# Propagators in noncommutative instantons

Bum-Hoon Lee[1,*] and Hyun Seok Yang[2,†]
[1]*Department of Physics, Sogang University, Seoul 121-742, Korea*
[2]*Department of Physics, National Taiwan University, Taipei 106, Taiwan, Republic of China*
(Received 11 June 2002; published 26 August 2002)

We explicitly construct Green's functions for a field in an arbitrary representation of a gauge group propagating in noncommutative instanton backgrounds based on the Atiyah-Drinfeld-Hitchin-Manin (ADHM) construction. The propagators for spinor and vector fields can be constructed in terms of those for the scalar field in a noncommutative instanton background. We show that the propagators in the adjoint representation are deformed by noncommutativity while those in the fundamental representation have exactly the same form as in the commutative case.

DOI: 10.1103/PhysRevD.66.045027 PACS number(s): 11.15.Tk, 02.40.Gh

## I. INTRODUCTION

Instantons were found by Belavin, Polyakov, Schwartz, and Tyupkin (BPST) [1] almost 30 years ago, as topologically nontrivial solutions of the duality equations of the Euclidean Yang-Mills theory with a finite action. Immediately instantons were realized to describe the tunneling processes between different $\theta$ vacua in Minkowski space and led to the strong $CP$ problem in QCD [2,3]. (For an earlier development of instanton physics, see the collection of papers [4].) The nonperturbative chiral anomaly in the instanton background led to baryon number violation and a solution to the U(1) problem [5,6]. These results revealed that instantons can have relevance to phenomenological models such as QCD and the standard model [7].

Instanton solutions also appear as Bogomol'nyi-Prasad-Sommerfield (BPS) states in string theory. They are described by D$p$-branes bound to D$(p+4)$-branes [8,9]. Subsequently, in [10,11], low-energy excitations of D-brane bound states were used to explain the microscopic degrees of freedom of black-hole entropy, for which the information on the instanton moduli space has a crucial role. In addition the multi-instanton calculus was used for a nonperturbative test of AdS/CFT correspondence [12–15], where the relation between Yang-Mills instantons and D-instantons was beautifully confirmed by the explicit form of the classical D-instanton solution in $\mathrm{AdS}_5\times\mathbf{S}^5$ background and its associated supermultiplet of zero modes.

Recently, instanton solutions on noncommutative spaces have turned out to have richer spectrums. While commutative instantons are always BPS states, noncommutative instantons admit both BPS and non-BPS states. In particular, instanton solutions can be found in U(1) gauge theory and the moduli space of non-BPS instantons is smooth, small instanton singularities being resolved by the noncommutativity [16,17]. Remarkably, instanton solutions in noncommutative gauge theory can also be studied by the Atiyah-Drinfeld-Hitchin-Manin (ADHM) equation [18] slightly modified by the noncommutativity [16]. The ADHM construction uses some quadratic matrix equations (hence noncommutative objects in nature) to construct (anti-)self-dual configurations of the gauge field. Thus the noncommutativity of space is not a serious obstacle for the ADHM construction of noncommutative instantons and indeed it turns out that it is a really powerful tool even for noncommutative instantons. Recently much progress has been made in this direction [16,17,19–35].

In order to calculate instanton effects in quantum gauge theory, it is important to know the Green's function in instanton backgrounds [6]. In this paper, based on the ADHM construction, we will construct the Green's functions for a field in an arbitrary representation of the gauge group propagating in noncommutative instanton backgrounds. Recently several papers [26,36–43] discussed the instanton moduli space and the instanton calculus in noncommutative spaces. This paper is organized as follows. In the next section we review briefly the Weyl ordering prescription for operators and the Green's function in noncommutative space, needed for later applications. In Sec. III, we generalize the argument in [44] to noncommutative space and show that the propagators for spinor and vector fields can be constructed in terms of those for the scalar field in a noncommutative instanton background. In Sec. IV, we explicitly construct the scalar propagators in the fundamental representation of $G$ and the tensor product $G_1\times G_2$ [45] where the adjoint representation is a special case. We observe that the propagator in the adjoint representation or the tensor product gauge group $G_1\times G_2$ is deformed by noncommutativity while that in the fundamental representation has exactly the same form as in the commutative case. In Sec. V we speculatively discuss some important issues such as the infrared divergence in the vector propagator, the zero modes for the tensor product gauge group, and the conformal property of instanton propagators.

## II. GREEN'S FUNCTION IN NONCOMMUTATIVE SPACE

In this section we review briefly the Weyl ordering prescription for operators and the Green's function in noncommutative space [46,47], needed for later applications.

Here we will work in general in flat noncommutative Euclidean space $\mathbf{R}^4$ represented by

*Electronic address: bhl@ccs.sogang.ac.kr
†Electronic address: hsyang@phys.ntu.edu.tw

 

$$[\hat{x}^\mu,\hat{x}^\nu]=i\theta^{\mu\nu} \tag{2.1}$$

where $\theta^{\mu\nu}=-\theta^{\nu\mu}$ and we use the caret to indicate operators in $\mathcal{A}_\theta$ for a moment. Since $\theta^{\mu\nu}$ is an antisymmetric tensor, let us decompose them into self-dual and anti-self-dual parts:

$$\theta_{\mu\nu}=\eta^a_{\mu\nu}\zeta^a+\overline{\eta}^a_{\mu\nu}\chi^a. \tag{2.2}$$

Since the self-duality condition is invariant under SO(4) rotations [or more generally SL(4,**R**) transformations], one can always make the matrix $\theta_{\mu\nu}$ into a standard symplectic form by performing the SO(4) transformation $R$:

$$\theta=R\tilde{\theta}R^T, \tag{2.3}$$

where we choose $\tilde{\theta}$ as

$$\tilde{\theta}_{\mu\nu}=\begin{pmatrix} 0 & \theta_1 & 0 & 0\\ -\theta_1 & 0 & 0 & 0\\ 0 & 0 & 0 & \theta_2\\ 0 & 0 & -\theta_2 & 0\end{pmatrix}. \tag{2.4}$$

There are four important cases to consider:

$$\theta_1=\theta_2=0: \quad \text{commutative } \mathbf{R}^4, \tag{2.5}$$

$$\theta_1=\theta_2=\frac{\zeta}{4}: \quad \text{self-dual } \mathbf{R}^4_{NC}, \tag{2.6}$$

$$\theta_1=-\theta_2=\frac{\zeta}{4}: \quad \text{anti-self-dual } \mathbf{R}^4_{NC}, \tag{2.7}$$

$$\theta_1\theta_2=0 \;\text{ but }\; \theta_1+\theta_2=\frac{\zeta}{2}: \quad \mathbf{R}^2_{NC}\times\mathbf{R}^2_C. \tag{2.8}$$

By noncommutative space $\mathbf{R}^4_{NC}$ one means the algebra $\mathcal{A}_\theta$ generated by the $\hat{x}^\mu$ satisfying Eq. (2.1). The commutation relation (2.1) in the basis (2.4) is equivalent to that of the annihilation and creation operators for a one-dimensional or two-dimensional harmonic oscillator:

$$[a_a,a_b^\dagger]=\delta_{ab}, \tag{2.9}$$

where $a=1,2$ for Eqs. (2.6) and (2.7) and $a=1$ for Eq. (2.8). Explicitly, for self-dual and anti-self-dual $\mathbf{R}^4_{NC}$ in Eqs. (2.6) and (2.7),

$$a_a^\dagger=\sqrt{\frac{2}{\zeta}}(\hat{x}^{2a}+i\epsilon^{a-1}\hat{x}^{2a-1}),$$

$$a_a=\sqrt{\frac{2}{\zeta}}(\hat{x}^{2a}-i\epsilon^{a-1}\hat{x}^{2a-1}), \tag{2.10}$$

where $a=1,2$ and $\epsilon=\theta_1/\theta_2$. So, for self-dual and anti-self-dual $\mathbf{R}^4_{NC}$, the representation space $\mathcal{H}$ of $\mathcal{A}_\theta$ can be identified with the Fock space $\mathcal{F}=\Sigma_{(n_1,n_2)\in\mathbf{Z}^2_{\geqslant 0}}\mathbf{C}|n_1,n_2\rangle$, where $n_1,n_2$ are occupation numbers in the harmonic oscillators. Thus the noncommutative space $\mathbf{R}^4_{NC}$ in the basis $\mathcal{F}$ becomes a two-dimensional integer lattice $\{(n_1,n_2)\in\mathbf{Z}^2_{\geqslant 0}\}$ and the integration on $\mathbf{R}^4_{NC}$ can be defined by the sum over the lattice,

$$\mathrm{Tr}_{\mathcal{H}}\mathcal{O}(x)\equiv\left(\frac{\zeta\pi}{2}\right)^2\sum_{(n_1,n_2)}\langle n_1,n_2|\mathcal{O}(x)|n_1,n_2\rangle \tag{2.11}$$

for an operator $\mathcal{O}(x)$ in $\mathcal{A}_\theta$. While, for $\mathbf{R}^2_{NC}\times\mathbf{R}^2_C$ in Eq. (2.8),

$$a^\dagger=\frac{\hat{x}^{2a}+i\hat{x}^{2a-1}}{\sqrt{\zeta}},\quad a=\frac{\hat{x}^{2a}-i\hat{x}^{2a-1}}{\sqrt{\zeta}}, \tag{2.12}$$

where $a=1$ for $\theta_1\neq 0$ and $a=2$ for $\theta_2\neq 0$. In this case, the representation space $\mathcal{H}$ is given by $\mathcal{F}=\Sigma_{n\in\mathbf{Z}_{\geqslant 0}}\mathbf{C}|n\rangle$ and the integration for an operator $\mathcal{O}(x)$ in $\mathcal{A}_\theta$ with $\theta_1\neq 0$, for example, can be replaced by

$$\int d^4x\mathcal{O}(x)\to\zeta\pi\sum_{n\in\mathbf{Z}_{\geqslant 0}}\int d^2x\langle n|\mathcal{O}(x)|n\rangle, \tag{2.13}$$

where $d^2x=dx^3dx^4$.

We introduce coherent states defined by

$$|\xi\rangle=e^{\xi a^\dagger}|0\rangle,\quad \langle\xi|=\langle 0|e^{\overline{\xi}a} \tag{2.14}$$

where $|0\rangle$ is a vacuum defined by $a|0\rangle=0$. For notational simplicity, we only present the construction for the algebra (2.12), but a similar construction can be given for Eq. (2.10), for which $|\xi\rangle=e^{\xi^a a_a^\dagger}|0\rangle$. The state $|\xi\rangle$ satisfies

$$a|\xi\rangle=\xi|\xi\rangle,\quad \langle\xi|a^\dagger=\langle\xi|\overline{\xi} \tag{2.15}$$

and

$$\langle\eta|\xi\rangle=e^{\overline{\eta}\xi},\quad \int\frac{d\overline{\xi}d\xi}{2\pi i}e^{-|\xi|^2}|\xi\rangle\langle\xi|=1. \tag{2.16}$$

Then we see that

$$\langle\eta|e^{i(k_1\hat{x}^1+k_2\hat{x}^2)}|\xi\rangle=e^{-\zeta k^2/8}e^{i(k_1z^1+k_2z^2)}e^{\overline{\eta}\xi} \tag{2.17}$$

where $k^2=k_1^2+k_2^2$ and $z^1=i(\sqrt{\zeta}/2)(\xi-\overline{\eta})$, $z^2=(\sqrt{\zeta}/2)(\xi+\overline{\eta})$.

