung spectrum for Gaussian beams is calculated in Sec. IV in the form of double integrals. In the specific case of narrow beams (the size of the beam is much smaller than the characteristic impact parameter) the formulas are simplified essentially (Sec. V). Experimental studies of the effect were performed with flat beams (the beam vertical size is much smaller than the horizontal one). This specific case is analyzed in Sec. VI, while comparison with the VEPP-4 and the DESY *ep* collider HERA data is given in Sec. VII. In Sec. VIII the possible application to the tuning of beams in a linear e^+e^- collider is discussed.

II. GENERAL ANALYSIS OF PROBABILITY OF INCOHERENT RADIATION

In this section we discuss in detail the conditions under which we consider the incoherent radiation. One can calculate the photon emission probability in the target rest frame, since the entering combinations ω/ε and $\gamma\vartheta$ (γ is the Lorentz factor $\gamma=\varepsilon/m$, and ϑ is the angle of photon emission) are invariant (within a relativistic accuracy) and a transfer to any frame is elementary. We use the operator quasiclassical method [12,13]. Within this method the photon formation length (time) is

$$l_f=\frac{\varepsilon'}{\varepsilon kv}=\frac{\varepsilon'}{\varepsilon\omega(1-\mathbf{nv})}\simeq\frac{l_{f0}}{\zeta},$$

$$l_{f0}=\frac{1}{q_{min}}=\frac{2\varepsilon\varepsilon'}{\omega m^2}=\frac{4\varepsilon'\gamma_c\varepsilon_r}{\omega m^2},$$

$$\zeta=1+\gamma^2\vartheta^2, \quad \varepsilon'=\varepsilon-\omega, \tag{2.1}$$

where $p_\mu=\varepsilon v_\mu$ [$v_\mu=(1,\mathbf{v})$] is the four-momentum of the radiating particle, $\gamma_c=\varepsilon_c/m_c$, ε_c is the energy of the target particle in the laboratory frame, m_c is its mass, ε_r is the energy of the radiating particle in the laboratory frame; $k_\mu=(\omega,\omega\mathbf{n})$ is the photon four-momentum, and ϑ is the angle between vectors $\mathbf{n}$ and $\mathbf{v}$.

In the case when the transverse dimension of the beam, σ, is $\sigma\gg l_{f0}$, the impact parameters $\varrho\leqslant\varrho_{max}=l_{f0}$ contribute. One can set the particle density in the target beam to a constant, so that the standard QED formulas are valid. Note that the value ϱ_{max} is the relativistic invariant, which is defined by the minimal value of the square of the invariant mass of the intermediate photon $|q^2|$. In the case when the characteristic size of the beams is smaller than the value ϱ_{max}, the lower value of $|q^2|$ is defined by this size.

In the target rest frame the scattering length of the emitting particle is of the order of the impact parameter ϱ. This length is much smaller than the longitudinal dimension of the target $\gamma_c l$ (l is the length of the target beam in the laboratory frame). So one can neglect the variation of the configuration of the beam during the scattering time. Possible variation of the particle configuration in the beam during a long time can be taken into account in the adiabatic approximation.

Another limitation is connected with the influence of the value of the transverse momentum arising from the electromagnetic field $E=|\mathbf{E}|$ of the colliding (target) beam over the photon formation length. This value should be smaller than the characteristic transverse momentum transfer $m\sqrt{\zeta}$ in the photon emission process:

$$\begin{aligned}\frac{eEl_f}{m\sqrt{\zeta}}&\sim\frac{\alpha N_c}{(\sigma_z+\sigma_y)l\gamma_c}\frac{1}{m\sqrt{\zeta}}\frac{4\varepsilon'\gamma_c\varepsilon_r}{\omega\zeta m^2}\\ &\sim\frac{2\alpha N_c}{(\sigma_z+\sigma_y)l}\frac{1}{m\sqrt{\zeta}}\frac{2\varepsilon'\varepsilon_r}{\omega\zeta m^2}\\ &=\frac{4\alpha N_c\gamma_r\lambda_c^2\varepsilon'}{(\sigma_z+\sigma_y)l\zeta^{3/2}\omega}\ll 1;\end{aligned} \tag{2.2}$$

here $\alpha=1/137$, N_c is the number of particles in the target beam, and σ_z and σ_y are the vertical and horizontal transverse dimensions of the target beam. Note that the ratio γ/l is the relativistic invariant. This condition can be presented in invariant form as

$$\frac{2\chi}{u\zeta^{3/2}}\ll 1, \tag{2.3}$$

where $\chi=(\gamma/E_0)|\mathbf{E}_\perp+\mathbf{v}\times\mathbf{H}|$, $u=\omega/\varepsilon'$, $E_0=m^2/e=1.32\times10^{16}$ V/cm. Since the main contribution to the spectral probability of radiation gives angles $\vartheta\sim 1/\gamma$ ($\zeta\sim 1$) this condition takes the form $\chi/u\ll 1$. For the case $\chi/u\gg 1$ the condition (2.3) can be satisfied for large photon emission angles $\zeta\simeq\gamma^2\vartheta^2>(\chi/u)^{2/3}\gg 1$. Under these conditions the formation length $l_f=l_{f0}/\zeta$ decreases as $(\chi/u)^{2/3}$. The same inhibition factor acquires the bremsstrahlung probability [14].

Now let us consider the spectral distribution of radiation probability in the case $\chi\ll 1$ (this condition is satisfied in all existing installations), so

$$\chi\sim\alpha N_c\gamma\frac{\lambda_c^2}{(\sigma_z+\sigma_y)l}\ll 1. \tag{2.4}$$

Only the soft photons ($\omega\leqslant\chi\varepsilon\ll\varepsilon$) contribute to the *coherent* radiation (the "beamstrahlung") while the hard photon region $\omega \gg \chi\varepsilon$ is suppressed exponentially as is known from classical radiation theory. As mentioned, in the soft photon region $(\omega \leq \chi\varepsilon \ll \varepsilon)$, the spectral probability of bremsstrahlung is suppressed by the factor $(\omega/\varepsilon\chi)^{2/3}$ only. On the contrary, the spectral probability of the bremsstrahlung is negligible compared with the beamstrahlung, taking into consideration that the mean square of the multiple scattering angle during the whole time of beam collisions is small compared with the value $1/\gamma^2$:

$$\gamma^2\langle \vartheta_s^2\rangle = \frac{\langle q_s^2\rangle}{m^2} \simeq \frac{8\alpha^2 N_c \lambda_c^2}{\sigma_z\sigma_y} L \ll 1, \tag{2.5}$$

where L is the characteristic logarithm of the scattering problem (in typical experimental conditions $L\sim 10$).

It was supposed in the above estimations of beamstrahlung probability that the radiation formation length is shorter than the target beam length:

$$\frac{l_f}{l} \sim \frac{1}{u}\left(1+\frac{\chi}{u}\right)^{-2/3}\frac{\gamma\lambda_c}{l} < 1. \tag{2.6}$$

In addition, it was supposed that one can neglect the variation of the impact parameter ϱ and therefore of the transverse electric field $\mathbf{E}_\perp(\varrho)$ during the beam collision. This is true when the disruption parameter is small:

$$D_i = \frac{2\alpha N_c \lambda_c l}{\gamma_r \sigma_i(\sigma_z+\sigma_y)} \ll 1 \quad (i=z,y). \tag{2.7}$$

So we consider incoherent bremsstrahlung under the following conditions:

$$\chi \ll 1, \quad \frac{\chi}{u} \ll 1, \quad D_i \ll 1. \tag{2.8}$$

III. SPECTRAL-ANGULAR DISTRIBUTION OF THE INCOHERENT BREMSSTRAHLUNG PROBABILITY

In this section we derive the basic expression for the incoherent bremsstrahlung probability in the collision of two beams with bounded transverse dimensions.

We consider first the photon emission in the collision of an electron with one particle with the transverse coordinate $\mathbf{x}$. We select an impact parameter $\varrho_0 = |\varrho_0|$ which is small compared with the typical transverse beam dimension σ but which is large compared with the electron Compton length λ_c $(\lambda_c \ll \varrho_0 \ll \sigma)$. In the interval of the impact parameter $\varrho = |\mathbf{r}_\perp - \mathbf{x}| \geq \varrho_0$, where $\mathbf{r}_\perp$ is the transverse coordinate of the emitting electron, the probability of radiation summed over the momenta of the final particle can be calculated using the classical trajectory of the particle. Indeed, one can neglect the value of the commutators $|[\hat{p}_{\perp i}, \varrho_j]| = \delta_{ij}$ compared with the value $p_\perp \varrho$ in this interval $(p_\perp \varrho \geq m\varrho_0 \gg 1)$. In this case the expression for the probability has the form [see [13], Eqs. (7.3) and (7.4)]

$$dw = |M(\vec{\varrho})|^2 w_r(\mathbf{r}_\perp) d^2 r_\perp d^3 k, \tag{3.1}$$

where

$$M(\vec{\varrho}) = \frac{e}{2\pi\sqrt{\omega}}\int_{-\infty}^{\infty} R(t)\exp[ik'x(t)]dt,$$

$$k' = \frac{\varepsilon}{\varepsilon'}k. \tag{3.2}$$

Here $w_r(\mathbf{r}_\perp)d^2r_\perp$ is the probability of finding the emitting particle with the impact parameter $\vec{\varrho} = \mathbf{r}_\perp - \mathbf{x}$ in the interval $d^2\varrho = d^2 r_\perp$, $R(t) = R(\mathbf{p}(t))$, $kx(t) = \omega t - \mathbf{kr}(t)$ (for details see [13], Sec. 7.1). Integrating by parts in the last equation and taking into account that $|\mathbf{q}_\perp(\vec{\varrho})| \leq 1/\varrho_0 \ll m$, we find

$$M(\vec{\varrho}) \simeq \frac{ie}{2\pi\sqrt{\omega}}\int_{-\infty}^{\infty}\exp(ik'vt)\frac{d}{dt}\frac{R(t)}{k'v(t)}dt$$

