Penrose limit, spontaneous symmetry breaking, and holography in a *pp*-wave background

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We argue that the gauge theory dual to the type IIB string theory in a ten-dimensional *pp*-wave background resides on a *Euclidean* subspace spanning four of the eight transverse coordinates. We then show that the evolution of the string along one of the light cone directions in the bulk is identifiable as the RG flow of the gauge theory, a relation facilitating the ''holography'' of the *pp*-wave background. The ''holography'' reorganizes the dual gauge theory into theories defined over Hilbert subspaces of fixed *R* charge. The reorganization breaks the SO(4,2)×SO(6) symmetry to a maximal subgroup SO(4)×SO(4) spontaneously. We argue that the low-energy string modes may be regarded as Goldstone modes resulting from such a symmetry breaking pattern.

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I. OVERVIEW

It has been known for some time that there is a certain limit, so-called the Penrose limit, that any spacetime which solves the Einstein's field equation reduces to a plane-wave background [1]. Roughly speaking, the plane-wave background refers to the spacetime close to a null geodesic. This assertion has been extended to supergravity backgrounds [2], involving, in addition to the metric, dilaton, *p*-form gauge fields and fermionic partner fields. It was also realized [3] that maximally supersymmetric *pp*-wave solutions [4–8] are obtainable as the Penrose limit of the $\mathrm{AdS}_p\times\mathrm{S}^q$ backgrounds in ten-dimensional type IIB supergravity and 11-dimensional supergravity. Remarkably, the first-quantized superstring is exactly solvable in the *pp*-wave background [9,10], as the Green-Schwarz string action is quadratic in the worldsheet variables.

Recently, Berenstein, Maldacena, and Nastase (BMN) [11] argued that type IIB string theory on such a *pp*-wave background with eight transverse directions is dual to the large *R*-charge sector of $\mathcal{N}=4$ supersymmetric gauge theory in the large *N* limit. They identified a certain class of long supermultiplet operators in the gauge theory with various string states. By summing over a class of Feynman diagrams, they claimed that anomalous contributions to the scaling dimension of these operators indeed reproduce the dispersion relations predicted by the light-cone quantization. More significantly, they proposed a concrete construction of the ten-dimensional string in terms of the four-dimensional gauge theory variables. If correct, the construction marks significant progress beyond the AdS conformal field theory (CFT) correspondence [14], as it provides a dictionary for associating the gauge theory operators for not just supergravity modes, but for higher string modes as well. The BMN proposal is also extended to backgrounds with less supersymmetry [12,13].

In this paper we substantiate aspects of the BMN proposal. Specifically, we clarify the holographic relation between the bulk string states and the boundary gauge theory operators. In doing so, we emphasize the crucial role played by the choice of the gauge theory vacuum, on which both the superconformal symmetry and the *R* symmetry are spontaneously broken. In Sec. II, we contrast the bulk-boundary relations displayed in the AdS/CFT correspondence and those in the *pp*-wave/Yang-Mills correspondence. In Sec. III, we illustrate this by working out a profile of the supergravity modes in the *pp*-wave background. In Sec. IV, we elaborate the pattern of the aforementioned spontaneous conformal and *R*-symmetry breaking. We emphasize that the dual gauge theory is a theory defined on Euclidean four-dimensional space. We argue that holography relates the light-cone time in the *pp*-wave background to the renormalization group scale in the dual gauge theory. We show that this newly identified holography facilitates the nature of the string in terms of the dual gauge theory. In Sec. V, we discuss aspects of the enhanced supersymmetry in the dual gauge theory. We conclude with remarks in Sec. VI.

II. AdS/CFT VERSUS *pp*-WAVE/YANG-MILLS

In the AdS/CFT correspondence, the dual conformal field theory resides on the boundary of the AdS space [14–16], and the radial direction of the AdS space plays the role of scale of the boundary theory [17–22]. Consider the global coordinates in $\mathrm{AdS}_{d+1}\times\mathrm{S}^{d+1}$ space with metric

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$$ds^2 = R^2\Big[-(1+\bar{r}^2)d\bar{t}^2 + \frac{d\bar{r}^2}{(1+\bar{r}^2)} + \bar{r}^2 d\Omega_{d-1}^2 + (1-\bar{\rho}^2)d\bar{\theta}^2 + \frac{d\bar{\rho}^2}{(1-\bar{\rho}^2)} + \bar{\rho}^2 d\Omega_{\bar{d}-1}^2\Big], \qquad (1)$$

where the first and the second parts express the AdS_{d+1} and the $\mathrm{S}^{\bar{d}+1}$ subspace, respectively. A bulk single particle state of a given mass and spin, satisfying classical field equation, is specified by several "momenta": angular momentum quantum numbers $(l, m_1, \dots, m_{d-2})$ for the S^{d-1} part in AdS_{d+1} space and $(\bar{l}, \bar{m}_1, \dots, \bar{m}_{\bar{d}})$ for the $\mathrm{S}^{\bar{d}+1}$, respectively, and a principal quantum number n for the remaining radial coordinate $\bar{r}$ in AdS_{d+1} space. The bulk energy ω is then given in terms of these quantum numbers by a dispersion relation. In the dual conformal field theory, we have composite operators $\mathcal{O}_{\{\bar{l},\bar{m}\}}(\bar{t}, \phi_1, \dots, \phi_{d-1})$, where $(\bar{t}, \phi_1, \dots, \phi_{d-1})$ denote coordinates of $\mathbf{R}\times \mathrm{S}^{d-1}$. These operators are decomposable into Fourier modes with a given energy ω and S^{d-1} spherical harmonics $(l, m_1, \dots, m_{d-2})$. The remaining quantum numbers $(\bar{l}, \bar{m}_1, \dots, \bar{m}_{\bar{d}})$ are encoded in the structure of the operators. For instance, in $\mathrm{AdS}_5\times S^5$, the S^5 quantum numbers are encoded in the manner the six Higgs fields $\Phi^1, \dots, \Phi^6$ of the $\mathcal{N}=4$ gauge theory appear in the operator. As a concrete example, a bulk dilaton mode in $\mathrm{AdS}_5\times \mathrm{S}^5$ with S^5 angular momentum $(\bar{l}, \bar{m}_1, \dots, \bar{m}_{\bar{d}})$ is described by a set of chiral primary operators whose bosonic component is given by

$$\mathrm{Tr}[F_{mn}F^{mn}\Phi^{(i_1}\cdots\Phi^{i_{\bar{l}})}](\bar{t}, \phi_i \cdots \phi_{d-1}), \qquad (2)$$

in which the indices $i_1 \cdots i_{\bar{l}}$ are decomposed into irreducible representations of $\mathrm{SO}(\bar{d})$. F_{mn} denotes the gauge field strength. Likewise, chiral primary operators

$$\mathrm{Tr}[\Phi^{(i_1}\cdots\Phi^{i_{\bar{l}})}](\bar{t}, \phi_i \cdots \phi_{d-1}) \qquad (3)$$

describe modes of a linear combination of the four-form self-dual potential and the trace of the longitudinal graviton in ten-dimensions.