It is well known that the Weyl or symmetric ordering prescription provides the procedure that maps commutative smooth functions onto operators acting on the Fock space $\mathcal{F}$ [48]:

$$f(x)\mapsto\hat{f}(\hat{x})=\int\frac{d^4k}{(2\pi)^4}f(k)e^{ik\cdot\hat{x}}, \tag{2.18}$$

where

$$f(k)=\int d^4xf(x)e^{-ik\cdot x}. \tag{2.19}$$

Using the prescription (2.18), it is easy to show that the operator multiplication in $\mathcal{A}_\theta$ is isomorphic to the Moyal product of functions:

$$\text{If } f(x)\mapsto\hat{f}(\hat{x}) \text{ and } g(x)\mapsto\hat{g}(\hat{x}),$$
$$\text{then } (f*g)(x)\mapsto(\hat{f}\hat{g})(\hat{x}), \tag{2.20}$$

where the Moyal product is defined as

$$(f*g)(x)=e^{(i/2)\theta^{\mu\nu}(\partial/\partial x^\mu)(\partial/\partial y^\nu)}f(x)g(y)\big|_{x=y}. \tag{2.21}$$

In order to discuss instanton propagators in the noncommutative space (2.1), we first should know the free Green's function $\hat{G}^{(0)}(\hat{x},\hat{y})$ for the ordinary Laplacian [46,47]:

$$-\hat{\partial}_\mu\hat{\partial}_\mu\hat{G}^{(0)}(\hat{x},\hat{y})=\hat{\delta}(\hat{x}-\hat{y}) \tag{2.22}$$

where the derivative for an operator $\hat{f}(\hat{x})$ is defined as

$$\hat{\partial}_\mu\hat{f}(\hat{x})=-i(\theta^{-1})_{\mu\nu}[\hat{x}^\nu,\hat{f}(\hat{x})]. \tag{2.23}$$

In commutative $\mathbf{R}^4$, it is given by

$$G^{(0)}(x,y)=\frac{1}{4\pi^2(x-y)^2}. \tag{2.24}$$

Here some comments should be made. In order to define the Green's function, we have introduced the tensor product $\mathcal{A}_\theta^{1,2}=\mathcal{A}_\theta^1\otimes\mathcal{A}_\theta^2$ of two copies of the algebra $\mathcal{A}_\theta$. We represent $\mathcal{A}_\theta^{1,2}$ as an algebra of operators on the tensor product $\mathcal{H}^{1,2}=\mathcal{H}^1\otimes\mathcal{H}^2$ of two Fock spaces. The functions $\hat{G}^{(0)}(\hat{x},\hat{y})$, $\hat{\delta}(\hat{x}-\hat{y})\in\mathcal{A}_\theta^{1,2}$, are operators acting on $\mathcal{H}^{1,2}$. We identify $\hat{x}^\mu=\hat{x}^\mu\otimes 1$ and $\hat{y}^\mu=1\otimes\hat{y}^\mu$ in the tensor product. Thus in the operator sense $[\hat{x}^\mu,\hat{y}^\nu]=0$.[1] Therefore, if we introduce the "center of mass coordinates" $\hat{R}^\mu$ and the "relative coordinates" $\hat{r}^\mu$ defined by

$$\hat{R}^\mu=\frac{\hat{x}^\mu+\hat{y}^\mu}{2}, \quad \hat{r}^\mu=\hat{x}^\mu-\hat{y}^\mu, \tag{2.25}$$

they satisfy the following commutation relations [46]:

$$[\hat{R}^\mu,\hat{R}^\nu]=\frac{i}{2}\theta^{\mu\nu}, \quad [\hat{r}^\mu,\hat{r}^\nu]=2i\theta^{\mu\nu}, \quad [\hat{R}^\mu,\hat{r}^\nu]=0. \tag{2.26}$$

The tensor product $\mathcal{A}_\theta^{1,2}$ can thus be decomposed in the form

$$\mathcal{A}_\theta^{1,2}\cong\mathcal{D}\otimes\mathcal{R} \tag{2.27}$$

where $\hat{R}^\mu$ acts on $\mathcal{D}$ and $\hat{r}^\mu$ on $\mathcal{R}$. Since the noncommutative space (2.1) is homogeneous and so always respects a global translation symmetry, it is reasonable to require translation invariance for the Green's function $\hat{G}^{(0)}(\hat{x},\hat{y})$. In other words the Green's function depends only on $\hat{r}^\mu$. Using the Weyl prescription (2.18), we see that[2]

$$G^{(0)}(x-y)=\int\frac{d^4k}{(2\pi)^4}G^{(0)}(k)e^{ik\cdot(x-y)}, \tag{2.28}$$

$$\delta(x-y)=\int\frac{d^4k}{(2\pi)^4}e^{ik\cdot(x-y)}.$$

Then the defining relation (2.22) implies that $G^{(0)}(k)=1/k^2$.

To discuss more general Green's functions, especially instanton propagators, let us describe the formal procedure defining the Green's function. Let $\Delta$ be a linear operator on $\mathcal{A}_\theta$ with a set of eigenvectors $\phi_r(x)\in\mathcal{A}_\theta$ and corresponding eigenvalues $\lambda_r$:

$$\Delta\phi_r(x)=\lambda_r\phi_r(x), \tag{2.29}$$

where the parameter $r$ can be either continuous or discrete. We shall assume the completeness of $\phi_r(x)$,

$$\mathrm{Tr}_{\mathcal{H}}\phi_r(x)^\dagger\phi_s(x)=\delta_{rs}, \tag{2.30}$$

in the Hilbert space $\mathcal{H}_p$ of one-particle states to be

$$\mathcal{H}_p=\left\{\phi(x)=\sum_r a_r\phi_r(x):\sum_r |a_r|^2<\infty\right\}. \tag{2.31}$$

As usual the $a_r$ become operators (for example, creation or annihilation operators of a particle with quantum number $r$) when the field is quantized. The Green's function is defined as the formal sum

$$G(x,y)=\sum_r \lambda_r^{-1}\phi_r(x)\phi_r(y)^\dagger. \tag{2.32}$$

For the free Green's function in Eq. (2.22), for example, $\phi_k(x)=e^{ik\cdot x}$ and $\lambda_k=k^2$ for the Laplacian $\Delta=-\partial_\mu\partial_\mu$. In this case, the sum over $r$ should be the integration over momenta $k^\mu$ as in Eq. (2.28).

## III. INSTANTON PROPAGATORS IN NONCOMMUTATIVE SPACE

In this section we will generalize the argument in [44] to noncommutative space to construct the propagators for spinor and vector fields in terms of those for the scalar field in a noncommutative instanton background. This generalization is straightforward so one may regard it as a review of Secs. II and III in [44]. This result definitely generalizes that for free fields [49]; the Green's functions for spinor and vector fields propagating in vacuum are determined by the corresponding scalar propagator.

[1]This is consistent with the fact that the Moyal brackets between two sets of independent variables vanish, that is, $x^\mu*y^\nu-y^\nu*x^\mu=0$ since $\partial y^\nu/\partial x^\mu=0$.

[2]From now on, we will delete the caret indicating operators in $\mathcal{A}_\theta$ for notational convenience as long as it does not cause any confusion.

To consider the spinor propagator, let us introduce quaternions defined by

$$\mathbf{x}=x_\mu\sigma^\mu, \quad \overline{\mathbf{x}}=x_\mu\overline{\sigma}^\mu, \tag{3.1}$$

where $\sigma^\mu=(i\tau^a,1)$ and $\overline{\sigma}^\mu=(-i\tau^a,1)=-\sigma^2\sigma^{\mu T}\sigma^2$. The quaternion matrices $\sigma^\mu$ and $\overline{\sigma}^\mu$ have the basic properties

$$\sigma^\mu\overline{\sigma}^\nu=\delta^{\mu\nu}+i\sigma^{\mu\nu}, \quad \sigma^{\mu\nu}=\eta^a_{\mu\nu}\tau^a={}^*\sigma^{\mu\nu}, \tag{3.2}$$

$$\overline{\sigma}^\mu\sigma^\nu=\delta^{\mu\nu}+i\overline{\sigma}^{\mu\nu}, \quad \overline{\sigma}^{\mu\nu}=\overline{\eta}^a_{\mu\nu}\tau^a=-{}^*\overline{\sigma}^{\mu\nu},$$

and

$$\overline{\sigma}^\mu_{\alpha\beta}\sigma^\mu_{\gamma\delta}=\sigma^\mu_{\alpha\beta}\overline{\sigma}^\mu_{\gamma\delta}=2\delta_{\alpha\delta}\delta_{\beta\gamma}, \tag{3.3}$$

$$\sigma^\mu_{\alpha\beta}\sigma^\mu_{\gamma\delta}=\overline{\sigma}^\mu_{\alpha\beta}\overline{\sigma}^\mu_{\gamma\delta}=2\varepsilon_{\alpha\gamma}\varepsilon_{\beta\delta},$$

where $\alpha,\beta,\gamma,\delta=1,2$ are quaternionic indices. The $\sigma^\mu$ and $\overline{\sigma}^\mu$ can be used to construct the Euclidean Dirac matrices as

$$\gamma^\mu=\begin{pmatrix}0 & \overline{\sigma}^\mu\\ \sigma^\mu & 0\end{pmatrix}, \quad \gamma_5=\gamma_1\gamma_2\gamma_3\gamma_4=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \tag{3.4}$$

$$\{\gamma^\mu,\gamma^\nu\}=2\delta^{\mu\nu}, \quad \gamma^{\mu\nu}=\frac{1}{2i}[\gamma^\mu,\gamma^\nu]=\begin{pmatrix}\overline{\sigma}^{\mu\nu} & 0\\ 0 & \sigma^{\mu\nu}\end{pmatrix}.$$

Thus Eqs. (3.2) and (3.4) show that

$${}^*\gamma^{\mu\nu}\frac{1\pm\gamma_5}{2}=\mp\gamma^{\mu\nu}\frac{1\pm\gamma_5}{2}. \tag{3.5}$$

We shall consider the propagator for spinor fields transforming in an arbitrary representation (fundamental, adjoint, etc.) of the U($N$) gauge group in the background of (anti-)self-dual instantons. The covariant derivative $D_\mu$ is defined by

$$D_\mu=\partial_\mu+A_\mu \tag{3.6}$$

and the field strength $F_{\mu\nu}$ is given by

$$F_{\mu\nu}=[D_\mu,D_\nu] =\partial_\mu A_\nu-\partial_\nu A_\mu+[A_\mu,A_\nu]. \tag{3.7}$$

Since we are interested in spinor fields propagating in the background of (anti-)self-dual instantons, we will assume that the field strength satisfies the (anti-)self-duality condition

$$F_{\mu\nu}=\pm{}^*F_{\mu\nu}=\pm\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F_{\rho\sigma}. \tag{3.8}$$

Then we have

$$(\gamma\cdot D)^2\frac{1\pm\gamma_5}{2}=D^2\frac{1\pm\gamma_5}{2}+\frac{i}{2}F^{\mu\nu}\gamma^{\mu\nu}\frac{1\pm\gamma_5}{2}. \tag{3.9}$$

Equation (3.5) forces the second term of the right-hand side of Eq. (3.9) to vanish for spinors with positive (negative) chirality in the self-dual (anti-self-dual) instanton background. In this case there is no zero mode solution satisfying

$$\gamma^\mu D_\mu\psi^{(0)}_\pm=0, \quad \gamma_5\psi^{(0)}_\pm=\pm\psi^{(0)}_\pm. \tag{3.10}$$

However, the second term in Eq. (3.9) does not vanish for a positive (negative) chirality spinor in anti-self-dual (self-dual) instantons. In this case a finite number of zero modes satisfying Eq. (3.10) can be found. In the background of $k$ instantons in U($N$) gauge theory, the number of zero modes is $k$ in the fundamental representation and $2Nk$ in the adjoint representation [42].