$$\simeq \frac{ie}{2\pi\sqrt{\omega}}\mathbf{m}(\vec{\varrho})\frac{\partial}{\partial \mathbf{p}_\perp}\frac{R(\mathbf{p}_\perp)}{k'v}, \tag{3.3}$$

where

$$\mathbf{p}_\perp = \mathbf{p} - \mathbf{n}(\mathbf{np}) \simeq \varepsilon(\mathbf{v}-\mathbf{n}),$$

$$\mathbf{m}(\vec{\varrho}) = \int_{-\infty}^{\infty}\exp(ik'vt)\dot{\mathbf{q}}(\vec{\varrho},t)dt$$

$$= -\frac{\partial}{\partial\vec{\varrho}}\int_{-\infty}^{\infty}\exp\left(\frac{it}{l_f}\right)V(\sqrt{\varrho^2+t^2})dt$$

$$= \frac{2\alpha}{l_f}\frac{\vec{\varrho}}{\varrho}K_1\left(\frac{\varrho}{l_f}\right)$$

$$= 2\alpha q_{min}\zeta K_1(\varrho q_{min}\zeta)\frac{\vec{\varrho}}{\varrho} \tag{3.4}$$

for the Coulomb potential, $K_1(z)$ is the modified Bessel function (the Macdonald function), $R(\mathbf{p}_\perp)$ has the form of a matrix element for the free particles:

$$R(\mathbf{p}_\perp) = \varphi_{s'}^{+}(A + i\boldsymbol{\sigma}\mathbf{B})\varphi_s, \quad A \simeq \frac{m(\varepsilon+\varepsilon')}{2\varepsilon\varepsilon'}(\mathbf{e}^*\mathbf{u}),$$

$$\mathbf{B} \simeq \frac{m\omega}{2\varepsilon\varepsilon'}[\mathbf{e}^*\times(\mathbf{n}-\mathbf{u})],$$

$$\mathbf{u} = \frac{\mathbf{p}_\perp}{m}, \quad \zeta = 1+\mathbf{u}^2, \quad k'v = q_{min}\zeta, \tag{3.5}$$

where the vector $\mathbf{e}$ describes the photon polarization and the spinors φ_s and $\varphi_{s'}$ describe the polarization of the initial and final electrons, respectively.

In the interval of impact parameters $\varrho \leq \lambda_c$ the expectation value $\langle\vec{\varrho}|M^+M|\vec{\varrho}\rangle$ cannot be written in the form (3.1) since the entering operators become noncommutative inside the expectation value. However, because of the condition $\lambda_c \ll \sigma$ in this interval $w_r(\mathbf{r}_\perp) \simeq w_r(\mathbf{x}) + O(\lambda_c/\sigma)$ and one can neglect the effect of the inhomogeneous distribution. For the same reason in the calculation of the correction to the probability of photon emission, which is defined as the difference between $dw(\sigma)$ and the probability of photon emission in an inhomogeneous medium, one can extend the integration interval into the region $\varrho \leqslant \varrho_0$.

In this paper we consider incoherent bremsstrahlung, which can be treated as the photon emission due to fluctuations of the potential V connected with the uncertainty of a particle position in the plane transverse to its momentum. Because of this we have to calculate the dispersion of the vector $\mathbf{m}(\vec{\varrho})$ with respect to the transverse coordinate $\vec{\varrho}$:

$$\langle m_i m_j \rangle - \langle m_i \rangle \langle m_j \rangle = \int m_i(\mathbf{r}_\perp - \mathbf{x}) m_j(\mathbf{r}_\perp - \mathbf{x}) w_c(\mathbf{x}) d^2x - \int m_i(\mathbf{r}_\perp - \mathbf{x}) w_c(\mathbf{x}) d^2x \times \int m_j(\mathbf{r}_\perp - \mathbf{x}) w_c(\mathbf{x}) d^2x, \tag{3.6}$$

where $w_c(\mathbf{x})$ is the distribution function of the target particles normalized to unity.

Finally, we obtain the following expression for the correction to the probability of photon emission connected with the restricted transverse dimensions of colliding beams of charged particles:

$$dw_1 = \frac{\alpha}{(2\pi)^2} \frac{d^3k}{\omega} T_{ij}(\mathbf{e}, \mathbf{p}_\perp, s, s') L_{ij},$$

$$T_{ij} = \left[\frac{\partial}{\partial p_{\perp i}} \frac{R^*(\mathbf{p}_\perp)}{k'v} \right] \left[\frac{\partial}{\partial p_{\perp j}} \frac{R^*(\mathbf{p}_\perp)}{k'v} \right],$$

$$L_{ij} = \int m_i(\vec{\varrho}) m_j(\vec{\varrho}) [w_r(\mathbf{x} + \vec{\varrho}) - w_r(\mathbf{x})] \times w_c(\mathbf{x}) d^2x d^2\varrho - \left(\int m_i(\vec{\varrho}) w_c(\mathbf{x} - \vec{\varrho}) d^2\varrho \right) \times \left(\int m_j(\vec{\varrho}) w_c(\mathbf{x} - \vec{\varrho}) d^2\varrho \right) w_r(\mathbf{x}) d^2x. \tag{3.7}$$

Averaging over the polarization of the initial electrons and summing over the polarization of the final electrons, we find

$$T_{ij} = \frac{l_f}{\varepsilon\varepsilon'} \left[e_i e_j - \frac{2\mathbf{eu}}{\zeta} (e_i u_j + u_i e_j) + \frac{4(\mathbf{eu})^2}{\zeta^2} u_i u_j + \frac{\omega^2}{4\varepsilon\varepsilon'} \delta_{ij} \right]. \tag{3.8}$$

Note that one can choose the real vector $\mathbf{e}$ since the linear polarization can arise only in the case of unpolarized electrons.

After the summation in Eq. (3.8) over the polarization of the emitted photon we have

$$T_{ij} = \frac{l_f}{2\varepsilon\varepsilon'} \left(v\,\delta_{ij} - \frac{8}{\zeta^2} u_i u_j \right),$$

$$v = \frac{\varepsilon}{\varepsilon'} + \frac{\varepsilon'}{\varepsilon}, \quad \zeta = 1 + \gamma^2 \vartheta^2. \tag{3.9}$$

Finally, averaging the last expression over the azimuthal angle of the emitted photon, we obtain

$$T_{ij} = \frac{l_f}{2\varepsilon\varepsilon'} U(\zeta) \delta_{ij}, \quad U(\zeta) = v - \frac{4(\zeta - 1)}{\zeta^2}. \tag{3.10}$$

Substituting the expression obtained into Eq. (3.7), we find the correction to the probability of photon emission connected with the restricted transverse dimensions of colliding beams of charged particles:

$$dw_1 = \frac{\alpha^3}{\pi m^2} \frac{\varepsilon'}{\varepsilon} \frac{d\omega}{\omega} U(\zeta) F(\omega, \zeta) d\zeta, \tag{3.11}$$

where

$$F(\omega,\zeta) = F^{(1)}(\omega,\zeta) - F^{(2)}(\omega,\zeta),$$

$$F^{(1)}(\omega,\zeta) = \frac{2\eta^2}{\zeta^2} \int K_1^2(\eta\varrho) [w_r(\mathbf{x} + \vec{\varrho}) - w_r(\mathbf{x})] \times w_c(\mathbf{x}) d^2x d^2\varrho,$$

$$F^{(2)}(\omega,\zeta) = \frac{2\eta^2}{\zeta^2} \int \left(\int K_1(\eta\varrho) \frac{\vec{\varrho}}{\varrho} w_c(\mathbf{x} - \vec{\varrho}) d^2\varrho \right)^2 \times w_r(\mathbf{x}) d^2x; \tag{3.12}$$

here $\eta = q_{min}\zeta$.

Using the integral

$$\int K_1^2(\eta\varrho) \varrho\, d\varrho = \frac{\varrho^2}{2} [K_1^2(\eta\varrho) - K_0(\eta\varrho) K_2(\eta\varrho)] \tag{3.13}$$

and integrating by parts we obtain

$$F(\omega,\zeta) = \frac{\eta^2}{\zeta^2} \left[\int [K_0(\eta\varrho) K_2(\eta\varrho) - K_1^2(\eta\varrho)] \varrho \frac{d\Phi(\vec{\varrho})}{d\varrho} d^2\varrho - 2 \int \left(\int K_1(\eta\varrho) \frac{\vec{\varrho}}{\varrho} w_c(\mathbf{x} - \vec{\varrho}) d^2\varrho \right)^2 w_r(\mathbf{x}) d^2x \right], \tag{3.14}$$

where

$$\Phi(\vec{\varrho})=\int w_r(\mathbf{x}+\vec{\varrho})w_c(\mathbf{x})d^2x. \qquad (3.15)$$

In the general case the axes of colliding beams are separated from each other in the transverse plane by the vector $\mathbf{x}_0$ with components z_0,y_0. In this case we have to consider

$$w_r(\mathbf{x})\to w_r(\mathbf{x}+\mathbf{x}_0),$$
$$F^{(1,2)}(\omega,\zeta)\to F^{(1,2)}(\omega,\zeta,\mathbf{x}_0),$$
$$\Phi(\vec{\varrho})\to\Phi(\vec{\varrho}+\mathbf{x}_0). \qquad (3.16)$$

The first term in the expression for $F(\omega,\zeta)$ in Eq. (3.14) coincides with the function $F(\omega,\zeta)$ defined in [7], Eq. (13). The second (subtraction) term in Eq. (3.14) which naturally arises in this derivation was missed in Eq. (13) of [7] as it was said above. The expression (3.11) is consistent with Eq. (21.6) in the book [13] [see also Eq. (2.2) in [11]] where another physical problem was analyzed. It is the incoherent bremsstrahlung in oriented crystals.

Below we restrict ourselves to the case of unpolarized electrons and photons. The influence of bounded transverse size on the probability of a process with polarized particles will be considered elsewhere.