One can obtain the Penrose limit of Eq. (1) along a generic null geodesic as follows. Boost along the two isometry directions:

$$t = \cosh\alpha\,\bar{t} - \sinh\alpha\,\bar{\theta}, \qquad \theta = -\sinh\alpha\,\bar{t} + \cosh\alpha\,\bar{\theta}, \qquad (4)$$

and rescale two "radial" and light-cone coordinates:

$$r = R\bar{r}, \quad \rho = R\bar{\rho}, \quad \text{and} \quad x^{\pm} = \frac{R}{\sqrt{2}}(\theta \pm t). \qquad (5)$$

Then, take the limit

$$R\to\infty \quad \text{and} \quad \alpha\to\infty \qquad (6)$$

while holding

$$x^{\pm}, r, \rho = \text{fixed} \quad \text{and} \quad \frac{e^{\alpha}}{\sqrt{2}R} \equiv \mu = \text{fixed}. \qquad (7)$$

The resulting spacetime is then reduced to

$$\begin{aligned} ds^2 &= 2dx^+dx^- - \mu^2(r^2+\rho^2)(dx^+)^2 + dr^2 + r^2 d\Omega_{d-1}^2 + d\rho^2 + \rho^2 d\Omega_{\bar{d}-1}^2 \\ &= 2dx^+dx^- - \mu^2(\mathbf{x}^2+\mathbf{y}^2)(dx^+)^2 + d\mathbf{x}\cdot d\mathbf{x} + d\mathbf{y}\cdot d\mathbf{y}, \end{aligned} \qquad (8)$$

where we have defined transverse coordinates $\mathbf{x}, \mathbf{y}$ which describe the $\mathbf{R}^d$ made out of r and S^{d-1}, and $\mathbf{R}^{\bar{d}}$ made out of ρ and $\mathrm{S}^{\bar{d}-1}$, respectively. Even though the metric exhibits SO $(d+\bar{d})$ isometry, it turns out the RR five-form field strengths break it to $\mathrm{SO}(d)\times \mathrm{SO}(\bar{d})$.

A novel feature of the pp-wave background is that the single particle bulk states are now given in terms of certain harmonic oscillator quantum numbers $(n_1 \cdots n_d)$ and $(m_1 \cdots m_{\bar{d}})$ for a given value of the momentum conjugate to x^-, which we call $p_- \equiv 2p^+$. The light-cone energy $p_+ \equiv 2p^-$ is then given by a dispersion relation. We will illustate this later in this section. According to the BMN proposal, with these harmonic oscillator quantum numbers, the chiral primary operators dual to a single-particle bulk state with the lowest light-cone energy, which turns out to be a linear combination of the self-dual RR four-form potential and trace of the graviton, take the form

$$\sum \mathrm{Tr}[Z\cdots ZZ(D_{i_1}Z)ZZ\cdots ZZ\Phi^{a_1}ZZ\cdots ZZ \times (D_{i_2}Z)ZZ\cdots ZZ\Phi^{a_2}ZZ\cdots]. \qquad (9)$$

Here, along a string of J factors of $Z\equiv(\Phi^5+i\Phi^6)$, one distributes n_i insertions of (D_iZ) and m_a insertions of "transverse" Higgs fields Φ^a $(a=7,\dots,10)$. Then, Φ^5 and Φ^6 are the two remaining, "longitudinal" Higgs fields in the $\mathcal{N}=4$ gauge theory. The sum is over all distinct (up to cyclic permutation) locations of the operators D_iZ and Φ^a in the string of Z's. The quantum number J is related to the light-cone momentum p^+ by the relation

$$p^+ = \frac{1}{2R^2}\left(2J + \sum_{i=1}^{d} n_i + \sum_{a=1}^{\bar{d}} m_a\right). \qquad (10)$$

For other single-particle supergravity states such as the dilaton, one needs to insert an operator $F_{mn}F^{mn}$ inside the "Z-string." For higher string-mode states, each term in the sum is weighted by a phase-factor, which depends on the location of the various operators in the string of Z's.

Note that *all* the bulk quantum numbers appear in the structure of the dual gauge theory operators, Eq. (9). This is in sharp contrast to the AdS/CFT correspondence, where only *half* of the quantum numbers reside in the operator structure. There, the remaining *half* were encoded as dependence of the operator on coordinates of the four-dimensional spacetime, the boundary of AdS_5, on which the dual gauge theory resides. Evidently, the operators in Eq. (9) cannot be regarded as functions of the coordinates of the four-dimensional spacetime, as that would result in more quantum numbers than needed for specifying a given single particle supergravity or string state in the bulk.

In subsequent sections, we will argue that the gauge theory dual to the pp-wave background Eq. (9) resides on a Euclidean four-dimensional space, which may be taken to be the $\mathbf{R}^4$ spanned by $\mathbf{x}$ coordinates. Precise form of the dual gauge theory operators are then given in terms of the Hermite transformation of local operators defined on $\mathbf{R}^4$. The fact that this space has to be Euclidean, rather than Minkowski spanned by light-cone coordinates and part of $\mathbf{R}^4$, follows from the correspondence between the operators Eq. (9) and the one-particle states of the bulk supergravity or string theory. The latter states are described in terms of the $(d+\bar{d})$ set of simple harmonic oscillator operators with indices in a Euclidean space. For a string theory defined in the bulk, these oscillators also carry a label for the level number [9]. As we will see, this observation leads naturally to an interpretation of x^+ as the holographic bulk coordinate in the Penrose limit, so that evolution in x^+ in the bulk generates scale transformation in the dual gauge theory.

From the point of view of the Yang-Mills theory we will argue that selecting a sector with a fixed SO(2) charge J is tantamount to a spontaneous breaking of the conformal group SO(4,2) to SO(4) and the R-symmetry group to SO(4) as well. The low-energy fluctuations are then the Goldstone modes of the broken symmetries. The representation of these operators in terms of Hermite transforms then follows in a natural fashion.

III. SUPERGRAVITY MODES IN pp-WAVE BACKGROUNDS

Let us first consider the equation satisfied by the dilaton in the $(d+\bar{d}+1)$ dimensional pp-wave background. Consider a minimally coupled, massless scalar field D whose field equation is given in the global coordinates Eq. (8) as

$$\left[2\partial_-\partial_+ - \mu^2(\mathbf{x}^2+\mathbf{y}^2)\partial_-^2 + \sum_{i=1}^{d} \partial_{\mathbf{x}^i}^2 + \sum_{j=1}^{\bar{d}} \partial_{\mathbf{y}^j}^2\right] \times D(x^+,x^-,\mathbf{x},\mathbf{y}) = 0. \tag{11}$$

The normal modes with $p_- > 0$ are given by

$$D_{p_+,p_-,\mathbf{n},\mathbf{m}}(x^+,x^-,\mathbf{x},\mathbf{y}) = e^{-(1/2)\mu p_-(\mathbf{x}^2+\mathbf{y}^2)} \prod_{i=1}^{d} H_{n_i}(\sqrt{\mu p_-}\mathbf{x}^i) \times \prod_{a=1}^{\bar{d}} H_{m_a}(\sqrt{\mu p_-}\mathbf{y}^a) \times \exp(ip_-x^- + p_+x^+), \tag{12}$$

where $H_{\mathbf{n}}(\mathbf{x}), H_{\mathbf{m}}(\mathbf{y})$ denote the Hermite polynomials.[1] The states of this scalar field theory are therefore created from the bulk Fock-space vacuum by creation operators $\mathbf{a}^\dagger(\mathbf{n}_i,\mathbf{m}_a,p^+)$ in a light cone quantization. In a first quantized theory of particles in this background, states are created by creation operators $\mathbf{c}^{i\dagger},\mathbf{c}^{a\dagger}$:

$$\mathbf{c}^i = (\mathbf{p}_i + i\mathbf{x}^i)/\sqrt{2} \quad \text{and} \quad \mathbf{c}^{i\dagger} = (\mathbf{p}_i - i\mathbf{x}^i)/\sqrt{2},$$

$$\mathbf{c}^a = (\mathbf{p}_a + i\mathbf{y}^a)/\sqrt{2} \quad \text{and} \quad \mathbf{c}^{a\dagger} = (\mathbf{p}_a - i\mathbf{y}^a)/\sqrt{2},$$

where the indices i,a refer to the transverse directions along $\mathbf{R}^d \times \mathbf{R}^{\bar{d}}$, spanning a $(d+\bar{d})$ dimensional transverse space. The bulk dispersion relation is then given by

$$p^- = \frac{1}{2}p_+ = \frac{\mu}{2}\left(\sum_{i=1}^{d} n_i + \sum_{a=1}^{\bar{d}} m_a + \frac{1}{2}(d+\bar{d})\right). \tag{13}$$

Note that the value of p_+ is independent of the value of p_-. This is because the supergravity modes are massless. For massive, string oscillation fields, the dispersion relation depends explicitly on p_-. The sum over zero-point energies is standard. We will see that, for the explicit example of a ten-dimensional pp-wave background, this zero-point energy is precisely what is required for precise correspondence with appropriate operators in the dual gauge theory. We note, for future reference, that the dispersion relation for the mode which is a linear combination of the four form RR potential and the trace of the longitudinal graviton does not contain this zero-point fluctuation.