We will now consider a spinor field in the background of $k$ anti-self-dual instantons. The self-dual case is obtained simply by changing the sign of $\gamma_5$, $\gamma_5\to-\gamma_5$. Let us introduce eigenfunctions $\psi_r(x)$ such that

$$\gamma^\mu D_\mu\psi_r(x)=\lambda_r\psi_r(x) \tag{3.11}$$

to define the spin-$\frac{1}{2}$ Green's function $S(x,y)$ which is described by the formal expression

$$S(x,y)=\sum_r{}'\ \lambda_r^{-1}\psi_r(x)\psi_r(y)^\dagger \tag{3.12}$$

where the prime means that the zero modes (states with $\lambda_r=0$) are excluded from the sum. It follows from Eq. (3.12) that the spin-$\frac{1}{2}$ propagator is orthogonal to all the zero modes in Eq. (3.10):

$$\mathrm{Tr}^x_{\mathcal{H}}(\psi^{(0)}_n(x)^\dagger S(x,y))=0. \tag{3.13}$$

Thus the spin-$\frac{1}{2}$ propagator obeys the following equation:

$$\gamma^\mu D_\mu S(x,y)=Q(x,y) \tag{3.14}$$

where

$$Q(x,y)=\delta(x,y)-\sum_n\ \psi^{(0)}_n(x)\psi^{(0)}_n(y)^\dagger \tag{3.15}$$

with the summation running over all the zero modes ($n=1,\ldots,k$ for spinors in the fundamental representation and $n=1,\ldots,2Nk$ in the adjoint representation). The quantity $Q(x,y)$ represents the projection operator, i.e., $\mathrm{Tr}^z_{\mathcal{H}}(Q(x,z)Q(z,y))=Q(x,y)$, into the subspace of all nonzero modes.

Using the same operator technique as in [44], the construction of $S(x,y)$ can be easily achieved. Let us introduce an operator $S$ whose matrix representation with regard to position eigenstates in $\mathcal{H}_p$ is $S(x,y)$,

$$\langle x|S|y\rangle=S(x,y). \tag{3.16}$$

Similarly, we write the corresponding spin-0 propagator $G(x,y)$, which is defined by

$$-D_\mu D_\mu G(x,y)=\delta(x-y), \tag{3.17}$$

as the matrix element of an operator $1/-D^2$,

$$\left\langle x \left| \frac{1}{-D^2} \right| y \right\rangle = G(x,y). \tag{3.18}$$

We will show that the operator expression of the spin-$\frac{1}{2}$ propagator is

$$S = -\gamma\cdot D \frac{1}{-D^2}\frac{1-\gamma_5}{2} - \frac{1}{-D^2}\gamma\cdot D \frac{1+\gamma_5}{2}. \tag{3.19}$$

First note that

$$\gamma\cdot DS = Q, \tag{3.20}$$

where

$$Q = \frac{1-\gamma_5}{2} - \gamma\cdot D \frac{1}{-D^2}\gamma\cdot D \frac{1+\gamma_5}{2}. \tag{3.21}$$

Equation (3.20) implies that $Q$ contains no zero modes since they are annihilated by $\gamma\cdot D$. On the other hand, we find that

$$\gamma\cdot DQ = \gamma\cdot D \tag{3.22}$$

and

$$QS = S. \tag{3.23}$$

It is easy to show that $Q^2 = Q$. Therefore Eq. (3.22) shows that $Q$ is the operator that projects into the subspace of all nonzero modes and so $\langle x|Q|y\rangle$ is the function defined in Eq. (3.15). Moreover, Eq. (3.23) implies that $S$ is orthogonal to all the zero modes. This ensures our claim in Eq. (3.19).

Let us consider Yang-Mills theory with gauge group U($N$) with action

$$S = -\frac{1}{2}\mathrm{Tr}_{\mathcal{H}}\mathrm{tr}\left( F_{\mu\nu}F_{\mu\nu} + \frac{1}{\xi}(D_\mu A_\mu)^2 \right) \tag{3.24}$$

and small fluctuations about a classical instanton solution $A_\mu(x)$

$$A'_\mu(x) = A_\mu(x) + \delta A_\mu(x). \tag{3.25}$$

If the action is expanded to second order in $\delta A_\mu$, one can find the following result:

$$S[A'_\mu] \approx S[A_\mu] - \mathrm{Tr}_{\mathcal{H}}\mathrm{tr}\, \delta A_\mu \left[ -D^2\delta_{\mu\nu} - 2F_{\mu\nu} + \left(1 - \frac{1}{\xi}\right) D_\mu D_\nu \right] \delta A_\nu. \tag{3.26}$$

In our previous paper [42] we showed that in a $k$ instanton background there are $4Nk$ adjoint zero modes $\phi_\mu^{(n)}$, $n = 1, \ldots, 4Nk$, satisfying $D_\mu \phi_\mu^{(n)} = 0$ and

$$(D^2\delta_{\mu\nu} + 2F_{\mu\nu})\phi_\nu^{(n)} = 0. \tag{3.27}$$

Thus to define the spin-1 propagator we should project out the zero modes just as in the spin-$\frac{1}{2}$ propagator (3.14). According to the action (3.26), the spin-1 propagator $G_{\mu\nu}(x,y)$ in the anti-self-dual $k$ instanton background is defined by

$$\left[ -D^2\delta_{\mu\lambda} - 2F_{\mu\lambda} + \left(1 - \frac{1}{\xi}\right) D_\mu D_\lambda \right] G_{\lambda\nu}(x,y) = Q_{\mu\nu}(x,y), \tag{3.28}$$

where

$$Q_{\mu\nu}(x,y) = \delta_{\mu\nu}\delta(x-y) - \sum_n \phi_\mu^{(n)}(x)\phi_\nu^{(n)}(y)^\dagger. \tag{3.29}$$

The quantity $Q_{\mu\nu}(x,y)$ is the projection operator, i.e., $\mathrm{Tr}^z_{\mathcal{H}} Q_{\mu\lambda}(x,z) Q_{\lambda\nu}(z,y) = Q_{\mu\nu}(x,y)$, onto the space of the nonzero modes.

Using the operator formalism used in the spin-$\frac{1}{2}$ propagator, one can show that the spin-1 propagator can also be constructed in terms of the corresponding scalar propagator. To proceed with the construction, define

$$q^{(\pm)}_{\mu\nu\lambda\kappa} = \delta_{\mu\nu}\delta_{\lambda\kappa} + \eta^{(\pm)a}_{\mu\nu}\eta^{(\pm)a}_{\lambda\kappa} = \delta_{\mu\lambda}\delta_{\nu\kappa} + \eta^{(\mp)a}_{\mu\lambda}\eta^{(\mp)a}_{\nu\kappa} \tag{3.30}$$

where $\eta^{(+)a}_{\mu\nu} = \eta^a_{\mu\nu}$ and $\eta^{(-)a}_{\mu\nu} = \overline{\eta}^a_{\mu\nu}$ are defined in Eq. (3.2). The tensor

$$\eta^{(\pm)a}_{\mu\nu}\eta^{(\pm)a}_{\lambda\kappa} = \frac{1}{4}(\delta_{\mu\lambda}\delta_{\nu\kappa} - \delta_{\mu\kappa}\delta_{\nu\lambda} \pm \varepsilon_{\mu\nu\lambda\kappa}) \tag{3.31}$$

projects out the self-dual or anti-self-dual part of the antisymmetric tensor since $\eta^{(\pm)a}_{\mu\nu}\eta^{(\mp)b}_{\mu\nu} = 0$. Following [44], let us introduce the bracket operation

$$\{X\}^{(\pm)}_{\mu\nu} = q^{(\pm)}_{\mu\nu\lambda\kappa} D_\lambda X D_\kappa \tag{3.32}$$

for an arbitrary operator $X$. Then it is easy to see that

$$D_\nu\{X\}^{(\pm)}_{\nu\mu} = D^2 X D_\mu + [F_{\mu\nu} \mp {}^*F_{\mu\nu}, X D_\nu]. \tag{3.33}$$

So if the field strength satisfies the self-duality condition (3.8), Eq. (3.33) reduces to

$$D_\nu\{X\}^{(\pm)}_{\nu\mu} = D^2 X D_\mu. \tag{3.34}$$

Similarly,

$$\{X\}^{(\pm)}_{\mu\nu} D_\nu = D_\mu D^2 X. \tag{3.35}$$

Let us quote the following algebraic relation [44]:

$$q^{(\pm)}_{\mu\lambda\nu\sigma} q^{(\pm)}_{\lambda\kappa\tau\rho} = \delta_{\sigma\tau} q^{(\pm)}_{\mu\kappa\nu\rho} + r^{(\pm)}_{\mu\kappa\nu\rho\sigma\tau} \tag{3.36}$$

where

$$r^{(\pm)}_{\mu\kappa\nu\rho\sigma\tau} = (\delta_{\mu\nu}\eta^{(\mp)c}_{\kappa\rho} - \delta_{\kappa\rho}\eta^{(\mp)c}_{\mu\nu} + \varepsilon_{abc}\eta^{(\mp)a}_{\mu\nu}\eta^{(\mp)b}_{\kappa\rho})\eta^{(\mp)c}_{\sigma\tau} \tag{3.37}$$

and thus has the following duality property:

$$\frac{1}{2}\varepsilon_{\sigma\tau\upsilon\lambda}r^{(\pm)}_{\mu\kappa\nu\rho\sigma\tau}=\mp r^{(\pm)}_{\mu\kappa\nu\rho\upsilon\lambda}. \tag{3.38}$$

In the derivation of Eq. (3.37), we used

$$\eta^{(\pm)a}_{\lambda\mu}\eta^{(\pm)b}_{\lambda\nu}=\delta_{ab}\delta_{\mu\nu}+\varepsilon_{abc}\eta^{(\pm)c}_{\mu\nu}. \tag{3.39}$$

Using these properties, the following bracket composition law can be derived:

$$\{X\}^{(\pm)}_{\mu\lambda}\{Y\}^{(\pm)}_{\lambda\nu}=\{XD^2Y\}^{(\pm)}_{\mu\nu}. \tag{3.40}$$

Now it is straightforward to see that $G_{\mu\nu}(x,y)$ has the following formal operator expression

$$G_{\mu\nu}=-\left\{\left(\frac{1}{D^2}\right)^2\right\}_{\mu\nu}+(1-\xi)D_\mu\left(\frac{1}{D^2}\right)^2D_\nu. \tag{3.41}$$

The reason is the following. First note that

$$\left[-D^2\delta_{\mu\lambda}-2F_{\mu\lambda}+\left(1-\frac{1}{\xi}\right)D_\mu D_\lambda\right]G_{\lambda\nu}=Q_{\mu\nu}, \tag{3.42}$$

where

$$Q_{\mu\nu}=\left\{\frac{1}{D^2}\right\}_{\mu\nu}. \tag{3.43}$$

We used Eqs. (3.34) and (3.35) and the bracket composition (3.40) in the derivation. Comparing with Eq. (3.28), we see that $Q_{\mu\nu}$ does not contain any zero modes. And, using the composition law (3.40), one can easily see that $Q_{\mu\nu}$ is a projection operator, i.e., $Q_{\mu\lambda}Q_{\lambda\nu}=Q_{\mu\nu}$. Indeed, $Q_{\mu\nu}$ is the projection operator onto all the nonzero modes in Eq. (3.29) and thus an operator realization of the projector $Q_{\mu\nu}(x,y)$ since it satisfies the following equations:

$$Q_{\mu\lambda}G_{\lambda\nu}=G_{\mu\nu},$$

$$\left[-D^2\delta_{\mu\lambda}-2F_{\mu\lambda}+\left(1-\frac{1}{\xi}\right)D_\mu D_\lambda\right]Q_{\lambda\nu}=\left[-D^2\delta_{\mu\nu}-2F_{\mu\nu}+\left(1-\frac{1}{\xi}\right)D_\mu D_\nu\right]. \tag{3.44}$$

Thus we complete the proof of our claim in Eq. (3.41).