IV. GAUSSIAN BEAMS

For calculation of the explicit expression for the bremsstrahlung cross section we have to specify the distributions of particles in the colliding beams. Here we consider the actual case of Gaussian beams. Using the Fourier transform we have

$$w(\mathbf{x})=\frac{1}{(2\pi)^2}\int d^2q\,\exp(-i\mathbf{q}\mathbf{x})w(\mathbf{q}),$$
$$w_r(\mathbf{q})=\exp\left[-\frac{1}{2}(q_z^2\Delta_z^2+q_y^2\Delta_y^2)\right],$$
$$w_c(\mathbf{q})=\exp\left[-\frac{1}{2}(q_z^2\sigma_z^2+q_y^2\sigma_y^2)\right], \qquad (4.1)$$

where as above the index r relates to the radiating beam and the index c relates to the target beam, and Δ_z and Δ_y (σ_z and σ_y) are the vertical and horizontal transverse dimensions of the radiating (target) beam. Substituting Eq. (4.1) into Eq. (3.15) we find

$$\Phi(\vec{\varrho})=\frac{1}{(2\pi)^2}\int d^2q\exp(-i\mathbf{q}\vec{\varrho})\exp\left[-\frac{q_z^2}{4\Sigma_z^2}-\frac{q_y^2}{4\Sigma_y^2}\right]$$
$$=\frac{\Sigma_z\Sigma_y}{\pi}\exp[-\varrho_z^2\Sigma_z^2-\varrho_y^2\Sigma_y^2],$$
$$\Sigma_z^2=\frac{1}{2(\sigma_z^2+\Delta_z^2)},\quad \Sigma_y^2=\frac{1}{2(\sigma_y^2+\Delta_y^2)}. \qquad (4.2)$$

Below we consider the general situation when the axes of colliding beams are separated from each other in the transverse plane by the vector $\mathbf{x}_0$ with components z_0,y_0. This separation has an essential influence on the luminosity. For processes where only short distances are essential (e.g., double bremsstrahlung [2]) the probability of the process is the product of the cross section and the luminosity. The geometrical luminosity per bunch, not taking into account the disruption effects, is given by

$$\mathcal{L}=N_cN_r\Phi(\mathbf{x}_0), \qquad (4.3)$$

where as above N_r and N_c are the number of particles in the radiating and target beams, respectively. We will use the same definition for our case. Then we have

$$dw_\gamma=\Phi(\mathbf{x}_0)d\sigma_\gamma,\quad d\sigma_1=\Phi^{-1}(\mathbf{x}_0)dw_1, \qquad (4.4)$$

where dw_1 is defined in Eq. (3.11).

We calculate first the function $F^{(1)}(\omega,\zeta)$ in Eq. (3.12) for the case of coaxial beams when $\mathbf{x}_0=\mathbf{0}$. Passing on to the momentum representation with the help of the formula (4.1) we find

$$F^{(1)}(\omega,\zeta)=-\frac{1}{2\pi\zeta^2}\int w_r(\mathbf{q})w_c(\mathbf{q})F_2\left(\frac{q}{2\eta}\right)q\,dq\,d\varphi, \qquad (4.5)$$

where $\eta=q_{min}\zeta$ is introduced in Eq. (3.12),

$$F_2\left(\frac{q}{2\eta}\right)=\frac{\eta^2}{\pi}\int K_1^2(\eta\varrho)[1-\exp(-i\mathbf{q}\vec{\varrho})]d^2\varrho,$$
$$F_2(x)=\frac{2x^2+1}{x\sqrt{1+x^2}}\ln(x+\sqrt{1+x^2})-1,$$
$$q_{min}=m^3\omega/4\varepsilon^2\varepsilon'; \qquad (4.6)$$

here the value q_{min} is defined in the c.m. frame of the colliding particles. The function $F_2(x)$ is encountered in radiation theory. To calculate the corresponding contribution in the radiation spectrum we have to substitute Eq. (4.5) into Eq. (3.11) and take the integrals. After substitution of variables in Eq. (4.5),

$$w=\frac{q}{2q_{min}\zeta}, \qquad (4.7)$$

we obtain the integral over ζ in Eq. (3.11):

$$\int_1^\infty\left(v-\frac{4}{\zeta}+\frac{4}{\zeta^2}\right)\exp(-s^2\zeta^2)d\zeta$$
$$\equiv f(s)=\frac{\sqrt{\pi}}{2s}(v-8s^2)\mathrm{Erfc}(s)+4e^{-s^2}+2\mathrm{Ei}(-s^2), \qquad (4.8)$$

where

$$s=wrq_{min}, \quad r^2=\Sigma_z^{-2}\cos^2\varphi+\Sigma_y^{-2}\sin^2\varphi. \tag{4.9}$$

Making use of Eq. (4.4) we find for the spectrum

$$d\sigma_1^{(1)}=\frac{2\alpha^3}{m^2}\frac{\varepsilon'}{\varepsilon}\frac{d\omega}{\omega}f^{(1)}(\omega),$$

$$f^{(1)}(\omega)=-\frac{1}{\pi\Sigma_z\Sigma_y}\int_0^{2\pi}\frac{d\varphi}{\Sigma_z^{-2}\cos^2\varphi+\Sigma_y^{-2}\sin^2\varphi}\times\int_0^\infty F_2(z)f(s)s\,ds,$$

$$z^2=\frac{s^2}{q_{min}^2}\frac{1}{\Sigma_z^{-2}\cos^2\varphi+\Sigma_y^{-2}\sin^2\varphi}. \tag{4.10}$$

This formula is quite convenient for numerical calculations.

In the case $\mathbf{x}_0\neq 0$ we will use Eqs. (3.12) and (4.4) straightforwardly. Taking into account Eq. (4.2) we have for the difference

$$\Delta^{(1)}(\mathbf{x}_0)\equiv\frac{1}{2\pi}[\Phi^{-1}(\mathbf{x}_0)F^{(1)}(\mathbf{x}_0)-\Phi^{-1}(0)F^{(1)}(0)]$$
$$=\frac{\eta^2}{\pi\zeta^2}\int K_1^2(\eta\varrho)\exp[-\varrho_z^2\Sigma_z^2-\varrho_y^2\Sigma_y^2]\times\{\exp[-2\varrho_z z_0\Sigma_z^2-2\varrho_y y_0\Sigma_y^2]-1\}d^2\varrho, \tag{4.11}$$

where the function $F^{(1)}(\mathbf{x}_0)$ is defined in Eqs. (3.12), (3.16).

Using Macdonald's formula (see, e.g., [15], p. 53)

$$2K_1^2(\eta\varrho)=\int_0^\infty\exp\left[-\varrho^2t-\frac{\eta^2}{2t}\right]K_1\left(\frac{\eta^2}{2t}\right)\frac{dt}{t} \tag{4.12}$$

and taking the Gaussian integrals over ϱ_z and ϱ_y we get

$$\Delta^{(1)}(\mathbf{x}_0)=\frac{1}{\zeta^2}\int_0^\infty\frac{\exp(-\eta^2/2t)K_1(\eta^2/2t)}{\sqrt{t+\Sigma_z^2}\sqrt{t+\Sigma_y^2}}\left\{\exp\left[\frac{z_0^2\Sigma_z^4}{t+\Sigma_z^2}+\frac{y_0^2\Sigma_y^4}{t+\Sigma_y^2}\right]-1\right\}\frac{\eta^2dt}{2t}. \tag{4.13}$$

For the correction to the cross section [see Eqs. (4.4) and (4.10)] we have, correspondingly,

$$d\sigma_1^{(1)}=\frac{2\alpha^3}{m^2}\frac{\varepsilon'}{\varepsilon}\frac{d\omega}{\omega}[f^{(1)}(\omega)+J^{(1)}(\omega,\mathbf{x}_0)], \tag{4.14}$$

where

$$J^{(1)}(\omega,\mathbf{x}_0)=\int_1^\infty U(\zeta)\Delta^{(1)}(\mathbf{x}_0)d\zeta. \tag{4.15}$$

Now we pass over to the calculation of the second (subtraction) term $F^{(2)}(\omega,\zeta)$ in Eq. (3.12). Using Eq. (4.1) we get

$$\mathbf{I}=\eta\int K_1(\eta\varrho)\frac{\vec{\varrho}}{\varrho}w_c(\mathbf{x}-\vec{\varrho})d^2\varrho$$
$$=\frac{\eta}{(2\pi)^2}\int S(q)\frac{\mathbf{q}}{q}\exp(-i\mathbf{qx})w_c(\mathbf{q})d^2q, \tag{4.16}$$

where

$$S(q)=\int K_1(\eta\varrho)\frac{\mathbf{q}\vec{\varrho}}{q\varrho}\exp(i\mathbf{q}\vec{\varrho})d^2\varrho$$
$$=2\pi i\int K_1(\eta\varrho)J_1(q\varrho)\varrho\,d\varrho$$
$$=2\pi i\frac{q}{\eta}\frac{1}{q^2+\eta^2}. \tag{4.17}$$