It is natural to expect that the dual gauge theory has operators which are Hermite transforms of local operators defined on the $\mathbf{R}^4$ spanned by $\mathbf{x}$.

$$\mathcal{O}[\mathbf{n}] = \text{H.T.}[\mathcal{O}], \tag{14}$$

where the Hermite transform of a generic operator $\mathcal{O}(\mathbf{x})$ on $\mathbf{R}^d$ is defined as

$$\text{H.T.}[\mathcal{O}] = \frac{1}{\mathcal{N}}\int d\mu[\mathbf{x}] \prod_{i=1}^{d} H_{n_i}(\sqrt{\mu p_-}x^i)\mathcal{O}(\mathbf{x}), \tag{15}$$

where $\mathcal{N}$ is a normalization factor, and the measure is given by

$$d\mu[\mathbf{x}] := d^d\mathbf{x}\, e^{-(1/2)\mu p_- \mathbf{x}^2}.$$

Using the standard recursion relation for Hermite polynomials these can be reduced to expressions with derivatives on $\mathcal{O}$ and no factors of the Hermite polynomials.

[1] Note that, because of the harmonic potential provided by the second term in Eq. (11), there is no real distinction between normalizable and non-normalizable modes for $p_- > 0$. The modes with $p_- = 0$ are not $\mathcal{L}_2$ normalizable, but are δ-function normalizable.

It may be useful to formally define operators on a $\mathbf{R}^d \times \mathbf{R}^{\bar{d}}$ by introducing a set of fiducial coordinates $\mathbf{y}$ for the $\mathbf{R}^{\bar{d}}$. Performing the Hermite transform on this eight-dimensional space,

$$\text{H.T.}[\mathcal{O}] = \frac{1}{\mathcal{N}} \int d\mu[\mathbf{x}] d\mu[\mathbf{y}] \prod_{i=1}^{d} H_{n_i}(\sqrt{\mu p_-} x^i) \times \prod_{a=1}^{\bar{d}} H_{m_a}(\sqrt{\mu p_-} y^a) \mathcal{O}(\mathbf{x},\mathbf{y}).$$

Using recursion relations obeyed by the Hermite polynomials, one can then express the Hermite transform in terms of derivatives with respect to $\mathbf{y}$, which in turn become commutators with Higgs fields inside the operator.

A. Dilaton

In the ten-dimensional pp-wave background, the dilaton field equation take the same form as Eq. (11). Thus the light-cone energy spectrum of the dilaton state is given by

$$E_{\text{dilaton}} = \frac{\mu}{2} \left(\sum_{i=1}^{4} n_i + \sum_{a=1}^{4} m_a + \frac{1}{2}(4+4) \right). \tag{16}$$

According to the BMN proposal, the light-cone energy (measured in units of $\mu/2$) ought to match with $(\Delta - J)$ of a gauge theory operator dual to the dilaton. A single insertion of $F_{mn}F^{mn}$, which carries $\Delta = 4$ and $J = 0$, inside the Z string in Eq. (14) is precisely what we need to match the zero-point light-cone energy. Interestingly, in providing the requisite zero-point energy $(4+4)/2 = 4$, four-dimensionality of the internal space $\mathbf{R}^{\bar{4}}$ has played a crucial role.

The ground-state of dilaton single particle states corresponds to $\mathbf{n} = \mathbf{m} = 0$. For the states with higher energy, using the recursion relation of Hermite polynomials, we deduce that the corresponding operators are precisely insertions of the "transverse" Higgs fields and covariant derivatives, viz. a set of operators of the form

$$\text{Tr}[F_{mn}F^{mn} ZZ \cdots ZZ \Phi^{a_1} ZZ \cdots ZZ \times (D_i Z) ZZ \cdots ZZ \Phi^{a_2} ZZ \cdots].$$

B. Longntudinal graviton and four-form potential

The pp-wave background is supported by a homogeneous RR 5-form field strength

$$F_{+1234} = + F_{+5678} = \mu,$$

giving rise to Eq. (1) through the Einstein's field equation. As such, degrees of freedom of the graviton and the four-form Ramond-Ramond (RR) potentials would mix each other. More precisely, expanding type IIB supergravity field equations of the metric and the RR four-form potential to linear order fluctuations, $h_{\mu\nu}, c_{\mu\nu\alpha\beta}$, and taking the light-cone gauge $h_{\mu -} = 0$, $c_{\mu\nu\alpha -} = 0$, we find that the mixing takes place between the traces of the graviton and the scalars of the RR four-form potential. We thus denote these modes as

$$h := h_{ij}\delta^{ij} \quad \text{and} \quad c := \frac{1}{4!} \epsilon^{ijkl} c_{ijkl},$$

$$\bar{h} := h_{mn}\delta^{mn} \quad \text{and} \quad \bar{c} := \frac{1}{4!} \epsilon^{mnpq} c_{mnpq}.$$

These fields are singlets of the two SO(4)'s on $\mathbf{R}^4 \times \mathbf{R}^{\bar{4}}$, respectively.

Then, the linearized field equations exhibiting the mode mixing are given by

$$\Delta_{\mathrm{L}} h - 16\mu \partial_- c = 0,$$

$$\nabla^2 c - 2\mu \partial_- h = 0,$$

where Δ_{L} stands for the Lichnerowitz operator for the spin-2 graviton. Utilizing the fact that $(\Lambda_{\mathrm{L}} h)_{ij} = -\frac{1}{2}\nabla^2 h_{ij}$ and diagonalizing the two coupled equations, we obtain scalar-mode field equations

$$[\nabla^2 - 8i\mu\partial_-](h + 4ic) = 0 \tag{17}$$

and its complex conjugated equation for $(h - 4ic)$. Exactly the same set of equations hold for $\bar{h}$ and $\bar{c}$ as well.

The field equation (17) is soluble exactly as in the dilaton case. We find that the light-cone spectrum of the $(h+4ic)$ complex "scalar" field is given by

$$E_{Z-\text{scalar}} = \frac{\mu}{2} \left(\sum_{i=1}^{4} n_i + \sum_{a=1}^{4} m_a + \frac{1}{2}(4+4) \right) - 2\mu = \frac{\mu}{2} \left(\sum_{i=1}^{4} n_i + \sum_{a=1}^{4} m_a \right) \geqslant 0.$$

On the right-hand side of the first expression, the first and the second terms are contributions from ∇^2 and $-8i\mu\partial_-$, respectively. Evidently, the zero-point energy arising from fluctuations along the eight transverse directions is cancelled precisely by the classical contribution -2μ to the light-cone energy. Hence along with the second set of complex "scalar;" field $(\bar{h} + 4i\bar{c})$, we conclude that there are two bulk "scalar" modes yielding the minimum of the light-cone energy (in units of $\mu/2$) to be zero. These bulk "scalar" fields are then identified with the dual gauge theory operators

$$\text{Tr}[ZZ \cdots ZZ \cdots ZZ],$$

viz. the Z-string, first introduced by BMN.