## IV. SCALAR INSTANTON PROPAGATORS

In order to calculate instanton effects in quantum gauge theory, it is important to know the Green's function in instanton backgrounds [6]. In the previous section, following the same method as in [44], we showed that the propagators for spinor and vector fields can be constructed in terms of those for the scalar field in a noncommutative instanton background. Thus, if we can find the scalar propagator $G(x,y)$ [Eq. (3.17)] for the fundamental representation or adjoint representation, we know the spin-$\frac{1}{2}$ propagator $S(x,y)$ for each representation in terms of Eq. (3.19) and the spin-1 propagator $G_{\mu\nu}(x,y)$ in terms of Eq. (3.41). In commutative space, the scalar propagator in the fundamental representation has a remarkably simple expression [50,51]:

$$G(x,y)=v(x)^\dagger G^{(0)}(x,y)v(y) \tag{4.1}$$

where $v(x)$ is a function determining the ADHM gauge field $A_\mu(x)$ by $A_\mu(x)=v(x)^\dagger\partial_\mu v(x)$. The scalar propagator in the adjoint representation has a more complicated expression of which we will present the explicit form. We will first show that the scalar propagator in the noncommutative instanton background has exactly the same form as Eq. (4.1).

To derive the above remarkable formula, we need the following basic properties in the ADHM construction [18,50,51]. The gauge field with instanton number $k$ for the U($N$) gauge group is given in the form

$$A_\mu(x)=v(x)^\dagger\partial_\mu v(x) \tag{4.2}$$

where $v(x)$ is the $(N+2k)\times N$ matrix defined by the equations

$$v(x)^\dagger v(x)=1, \tag{4.3}$$

$$v(x)^\dagger\Delta(x)=0. \tag{4.4}$$

In Eq. (4.4), $\Delta(x)$ is an $(N+2k)\times 2k$ matrix, linear in the position variable $x$, having the structure

$$\Delta(x)=\begin{cases} a-b\mathbf{x}, & \text{self-dual instantons,}\\ a-b\bar{\mathbf{x}}, & \text{anti-self-dual instantons,}\end{cases} \tag{4.5}$$

where $a,b$ are $(N+2k)\times 2k$ matrices. $v(x)$ can be thought of as a map from an $N$-complex dimensional space $W$ to an $(N+2k)$-complex dimensional space $V$. Thus $\Delta(x)$ must obey the completeness relation

$$P(x)+\Delta(x)f(x)\Delta(x)^\dagger=1 \tag{4.6}$$

where $P(x)=v(x)v(x)^\dagger$. The matrices $a,b$ are constrained to satisfy the conditions that $\Delta(x)^\dagger\Delta(x)$ be invertible and that it commutes with the quaternions. These conditions imply that $\Delta(x)^\dagger\Delta(x)$ as a $2k\times 2k$ matrix has to be factorized as follows:

$$\Delta(x)^\dagger\Delta(x)=f^{-1}(x)\otimes 1_2 \tag{4.7}$$

where $f^{-1}(x)$ is a $k\times k$ matrix and $1_2$ is a unit matrix in quaternion space.

Given a pair of matrices $a,b$, Eqs. (4.3) and (4.4) define $A_\mu$ up to gauge equivalence. Different pairs of matrices $a,b$ may yield gauge equivalent $A_\mu$ since Eqs. (4.3) and (4.4) are invariant under

$$a\to QaK, \quad b\to QbK, \quad v\to Qv \tag{4.8}$$

where $Q\in U(N+2k)$ and $K\in GL(k,\mathbf{C})$. This freedom can be used to put $a,b$ in the canonical forms

$$a=\begin{pmatrix}\lambda\\ \xi\end{pmatrix}, \quad b=\begin{pmatrix}0\\ 1_{2k}\end{pmatrix}, \tag{4.9}$$

where $\lambda$ is an $N\times 2k$ matrix and $\xi$ is a $2k\times 2k$ matrix. Here we decompose the matrix $\xi$ in the quaternionic basis $\bar{\sigma}^\mu$ as a matter of convenience,

$$\xi=\xi_\mu\bar{\sigma}^\mu, \tag{4.10}$$

where the $\xi_\mu$'s are $k\times k$ matrices. In the basis (4.9), the constraint (4.7) boils down to

$$\mathrm{tr}_2\tau^a a^\dagger a=\begin{cases}\theta^{\mu\nu}\bar{\eta}^a_{\mu\nu}, & \text{self-dual instantons,}\\ \theta^{\mu\nu}\eta^a_{\mu\nu}, & \text{anti-self-dual instantons,}\end{cases} \tag{4.11}$$

$$\xi^\dagger_\mu=\xi_\mu, \tag{4.12}$$

where $\mathrm{tr}_2$ is the trace over the quaternionic indices.

### A. Scalar propagator in fundamental representation

Now we will explain how to derive the formula (4.1). First note that the covariant derivative for a field $\Phi$ in the fundamental representation of U($N$) has the simple expression

$$D_\mu\Phi=(\partial_\mu+v^\dagger\partial_\mu v)\Phi=v^\dagger\partial_\mu(v\Phi). \tag{4.13}$$

Using this relation, let us calculate $-D_\mu D_\mu G(x,y)$,

$$\begin{aligned}-D_\mu D_\mu&(v(x)^\dagger G^{(0)}(x,y)v(y))\\ &=-v(x)^\dagger\partial_\mu(P(x)\partial_\mu(P(x)G^{(0)}(x,y)v(y))).\end{aligned} \tag{4.14}$$

Note that $v(x)^\dagger P(x)=v(x)^\dagger$, so

$$\begin{aligned}-D_\mu D_\mu G(x,y)=&-v(x)^\dagger\partial_\mu(P(x)\partial_\mu P(x))G^{(0)}(x,y)v(y)\\ &-2v(x)^\dagger(\partial_\mu P(x)\partial_\mu G^{(0)}(x,y))v(y)\\ &-v(x)^\dagger\partial_\mu\partial_\mu G^{(0)}(x,y)v(y).\end{aligned} \tag{4.15}$$

Let us calculate the first term of the right-hand side in Eq. (4.15):

$$\begin{aligned}v(x)^\dagger&\partial_\mu(P(x)\partial_\mu P(x))\\ &=-v(x)^\dagger\partial_\mu(P(x)\partial_\mu\Delta(x)f(x)\Delta(x)^\dagger)\\ &=v(x)^\dagger\partial_\mu\Delta(x)f(x)\Delta(x)^\dagger\partial_\mu\Delta(x)f(x)\Delta(x)^\dagger\\ &\quad-v(x)^\dagger\partial_\mu\Delta(x)\partial_\mu f(x)\Delta(x)^\dagger\\ &\quad-v(x)^\dagger\partial_\mu\Delta(x)f(x)\partial_\mu\Delta(x)^\dagger\end{aligned} \tag{4.16}$$

where we used Eqs. (4.4) and (4.6). Also note that

$$\partial_\mu f(x)=-f(x)(\partial_\mu\Delta(x)^\dagger\Delta(x)+\Delta(x)^\dagger\partial_\mu\Delta(x))f(x) \tag{4.17}$$

from Eq. (4.7).

For explicit calculation, let us take the anti-self-dual instanton with $\Delta(x)=a-b\bar{\mathbf{x}}$. The self-dual case can be similarly calculated. Using $\partial_\mu\Delta(x)=-b\bar{\sigma}_\mu$ and $\partial_\mu\Delta(x)^\dagger=-\sigma_\mu b^\dagger$ and the formulas

$$\bar{\sigma}_\mu\Delta(x)^\dagger b\bar{\sigma}_\mu=-2b^\dagger\Delta(x),\quad \bar{\sigma}_\mu f(x)\sigma_\mu=4f(x), \tag{4.18}$$

we arrive at

$$v(x)^\dagger\partial_\mu(P(x)\partial_\mu P(x))=-4v(x)^\dagger bf(x)b^\dagger, \tag{4.19}$$

where we used the fact that the function $f(x)$ commutes with $\bar{\sigma}_\mu$ and $\sigma_\mu$. Then our original equation (4.15) reduces to

$$\begin{aligned}-D_\mu D_\mu G(x,y)=&2v(x)^\dagger(2bf(x)b^\dagger G^{(0)}(x,y)\\ &-\partial_\mu P(x)\partial_\mu G^{(0)}(x,y))v(y)+\delta(x-y)\end{aligned} \tag{4.20}$$

where $-\partial_\mu\partial_\mu G^{(0)}(x,y)=\delta(x-y)$ is used. Note that the whole procedure above until Eq. (4.20) is totally valid even for noncommutative space.

To arrive at our final destination (3.17), we must show that

$$v(x)^\dagger(2bf(x)b^\dagger G^{(0)}(x,y)-\partial_\mu P(x)\partial_\mu G^{(0)}(x,y))v(y)=0. \tag{4.21}$$

First let us show Eq. (4.21) in commutative $\mathbf{R}^4$, where we do not have to worry about the ordering problem, which will also be helpful in finding the noncommutative version. If one notices that $\Delta(y)^\dagger v(y)=0$ and

$$\partial_\mu G^{(0)}(x,y)=-G^{(0)}(x,y)\frac{2(x-y)_\mu}{(x-y)^2}, \tag{4.22}$$

the second term of Eq. (4.21) can be written as

$$\begin{aligned}v(x)^\dagger&\partial_\mu P(x)\partial_\mu G^{(0)}(x,y)v(y)\\ &=v(x)^\dagger b\bar{\sigma}_\mu f(x)\partial_\mu G^{(0)}(x,y)(\Delta(x)^\dagger-\Delta(y)^\dagger)v(y)\\ &=2v(x)^\dagger bf(x)\bar{\sigma}_\mu\sigma_\nu b^\dagger G^{(0)}(x,y)\frac{(x-y)_\mu}{(x-y)^2}(x-y)_\nu v(y).\end{aligned} \tag{4.23}$$

Since $\bar{\sigma}_\mu\sigma_\nu=\delta_{\mu\nu}+i\bar{\sigma}_{\mu\nu}$, Eq. (4.23) exactly cancels the first term in Eq. (4.21). Thus we proved Eq. (4.1) in commutative $\mathbf{R}^4$.