Using the exponential parametrization

$$\frac{1}{q^2+\eta^2}=\frac{1}{4}\int_0^\infty\exp\left[-\frac{s}{4}(q^2+\eta^2)\right]ds \tag{4.18}$$

and taking the Gaussian integrals over q_z and q_y we obtain

$$\mathbf{I}=\int_0^\infty\exp\left[-\frac{\eta^2s}{4}-\frac{z^2}{s+2\sigma_z^2}-\frac{y^2}{s+2\sigma_y^2}\right]\left[\frac{z\mathbf{e}_z}{s+2\sigma_z^2}+\frac{y\mathbf{e}_y}{s+2\sigma_y^2}\right]\frac{ds}{\sqrt{s+2\sigma_z^2}\sqrt{s+2\sigma_y^2}}, \tag{4.19}$$

where $\mathbf{e}_z$ and $\mathbf{e}_y$ are unit vectors along the axes z and y. Substituting Eq. (4.19) into Eq. (3.12), taking the Gaussian integrals over z and y, and using Eq. (4.4) we get the correction to the cross section

$$d\sigma_1^{(2)}=-\frac{2\alpha^3}{m^2}\frac{\varepsilon'}{\varepsilon}\frac{d\omega}{\omega}J^{(2)}(\omega,\mathbf{x}_0), \tag{4.20}$$

where

$$J^{(2)}(\omega,\mathbf{x}_0)=\frac{\sqrt{ab}}{\Sigma_z\Sigma_y}\exp(z_0^2\Sigma_z^2+y_0^2\Sigma_y^2)\times\int_0^\infty ds_1\int_0^\infty ds_2 g\left(\frac{q_{min}\sqrt{s}}{2}\right)G(s_1,s_2,\mathbf{x}_0),$$

$$G(s_1,s_2,\mathbf{x}_0)=\left(\frac{a_1a_2b_1b_2}{AB}\right)^{1/2}\left[\frac{a_1a_2}{A}\left(\frac{1}{2}+\frac{z_0^2a^2}{A}\right)+\frac{b_1b_2}{B}\left(\frac{1}{2}+\frac{y_0^2b^2}{B}\right)\right]\exp\left[-\frac{z_0^2a}{A}(a_1+a_2)-\frac{y_0^2b}{B}(b_1+b_2)\right]. \tag{4.21}$$

Here the function g appears as a result of integration over ζ:

$$g(q)=\int_1^\infty\left(v-\frac{4}{\zeta}+\frac{4}{\zeta^2}\right)\exp(-q^2\zeta^2)\frac{d\zeta}{\zeta^2}$$
$$=\left(v-\frac{2}{3}\right)\exp(-q^2)-2q^2\int_1^\infty\left(v-\frac{2}{\zeta}+\frac{4}{3\zeta^2}\right)$$
$$\times\exp(-q^2\zeta^2)d\zeta$$
$$=\left(v-\frac{2}{3}\right)\exp(-q^2)-2q^2\left[\frac{\sqrt{\pi}}{2q}\left(v-\frac{8}{3}q^2\right)\right.$$
$$\left.\times\mathrm{Erfc}(q)+\frac{4}{3}e^{-q^2}+\mathrm{Ei}(-q^2)\right]. \tag{4.22}$$

In Eq. (4.21) we introduced the following notation:

$$a=\frac{1}{2\Delta_z^2},\quad b=\frac{1}{2\Delta_y^2},$$
$$a_{1,2}=\frac{1}{s_{1,2}+2\sigma_z^2},\quad b_{1,2}=\frac{1}{s_{1,2}+2\sigma_y^2},$$
$$A=a_1+a_2+a,\quad B=b_1+b_2+b,$$
$$s=s_1+s_2. \tag{4.23}$$

V. NARROW BEAMS

This is the case when the ratio $q_{min}/(\Sigma_z+\Sigma_y)\ll 1$, so that the main contribution to the integral (4.10) gives the region $s\sim\zeta\sim 1$, $z\gg 1$. Using the asymptotics of the function $F_2(z)$ at $z\gg 1$

$$F_2(z)\simeq\ln(2z)^2-1 \tag{5.1}$$

and the following integrals:

$$\frac{1}{2\pi\Sigma_z\Sigma_y}\int_0^{2\pi}\frac{d\varphi}{\Sigma_z^{-2}\cos^2\varphi+\Sigma_y^{-2}\sin^2\varphi}=1,$$
$$\frac{1}{2\pi\Sigma_z\Sigma_y}\int_0^{2\pi}\frac{d\varphi}{\Sigma_z^{-2}\cos^2\varphi+\Sigma_y^{-2}\sin^2\varphi}$$
$$\times\ln\frac{4}{\Sigma_z^{-2}\cos^2\varphi+\Sigma_y^{-2}\sin^2\varphi}=\ln(\Sigma_z+\Sigma_y)^2,$$
$$\int_1^\infty ds^2(\alpha-\beta\ln s^2)$$
$$\times\int_1^\infty\left(v-\frac{4}{\zeta}+\frac{4}{\zeta^2}\right)\exp(-s^2\zeta^2)d\zeta$$
$$=\left(v-\frac{2}{3}\right)[\alpha+\beta(2+C)]+\frac{2}{9}\beta, \tag{5.2}$$

where C is Euler's constant $C=0.577\ldots$, we get for the function $f^{(1)}(\omega)$ [Eq. (4.14)] the following expression:

$$f^{(1)}(\omega)\simeq\left(v-\frac{2}{3}\right)\left(2\ln\frac{q_{min}}{\Sigma_z+\Sigma_y}+3+C\right)+\frac{2}{9},$$
$$q_{min}\ll(\Sigma_z+\Sigma_y). \tag{5.3}$$

This expression agrees with Eq. (24) of [7].

Under the assumption used in Eq. (5.3) and the additional condition $q_{min}(z_0+y_0)\ll 1$ the main contribution to the integral in Eq. (4.13) gives the region $t\gg\eta^2$. In this case one can use the asymptotic expansion $K_1(z)\simeq 1/z$ $(z\ll 1)$. Then we have for the function $J^{(1)}(\omega,\mathbf{x}_0)$ in Eq. (4.14) the following expression:

$$J^{(1)}(\omega,\mathbf{x}_0)\simeq\left(v-\frac{2}{3}\right)J,$$
$$J=\int_0^\infty\left[\exp\left(\frac{z_0^2\Sigma_z^4}{t+\Sigma_z^2}+\frac{y_0^2\Sigma_y^4}{t+\Sigma_y^2}\right)-1\right]$$
$$\times\frac{dt}{\sqrt{t+\Sigma_z^2}\sqrt{t+\Sigma_y^2}}. \tag{5.4}$$

The expression (5.4) is consistent with Eq. (26) of [7].

In the case $(\mathbf{x}_0^2+\sigma_z^2+\sigma_y^2)q_{min}^2\ll 1$ the main contribution to the integral in Eq. (4.21) gives the interval $sq_{min}^2\sim(\mathbf{x}_0^2+\sigma_z^2+\sigma_y^2)q_{min}^2\ll 1$. Keeping the main term of the expansion over q^2 in Eq. (4.22) we get

$$g\left(\frac{q_{min}\sqrt{s}}{2}\right)\simeq v-\frac{2}{3}. \tag{5.5}$$

The same result can be obtained if one neglects the term containing η^2 in the exponent of the integrand in Eq. (4.19).

Summing the cross section $d\sigma=d\sigma_1^{(1)}+d\sigma_1^{(2)}$ with the standard QED bremsstrahlung cross section

$$d\sigma_0=\frac{2\alpha^3}{m^2}\frac{\varepsilon'}{\varepsilon}\frac{d\omega}{\omega}\left(v-\frac{2}{3}\right)\left(\ln\frac{m^2}{q_{min}^2}-1\right), \tag{5.6}$$

we get the cross section for the case of interaction of narrow beams:

$$d\sigma_\gamma=d\sigma_0+d\sigma_1=\frac{2\alpha^3}{m^2}\frac{\varepsilon'}{\varepsilon}\frac{d\omega}{\omega}\left\{\left(v-\frac{2}{3}\right)\right.$$
$$\left.\times\left[2\ln\frac{m}{\Sigma_z+\Sigma_y}+C+2+J-J_-\right]+\frac{2}{9}\right\},$$
$$v=\frac{\varepsilon}{\varepsilon'}+\frac{\varepsilon'}{\varepsilon},\quad\varepsilon'=\varepsilon-\omega, \tag{5.7}$$

where J is given in Eq. (5.4),

$$J_-=\frac{\sqrt{ab}}{\Sigma_z\Sigma_y}\exp(z_0^2\Sigma_z^2+y_0^2\Sigma_y^2)\int_0^\infty ds_1\int_0^\infty ds_2 G(s_1,s_2,\mathbf{x}_0), \tag{5.8}$$

where the entering functions are defined in Eqs. (4.21) and (4.23).

In the case of coaxial beams $\mathbf{x}_0=\mathbf{0}$, $J=0$ one can take the integral in Eq. (5.8) over one of the variables (for definiteness over s_2) using the formula

$$\int_0^\infty \frac{dx}{(a_z+b_zx)^{3/2}(a_y+b_yx)^{1/2}} = \frac{2}{a_z\sqrt{b_zb_y}+b_z\sqrt{a_za_y}}. \tag{5.9}$$

After this we have the simple integral over $s\equiv s_1$

$$J_-(0)=\sqrt{1+\delta_z}\sqrt{1+\delta_y}(J_z+J_y),$$

$$J_{z,y}=\int_0^\infty D_{z,y}(s)ds,$$

$$D_{z,y}=\frac{1}{a_{z,y}\sqrt{b_zb_y}+b_{z,y}\sqrt{a_za_y}}, \tag{5.10}$$

where

$$a_{z,y}=s(1+\delta_{z,y})+2\sigma_{z,y}^2(2+\delta_{z,y}),$$

$$b_{z,y}=\frac{s}{2\Delta_{z,y}^2}+1+\delta_{z,y}, \quad \delta_{z,y}=\frac{\sigma_{z,y}^2}{\Delta_{z,y}^2}. \tag{5.11}$$

The cross section (5.7) differs from Eq. (24) of [7] because the subtraction term J_- is included. Without this term, generally speaking, the bremsstrahlung cross section would be qualitatively erroneous. In particular, the appearance of the term J_- violates, generally speaking, the symmetry of the radiation cross section in opposite directions in e^-e^- (e^-e^+) collisions.