In contrast, the complex-conjugate "scalar" fields $(h - 4ic)$ and $(\bar{h} - 4i\bar{c})$ are subject to the classical contribution $+2\mu$ to the light-cone energy. It implies that the minimum of the light-cone energy is (in units of $\mu/2$) +8, instead of 0, rendering the corresponding dual gauge theory operator involving eight powers of Φ^a's distributed along the Z-string.

IV. PENROSE CONTRACTION, SPONTANEOUS SYMMETRY BREAKING, AND EUCLIDEAN DUAL GAUGE THEORY

We now turn to the dual $\mathcal{N}=4$ supersymmetric gauge theory. This theory is invariant under SO(4,2)$\otimes$SO(6), where SO(6) refers to the internal R symmetry. We denote the generators of SO(4,2) as J_{AB} with $A,B=1,\ldots,6$, where 5,6 are the directions with negative signature, and those of SO(6) as J_{UV} with $U,V=7,\ldots,12$. In terms of J_{AB}, the generators of the conformal group are

$$J_{ij}, \quad P_i=J_{5i}+J_{6i}, \quad K_i=J_{5i}-J_{6i}, \quad D_1=J_{56} \tag{18}$$

with $i,j=1,\ldots,4$. The same can be done for the generators of SO(6), and we define J_{ab}, P_a, K_a, D_2 accordingly, where $a,b=7,\ldots,10$ and $D_2=J_{11,12}$.

Let us now assume that there exists a vacuum state, on which the SO(4,2)$\otimes$SO(6) is broken spontaneously to SO(3,1)$\otimes$SO(4), viz. standard symmetry breaking pattern preserving Lorentz plus "transverse" internal symmetries. The number of generators of broken symmetries is 18, viz. nine nonlinearly realized symmetries for each product group. The generators of the broken symmetries are $P_i, K_i, P_a, K_a, D_1, D_2$ and the generators of the unbroken symmetries are the J_{ij} for the Lorentz group SO(3,1) and the J_{ab} for the internal symmetry group SO(4). One easily finds that generators of the broken symmetries satisfy the following commutation relations

$$[P_i,K_i]=D_1 \quad \text{and} \quad [P_a,K_a]=D_2.$$

These commutation relations are very suggestive. If one were to put aside the fact that D_1 and D_2 do not commute with the P's and K's, one may try to interpret the previous commutation relations as defining two Heisenberg algebras h(4)$\oplus$h(4), each one with eight generators, for which D_1 and D_2 are the two central extensions. This interpretation, as it stands, is not viable if one just considers the standard symmetry breaking pattern SO(4,2)$\otimes$SO(6) to SO(3,1)$\otimes$SO(4): D_1 and D_2 are not central terms and we cannot organize the generators of the broken symmetries in terms of two Heisenberg algebras. It is precisely at this point where the existence of supergravity/string duals and the concept of the Penrose contraction can help us to define a different pattern of the symmetry breaking.

A. Penrose contraction

As is well known, the symmetry algebra SO(4,2)$\otimes$SO(6) of $\mathcal{N}=4$ gauge theory are realizable as isometries of the $AdS_5\times S^5$ spacetime. The Penrose limit recapitulated in Sec. II preserves the total number of Killing vectors but can change their algebraic relations. In particular, if we perform the Penrose limit on a generic light geodesic in $AdS_5\times S^5$ the Killing vectors define the algebra [h(4)$\oplus$h(4)]$\oplus$so(4)$\oplus$so(4), where the bracket is to emphasize the fact that two Heisenberg algebras share the same central extension. The extra Killing vector defines an outer-automorphism of the Heisenberg algebras. We interpret the Penrose limit as defining a sort of spontaneous symmetry breaking from SO(4,2)$\otimes$SO(6) to SO(4)$\otimes$SO(4) with the 18 generators of the broken symmetries defining the two Heisenberg algebras h(4)'s and the outer automorphism.

As the simplest illustration, consider $AdS_2\times S^2$, relevant for the near-horizon geometry of four-dimensional Bogemol'nyi-Prasad-Sommerfield (BPS) black holes. In this case, the symmetry group is SO(1,2)$\otimes$SO(3) with six generators that we will denote P_1,K_1,P_2,K_2,D_1,D_2. They satisfy, in particular, $[P_1,K_1]=D_1$ and $[P_2,K_2]=D_2$.[2] In the Penrose limit, P_i,K_i become the generators of two Heisenberg algebras and the D_i's produce the common central term and the outer automorphism. In fact, denoting the Penrose scaling by Ω, we get $D_i(\Omega)=d^{i,0}\Omega^{-2}+d^{i,1}+d^{i,2}\Omega^2+\cdots$ with $d^{1,0}=d^{2,0}$. The central term is defined by $d^{i,0}$ and the outer automorphism by $(d^{i,1}-d^{i,2})$. Expansion of $D^i(\Omega)$ is then interpretable as a perturbative expansion in powers of the Penrose scaling parameter, Ω.

An important aspect of the Penrose limit in the case of $AdS_5\times S^5$ considered by BMN is that the unbroken symmetry is SO(4)$\otimes$SO(4). In other words, if we want to use the Penrose contraction as a pattern of the symmetry breaking for the dual $\mathcal{N}=4$ gauge theory, we should assume that the vacuum is invariant not under the Lorentz group but under a rotation group in a four-dimensional Euclidean space. Insight to this possiblity can be gained by recalling aspects of spontaneous conformal symmetry breaking, studied thoroughly some time ago [23,24]. The idea was to assume an underlying theory invariant under the conformal group and, after spontaneous conformal symmetry breaking, to study the low-energy physics of the corresponding Goldstone bosons. The first peculiar aspect of the spontaneous conformal symmetry breaking, SO(4,2) to SO(1,3), is that the generators of translations are part of the broken symmetries. Being so, only the generators of special conformal transformations and dilatations were considered [23] as real Goldstone bosons. A consequence of this is that these Goldstone bosons, contrary to the standard case, are not massless as the broken symmetries do not commute with the Hamiltonian, viz. with translations in time. In the Penrose contraction, we are facing a similar problem. If we consider $\mathcal{N}=4$ gauge theory and the standard spontaneous breaking pattern to SO(3,1)$\otimes$SO(4), we are including among the broken symmetry generators the translation generators in physical time as well as the spatial translation generators. If we try to understand this breaking in the old-fashioned approach [23], we need to organize the 18 broken symmetries into a set of nine massless Goldstone bosons, corresponding to the spontaneous breakdown of the internal symmetry SO(6), five massive Goldstone bosons corresponding to the special conformal transformations and dilatations that do not commute with the Hamiltonian and four translations. This is certainly not the picture we get if we use Penrose contraction. In the Penrose contraction, we organize the 18 broken symmetries into a Heisenberg algebra h(8) and an outer automorphism. What now remains is a

[2]In [6] the generators P_i,K_i,D_i correspond, respectively, to the Killing vectors E_i,E_i^*,ϵ_i.

concrete interpretation of the Heisenberg algebra h(8) and the outer automorphism entirely within the dual gauge theory formulation.