Before going on to noncommutative space, let us explain why we expect Eq. (4.1) even for noncommutative space. The relation (3.6) implies that, if we define

$$\hat{\Phi}=v\Phi\qquad\text{and}\qquad D_\mu\hat{\Phi}=\widehat{D_\mu\Phi}, \tag{4.24}$$

we get

$$D_\mu\hat{\Phi}=P\partial_\mu\hat{\Phi}. \tag{4.25}$$

We may interpret this result as follows [50]. The matrix $v$: $W\rightarrow V$ maps $\Phi$ in the $N$-dimensional complex vector space

$W$ to $\hat{\Phi}$ in an $(N+2k)$-dimensional complex vector space $V$ which lies in an $N$-dimensional subspace of $\mathbf{C}^{N+2k}$, i.e., the subspace

$$E_x=\{\xi|P(x)\xi=\xi\} \tag{4.26}$$

orthogonal to $\Delta(x)$ onto which $P$ is the projection operator. The collection of spaces $\{E_x\}$ as $x$ varies over $\mathbf{R}^4$ or $\mathbf{R}^4_{NC}$ forms a vector bundle and this vector bundle precisely defines the ADHM gauge fields $A_\mu(x)$ through Eq. (4.25). This is a statement of the Serre-Swan theorem [52]; the vector bundle over a $C^*$ algebra $\mathcal{A}$ (which is a complex Banach algebra with adjoint operation) is a finitely generated projective module. (A module $\mathcal{E}$ is projective if there exists another module $\mathcal{F}$ such that the direct sum $\mathcal{E}\oplus\mathcal{F}$ is free, i.e., $\mathcal{E}\oplus\mathcal{F}\cong\mathcal{A}\otimes\cdots\otimes\mathcal{A}$ as right $\mathcal{A}$ module.) Thus one can imagine that the Green's function $G(x,y)$ for the field $\Phi$ living in the "nontrivial" $N$-dimensional vector space $W$, which is defined as $G(x,y)=\langle\Phi(x),\Phi(y)\rangle$, is obtained by the map $v$: $W\to V$ from the Green's function $G^{(0)}(x,y)\equiv\langle\hat{\Phi}(x),\hat{\Phi}(y)\rangle$ for the field $\hat{\Phi}$ living in the "free" $(N+2k)$-dimensional vector space $V$. This is precisely the content of Eq. (4.1). Note, however, that this argument should also be valid for a noncommutative space. This is the reason why we expect the propagator (4.1) even for a noncommutative instanton background.

To show Eq. (4.21) in the noncommutative space (2.1), let us follow the previous commutative calculation keeping in mind the ordering due to the noncommutativity. The second term in Eq. (4.21) can be written as

$$\begin{aligned}
v(x)^\dagger\partial_\mu P(x)\partial_\mu G^{(0)}(x,y)v(y)&=v(x)^\dagger b\bar{\sigma}_\mu f(x)\Delta(x)^\dagger\partial_\mu G^{(0)}(x,y)v(y)\\
&=v(x)^\dagger bf(x)\bar{\sigma}_\mu\partial_\mu G^{(0)}(x,y)(\Delta(x)^\dagger-\Delta(y)^\dagger)v(y)+v(x)^\dagger bf(x)\bar{\sigma}_\mu[\Delta(x)^\dagger,\partial_\mu G^{(0)}(x,y)]v(y)\\
&=-v(x)^\dagger bf(x)\bar{\sigma}_\mu\sigma_\nu b^\dagger\partial_\mu G^{(0)}(x,y)(x-y)_\nu v(y)-v(x)^\dagger bf(x)\bar{\sigma}_\mu\sigma_\nu b^\dagger[x_\nu,\partial_\mu G^{(0)}(x,y)]v(y)\\
&=-v(x)^\dagger bf(x)\bar{\sigma}_\mu\sigma_\nu b^\dagger(x-y)_\nu\partial_\mu G^{(0)}(x,y)v(y)-v(x)^\dagger bf(x)\bar{\sigma}_\mu\sigma_\nu b^\dagger[y_\nu,\partial_\mu G^{(0)}(x,y)]v(y)\\
&=-\frac{1}{2}v(x)^\dagger bf(x)\bar{\sigma}_\mu\sigma_\nu b^\dagger(\partial_\mu G^{(0)}(x,y)(x-y)_\nu+(x-y)_\nu\partial_\mu G^{(0)}(x,y))v(y)\\
&\quad-\frac{1}{2}v(x)^\dagger bf(x)\bar{\sigma}_\mu\sigma_\nu b^\dagger[(x+y)_\nu,\partial_\mu G^{(0)}(x,y)]v(y)\\
&=-\frac{1}{2}v(x)^\dagger bf(x)\bar{\sigma}_\mu\sigma_\nu b^\dagger(\partial_\mu G^{(0)}(x,y)(x-y)_\nu+(x-y)_\nu\partial_\mu G^{(0)}(x,y))v(y).
\end{aligned}\tag{4.27}$$

In the last step, we used the fact that the function $\partial_\mu G^{(0)}(x,y)$ depends only on the combination $(x-y)$ because of translation invariance and Eq. (2.26).

In order to calculate the right-hand side of Eq. (4.27), we will use the Weyl symmetric prescription (2.18):

$$\begin{aligned}
&\frac{1}{2}\bar{\sigma}_\mu\sigma_\nu(\partial_\mu G^{(0)}(x,y)(x-y)_\nu+(x-y)_\nu\partial_\mu G^{(0)}(x,y))\\
&=\int\frac{d^4k}{(2\pi)^4}\bar{\sigma}_\mu\sigma_\nu\frac{k_\mu}{k^2}\frac{\partial}{\partial k^\nu}e^{ik\cdot(x-y)}\\
&=\int\frac{d^4k}{(2\pi)^4}\bar{\sigma}_\mu\sigma_\nu\frac{\partial}{\partial k^\nu}\left(\frac{k_\mu}{k^2}e^{ik\cdot(x-y)}\right)\\
&\quad-\int\frac{d^4k}{(2\pi)^4}\bar{\sigma}_\mu\sigma_\nu\frac{1}{k^2}\left(\delta_{\mu\nu}-2\frac{k_\mu k_\nu}{k^2}\right)e^{ik\cdot(x-y)}\\
&=\int\frac{d^4k}{(2\pi)^4}\bar{\sigma}_\mu\sigma_\nu\frac{\partial}{\partial k^\nu}\left(\frac{k_\mu}{k^2}e^{ik\cdot(x-y)}\right)-2G^{(0)}(x,y).
\end{aligned}\tag{4.28}$$

Thus if we can show that the total divergence

$$\int\frac{d^4k}{(2\pi)^4}\bar{\sigma}_\mu\sigma_\nu\frac{\partial}{\partial k^\nu}\left(\frac{k_\mu}{k^2}e^{ik\cdot(x-y)}\right)\equiv K(x-y)$$

vanishes on $\mathbf{S}^3$ in the large $k$ limit, we can finally achieve our goal (4.21) in the noncommutative space. It is easy to show directly in the basis of the tensor product $|\xi_1,\xi_2\rangle=|\xi_1\rangle\otimes|\xi_2\rangle$ of coherent states such as (2.14), using (2.17), that the function $\langle\xi_1,\xi_2|K(x-y)|\xi_1,\xi_2\rangle$ vanishes for any $|\xi_1,\xi_2\rangle$. This means that the operator function $K(x-y)$ should vanish even in noncommutative space.

### B. Scalar propagator in adjoint representation

Next let us consider the scalar propagator in the adjoint representation. If $q$ denotes the fundamental representation of U$(N)$, the adjoint representation can be obtained by the tensor product $q\otimes\bar{q}$, for which

$$D_\mu=\partial_\mu+A_\mu\otimes1+1\otimes\bar{A}_\mu. \tag{4.29}$$

In other words, we regard a field in the adjoint representation as a two-index object, one index transforming according to the fundamental representation and the other its complex conjugate. Motivated by this fact and to follow the method in [45], we treat this problem in a more general context. Consider the direct product $G_1\times G_2$ and suppose we have instanton solutions

$$A^1_\mu(x)=v_1(x)^\dagger\partial_\mu v_1(x),\quad A^2_\mu(x)=v_2(x)^\dagger\partial_\mu v_2(x), \tag{4.30}$$

described in the ADHM way for each gauge group. Also consider a field transforming under the fundamental representation of each; thus its covariant derivative is defined by

$$D_\mu=\partial_\mu+A^1_\mu\otimes 1+1\otimes A^2_\mu. \tag{4.31}$$

The adjoint representation of $\mathrm{U}(N)$ would be obtained by taking $G_1=G_2=\mathrm{U}(N)$ and $A^1_\mu=A^2_\mu=A_\mu$.

The Green's function for a tensor product should be obtained by solving Eq. (3.17) with $D_\mu$ defined by Eq. (4.31). Thus we will also consider the tensor product

$$v(x)=v_1(x)\otimes v_2(x) \tag{4.32}$$

of two independent fields $v_1(x),v_2(x)$ in the fundamental representation of $G_1$ and $G_2$, respectively. To preserve the group structure for $G_1\times G_2$, i.e., $(g_1,g_2)(h_1,h_2)=(g_1h_1,g_2h_2)\in G_1\times G_2$ for all $g_1,h_1\in G_1$ and $g_2,h_2\in G_2$, we define a (unique) multiplication between elements of $G_1\times G_2$ such that

$$(\phi_1(x)\otimes\phi_2(x))(\chi_1(x)\otimes\chi_2(x))=(\phi_1(x)\chi_1(x))\otimes(\phi_2(x)\chi_2(x)) \tag{4.33}$$

for all $\phi_1(x),\chi_1(x)\in G_1$ and $\phi_2(x),\chi_2(x)\in G_2$. This multiplication law will be crucial for our calculation of the adjoint Green's function. The commutative Green's function $G(x,y)$ satisfying Eq. (3.17) for $G_1\times G_2$ was previously constructed in [45] and is of the form

$$G(x,y)=[v_1(x)\otimes v_2(x)]^\dagger G^{(0)}(x,y)[v_1(y)\otimes v_2(y)]+\frac{1}{4\pi^2}C(x,y). \tag{4.34}$$

We will specify the explicit form of $C(x,y)$ later.