To elucidate the qualitative features of narrow beam bremsstrahlung processes we consider the case of round beams where the calculation becomes simpler:

$$\sigma_z=\sigma_y=\sigma, \quad \Delta_z=\Delta_y=\Delta,$$

$$\Sigma_z^2=\Sigma_y^2=\Sigma^2=\frac{1}{2(\sigma^2+\Delta^2)},$$

$$b=a, \quad b_{1,2}=a_{1,2}, \quad B=A, \quad \delta=\frac{\sigma^2}{\Delta^2}. \tag{5.12}$$

We consider first the case of coaxial beams ($\mathbf{x}_0=0,\ J=0$),

$$J_-=(1+\delta)\int_0^\infty \frac{ds}{[s(1+\delta)+2+\delta][s\delta+1+\delta]} =(1+\delta)\ln\frac{(1+\delta)^2}{\delta(2+\delta)}. \tag{5.13}$$

In the limiting cases the function J_- has the form

$$J_-(\delta\gg1)\simeq\frac{1}{\delta}, \quad J_-(\delta=1)=2\ln\frac{4}{3},$$

$$J_-(\delta\ll1)\simeq\ln\frac{1}{2\delta}. \tag{5.14}$$

In the first case the subtraction term J_- is small. For beams of the same size the subtraction term J_- contributes to the constant entering into the expression for the cross section. The subtraction term J_- essentially modifies the cross section in the case when the radius of the target beam is much smaller than the radius of the radiating beam. In this case the cross section (5.7) contains the combination

$$\ln\frac{m^2}{4\Sigma^2}-J_-\simeq\ln\frac{m^2\Delta^2}{2}-\ln\frac{\Delta^2}{2\sigma^2}=\ln(m\sigma)^2. \tag{5.15}$$

So in all the cases considered above the cross section defines the transverse dimension of the target beam.

When the axes of round beams are separated with respect to each other in the transverse plane the integral in Eq. (5.4) is

$$J=\int_0^\infty\left[\exp\left(\frac{d}{x+1}\right)-1\right]\frac{dx}{x+1} =\mathrm{Ei}(d)-C-\ln d,$$

$$d=\mathbf{x}_0^2\Sigma^2=\frac{x_0^2+y_0^2}{2(\Delta^2+\sigma^2)}. \tag{5.16}$$

It is convenient in this case to calculate the function J_- using straightforwardly Eq. (4.19) where we omit the term with η^2 in the exponent of the integrand:

$$\mathbf{I}_{cr}=\vec{\varrho}\int_0^\infty\exp\left(-\frac{\vec{\varrho}^2}{s+2\sigma^2}\right)\frac{ds}{(s+2\sigma^2)^2} =\frac{\vec{\varrho}}{\varrho^2}\left[1-\exp\left(-\frac{\vec{\varrho}^2}{2\sigma^2}\right)\right]. \tag{5.17}$$

Substituting this expression [$\mathbf{I}$ is defined in Eq. (4.16)] into the subtraction term Eq. (3.12) and using the exponential parametrization

$$\frac{1}{\vec{\varrho}^2}=\int_0^\infty\exp(-\vec{\varrho}^2s)ds,$$

we obtain

$$J_-=\frac{ae^d}{\pi\Sigma^2}\int_0^\infty ds\int d^2\varrho\,\exp(-\vec{\varrho}^{\,2}s)\times\left[1-\exp\left(-\frac{\vec{\varrho}^{\,2}}{2\sigma^2}\right)\right]\exp[-a(\vec{\varrho}+\mathbf{x}_0)^2]$$
$$=\frac{ae^{d-d_1}}{\Sigma^2}\int_0^\infty\left[\frac{1}{s+a}\exp\left(\frac{d_1a}{s+a}\right)-2\frac{1}{s+a+\sigma^{-2}/2}\exp\left(d_1\frac{a}{s+a+\sigma^{-2}/2}\right)+\frac{1}{s+a+\sigma^{-2}}\exp\left(d_1\frac{a}{s+a+\sigma^{-2}}\right)\right]ds$$
$$=\frac{ae^{d-d_1}}{\Sigma^2}\left[\mathrm{Ei}(d_1)-2\,\mathrm{Ei}\left(d_1\frac{\sigma^2}{\sigma^2+\Delta^2}\right)+\mathrm{Ei}\left(d_1\frac{\sigma^2}{\sigma^2+2\Delta^2}\right)\right],\quad d_1=a\mathbf{x}_0^2=\frac{z_0^2+y_0^2}{2\Delta^2}.\tag{5.18}$$

In the limit $d_1\to 0$ the last expression goes over to Eq. (5.13).

When the separation of the axes of the colliding beams is large enough ($\mathbf{x}_0^2\gg\sigma^2+\Delta^2$) one can use the asymptotic expansion of the function $\mathrm{Ei}(z)$ in Eq. (5.18):

$$\mathrm{Ei}(z)\simeq\frac{e^z}{z}\left(1+\frac{1}{z}\right),\quad z\gg 1.\tag{5.19}$$

In this case the main terms in the difference $J-J_-$ in Eq. (5.7) are canceled:

$$J-J_-\simeq\frac{e^d}{d}\left(\frac{1}{d}-\frac{1}{d_1}\right)=\frac{2e^d}{d}\frac{\sigma^2}{\mathbf{x}_0^2}.\tag{5.20}$$

The compensation of the main terms in Eq. (5.19) is due to the fact that the incoherent scattering originates from the fluctuations of the target (scattering) beam potential. Correspondingly we have for the mean square of the momentum transfer dispersion at large distance from the target beam

$$\langle\mathbf{q}^2(\vec{\varrho})\rangle-\langle\mathbf{q}(\vec{\varrho})\rangle^2\propto\left\langle\frac{1}{(\mathbf{x}_0+\vec{\varrho})^2}-\frac{1}{\mathbf{x}_0^2}\right\rangle\simeq\left\langle\frac{4(\mathbf{x}_0\vec{\varrho})^2}{\mathbf{x}_0^6}-\frac{\vec{\varrho}^{\,2}}{\mathbf{x}_0^4}\right\rangle=\frac{\langle\vec{\varrho}^{\,2}\rangle}{\mathbf{x}_0^4}=\frac{2\sigma^2}{\mathbf{x}_0^4}.\tag{5.21}$$

Substituting Eq. (5.20) into Eq. (5.7) and multiplying the result by the luminosity (4.3)

$$\mathcal{L}=N_cN_r\frac{\Sigma^2}{\pi}\exp(-\mathbf{x}_0^2\Sigma^2),\tag{5.22}$$

we have for the probability of bremsstrahlung of round beams moving apart at a large distance

$$dw_\gamma\simeq 4N_cN_r\frac{\alpha^3}{\pi}\lambda_c^2\Sigma^2\frac{\varepsilon'}{\varepsilon}\frac{d\omega}{\omega}\left(v-\frac{2}{3}\right)\left[\exp(-\mathbf{x}_0^2\Sigma^2)\ln\frac{m}{\Sigma}+\frac{\sigma^2\Sigma^2}{(\mathbf{x}_0^2\Sigma^2)^2}+O[\exp(-\mathbf{x}_0^2\Sigma^2)]\right],$$
$$\Sigma^2=\frac{1}{2(\Delta^2+\sigma^2)},\quad \mathbf{x}_0^2\Sigma^2=\frac{z_0^2+y_0^2}{2(\Delta^2+\sigma^2)}\gg 1,$$
$$q_{min}^2(z_0^2+y_0^2)\ll 1.\tag{5.23}$$

According to Eq. (5.23) when $\mathbf{x}_0^2$ increases so that one can neglect the first term in square brackets, the probability of bremsstrahlung of the round beams diminishes as a power of the distance between the beams ($\propto\sigma^2/\mathbf{x}_0^4$). The cross section Eq. (5.7) in this case grows exponentially as e^d/d^2. Let us note that without the subtraction term one has erroneous qualitative behavior of the probability ($\propto 1/\mathbf{x}_0^2$). These circumstances also explain Eq. (5.15) for the coaxial beams: on integration over $d^2\varrho$ the region contributes where $\langle\mathbf{q}^2(\varrho)\rangle-\langle\mathbf{q}(\varrho)\rangle^2\propto 1/\varrho^2$, so that $\varrho\leqslant\sigma$.

Now let us consider the general case $\Sigma_z\neq\Sigma_y$ for enough large separation of beams $\mathbf{x}_0^2\gg\Sigma_{z,y}^{-2}$. In this case the main contribution to the integral $\mathbf{I}(\mathbf{x})$ (for $\eta^2=0$) in Eqs. (4.16),(4.19) at large $|\mathbf{x}|\simeq|\mathbf{x}_0|$ [see Eq. (3.12)] are given by large values $s\sim\mathbf{x}_0^2\gg\sigma_{z,y}^2$. Expanding the integrand over the powers $\sigma_{z,y}^2/s$ and keeping after integration the two main terms of the decomposition over $1/\mathbf{x}^2$ we get

$$\mathbf{I}^2(\mathbf{x})\simeq\frac{1}{\mathbf{x}^2}\left[1+\frac{2}{(\mathbf{x}^2)^2}(y^2-z^2)(\sigma_y^2-\sigma_z^2)\right].\tag{5.24}$$

Expanding the function $1/(\mathbf{x}_0+\boldsymbol{\xi})^2$ over the powers ξ/x_0 during integration over $\boldsymbol{\xi}=\mathbf{x}-\mathbf{x}_0$ in Eq. (3.12) we find

$$\int\mathbf{I}^2(\mathbf{x}_0+\boldsymbol{\xi})w_r(\boldsymbol{\xi})d^2\xi\simeq\frac{1}{\mathbf{x}_0^2}\left[1+\frac{4}{(\mathbf{x}_0^2)^2}(z_0^2\Delta_z^2+y_0^2\Delta_y^2)-\frac{\boldsymbol{\Delta}^2}{\mathbf{x}_0^2}+\frac{2}{(\mathbf{x}_0^2)^2}(y_0^2-z_0^2)\times(\sigma_y^2-\sigma_z^2)\right],$$
$$\boldsymbol{\Delta}^2=\Delta_z^2+\Delta_y^2.\tag{5.25}$$