B. Dual gauge theory is Euclidean

Let Φ_i $i=5\cdots10$ be the Higgs fields of $\mathcal{N}=4$ super Yang Mills theory. Following BMN, define the field $Z=(\Phi_5+i\Phi_6)$, and denote by J the SO(2) R charge corresponding to rotations in the internal (5,6)-plane. Consider decomposing the gauge theory Hilbert space into infinite towers of Hilbert subspaces of definite J quantum number. Evidently, on each subspace, Fock-space "ground state" breaks the internal SO(6) spontaneously to SO(4). We denote the Fock-space vacuum with R charge equal to J as $|0\rangle_J$. We will be interested in the Hilbert space of quantum fluctuations around this vacua. The first thing to be done is characterizing the state $|0\rangle_J$. The simplest way to define this state is

$$\mathrm{Tr}(Z^J)(\mathbf{x}=0)|0\rangle_{\mathrm{YM}},$$

where $|0\rangle_{\mathrm{YM}}$ refers to the perturbative vacuum of the dual $\mathcal{N}=4$ gauge theory. The dual gauge theory is defined on the Euclidean space, $\mathbf{R}^4$, and is not related *a priori* to Euclideanized $\mathcal{N}=4$ super Yang-Mills theory defined on $\mathbf{R}_t\times\mathbf{R}^3$ after the Wick rotation. On $\mathbf{R}^4$, a *local* operator $\mathrm{Tr}(Z^J)(\mathbf{x})$ is expandable in a complete basis of the Hermite polynomials

$$\mathrm{Tr}(Z^J)(\mathbf{x})=\sum_{\{\mathbf{n}\}} c_{\mathbf{n}}\prod_{i=1}^{4}\,[e^{-\Lambda^2x_i^2}H_{n_i}(\Lambda x_i)],\tag{19}$$

where Λ is a scale defined within the dual gauge theory, which will be determined later. Thus we can write

$$|0\rangle_J=c_{\mathbf{n}=0}|0\rangle_{\mathrm{YM}}.\tag{20}$$

The Hilbert space of quantum fluctuations is generated by states $|\mathbf{n}\rangle_J=c_{\mathbf{n}}|0\rangle_{\mathrm{YM}}$. For instance, we get

$$\sum_l \mathrm{Tr}(Z^l(D_iZ)Z^{J-l})(0)|0\rangle_{\mathrm{YM}}=|\mathbf{n}_i=1\rangle_J.$$

One can define creation and annhilation operators b_0^i and $b_0^{i\dagger}$ obeying the canonical commutation relation $[b_0^i,b_0^{j\dagger}]=\delta^{i,j}$ such that

$$b_0^{i\dagger}|0\rangle_J=|\mathbf{n}_i=1\rangle_J.$$

These operators generate the Heisenberg algebra h(4). As we are working in $\mathcal{N}=4$ gauge theory, we can also consider fluctuations with respect to the internal directions, namely,

$$\sum_l \mathrm{Tr}(Z^l(D_aZ)Z^{J-l})(0)|0\rangle_{\mathrm{YM}}$$

$$\sim\sum_l \mathrm{Tr}(Z^l\Phi_aZZ^{J-l})(0)|0\rangle_{\mathrm{YM}}$$

with $a=7,\ldots,10$. In the large-J limit, one can represent these states in terms of the same type of creation and annihilation operators as before, viz. $|\mathbf{n}_a=1\rangle Js=b_0^{a\dagger}|0\rangle_J$. Both b_0^i with $i=1,\ldots,4$ and b_0^a with $a=7,\ldots,10$ transform as vectors under the two SO(4)'s, respectively. From now on, we will denote them collectively as $b_0^i,b_0^{i\dagger}$ with $i=1,\ldots,4,7,\ldots,10$. These operators generate the Heisenberg algebra h(8). Note that this is true only in the large-J limit and for Euclidean gauge theory. In view of the BMN proposal, it is quite natural to identify this Heisenberg algebra h(8) with the similar Heisenberg algebra encountered in the Penrose contraction of $\mathrm{SO}(4,2)\otimes\mathrm{SO}(6)$.

The next step would be to identify, within the dual gauge theory, the physical meaning of both the central extension and the outer automorphism. In the original theory invariant under $\mathrm{SO}(4,2)\otimes\mathrm{SO}(6)$, there are two generators of the symmetry algebra that are of special importance, viz. the generator of dilatations of the space-time coordinates and the generator J of the SO(2) R symmetry. Inferring the discussion on the Penrose contraction in Sec. II, we ought to expect that both the central extension and the outer automorphism are associated with these two generators. In the dual gauge theory, these generators have a very clear physical meaning: the generator of space-time dilatations will define the *scaling dimension* Δ of operators and the generator J the corresponding R charge. Note that in the AdS_5 realization of SO(4,2) the embedding coordinates X^A, $A=1,\ldots,6$ [with $(X^1)^2+(X^2)^2-(X^3)^2-\cdots(X^6)^2=R^2$] are given in terms of the global coordinates $(\overline{t},\overline{r},\phi_i)$ as

$$X^1=R\sqrt{1+\overline{r}^2}\cos\overline{t},$$

$$X^2=R\sqrt{1+\overline{r}^2}\sin\overline{t},\tag{21}$$

$$X^\alpha=R\overline{r}\omega^\alpha\quad(\alpha=3,\ldots,6),$$

where ω^α denotes the embedding coordinates of a unit S^3. The standard "dilatation" generator of the SO(4,2) group, which is J_{12}, generates translations in global time $\overline{t}$. In the Penrose limit, $\overline{r}\to0$ with $r=R\overline{r}$ held fixed. Thus $X^\alpha(\alpha=3,\ldots,6)$ are $\mathcal{O}(1)$ and become *unconstrained*, while X^1,X^2 are of $\mathcal{O}(R)$. The dual gauge theory is now defined on the Euclidean plane $X^3,\ldots,X^6$ and J_{12} generates the scale transformations on this $\mathbf{R}^4$ subspace as in the standard realization of the SO(4,2) group. Let us denote the eigenvalues of these generators, for a given operator, as Δ and J, respectively. For the state $|0\rangle_J$, we have

$$\Delta|0\rangle_J=J|0\rangle_J\quad\text{and}\quad J|0\rangle_J=J|0\rangle_J,$$

while, for states of type $b_0^{i\dagger}\cdots b_0^{i\dagger}|0\rangle_J$, we have $\Delta=(J+n)$, where n refers to the number of $b^\dagger$ oscillators and $J=J$. Thus, on these states, we have $(\Delta+J)=2J+n$ and $(\Delta-J)=n$. If we work in the limits of large-J, large-N, and small-n limits with

$$g^2N:=\lambda^2\to\infty,\quad J^2\to\infty,\quad\text{and}\quad\frac{\lambda^2}{J^2}:=g_{\mathrm{eff}}^2\to\text{finite},\tag{22}$$

we observe that $(\Delta+J)\lambda$ becomes the true central extension commuting with the b_0^i and $b_0^{i\dagger}$ operators,[3] and that $(\Delta-J)$ is simply the number operator for the $b_0^i, b_0^{i\dagger}$ oscillators and therefore is a true outer automorphism of the Heisenberg algebra.

In summary, built only upon Euclidean gauge theory residing on $\mathbf{R}^4$ subspace, we succeeded in finding a $SO(4)\otimes SO(4)$ invariant vacuum and a representation of the Heisenberg group H(8) in terms of creation and annhilation operators acting on the Hilbert space of small quantum fluctuations. The corresponding outer automorphism is just the number operator. Note that the vacuum state $|0\rangle_J$ is not only invariant under $SO(4)\otimes SO(4)$ but also with respect to the one-parameter group generated by the outer automorphism.

So far, we have considered only the modes which are chiral primaries. The scaling dimension Δ of the corresponding operator is

$$\Delta=\left(J+\sum_{i=1}^{4} n_i+\sum_{a=1}^{4} m_a\right), \tag{23}$$

where there are n_i insertions of D_iZ, and m_j insertions of Φ^j. Supersymmetry descendants of these would contain factors involving the gauge field, as discussed in Sec. III. Consider, for example, the dilaton. The operator dual to this should be the integral of

$$\mathrm{Tr}[F_{mn}F^{mn}\Phi^{(i_1}\cdots\Phi^{i_{\bar{l}})}](t,\phi_i\cdots\phi_{d-1}).$$

For such operators, the scaling dimension Δ is given by

$$\Delta=J+\sum_{i=1}^{4} n_i+\sum_{m=1}^{4} m_a+4. \tag{24}$$

These relations are consistent with our interpretation of the holographic coordinate.