We will try the same expression as (4.34) to construct the noncommutative instanton propagator. To calculate Eq. (3.17) for the ansatz (4.34), first note that, using the multiplication law (4.33),

$$D_\mu v(x)^\dagger=v_1(x)^\dagger\partial_\mu P_1(x)\otimes v_2(x)^\dagger+v_1(x)^\dagger\otimes v_2(x)^\dagger\partial_\mu P_2(x), \tag{4.35}$$

$$D_\mu D_\mu v(x)^\dagger=v_1(x)^\dagger\partial_\mu(v_1(x)^\dagger\partial_\mu P_1(x))\otimes v_2(x)^\dagger+v_1(x)^\dagger\otimes v_2(x)^\dagger\partial_\mu(v_2(x)^\dagger\partial_\mu P_2(x))+2v_1(x)^\dagger\partial_\mu P_1(x)\otimes v_2(x)^\dagger\partial_\mu P_2(x), \tag{4.36}$$

where we defined

$$P_a(x)=v_a(x)v_a(x)^\dagger,\qquad a=1,2. \tag{4.37}$$

If we further define

$$P(x)=P_1(x)\otimes P_2(x), \tag{4.38}$$

the above covariant derivatives can be rewritten as, using Eq. (4.33) again,

$$D_\mu v(x)^\dagger=v(x)^\dagger\partial_\mu P(x), \tag{4.39}$$

$$D_\mu D_\mu v(x)^\dagger=v(x)^\dagger\partial_\mu(P(x)\partial_\mu P(x)), \tag{4.40}$$

where we used $v_1^\dagger(x)P_1(x)=v_1^\dagger(x)$ and $v_2^\dagger(x)P_2(x)=v_2^\dagger(x)$. Thus we can proceed with the calculation of $-D_\mu D_\mu(v(x)^\dagger G^{(0)}(x,y)v(y))$ in the same way as in Sec. III A. Consequently, we get

$$-D_\mu D_\mu(v(x)^\dagger G^{(0)}(x,y)v(y))=\delta(x-y)-2(v_1(x)^\dagger\partial_\mu P_1(x)\otimes v_2(x)^\dagger\partial_\mu P_2(x))\times G^{(0)}(x,y)(v_1(y)\otimes v_2(y)). \tag{4.41}$$

Let us calculate the second term in Eq. (4.41). For explicit calculation, let us take the anti-self-dual instanton with $\Delta_a(x)=a_a-b_a\bar{\mathbf{x}}$. The self-dual case can be similarly done. Note that, according to ADHM construction, $v_a(x)^\dagger\partial_\mu P_a(x)=v_a(x)^\dagger b_a\bar{\sigma}_\mu f_a(x)\Delta_a(x)^\dagger$. Again the multiplication law (4.33) will have a crucial role in the calculation below:

$$\begin{aligned}
&-2(v_1(x)^\dagger\partial_\mu P_1(x)\otimes v_2(x)^\dagger\partial_\mu P_2(x))G^{(0)}(x,y)v(y)\\
&=-2(v_1(x)^\dagger b_1\bar{\sigma}_\mu f_1(x)\Delta_1(x)^\dagger\otimes v_2(x)^\dagger\partial_\mu P_2(x))G^{(0)}(x,y)v(y)\\
&=-2(v_1(x)^\dagger b_1\bar{\sigma}_\mu f_1(x)\otimes v_2(x)^\dagger\partial_\mu P_2(x))([\Delta_1(x)^\dagger\otimes 1,G^{(0)}(x,y)]+G^{(0)}(x,y)(\Delta_1(x)^\dagger-\Delta_1(y)^\dagger)\otimes 1)v(y)\\
&=-2(v_1(x)^\dagger b_1\bar{\sigma}_\mu f_1(x)\otimes v_2(x)^\dagger\partial_\mu P_2(x))([\sigma_\nu b_1^\dagger y_\nu\otimes 1,G^{(0)}(x,y)]+(\sigma_\nu b_1^\dagger(x-y)_\nu\otimes 1)G^{(0)}(x,y))v(y)\\
&=-(v_1(x)^\dagger b_1\bar{\sigma}_\mu f_1(x)\otimes v_2(x)^\dagger\partial_\mu P_2(x))(G^{(0)}(x,y)(\sigma_\nu b_1^\dagger(x-y)_\nu\otimes 1)+(\sigma_\nu b_1^\dagger(x-y)_\nu\otimes 1)G^{(0)}(x,y))v(y).
\end{aligned} \tag{4.42}$$

In the last step, we used the fact that the function $\partial_\mu G^{(0)}(x,y)$ depends only on the combination $(x-y)$ and Eq. (2.26). We can repeat the same procedure as in Eq. (4.42) for the term $v_2(x)^\dagger \partial_\mu P_2(x)$. Then we get

$$\begin{aligned}
&-2(v_1(x)^\dagger \partial_\mu P_1(x)\otimes v_2(x)^\dagger \partial_\mu P_2(x))G^{(0)}(x,y)v(y)\\
&\quad=-\frac{1}{2}(v_1(x)^\dagger b_1\bar{\sigma}_\mu f_1(x)\otimes v_2(x)^\dagger b_2\bar{\sigma}_\mu f_2(x))\\
&\qquad\times(F^{(0)}(x,y)(1\otimes\sigma_\lambda b_2^\dagger(x-y)_\lambda)\\
&\qquad+(1\otimes\sigma_\lambda b_2^\dagger(x-y)_\lambda)F^{(0)}(x,y))v(y),
\end{aligned}\tag{4.43}$$

where $F^{(0)}(x,y)=G^{(0)}(x,y)(\sigma_\nu b_1^\dagger(x-y)_\nu\otimes 1)+(\sigma_\nu b_1^\dagger(x-y)_\nu\otimes 1)G^{(0)}(x,y)$.

In order to calculate the right-hand side of Eq. (4.43), we will again use the Weyl symmetric prescription (2.18). First note that

$$F^{(0)}(x,y)=-2i(\sigma_\nu b_1^\dagger\otimes 1)\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2}\frac{\partial}{\partial k^\nu}e^{ik\cdot(x-y)}\tag{4.44}$$

and

$$\begin{aligned}
&F^{(0)}(x,y)(1\otimes\sigma_\lambda b_2^\dagger(x-y)_\lambda)+(1\otimes\sigma_\lambda b_2^\dagger(x-y)_\lambda)F^{(0)}(x,y)\\
&\quad=-4(\sigma_\nu b_1^\dagger\otimes\sigma_\lambda b_2^\dagger)\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2}\frac{\partial^2}{\partial k^\nu\partial k^\lambda}e^{ik\cdot(x-y)}\\
&\quad\equiv-4(\sigma_\nu b_1^\dagger\otimes\sigma_\lambda b_2^\dagger)H_{\nu\lambda}(x,y).
\end{aligned}\tag{4.45}$$

Then we arrive at

$$\begin{aligned}
&-2(v_1(x)^\dagger \partial_\mu P_1(x)\otimes v_2(x)^\dagger \partial_\mu P_2(x))G^{(0)}(x,y)v(y)\\
&\quad=2(v_1(x)^\dagger b_1\bar{\sigma}_\mu\sigma_\nu f_1(x)b_1^\dagger\otimes v_2(x)^\dagger b_2\bar{\sigma}_\mu\sigma_\lambda f_2(x)b_2^\dagger)\\
&\qquad\times H_{\nu\lambda}(x,y)v(y),\\
&\quad=2(v_1(x)^\dagger b_1\bar{\sigma}_\mu f_1(x)b_1^\dagger\otimes v_2(x)^\dagger b_2\bar{\sigma}_\mu f_2(x)b_2^\dagger)\\
&\qquad\times H_{\nu\nu}(x,y)v(y).
\end{aligned}\tag{4.46}$$

To derive the last result in Eq. (4.46), we used Eq. (3.2), Eq. (3.39), and the fact that $H_{\nu\lambda}(x,y)=H_{\lambda\nu}(x,y)$.

In order to calculate $H_{\nu\nu}(x,y)$, we introduce an infrared cutoff $\epsilon$, i.e., $1/k^2\to 1/k^2+\epsilon$. After performing the integral, we will take the limit $\epsilon\to 0$:

$$\begin{aligned}
H_{\nu\nu}(x,y)&=\lim_{\epsilon\to 0}\int\frac{d^4k}{(2\pi)^4}\frac{\partial}{\partial k_\nu}\left(\frac{1}{k^2+\epsilon}\frac{\partial}{\partial k^\nu}e^{ik\cdot(x-y)}\right.\\
&\quad\left.+\frac{2k^\nu}{(k^2+\epsilon)^2}e^{ik\cdot(x-y)}\right)\\
&\quad-\lim_{\epsilon\to 0}\int\frac{d^4k}{(2\pi)^4}\frac{8\epsilon}{(k^2+\epsilon)^3}e^{ik\cdot(x-y)}.
\end{aligned}\tag{4.47}$$

One can check that

$$\lim_{\epsilon\to 0}\frac{2}{\pi^2}\frac{\epsilon}{(k^2+\epsilon)^3}=\delta(k).\tag{4.48}$$

Using the coherent state basis $|\xi_1,\xi_2\rangle=|\xi_1\rangle\otimes|\xi_2\rangle$ as in Eq. (4.28), it is easy to see that the total divergence in Eq. (4.47) vanishes on $\mathbf{S}^3$ in the large $k$ limit. So we have

$$H_{\nu\nu}(x,y)=-\frac{1}{4\pi^2}.\tag{4.49}$$

Finally, we get

$$\begin{aligned}
&-D_\mu D_\mu(v(x)^\dagger G^{(0)}(x,y)v(y))\\
&\quad=\delta(x-y)-\frac{1}{2\pi^2}(v_1(x)^\dagger b_1\bar{\sigma}_\mu f_1(x)b_1^\dagger\\
&\qquad\otimes v_2(x)^\dagger b_2\bar{\sigma}_\mu f_2(x)b_2^\dagger)v(y).
\end{aligned}\tag{4.50}$$

Note that the above expression has exactly the same form as in the commutative case.

Therefore, in order to get the answer (3.17) for the spin-1 propagator (4.34), we must show that

$$\begin{aligned}
-D_\mu D_\mu C(x,y)&=4((v_1(x)^\dagger b_1 f_1(x))_\alpha\otimes(v_2(x)^\dagger b_2 f_2(x))_\beta)\\
&\quad\times((b_1^\dagger v_1(y))_\gamma\otimes(b_2^\dagger v(y))_\delta)\varepsilon_{\alpha\beta}\varepsilon_{\gamma\delta},
\end{aligned}\tag{4.51}$$

where we used Eqs. (3.3) and (4.33). Since the right-hand side of Eq. (4.51) has exactly the same form as the commutative one, we will take the same ansatz for $C(x,y)$ as in the commutative case [45]:

$$\begin{aligned}
C_{us,vt}(x,y)&=M_{ij,lm}(v_1(x)^\dagger b_1)_{u,i\alpha}(v_2(x)^\dagger b_2)_{s,j\beta}\\
&\quad\times(b_1^\dagger v_1(y))_{l\gamma,v}(b_2^\dagger v_2(y))_{m\delta,t}\varepsilon_{\alpha\beta}\varepsilon_{\gamma\delta},
\end{aligned}\tag{4.52}$$

or in tensor notation

$$\begin{aligned}
C(x,y)&=((v_1(x)^\dagger b_1)_\alpha\otimes(v_2(x)^\dagger b_2)_\beta)\\
&\quad\times M((b_1^\dagger v_1(y))_\gamma\otimes(b_2^\dagger v_2(y))_\delta)\varepsilon_{\alpha\beta}\varepsilon_{\gamma\delta},
\end{aligned}\tag{4.53}$$