In this case the region $t\sim 1/\mathbf{x}_0^2\ll\Sigma_{z,y}^2$ contributes to the integral J Eq. (5.4). Expanding the integrand over the powers $t\Sigma_{z,y}^{-2}$ and keeping the two main terms of decomposition over $1/\mathbf{x}_0^2$ we have

$$J \simeq \frac{1}{\Sigma_z \Sigma_y \mathbf{x}_0^2} \exp(z_0^2 \Sigma_z^2 + y_0^2 \Sigma_y^2) \left\{ 1 - \frac{\boldsymbol{\sigma}^2 + \boldsymbol{\Delta}^2}{\mathbf{x}_0^2} + \frac{4}{(\mathbf{x}_0^2)^2} \right.$$
$$\left. \times [z_0^2(\sigma_z^2 + \Delta_z^2) + y_0^2(\sigma_y^2 + \Delta_y^2)] \right\},$$
$$\boldsymbol{\sigma}^2 = \sigma_z^2 + \sigma_y^2. \tag{5.26}$$

For the difference $J - J_-$ we obtain finally

$$J - J_- = \frac{1}{\Sigma_z \Sigma_y} \exp(z_0^2 \Sigma_z^2 + y_0^2 \Sigma_y^2) \frac{\boldsymbol{\sigma}^2}{(\mathbf{x}_0^2)^2}. \tag{5.27}$$

VI. NARROW FLAT BEAMS ($\boldsymbol{\sigma_z \ll \sigma_y, \Delta_z \ll \Delta_y}$)

Let us begin with coaxial beams. We consider first the case where the size of radiating beam is much larger than the size of the target beam ($\delta_{z,y} \ll 1$). In this case one can neglect the terms proportional to $\delta_{z,y}$, σ_z^2, Δ_y^{-2} in the functions $a_{z,y}$ and $b_{z,y}$ in the integral in Eq. (5.10). Within this accuracy

$$a_z \simeq s, \quad a_y \simeq s + 4\sigma_y^2, \quad b_z \simeq \frac{s}{2\Delta_z^2} + 1, \quad b_y \simeq 1. \tag{6.1}$$

After substitution in the integral J_y in Eq. (5.10) $s \to 4\sigma_y^2 s$ one gets

$$J_y(\kappa) = \int_0^\infty \frac{ds}{\sqrt{s+1}(\sqrt{s} + \sqrt{s+1}\sqrt{1+2\kappa s})},$$
$$\kappa = \frac{\sigma_y^2}{\Delta_z^2}. \tag{6.2}$$

After substitution in the integral J_z in Eq. (5.10) $s \to 2\Delta_z^2/s$ one gets $J_z = J_y$ so that

$$J_-(\kappa) = 2\sqrt{1+\delta_z}\sqrt{1+\delta_y} J_y(\kappa) \simeq 2 J_y(\kappa),$$
$$J_-(\kappa \ll 1) \simeq \ln\frac{8}{\kappa}, \quad J_-(\kappa \gg 1) \simeq \pi\sqrt{\frac{2}{\kappa}}. \tag{6.3}$$

It is seen from the last equation that at $\Delta_z \ll \sigma_y$ the contribution of the term J_- to the cross section Eq. (5.7) is relatively small. In the opposite case $\Delta_z \gg \sigma_y$ this contribution leads to a change of the logarithm argument in Eq. (5.7):

$$2\ln\frac{m}{(\Sigma_z + \Sigma_y)} - \ln\frac{8}{\kappa} \simeq 2\left[\ln(\sqrt{2} m \Delta_z) - \ln\left(2\sqrt{2}\frac{\Delta_z}{\sigma_y}\right)\right]$$
$$= 2\ln\frac{m\sigma_y}{2}. \tag{6.4}$$

This is a new qualitative result.

In the opposite case when the size of the radiating beam is smaller than or of the order of the size of the target beam ($\delta_{z,y} \gg 1$) the contribution to the integral J_z in Eq. (5.10) gives the region $s \sim \sigma_z^2$ and to the integral J_y the region $s \sim \sigma_y^2$. Performing in the integral J_z the substitution $s \to 2\sigma_z^2 s$ and in the integral J_y the substitution $s \to 2\sigma_y^2/s$, one gets

$$J_z \simeq \frac{\sigma_z}{\sqrt{2+\delta_y}\sigma_y} \int_0^\infty \frac{ds}{[(s+1)\delta_z + 1]\sqrt{s(1+\delta_z) + 2 + \delta_z}}$$
$$= \frac{2}{\sqrt{2+\delta_y}} \frac{\Delta_z}{\sigma_y} \arctan\frac{1}{\sqrt{\delta_z(2+\delta_z)}},$$
$$J_y \simeq \frac{\Delta_z}{\sigma_y} \int_0^\infty \frac{ds}{[(s+1)(\delta_y + 1) + s]\sqrt{(s+1)\delta_y + s}}$$
$$= \frac{2}{\sqrt{2+\delta_y}} \frac{\Delta_z}{\sigma_y} \arctan\frac{1}{\sqrt{\delta_y(2+\delta_y)}},$$
$$J_- = \sqrt{1+\delta_z}\sqrt{1+\delta_y}(J_z + J_y)$$
$$= \frac{2\sqrt{1+\delta_z}\sqrt{1+\delta_y}}{\sqrt{2+\delta_y}} \frac{\Delta_z}{\sigma_y} \left(\arctan\frac{1}{\sqrt{\delta_z(2+\delta_z)}} \right.$$
$$\left. + \arctan\frac{1}{\sqrt{\delta_y(2+\delta_y)}} \right). \tag{6.5}$$

In the case $\delta_{z,y} \ll 1$, $\Delta_z \ll \sigma_y$ this formula is consistent with Eq. (6.3).

Now we go over to the case of separated beams. For large enough separation of the beams the formulas (5.7) and (5.27) are valid. So the intermediate case is of interest. As an example we consider the case $\sigma_y^2 \gg z_0^2 \gg \sigma_z^2 + \Delta_z^2$, $y_0^2 \ll \sigma_y^2$. In this case the contribution to the integral in Eq. (5.4) gives the interval $\Sigma_y^2 \ll t \sim z_0^{-2} \ll \Sigma_z^2$. Keeping the main terms of decomposition over $t\Sigma_z^{-2} \ll 1$ and $t\Sigma_y^{-2} \gg 1$ we have

$$J \simeq \frac{1}{\Sigma_z} \int_0^\infty \exp(z_0^2 \Sigma_z^2 - z_0^2 t) \frac{dt}{\sqrt{t}}$$
$$= \frac{\sqrt{\pi}}{z_0 \Sigma_z} \exp(z_0^2 \Sigma_z^2). \tag{6.6}$$

Under these conditions ($\mathbf{x}_0^2 \ll \sigma_y^2$) the contribution to the integral for J_- in Eq. (5.8) of the term in the function $G(s_1, s_2, \mathbf{x}_0)$ of Eq. (4.21) containing $b_1 b_2/B$ in the square brackets is defined by the function J_y in Eq. (6.5) to within the terms $\sim z_0/\sigma_y$. In the term containing $a_1 a_2/A$ (which we denote by $J_-^{(z)}$) the main contribution gives the summand $z_0^2 a^2/A^2$ in the interval $\sigma_y^2 \gg s_{1,2} \sim z_0^2 \gg \sigma_z^2$ where

$$a_{1,2} \simeq \frac{1}{s_{1,2}}, \quad b_{1,2} \simeq \frac{1}{2\sigma_y^2},$$
$$A \simeq a, \quad B \simeq \frac{1}{\sigma_y^2} + \frac{1}{2\Delta_y^2}. \tag{6.7}$$

As a result we obtain

$$J_{-}^{(z)}(\mathbf{x}_0)\simeq\frac{z_0^2}{2\Sigma_z\Sigma_y\sigma_y^2}\sqrt{\frac{b}{B}}e^{d_z}\times\int_0^\infty\frac{ds_1}{s_1^{3/2}}\int_0^\infty\exp\left[-z_0^2\left(\frac{1}{s_1}+\frac{1}{s_2}\right)\right]\frac{ds_2}{s_2^{3/2}}$$

$$=\pi\frac{\Delta_z}{\sigma_y}\frac{\sqrt{1+\delta_z}\sqrt{1+\delta_y}}{\sqrt{2+\delta_y}}e^{d_z},$$

$$J-J_{-}\simeq\sqrt{\frac{\pi}{d_z}}e^{d_z}h(z_0),\quad d_z=z_0^2\Sigma_z^2,$$

$$h(z_0)=1-\frac{\sqrt{\pi(1+\delta_y)}}{\sqrt{2(2+\delta_y)}}\frac{z_0}{\sigma_y}\left(1+\frac{2}{\pi}\arctan\frac{1}{\sqrt{\delta_y(2+\delta_y)}}\right). \tag{6.8}$$

It should be noted that for flat beams the probability of radiation as a function of the distance between beams (for the considered interval) decreases more slowly (proportional to $1/\sqrt{d_z}$) than for the round beams given in Eq. (5.23):

$$dw_\gamma^{fl}\simeq 4N_cN_r\frac{\alpha^3}{\pi}\lambda_c^2\Sigma_z\Sigma_y\frac{\varepsilon'}{\varepsilon}\frac{d\omega}{\omega}\left(v-\frac{2}{3}\right)\left[e^{-d_z}\ln\frac{m}{\Sigma_z}+\frac{1}{2}\sqrt{\frac{\pi}{d_z}}h(z_0)\right]. \tag{6.9}$$

Compensation in the difference $J-J_{-}$ begins in the region $z_0\sim\sigma_y+\Delta_y$ where Eq. (6.8) is not valid and one has to use the more accurate Eq. (5.8). In the region $z_0\gg\sigma_y+\Delta_y$ the probability of radiation decreases as $1/z_0^4$ according to Eqs. (4.4), (5.7), (5.27) provided that one can neglect the exponential term in the square brackets in Eq. (6.9) [compare with Eq. (5.23)]:

$$dw_\gamma^{fl}(z_0)\simeq 2N_cN_r\frac{\alpha^3}{\pi}\frac{\lambda_c^2\sigma_y^2}{z_0^4}\frac{\varepsilon'}{\varepsilon}\left(v-\frac{2}{3}\right)\frac{d\omega}{\omega},\quad z_0\gg y_0. \tag{6.10}$$

VII. OBSERVATION OF BEAM-SIZE EFFECT

Above we calculated the incoherent bremsstrahlung spectrum in the collision of electron and positron beams with finite transverse dimensions. This spectrum differs from the spectrum found previously in [7–9] because here (in contrast to previous papers) we subtract the coherent contribution. In the general expression for the correction to the probability of photon emission [Eq. (3.11)] the subtraction term is $F^{(2)}(\omega,\zeta)$. For numerical calculation in the case of coaxial beams it is convenient to use Eqs. (4.10), (4.20), and (4.21). In the last equation one has to put $y_0=z_0=0$. In the case of collision of narrow beams the subtraction term in the bremsstrahlung spectrum (5.7) is J_{-}. The dimensions of the beams in the experiment [6] were $\sigma_z=\Delta_z=24$ μm, $\sigma_y=\Delta_y=450$ μm, so this is the case of flat beams. The estimate for this case [Eq. (6.5)] gives $J_{-}\simeq(4/3\sqrt{3})\pi\sigma_z/\sigma_y\ll 1$. This term is much smaller than other terms in Eq. (5.7). This means that for this case the correction to the spectrum calculated in [7] is very small.