The scaling dimension Δ of the dual operator is, however, the eigenvalue of the operator $i\partial_t$ in the bulk theory. The R charge is of course the eigenvalue of $i\partial_\theta$. These relations are in accord with the solutions of the bulk wave equations. Take the dilaton as an example. From Eq. (13) with $d=4$ and $\bar{d}=4$, we have

$$p^-=(\Delta-J)=\frac{\mu}{2}\left(\sum_{i=1}^{4} n_i+\sum_{a=1}^{4} m_a+4\right) \tag{25}$$

and find precise agreement with Eq. (24). From the bulk point of view, the additive factor 4 appears as a zero point energy. From the gauge theory viewpoint, this reflects the presence of $F_{mn}F^{mn}$ in the operator. For the bulk mode which is a fluctuation of the four form RR potential, this zero point energy is absent, which is consistent with the absence of any factor of gauge field strength in the dual operator. In the $J\to\infty$ limit, $\Delta\to\infty$ as well. However, $p_+=(\Delta-J)$ remains finite. This is the reason why, though it appears natural to consider t as a holographic direction from the bulk point of view, it is actually more natural to consider x^+ as the holographic direction from the gauge theory point of view.

The relation Eq. (25) reflects an important difference between the pp wave and AdS backgrounds. In the standard AdS_5/CFT_4 correspondence, the gauge theory operators are labeled by the angular momenta (l,m_1,m_2) on the S^3 where the gauge theory lives, the R symmetry quantum numbers $(\bar{l},\bar{m}_1\cdots\bar{m}_4)$, and the energy ω. The operators which are dual to the single-particle modes in $AdS_5\times S^5$ have the property that their conformal dimension depends on the R symmetry quantum numbers $(\bar{l},\bar{m}_1\cdots\bar{m}_4)$, but *not* on ω or (l,m_1,m_2). This is reflected in the bulk wave function as well. Here, the radial coordinate is identified with the RG direction and, to read off the dimension of the corresponding gauge theory operator, one has to look at the radial dependence of the wave function near the boundary. As is well known (and is in fact a consequence of conformal invariance), the radial dependence is *independent* of the angular momentum in AdS_5 and the energy, but does depend on the angular momentum on S^5. The previous discussion shows that, in order to read off the conformal dimension in the Penrose limit, one has to look at the *time* dependence of the bulk wave function. It is clear from Eqs. (12) and (13) that this depends on *all* the quantum numbers n_i and m_a. In the gauge theory, this is reflected by the fact that the dimension of the dual gauge theory operators also depends on all of the n_i, m_a quantum numbers.

Finally, it should be clear from the definition of the operators that one needs to introduce a cutoff mass scale Λ. Inferring expressions for the normal modes of the bulk fields, it is natural to choose the scale to be given, up to $\mathcal{O}(1)$ numerical factor, by

$$\Lambda=\mu p_-. \tag{26}$$

At first sight, this identification appears strange. It would mean that one needs *a priori* a different scale for each operator as p_- is defined by $(\Delta+J)/\lambda$. However, in the approximation adpoted Eq. (22), $n_i, m_a\ll J$. As such, $p_-\sim J/\lambda\sim\mathcal{O}(1)$. This implies that all the operators involved are governed universally by a common renormalization scale.

C. Light-cone holography

Once we have defined the outer automorphism, call it H, we can trivially use it to define a one parameter family of operators. In fact, we can introduce a formal parameter x^+ as a conjugate variable to H and define

$$\sum_l \mathrm{Tr}(Z^l(D_iZ)Z^{J-l})(0,x^+)|0\rangle=e^{-ix^+H}b_0^{i\dagger}|0\rangle_J.$$

[3] Notice that once we identify this term with the central extension of the Heisenberg algebra we need to normalize the b_0^i and $b_0^{i\dagger}$ so that they obey the canonical commutation relations.

This is a trivial and certainly not an interesting way to introduce an extra *holographic* coordinate in the Euclidean gauge theory. In fact, what we get is simply

$$\sum_l \operatorname{Tr}(Z^l(D_iZ)Z^{J-l})(0,x^+)|0\rangle_{\rm YM}$$
$$=e^{-ix^+(\Delta-J)}\sum_l \operatorname{Tr}(Z^l(D_iZ)Z^{J-l})(0)|0\rangle_{\rm YM} \quad (27)$$

and the outer automorphism $H=i\partial_{x^+}$. If we wish, we can also introduce another holographic coordinate, say x^-, conjugate to the central extension $C=(\Delta+J)/\lambda$ by

$$\sum_l \operatorname{Tr}(Z^l(D_iZ)Z^{J-l})(0,x^+,x^-)|0\rangle_{\rm YM}$$
$$=e^{ix^-C}e^{-ix^+H}b_0^{i\dagger}|0\rangle_J.$$

This dependence on x^- is trivial, as it is the same for all operators. It depends only on the particular finite value we choose for g^2N/J^2 in the double large-J and large-N limit. Once we introduce the coordinate x^-, the central extension C can be represented as $-i\partial_{x^-}$.

We can readily make contact with the spacetime Killing vectors in the Penrose limit. To illustrate this, consider the case of $AdS_3\times S^3$. In the Penrose limit, the isometry group $SO(2,2)\otimes SO(4)$ is contracted to $[H(2)\otimes H(2)]\otimes SO(2)\otimes SO(2)$. We are interested in the generators of the Heisenberg groups H(2)'s:

$$P_i(x^+)=\cos(\mu x^+)\frac{\partial}{\partial x^i}+\mu x^i\sin(\mu x^+)\frac{\partial}{\partial x^-},$$

$$K_i(x^+)=\sin(\mu x^+)\frac{\partial}{\partial x^i}-\mu x^i\cos(\mu x^+)\frac{\partial}{\partial x^-}.$$

These are the generators that we should put in correspondence with the operators b_0^i and $b_0^{i\dagger}$ in the dual gauge theory. As discussed above, the operator $-i\partial_{x^-}$ plays the role of the central extension C of the Heisenberg algebra. In the standard coordinate-momentum notation, we can write

$$P_i(x^+)=p_i\cos(\mu x^+)+\mu Cx^i\sin(\mu x^+),$$

$$K_i(x^+)=p_i\sin(\mu x^+)-\mu Cx^i\cos(\mu x^+),$$

where the parameter C represents the central extension term, and

$$[p_i,x^j]=-i\delta_i^j.$$

Evidently, we can write

$$P_i(x^+)=e^{-ix^+H}P_i(0)e^{+ix^+H} \quad (28)$$

for H the light-cone Hamiltonian defined by $H=i\partial_{x^+}$. This shows that the Hamiltonian H is the outer automorphism of the Heisenberg algebra.