where $M_{ij,lm}$ is a constant matrix to be determined later, $u,v=1,\ldots,N_1=\dim G_1$, $s,t=1,\ldots,N_2=\dim G_2$ are group indices, and $i,l=1,\ldots,k_1,j,m=1,\ldots,k_2$ are instanton number indices. Using the formulas [see Eqs. (3.6) and (4.19)]

$$\begin{aligned}
D_\mu v_a(x)^\dagger&=v_a(x)^\dagger b_a\bar{\sigma}_\mu f_a(x)\Delta_a(x)^\dagger,\\
D_\mu D_\mu v_a(x)^\dagger&=-4v_a(x)^\dagger b_a f_a(x)b_a^\dagger,
\end{aligned}\tag{4.54}$$

and the multiplication (4.33), it is straightforward to show that

$$-D_\mu D_\mu C(x,y)=4((v_1(x)^\dagger b_1 f_1(x))_\alpha\otimes(v_2(x)^\dagger b_2 f_2(x))_\beta)(b_1^\dagger b_1\otimes f_2^{-1}(x)+f_1^{-1}(x)\otimes b_2^\dagger b_2-(\Delta_1(x)^\dagger b_1)_{\eta\xi}$$
$$\otimes(\Delta_2(x)^\dagger b_2)_{\chi\zeta}\varepsilon_{\eta\chi}\varepsilon_{\xi\zeta})M((b_1^\dagger v_1(y))_\gamma\otimes(b_2^\dagger v_2(y))_\delta)\varepsilon_{\alpha\beta}\varepsilon_{\gamma\delta}, \tag{4.55}$$

where we used Eq. (3.3) in the final stage. Thus if the matrix $M$ satisfies

$$(b_1^\dagger b_1\otimes f_2^{-1}(x)+f_1^{-1}(x)\otimes b_2^\dagger b_2 -(\Delta_1(x)^\dagger b_1)_{\eta\xi}\otimes(\Delta_2(x)^\dagger b_2)_{\chi\zeta}\varepsilon_{\eta\chi}\varepsilon_{\xi\zeta})=M^{-1}, \tag{4.56}$$

we finally prove Eq. (4.51) and so Eq. (3.17) for the tensor product $G_1\times G_2$, where the adjoint representation is a special case. However, to achieve this final goal, we should go further since Eq. (4.56) appears to state that the constant matrix $M$ is the inverse of an $x$-dependent matrix. In commutative space, as a result of conformal invariance of the matrix $M$ [45], the $x$-dependent parts of the left-hand side in Eq. (4.56) are completely canceled. We will show that this also the case even for noncommutative space, but the matrix $M$ is slightly modified by the noncommutativity.

First note that, in the canonical basis (4.9),

$$f_a^{-1}(x)=\Delta_a(x)^\dagger\Delta_a(x) =a_a^\dagger a_a-\xi_a^\mu x_\mu+x^2-\frac{1}{2}\sigma_{\mu\nu}\theta^{\mu\nu} \tag{4.57}$$

and

$$(\Delta_1(x)^\dagger b_1)_{\eta\xi}\otimes(\Delta_2(x)^\dagger b_2)_{\chi\zeta}\varepsilon_{\eta\chi}\varepsilon_{\xi\zeta} =(a_1^\dagger b_1)_{\eta\xi}\otimes(a_2^\dagger b_2)_{\chi\zeta}\varepsilon_{\eta\chi}\varepsilon_{\xi\zeta}-2x_\mu(\xi_1^\mu\otimes 1) -2x_\mu(1\otimes\xi_2^\mu)+2x^2. \tag{4.58}$$

Using these results, it is easy to see that the $x$-dependent parts of the left-hand side in Eq. (4.56) are completely canceled and the matrix $M$ is defined by

$$\left(b_1^\dagger b_1\otimes\left(a_2^\dagger a_2-\frac{1}{2}\sigma_{\mu\nu}\theta^{\mu\nu}\right)+\left(a_1^\dagger a_1-\frac{1}{2}\sigma_{\mu\nu}\theta^{\mu\nu}\right)\otimes b_2^\dagger b_2-(a_1^\dagger b_1)_{\eta\xi}\otimes(a_2^\dagger b_2)_{\chi\zeta}\varepsilon_{\eta\chi}\varepsilon_{\xi\zeta}\right)=M^{-1}. \tag{4.59}$$

Note that $a_a^\dagger a_a-\frac{1}{2}\sigma_{\mu\nu}\theta^{\mu\nu}$ is proportional to the identity matrix in quaternionic space while $a_a^\dagger a_a$ is not, as seen from Eq. (4.57). We see that the matrix $M$ is deformed by the noncommutativity, but only for a non-BPS instanton background, that is for anti-self-dual (self-dual) instantons in self-dual (anti-self-dual) $\mathbf{R}^4_{NC}$ and all instantons for Eq. (2.8), since $\eta^a_{\mu\nu}\overline{\eta}^b_{\mu\nu}=0$. If $\theta^{\mu\nu}=0$, of course, we recover the result in commutative space.

In order to construct the propagator for $q\otimes\overline{q}$ in Eq. (4.29), that is, the adjoint representation of U($N$), we take [45]

$$a_1=a,\quad a_2=a^*\sigma^2,\quad b_1=b,\quad b_2=b^*\sigma^2,$$
$$v_1(x)=v_2(x)^*=v(x) \tag{4.60}$$

and anti-Hermitian generators of U($N$) as $T^A$, $A=1,\dots,N^2$ which are normalized as

$$\mathrm{tr}(T^AT^B)=-\frac{1}{2}\delta_{AB} \tag{4.61}$$

where $T^{N^2}=(1/i\sqrt{2N})\mathbf{1}_N$. Then the propagator $G^{AB}(x,y)$ for the adjoint representation can be obtained by multiplying Eq. (4.34) by $T^A_{us},T^B_{vt}$ and summing over $u,s,v,t=1,\dots,N$:

$$G_{AB}(x,y)=[v(x)^\dagger]_{u,\lambda}T^A_{us}[v(x)]_{\rho,s}G^{(0)}(x,y)\times[v(y)]_{\lambda,v}T^B_{vt}[v(y)^\dagger]_{t,\rho}+\frac{1}{4\pi^2}M_{ij,lm}[w(x)^\dagger]_{u,i\alpha}T^A_{us}[w(x)]_{j\alpha,s}\times[w(y)]_{l\beta,v}T^B_{vt}[w(y)^\dagger]_{t,m\beta}, \tag{4.62}$$

where $\lambda,\rho=1,\dots,N+2k$ are ADHM indices and we introduced a $2k\times N$ matrix $w(x)=b^\dagger v(x)$.

## V. DISCUSSION

We explicitly constructed Green's functions for a scalar field in an arbitrary representation of a gauge group propagating in a noncommutative instanton background. We showed that the propagators in the adjoint representation are deformed by noncommutativity while those in the fundamental representation have exactly the same form as in the commutative case.

We showed, generalizing the argument in [44] to noncommutative space, that the propagators for spinor and vector fields can be constructed in terms of those for the scalar field in a noncommutative instanton background. However, it was pointed out in [44] that the vector propagator suffers from an infrared divergence. Let us discuss this problem in our context. The vector propagator can be constructed by the operator expression (3.41) which is involved in the convolution integral over $z$ coordinates

$$\left\langle x\left|\left(\frac{1}{D^2}\right)^2_{AB}\right|y\right\rangle=\mathrm{Tr}^z_{\mathcal{H}}G_{AC}(x,z)G_{CB}(z,y), \tag{5.1}$$

where $G_{AB}(x,y)$ is defined by Eq. (4.62). Using the asymptotic behavior [53,14] of several ADHM quantities in the large $z$ limit, in which the noncommutativity of space is irrelevant, one can check that the integral (5.1) is logarithmically divergent or a logarithmically divergent sum, e.g., $\Sigma_n n^{-1}$, in the noncommutative case. According to [44], one can see that this divergence is coming from the zero mode fluctuations corresponding to global gauge rotations and an overall scale change, which are already contained in the zero mode sum (3.29), where the zero mode for the scale change is generated by the Lorentz rotation of the gauge zero modes. However, note that the global gauge zero modes are not normalizable on $\mathbf{R}^4$ or $\mathbf{R}^4_{NC}$ since it is noncompact space [54,51]. A natural way to remove this logarithmic divergence is to put the theory on compactified Euclidean space, i.e., $\mathbf{S}^4$, as in [55,56]. As shown in [56], this compactified Euclidean formalism provides a gauge invariant normalization for the global gauge zero modes since the volume of $\mathbf{S}^4$ is now finite. Thus the divergence in the vector propagator may be cured in this way because the convolution integral (5.1) on $\mathbf{S}^4$ can be finite. It will be interesting to see if the infrared divergence in the vector propagator can be cured by an ''appropriate'' compactification [28] of noncommutative space in the same way.

Let us consider the massless Dirac equation defined by the covariant derivative (4.29) in the background of anti-self-dual instantons. In this case it has only positive chirality solutions [42] described by two spinors $\psi_R=\psi_{us,\alpha}$ satisfying

$$\sigma^\mu D_\mu \psi_R=0. \tag{5.2}$$

Take the same ansatz as in the commutative case [45]

$$\psi_{us,\alpha}=[v_1(x)^\dagger b_1 \sigma^2 f_1(x)]_{u,i\alpha}[v_2(x)^\dagger d]_{s,i} + [v_1(x)^\dagger c]_{u,i}[v_2(x)^\dagger b_2 \sigma^2 f_2(x)]_{s,i\alpha}, \tag{5.3}$$

where $c,d$ are constant $(N_1+2k_1)\times k_2$ and $(N_2+2k_2)\times k_1$ matrices to be determined. Here we are using ordinary multiplication rather than the tensor product (4.33) since $c,d$ are coupling two spaces $G_1$ and $G_2$ together. In the case of the adjoint representation, using Eq. (4.60), the ansatz (5.3) can be arranged in the form

$$\psi_R=v(x)^\dagger \mathcal{M} f(x) b^\dagger v(x) - v(x)^\dagger b f(x) \mathcal{M}^\dagger v(x), \tag{5.4}$$

with the $(N+2k)\times k$ matrix $\mathcal{M}$. Using the formula (4.54) and

$$\bar{\sigma}^\mu(\Delta_a(x)^\dagger b_\alpha)\bar{\sigma}^\mu = -2 b_a^\dagger \Delta_a(x),$$
$$\sigma^\mu(b_a^\dagger \Delta_a(x))\bar{\sigma}^\mu = 2(\Delta_a(x)^\dagger b_a + b_a^\dagger \Delta_a(x)), \tag{5.5}$$

it is straightforward to calculate Eq. (5.2):

$$\sigma^\mu D_\mu \psi_R = 2\varepsilon_{\beta\gamma}[v_1(x)^\dagger b_1 f_1(x)]_{u,i\beta} \times([v_2(x)^\dagger b_2 f_2(x)]_{s,j\gamma}[\Delta_2(x)^\dagger d]_{j\alpha,i} - [\Delta_1(x)^\dagger c]_{i\alpha,j}[v_2(x)^\dagger b_2 f_2(x)]_{s,j\gamma}). \tag{5.6}$$

In the commutative case [45], Eq. (5.2) requires that

$$[a_1^\dagger c]_{i\alpha,j}=[a_2^\dagger d]_{j\alpha,i}, \qquad [b_1^\dagger c]_{i\alpha,j}=[b_2^\dagger d]_{j\alpha,i}. \tag{5.7}$$

However, in noncommutative space, we cannot say that the solution of Eq. (5.2) would be Eq. (5.7) since $\mathbf{x}$ does not necessarily commute with $v_2(x)^\dagger b_2 f_2(x)$. So the simple minded ansatz (5.3) does not work for noncommutative space. To find the fermionic zero modes for the gauge group $G_1\times G_2$, it may be necessary to apply a systematic method for the tensor product of instantons as was done in [45]. Unfortunately, the ADHM construction for the tensor product involves tedious and complicated manipulations even for commutative space. We did not succeed in generalizing to noncommutative space yet. We leave this problem for future work.