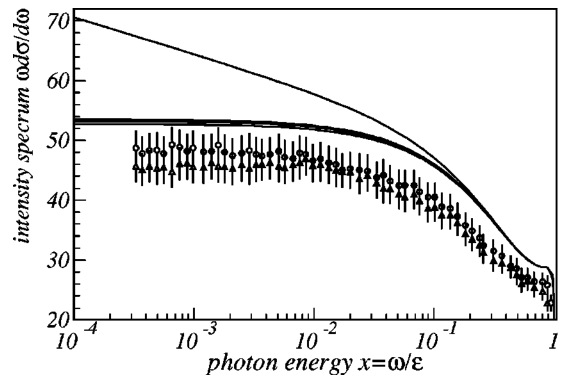

FIG. 1. The bremsstrahlung intensity spectrum $\omega d\sigma/d\omega$ in units of $2\alpha r_0^2$ versus the photon energy in units of the initial electron energy ($x=\omega/\varepsilon$) for the VEPP-4 experiment. The top curve is the standard QED spectrum; the three close curves below are calculated for different vertical dimensions of the colliding beams (equal for two colliding beams $\sigma=\sigma_z=\Delta_z$): $\sigma=20$ μm (bottom), $\sigma=24$ μm (middle), $\sigma=27$ μm (top). The data measured in [6] are presented as circles (the experiment in 1980) and as triangles (the experiment in 1981) with 6% systematic error as obtained in [6].

The results of calculation and the VEPP-4 (INP, Novosibirsk) data are presented in Fig. 1 where the bremsstrahlung intensity spectrum $\omega d\sigma/d\omega$ is given in units of $2\alpha r_0^2$ versus the photon energy in units of the initial electron energy ($x=\omega/\varepsilon$). The upper curve is the standard QED spectrum; the three close curves below are calculated using Eqs. (4.10) and (4.20) for different vertical dimensions of the colliding beams (equal for both colliding beams $\sigma=\sigma_z=\Delta_z$): $\sigma=20$ μm (bottom), $\sigma=24$ μm (middle), $\sigma=27$ μm (top) (this is just the 1σ dispersion for the beams used in the experiment). We want to emphasize that all the theoretical curves are calculated to within the relativistic accuracy (the discarded terms are of the order m/ε). It is seen that the effect of the small transverse dimensions is essential in the soft part of the spectrum (at $\omega/\varepsilon=10^{-4}$ the spectral curve is diminished by 25%), while for $\omega/\varepsilon>10^{-1}$ the effect becomes negligible. The data measured in [6] are presented as circles (experiment in 1980) and as triangles (experiment in 1981) with 6% systematic error as obtained in [6] (while the statistical errors are negligible). This presentation is somewhat different from that in [6]. It is seen that the data points are situated systematically below the theory curves but the difference does not exceed the 2σ level [6]. It should be noted that this is true also in the hard part of the spectrum where the beam-size effect is very small.

The last remark is connected with the radiative corrections (RC). The RC to the spectrum of double bremsstrahlung [16] (this was the normalization process) are essential (of the order of 10%) and were taken into account. The RC to the bremsstrahlung spectrum [17] are very small (less than 0.4%) and may be neglected. It should be noted that the RC

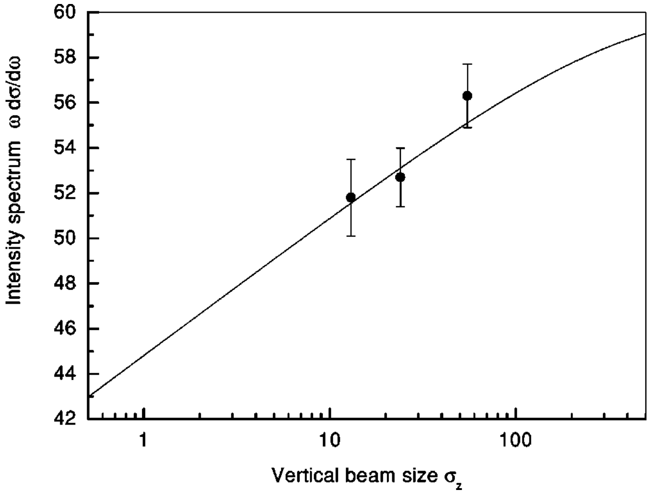

FIG. 2. The bremsstrahlung intensity spectrum $\omega d\sigma/d\omega$ in units of $2\alpha r_0^2$ versus the vertical sizes of the beams σ_z (in μm). The data are taken from [6].

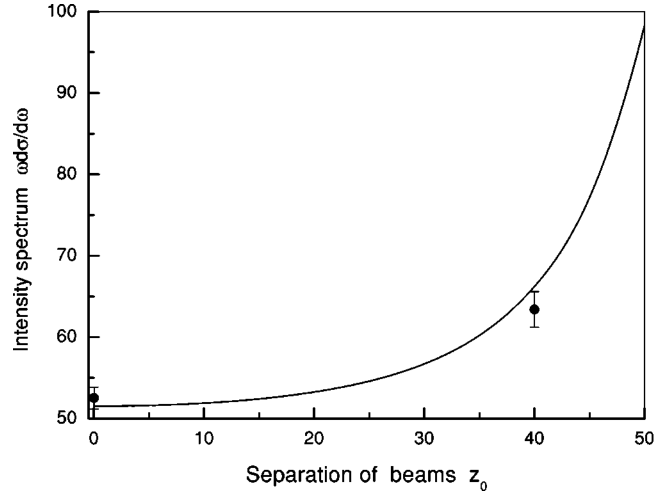

FIG. 3. The bremsstrahlung intensity spectrum $\omega d\sigma/d\omega$ normalized to luminosity $\mathcal{L}$ in units of $2\alpha r_0^2$ versus the vertical separation of the beams z_0 (in μ m). The data are taken from [6].

to the bremsstrahlung spectrum are insensitive to the effect of small transverse dimensions.

The dependence of the bremsstrahlung spectrum on beam characteristics was measured specially in [6]. The first effect is the dependence of the bremsstrahlung spectrum on the vertical sizes of the beams σ_z. It is calculated using Eqs. (4.10) and (4.20) for $\omega/\varepsilon=10^{-3}$. The result is shown in units of $2\alpha r_0^2$ in Fig. 2. The data are taken from Fig. 7 in [6]. The second is the measurement of the dependence of the bremsstrahlung spectrum on the vertical separation of the beams z_0. It is calculated using Eqs. (5.4) and (5.8) for $\omega/\varepsilon=10^{-3}$. Because of the separation it is necessary to normalize the spectrum to the luminosity

$$\mathcal{L}=N_cN_r\frac{\Sigma_z\Sigma_y}{\pi}\exp(-z_0^2\Sigma_z^2)$$

[see Eq. (4.3)]. This means that when we compare the bremsstrahlung process (where the beam-size effect is essential) with some other process like the double bremsstrahlung used in [6] (which is insensitive to the effect) we have to multiply the cross section of the last process by the luminosity $\mathcal{L}$. This is seen in the estimate Eq. (6.9): after taking out the exponent e^{-d_z} we have the luminosity as the external factor and in the expression for the ratio $N_\gamma/N_{2\gamma}$ (which was observed in [6]) the cross section of double bremsstrahlung will be multiplied by the luminosity. After this operation the second term in square brackets will contain the combination $e^{d_z}h(z_0)/\sqrt{d_z}$ which grows exponentially with increase in the separation z_0. The normalized bremsstrahlung spectrum is shown in units of $2\alpha r_0^2$ in Fig. 3. So the very fast (exponential) increase with z_0 is due to the fast decrease with z_0 of the double bremsstrahlung probability for the separated beams. The data are taken from Fig. 8 in [6]. It should be noted that in the soft part of the spectrum the dependence on photon energy ω is very weak. It is seen in these figures that there is quite reasonable agreement between theory and data just as in [6]. This shows that the contribution of the J_- term, which is calculated only in the present paper, is relatively small.