We close this section with comments. Quantum mechanically, the value Δ of the scaling dimension is corrected by the contribution of anomalous dimensions, whereas the value J of the R-symmetry charge remains unchanged. As such, one ought to expect quantum corrections to the outer automorphism $(\Delta-J)$. How is it corrected? The interaction part of $\mathcal{N}=4$ gauge theory contains a term of the type $(1/2\pi g^2)\operatorname{Tr}[Z,\Phi_a][Z,\Phi_a]$. This term induces nonvanishing transitions of the type:

$$\langle \operatorname{Tr}(Z^l\Phi_aZ^{J-l})(x)\operatorname{Tr}(Z^{l+1}\Phi_bZ^{J-(l+1)})(0)\rangle$$
$$=\delta_{ab}2\pi g^2N\mathcal{I}(x)\frac{1}{4\pi^2x^{2J-2}} \quad (29)$$

and

$$\langle \operatorname{Tr}(Z^l\Phi_aZ^{J-l})(x)\operatorname{Tr}(Z^l\Phi_bZ^{J-l})(0)\rangle$$
$$=\delta_{ab}2\pi g^2NI(x)\frac{1}{4\pi^2x^{2J-2}}.$$

The function $\mathcal{I}(x)$ is given by

$$I(x)=\frac{1}{4\pi^2}\log|x|\Lambda+\text{finite},$$

where Λ defines the ultraviolet cutoff. Both contributions cancel each other, meaning that, for operators of the type $\Sigma_l \operatorname{Tr}(Z^l(D_iZ)Z^{J-l})$ or $\Sigma_l \operatorname{Tr}(Z^l\Phi_aZ^{J-l})$, the value of $(\Delta-J)$ is not corrected at least at first-order in the weak coupling perturbation theory. As pointed out by BMN already, the situation is different for operators of the type

$$\sum_l e^{i(2\pi nl/J)}\operatorname{Tr}(Z^l(D_iZ)Z^{J-l}) \quad (30)$$

or

$$\sum_l e^{i(2\pi nl/J)}\operatorname{Tr}(Z^l\Phi_iZ^{J-l}) \quad (31)$$

modulated by the "separation-dependent" phase-factors. We will discuss aspects of these corrections in the next section.

D. Strings out of dual gauge theory

Let us first recapitulate what we have done so far. We have considered Euclidean $\mathcal{N}=4$ gauge theory around a vacuum state invariant under $SO(4)\otimes SO(4)$ and under the outer automorphism H. The Hilbert space of small fluctuations around this vacua define a representation of the Heisenberg algebra h(8). The outer automorphism H is simply the number operator associated with the creation and annihilation operators generating h(8). In addition, using the gravity dual of the Minkowskian $\mathcal{N}=4$ gauge theory, we can define a precise map between the creation anhilation operators and the outer automorphism H on the dual gauge theory side and the Killing vectors in the Penrose limit of $AdS_5\times S^5$ on the gravity side. The formal conjugate variables of the Hamiltonian H and the central extension C become the coordinates x^+ and x^- of the Penrose limit of the bulk spacetime.

Probably the most surprising result of all this is the connection between the Penrose limit of Minkowskian $\mathcal{N}=4$ gauge theory and the Euclidean $\mathcal{N}=4$ gauge theory around a particular vacua. The main reason for this strange connection has to do with the peculiarities of spontaneous breakdown of conformal invariance. In fact, one can in principle think of this as a spontaneous breakdown of SO(4,2) to the Euclidean subgroup SO(4). In this case, the Lorentz invariance of the original Minkowskian theory should be hidden somehow in the dynamics of the Goldstone bosons around the vacua used to break the conformal symmetry spontaneously.

How then is the Lorentz invariance realized in the Hilbert space of small fluctuations around the chosen vacua $|0\rangle_J$? The answer descending from the BMN proposal is quite surprising and in fact extremely interesting. It asserts that the Hilbert space of small fluctuations of the Euclidean $\mathcal{N}=4$ gauge theory around the vacua $|0\rangle_J$ in the large-N and large-J limit is the Hilbert space of a ten-dimensional string theory in a suitable Minkowskian background.

To understand this, we begin with recalling some salient facts of string dynamics in the light-cone gauge. In flat ten-dimensional spacetime and for the bosonic sector, the light-cone gauge-fixed string is defined by:

(i) string oscillators: infinite tower of Heisenberg algebras

$$[a_n^i, a_m^{j\dagger}] = \delta_{n,m}\delta_{i,j}$$

with $i,j=1, \ldots ,8$ the transversal coordinates;

(ii) light-cone Hamiltonian:

$$H_{\rm LC}=\sum_n \frac{n}{2\alpha' p^+} a_n^\dagger a_n + ({\rm H.c.});$$

(iii) string parameter space: total length of the light-cone string is given by p^+; and

(iv) Virasoro constraint: infinite tower of constraints satisfying the Virasoro algebra.

The way a string dynamics emerges out of the Euclidean $\mathcal{N}=4$ gauge theory around the vacuum $|0\rangle_J$ relies crucially on the existence of the Heisenberg algebra h(8) and of the outer automorphism H. In order to establish a connection between the two structures, the first thing we should do is to extend the Heisenberg algebra h(8) to an infinite family of Heisenberg algebras of the type displayed in (i). Remarkably, this is achieved by the phase-factor-modulated operators, introduced first by BMN:

$$\sum_l e^{i(2\pi nl/J)} {\rm Tr}(Z^l \Phi_a Z^{J-l})(0)|0\rangle_{\rm YM} := b_n^{i\dagger}|0\rangle_J .$$

One can show readily that the newly introduced creation and annihilation operators obey the requisite infinite towers of Heisenberg algebras:

$$[b_n^i, b_m^{j\dagger}] = \delta_{n,m}\delta_{i,j} \quad (m,n=0,1,2, \ldots).$$

Compared to the light-cone string in flat spacetime, the main difference is the existence of the Heisenberg algebra for the $b_0^i, b_0^{i\dagger}$ harmonic oscillator operators. In fact, in flat spacetime, one only has the Heisenberg algebra for the center-of-mass part

$$[x_0^-, p^+] = i.$$

Let us now see how the outer automorphism H is modified by quantum effects. As computed by BMN, the scaling dimension Δ for the state $b_n^{i\dagger}|0\rangle_J$ is given in first-order in perturbation theory by

$$\Delta = J + 1 + 8\pi g^2 \left[\cos\left(\frac{2\pi n}{J}\right) - 1\right]. \tag{32}$$

Thus the outer automorphism at the quantum level is given by

$$H = (\Delta - J) = 1 + 8\pi^2 g^2 N \left[\cos\left(\frac{2\pi n}{J}\right) - 1\right]$$

$$= 1 + \frac{2\pi g^2 N}{J^2} n^2 + \cdots ,$$

where, in obtaining the second expression, we have taken the large-J and large-N limit. We thus find that

$$H_{\rm LC} = \sum_n \sqrt{1 + \frac{n^2}{(p^+)^2}} b_n^\dagger b_n \tag{33}$$

provided the parameters are identified as

$$(p^+)^2 = \frac{J^2}{g^2 N} = \frac{1}{g_{\rm eff}^2}.$$

This is the light-cone Hamiltonian of a string in the Penrose limit of $AdS_5 \times S^5$ with a nonvanishing RR five-form field strength background. This RR background is in fact crucial to match the units consistently. The light-cone Hamiltonian in the pp-wave background with constant RR field is

$$\sqrt{\mu^2 + \frac{n^2}{\alpha'(p^+)^2}}.$$

In order to make contact with what we get from the $\mathcal{N}=4$ gauge theory, we need to use dimensionless quantities, namely, $\alpha' \cdot \mu$. In other words, in order to map the string theory into the dual gauge theory, we need two independent scales in the string theory. In addition to the slope parameter, α', this requires turning on the RR five-form field strength background, μ.

One thing we have not elaborated on in detail is the string Virasoro constraints. We end this section with brief remarks on this issue. By inserting phases into the dual gauge theory operators, the Heisenberg algebra h(8) is extended to the family of Heisenberg algebras Eq. (32). It is also natural to extend the outer automorphism to this collection of Heisenberg algebras by introducing operators H_n ($n=0,1,2,\dots$) such that $[H_n,b_m]=b_{n+m}$ and $H_0=H$ for the light-cone Hamiltonian H. These outer automorphisms then generate the string Virasoro algebra. A very interesting question left for the future would be to uncover meaning of these Virasoro constraints entirely within the dual gauge theory viewpoint.

E. Light-cone Hamiltonian and renormalization group flow

The renormalization group equation for the correlators $\langle\mathcal{O}(x)\mathcal{O}^*(0)\rangle$ is

$$[\mu\partial_\mu+2\gamma(\mathcal{O})]\langle\mathcal{O}(x)\mathcal{O}^*(0)\rangle=0,$$

where μ refers to the renormalization group scale. If we consider the phase-modulated operator $\mathcal{O}=\Sigma_n e^{2\pi nl}\,\mathrm{Tr}(Z^l\Phi_a Z^{J-l})$, in the large-$J$ and large-N limit, we get

$$2\gamma(\mathcal{O})=(\Delta-J)[\mathcal{O}]-1, \tag{34}$$

where $(\Delta-J)[\mathcal{O}]$ is the value of $(\Delta-J)$ for the operator $\mathcal{O}$. This equation is reexpressible in a more suggestive form:

$$2\gamma(\mathcal{O})\mathcal{O}=[H,\mathcal{O}]-\mathcal{O},$$

where H is, as usual, the string light-cone Hamiltonian $(L_0+\overline{L}_0)$.

Note that the anomalous dimension $\gamma(\mathcal{O})$ appearing in Eq. (34) is, for the operator $\mathcal{O}=\Sigma_n e^{2\pi inl}\,\mathrm{Tr}(Z^l\Phi_a Z^{J-l})$, given by

$$\gamma(\mathcal{O})=J\gamma(Z)+\gamma(\Phi_a),$$

where $\gamma(Z)$ represents the anomalous dimension of the operator Z. Generically, the anomalous dimension $\gamma(Z)$ is affected by radiative effects through the self-energy corrections.[4] If not protected by supersymmetry, these contributions would go as g^2N and the connection with the string light-cone Hamiltonian would be lost. Moreover, the scaling dimension of the operators $\mathcal{O}$, in that case, would grow with 't Hooft's coupling constant and eventualy disappear out of the physical spectrum. In the supersymmetric case we are considering, supersymmetry renders $\gamma(Z)=0$. This is the reason behind regarding $\gamma(\mathcal{O})$ as the anomalous dimension of the field Φ_a's. From the previous discussion, it should be evident that changes in the holographic coordinate x^+ are equivalent to changes of the renormalization group scale μ if we interpret $(\Delta-J)$ as the anomalous dimension.[5]

[4]These are so-called zero-momentum effects in the nomenclature of BMN.

[5]Definition of the renormalization group $\gamma(\mathcal{O})$ as $(\Delta-J)$ or as $(\Delta-J)-1$ depends on whether one adpots the canonical dimensions for the fields and masses or not. For a lucid discussion on this point, see [25].

V. SYMMETRY ENHANCEMENT AS "*POST MORTEM*" EFFECT

In recent papers [12,13], the results of BMN have been generalized to gauge theories with less supersymmetry, in particular, to the gravity duals of $\mathrm{AdS}_5\times T^{1,1}$. The fact that the Penrose limit of this space is the same as the one of $\mathrm{AdS}_5\times\mathrm{S}^5$ raises the question regarding the reason behind supersymmetry enhancement from $\mathcal{N}=1$ to $\mathcal{N}=4$. The point of this seemingly mysterious result goes back to the fact that, in the *strict* Penrose limit, there always appear extra isometries. In [6], these extra isometries are referred to as a *post mortem* effect. These isometries always define a Heisenberg algebra. In the case of $\mathrm{AdS}_5\times T^{1,1}$, the isometries are $\mathrm{SO}(4,2)\otimes[\mathrm{SU}(2)\otimes\mathrm{SU}(2)\otimes\mathrm{U}(1)]$, and are in correspondence, respectively, with the conformal invariance and the R symmetry of the dual gauge theory. Following our approach, we can think in terms of a spontaneous breakdown of this symmetry to $\mathrm{SO}(4)\otimes\mathrm{SO}(4)$. The difference with the case of $\mathcal{N}=4$ supersymmetric gauge theory is that we now have a smaller number of Goldstone bosons associated with the broken symmetries, viz. ten instead of 18 in the $\mathcal{N}=4$ supersymmetric case. The deficit eight Goldstone bosons are precisely the ones associated with the Higgs fields in the $\mathcal{N}=4$ supersymmetric gauge theory. These are the fields that would render the theory $\mathcal{N}=4$ supersymmetric. In the Penrose limit of any ten-dimensional background, we always have a Heisenberg algebra h(8) of the isometries. In the $\mathrm{AdS}_5\times T^{1,1}$ case, the Heisenberg algebra is composed of two Heisenberg subalgebras h(4) with a common central extension. The eight generators of one of the two algebras are the ones we are going to use as the eight missing broken symmetries. These "*post mortem*" Goldstone bosons are simply states that, in the $\mathcal{N}=1$ gauge theory, are degenerate in the light-cone mass (the eigenvalue of the outer automorphism or the light-cone Hamiltonian) with the real Goldstone bosons. As the enhancement of $\mathcal{N}=4$ supersymmetry is true only in the strict Penrose limit, viz. for the Penrose scaling factor $1/\lambda\to0$, we expect this enhancement to be violated by $\mathcal{O}(1/N)$ corrections.

VI. CONCLUSIONS

Our main conclusion is that the supersymmetric gauge theory dual to the type IIB string theory in a ten-dimensional pp-wave background lives on a *Euclidean* four-dimensional space. The indications has come from several corners. The most direct reason is that the dual operators are in one-to-one correspondence with the states of the string theory which are created by *transverse* oscillators in the light-cone gauge. This is in fact apparent in the zero-mode sector of the string relevant for the supergravity modes and follows from field equations in this background. Furthermore, understanding of the low-energy Nambu-Goldstone modes resulting from symmetry breaking of the original symmetry group $\mathrm{SO}(4,2)\times\mathrm{SO}(6)$ to $\mathrm{SO}(4)\times\mathrm{SO}(4)\times\mathrm{H}(4)\times\mathrm{H}(4)$ also indicates that the dual theory lives in a Euclidean space.

One may think of this Euclidean space as the space spanned by four of the transverse directions. In this case, the light-cone time has a natural explanation as the holographic coordinate representing the scale of the dual gauge theory. Details of the proposed holographic correspondence remain to be understood better. The fact that a Euclidean theory can give rise to strings living in a spacetime with Lorentzian signature is intriguing and deserves better understanding. We expect that this would enhance our understanding of holography in general.

Extending the results of this work, one can abstract the main ingredients needed in order to generalize the BMN proposal to generic situations. The minimal starting point would be a four-dimensional gauge theory invariant under the conformal group and with a nonanomalous global U(1) symmetry. The spontaneous breakdown of $SO(4,2)\otimes U(1)$ to SO(4) provides ten broken generators that we can try to organize, taking Euclidean signature dual theory, into a Heisenberg algebra h(4) and an outer automorphism H. This is the structure that we can try to map into the Hilbert space of a noncritical six-dimensional string theory in the light-cone gauge with the SO(4) rotational invariance acting on the transversal coordinates. Of course, we also need controllable, finite contributions—in the large 't Hooft coupling limit—to the anomalous dimensions of the fields representing the small quantum fluctuations around the selected vacuum or, equivalently, finite light-cone mass (defined by the outer automorphism) for the Goldstone bosons. Taking the real-world QCD, one now has *ab initio* two related problems: anomaly for the conformal invariance (viz. a nonvanishing beta function) and anomaly for the axial U(1) symmetry, which would serve as a natural candidate for the global U(1) symmetry. We can try to solve these problems by introducing two extra scalar fields, namely the dilaton D and the axion A. We then find an indication that the field $(D+iA)$ is a natural candidate to play the role of the Z-field in the BMN proposal.

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