One more measurement of beam-size effect was performed at the HERA electron-proton collider (DESY, Germany) [18]. The electron beam energy was $\varepsilon=27.5$ GeV; the proton beam energy was $\varepsilon_p=820$ GeV. The standard bremsstrahlung spectrum for this case is given by Eq. (5.6) where q_{min} should be substituted by

$$q_{min}\rightarrow q_{min}^D=\frac{\omega m^2 m_p}{4\varepsilon_p\varepsilon\varepsilon'}; \tag{7.1}$$

here m_p is the proton mass. In this situation the formation length is $l_{f0}^D=1/q_{min}^D$ and at the photon energy $\omega=1$ GeV one has $l_{f0}^D\sim 2$ mm. Since the beam sizes at HERA are much smaller than this formation length, the beam-size effect can be observed at HERA. The parameters of the beam in this experiment were (in our notation) $\sigma_z=\Delta_z=50$–58 μm, $\sigma_y=\Delta_y=250$–290 μm. In some runs separated beams were used with $z_0=20$ μm and $y_0=100$ μm. The bremsstrahlung intensity spectrum $\omega d\sigma/d\omega$ in units of $2\alpha r_0^2$ versus the photon energy in units of the initial electron energy ($x=\omega/\varepsilon$) for the HERA experiment is given in Fig. 4. The upper curve is the standard QED spectrum. We calculated the spectrum with beam-size effect taken into account for three sets of beam parameters: set 1, $\sigma_z=\Delta_z=50$ μm, $\sigma_y=\Delta_y=250$ μm, $z_0=y_0=0$; set 2, $\sigma_z=\Delta_z=50$ μm, $\sigma_y=\Delta_y=250$ μm, $z_0=20$ μm, $y_0=0$; set 3, $\sigma_z=\Delta_z=54$ μm, $\sigma_y=\Delta_y=250$ μm, $z_0=y_0=0$. The result of the calculation is seen as the two close lower curves, the top curve being for set 3, while the bottom curve is actually two merged curves for sets 1 and 2. Since the ratio of the vertical and horizontal dimensions is not very small, the general formulas were used in the calculation: for coaxial beams Eqs. (4.11) and (4.20), and for separated beams Eqs. (4.14) and (4.20). It should be noted that the contribution of the subtraction term [Eq. (4.20)] is quite essential (more than 10%) for the beam parameters used at HERA. The data are taken from Fig. 5c in

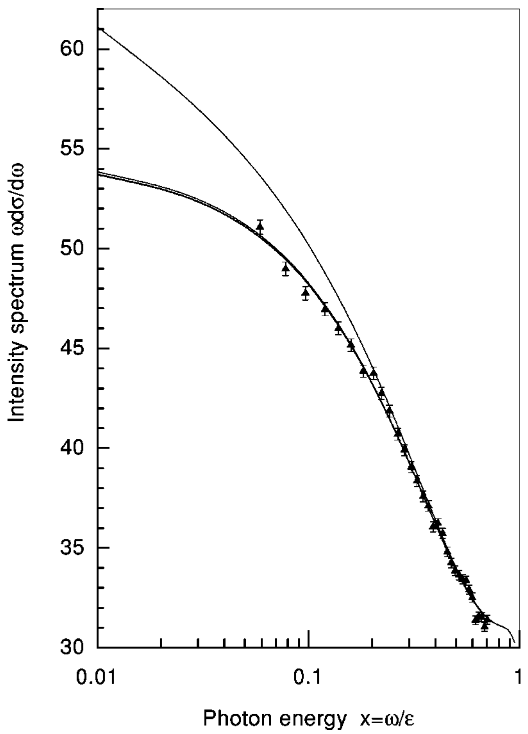

FIG. 4. The bremsstrahlung intensity spectrum $\omega d\sigma/d\omega$ in units of $2\alpha r_0^2$ versus the photon energy in units of the initial electron energy ($x=\omega/\varepsilon$) for the HERA experiment. The top curve is the standard QED spectrum. The two close curves below are calculated with the beam-size effect taken into account, The bottom curve is actually two merged curves for sets 1 and 2 (set 1 is $\sigma_z=\Delta_z=50\ \mu$m, $\sigma_y=\Delta_y=250\ \mu$m, $z_0=y_0=0$, set 2 is $\sigma_z=\Delta_z=50\ \mu$m, $\sigma_y=\Delta_y=250\ \mu$m, $z_0=20\ \mu$m, $y_0=0$), while the top curve is for set 3 ($\sigma_z=\Delta_z=54\ \mu$m, $\sigma_y=\Delta_y=250\ \mu$m, $z_0=y_0=0$). The data are taken from Fig. 5c in [18].

[18]. The errors are the recalculated overall systematic errors given in [18]. It is seen that there is a quite satisfactory agreement of theory and data. The data are given in [18] also as the averaged relative difference $\delta=(d\sigma_{QED}-d\sigma_{bs})/d\sigma_{QED}$ (where $d\sigma_{QED}$ is the standard QED spectrum and σ_{bs} is the result of a calculation with the beam-size effect taken into account) over the whole interval of photon energies (2–8 GeV), e.g., for set 1 $\delta_{ex}=(3.28\pm0.7)\%$, for set 2 $\delta_{ex}=(3.57\pm0.7)\%$, and for set 3 $\delta_{ex}=(3.06\pm0.7)\%$ [18]. The averaged $\langle\delta\rangle$ over the interval $0.07\leqslant x\leqslant0.28$ (or 1.95 GeV$\leqslant\omega\leqslant$7.7 GeV) in our calculation for set 1 is $\langle\delta\rangle=2.69\%$, for set 2 is $\langle\delta\rangle=2.65\%$, and for set 3 is $\langle\delta\rangle=2.54\%$. So for these data there is also a satisfactory agreement of data with theory (at the 1σ level, except for set 2 where the difference is slightly larger).

So the beam-size effect discovered at BINP (Novosibirsk) was confirmed at DESY (Germany). Of course, more accurate measurement is desirable to verify that we entirely understand this mechanism of deviation from standard QED.

VIII. CONCLUSION

In this paper the influence of the finite transverse size of colliding beams on the incoherent bremsstrahlung process is investigated. Previously (see papers [7–10]) an incomplete expression for the bremsstrahlung intensity spectrum was used for analysis of this effect because a subtraction was not carried out. It is necessary to carry out this subtraction to extract the pure fluctuation process which is just the incoherent bremsstrahlung. We implement this procedure in the present paper. The cases are indicated where the results without the subtraction term are qualitatively erroneous. The first is the case when the transverse sizes of the scattering beam are much smaller than the corresponding sizes of the radiating beam. For coaxial round beams, see, e.g., Eq. (5.15) and for flat beams Eq. (6.4). In contrast to previous papers here we draw the conclusion that the bremsstrahlung cross section is determined by the transverse sizes of the scattering beam.

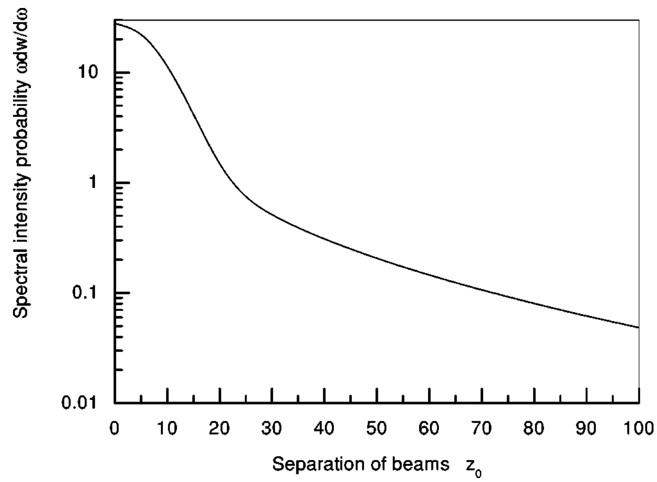

FIG. 5. The spectral intensity probability $\omega dw_\gamma/d\omega$ normalized to one particle in the beam in units of $2\alpha r_0^2\Sigma_z\Sigma_y/\pi$ versus the vertical separation of the beams z_0 (in nm).

A new qualitative result is deduced for the case when the separation of beams is large enough. Then the dispersior of the square of the momentum transfer, which determines the bremsstrahlung cross section, decreases with increasing separation distance faster than the mean square of the momentum transfer [see Eqs. (5.21),(5.27)]. As was noted in Sec. VII, it is necessary to normalize the spectrum to the luminosity for the separated beams. Then the bremsstrahlung cross section grows exponentially with increasing separation z_0. This very fast (exponential) increase is due to fast decrease in the normalization process probability for the separated beams.

For Gaussian beams the expression for the bremsstrahlung spectrum is obtained in the form of double integrals convenient for numerical calculations [see Eqs. (4.10), (4.20), and (4.21)]. For the soft part of the spectrum we deduced a general expression which is independent of the minimal momentum transfer q_{min} and is defined by the transverse sizes of the beams only [see Eqs. (5.3), (5.4), and (5.7)–(5.11)].

The important feature of the considered beam-size effect is the smooth decrease of radiation probability with growth in the beam separation. For flat beams we see in Eqs. (6.9), (6.10) that the main (logarithmic) term in the expression for the probability decreases exponentially [$\propto\exp(-z_0^2\Sigma_z^2$ as luminosity], but there is a specific long-range term proportional to $1/z_0$ which results in quite appreciable radiation probability even in the case when the separation of the beams is large. This phenomenon may be helpful for tuning high-energy electron-positron colliders. As an example we consider the "typical" collider where the beam energy is $\varepsilon = 500$ GeV, the beam dimensions are equal, and $\sigma_z = 5$ nm and $\sigma_y = 100$ nm. The beam-size effect in this collider is very strong and for $x = 10^{-3}$ the intensity spectrum is only ~ 0.3 of the standard $\omega d\sigma_{QED}(\omega)/d\omega$. The dependence of the bremsstrahlung probability on the separation distance z_0 (in nanometers) is shown in Fig. 5. It is calculated using Eqs. (5.6)–(5.8) for soft photons with $x = 10^{-3}$ [the asymptotic formulas (6.9), (6.10) are not accurate enough in this case]. Actually, the dependence on photon energy is contained with good accuracy in the external factor $f(x) = (1-x)[v(x) - 2/3]$ if the condition $(z_0^2 + \sigma_z^2 + \sigma_y^2)q_{min}^2 \ll 1$ is satisfied. This means that one can obtain the curve for any x by multiplying the ordinate in Fig. 5 by the factor $f(x)/f(10^{-3})$. The curve in Fig. 5 reflects the main features mentioned above. One can see that even for $z_0 = 100$ ($z_0 = 20\sigma_z$) the cross section is ~ 0.002 of the very large bremsstrahlung probability for head-on collision of beams. So by measuring the radiation from separated beams one can estimate the distance between the beams. This information may be useful for the tuning of beams.

ACKNOWLEDGMENTS

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