In Sec. IV, we observed that the $x$-dependent matrix $M$ (4.56) is equal to the constant matrix $M$ (4.59) and the matrix $M$ is deformed by noncommutativity only for non-BPS instantons. In commutative space, the $x$ independence of the matrix $M$ is a result of conformal invariance and the conformal invariance has an important role in calculating multi-instanton determinants [57–60]. Since, for BPS instantons (commutative instantons are always BPS), the matrix $M$ has the same form as the commutative one, the conformal invariance for this background should be manifest. Although the matrix $M$ is deformed by the noncommutativity for non-BPS instantons, it is still a constant matrix. Thus one may expect (deformed) conformal invariance even for the non-BPS background. In [42], we observed that the conformal zero modes have a similar deformation because of the noncommutativity and we speculated that the conformal symmetry has to act nontrivially only on the SU($N$) instanton sector. These deformations of conformal symmetry in zero modes and propagators should be related to each other.

Let us briefly discuss the conformal property of the matrix $M$ (4.59) in the BPS background, in which the $\sigma_{\mu\nu}\theta^{\mu\nu}$ term vanishes. From Eq. (4.56) and Eq. (4.59), we see that the matrix $M$ in (4.59) is invariant under the transformations

$$a_a \to a_a + b_a \bar{\mathbf{p}}, \qquad b_a \to b_a, \qquad a=1,2. \tag{5.8}$$

Since it is symmetric under interchange of $a_a$ and $b_a$, it is also invariant under the transformations [use $\sigma^\mu\sigma^\nu+\bar{\sigma}^\mu\bar{\sigma}^\nu = \mathrm{tr}_2(\sigma^\mu\sigma^\nu)=\mathrm{tr}_2(\bar{\sigma}^\mu\bar{\sigma}^\nu)$ to check]

$$a_a \to a_a, \qquad b_a \to b_a + a_a \bar{\mathbf{p}}. \tag{5.9}$$

While, under the transformations

$$a_a \to a_a \bar{\mathbf{p}}, \qquad b_a \to b_a \bar{\mathbf{q}}, \tag{5.10}$$

it changes by a factor $p^2 q^2$ for any quaternions $\bar{\mathbf{p}},\bar{\mathbf{q}}$ [use Eq. (3.2) to check]. This factor can be scaled to unity in terms of simultaneous global scaling of $a_a, b_a$ by a real number. The above transformations (5.8)–(5.10) actually correspond to a unimodular conformal group [45] (so a 15-parameter group).

If we try to generalize the above consideration to non-BPS instantons, in which we have the $\sigma_{\mu\nu}\theta^{\mu\nu}$ term, we immediately meet some nontrivial problems. The main source of this difficulty is that the matrix $M$ (4.59) is asymmetric under interchange of $a_a$ and $b_a$ due to the presence of the inhomogeneous term, i.e., $\sigma_{\mu\nu}\theta^{\mu\nu}$. The scale transformation (5.10) does not generate an overall scale either, because of the inhomogeneous term; thus some modified transformation would be genuinely required. Currently, we do not know how to modify the conformal transformations (5.8)–(5.10). We hope to report some progress along this line in the near future.

**ACKNOWLEDGMENTS**

We thank Keun-Young Kim for helpful discussions and a careful reading of this manuscript. B.H.L. is supported by the Ministry of Education, BK21 Project No. D-0055 and by Grant No. R01-1999-00018 from the Interdisciplinary Research Program of the KOSEF. H.S.Y. is supported by NSC (NSC90-2811-M-002-019). He also acknowledges NCTS as well as CTP at NTU for partial support.

[1] A.A. Belavin, A.M. Polyakov, A.S. Schwartz, and Yu.S. Tyupkin, Phys. Lett. **59B**, 85 (1975).
[2] R. Jackiw and C. Rebbi, Phys. Rev. Lett. **37**, 172 (1976).
[3] C.G. Callan, R.F. Dashen, and D.J. Gross, Phys. Lett. **63B**, 334 (1976).
[4] *Instantons in Gauge Theories*, edited by M. Shifman (World Scientific, Singapore, 1994).
[5] G. 't Hooft, Phys. Rev. Lett. **37**, 8 (1976).
[6] G. 't Hooft, Phys. Rev. D **14**, 3432 (1976); **18**, 2199(E) (1978).
[7] For a recent review, for example, see T. Schafer and E.V. Shuryak, Rev. Mod. Phys. **70**, 323 (1998).
[8] E. Witten, Nucl. Phys. **B460**, 541 (1996).
[9] M.R. Douglas, ''Branes within Branes,'' hep-th/9512077; J. Petrol. **28**, 255 (1998).
[10] A. Strominger and C. Vafa, Phys. Lett. B **379**, 99 (1996).
[11] C.G. Callan and J.M. Maldacena, Nucl. Phys. **B472**, 591 (1996).
[12] T. Banks and M.B. Green, J. High Energy Phys. **05**, 002 (1998).
[13] M. Bianchi, M.B. Green, S. Kovacs, and G. Rossi, J. High Energy Phys. **08**, 013 (1998).
[14] N. Dorey, T.J. Hollowood, V.V. Khoze, M.P. Mattis, and S. Vandoren, Nucl. Phys. **B552**, 88 (1999).
[15] For a review on Yang-Mills and D-instantons, see A.V. Belitsky, S. Vandoren, and P. van Nieuwenhuizen, Class. Quantum Grav. **17**, 3521 (2000).
[16] N.A. Nekrasov and A. Schwarz, Commun. Math. Phys. **198**, 689 (1998).
[17] N. Seiberg and E. Witten, J. High Energy Phys. **09**, 032 (1999).
[18] M.F. Atiyah, N.J. Hitchin, V.G. Drinfeld, and Yu.I. Manin, Phys. Lett. **65A**, 185 (1978).
[19] K. Lee and P. Yi, Phys. Rev. D **61**, 125015 (2000).
[20] H.W. Braden and N.A. Nekrasov, ''Spacetime Foam from Noncommutative Instantons,'' hep-th/9912019.
[21] K. Furuuchi, Prog. Theor. Phys. **103**, 1041 (2000).
[22] A. Kapustin, A. Kuznetsov, and D. Orlov, Commun. Math. Phys. **221**, 385 (2001).
[23] P.-M. Ho, ''Twisted Bundle on Noncommuative Space and $U(1)$ Instanton,'' hep-th/0003012.
[24] K.-Y. Kim, B.-H. Lee, and H.S. Yang, ''Comments on Instantons on Noncommutative $\mathbf{R}^4$,'' hep-th/0003093.
[25] K. Furuuchi, Commun. Math. Phys. **217**, 579 (2001).
[26] K. Lee, D. Tong, and S. Yi, Phys. Rev. D **63**, 065017 (2001).
[27] K. Furuuchi, J. High Energy Phys. **03**, 223 (2001).
[28] A. Schwarz, Commun. Math. Phys. **221**, 433 (2001).
[29] D.H. Correa, G.S. Lozano, E.F. Moreno, and F.A. Schaposnik, Phys. Lett. B **515**, 206 (2001).
[30] C.-S. Chu, V.V. Khoze, and G. Travaglini, Nucl. Phys. **B621**, 101 (2002).
[31] M. Hamanaka, Phys. Rev. D **65**, 085022 (2002).
[32] T. Ishikawa, S.-I. Kuroki, and A. Sako, J. High Energy Phys. **11**, 068 (2001).
[33] K.-Y. Kim, B.-H. Lee, and H.S. Yang, Phys. Lett. B **523**, 357 (2001).
[34] O. Lechtenfeld and A.D. Popov, J. High Energy Phys. **03**, 040 (2002).
[35] T. Ishikawa, S.-I. Kuroki, and A. Sako, ''Instanton Number Calculus on Noncommutative $\mathbf{R}^4$,'' hep-th/0201196.
[36] N. Dorey, T.J. Hollowood, and V.V. Khoze, J. High Energy Phys. **03**, 040 (2001).
[37] T.J. Hollowood, V.V. Khoze, and G. Travaglini, J. High Energy Phys. **05**, 051 (2001).
[38] N. Dorey, T.J. Hollowood, and V.V. Khoze, ''Notes on Soliton Bound-State Problems in Gauge Theory and String Theory,'' hep-th/0105090.
[39] C. Kim, K. Lee, and P. Yi, Phys. Rev. D **65**, 065024 (2002).
[40] T.J. Hollowood, J. High Energy Phys. **03**, 038 (2002).
[41] N.A. Nekrasov, ''Lectures on Open Strings and Noncommutative Gauge Theories,'' hep-th/0203109.
[42] K.-Y. Kim, B.-H. Lee, and H.S. Yang, Phys. Rev. D **66**, 025034 (2002).
[43] Y. Hiraoka, ''BPS Solutions of Noncommutative Gauge Theories in Four and Eight Dimensions,'' hep-th/0205283.
[44] L.S. Brown, R.D. Carlitz, D.B. Creamer, and C. Lee, Phys. Rev. D **17**, 1583 (1978).
[45] E.F. Corrigan, P. Goddard, and S. Templeton, Nucl. Phys. **B151**, 93 (1979).
[46] S. Cho, R. Hinterding, J. Madore, and H. Steinacker, Int. J. Mod. Phys. D **9**, 161 (2000).
[47] D.J. Gross and N.A. Nekrasov, J. High Energy Phys. **07**, 034 (2000).
[48] R. Gopakumar, S. Minwalla, and A. Strominger, J. High Energy Phys. **05**, 020 (2000).
[49] For example, see N.N. Bogoliubov and D.V. Shirkov, *Introduction to The Theory of Quantized Fields*, 3rd ed. (John Wiley & Sons, New York, 1980), Chap. 3.
[50] E.F. Corrigan, D.B. Fairlie, P. Goddard, and S. Templeton, Nucl. Phys. **B140**, 31 (1978).
[51] N.H. Christ, N.K. Stanton, and E.J. Weinberg, Phys. Rev. D **18**, 2013 (1978).

[52] N.E. Wegge-Olsen, *K-Theory and C*-Algebras* (Oxford University Press, Oxford, 1993).
[53] E. Corrigan and P. Goddard, Ann. Phys. (N.Y.) **154**, 253 (1984).
[54] C.W. Bernard, N.H. Christ, A.H. Guth, and E.J. Weinberg, Phys. Rev. D **16**, 2967 (1977).
[55] R. Jackiw and C. Rebbi, Phys. Rev. D **14**, 517 (1976).
[56] N.K. Nielsen and B. Schroer, Nucl. Phys. **B127**, 493 (1977).
[57] E. Corrigan, P. Goddard, H. Osborn, and S. Templeton, Nucl. Phys. **B159**, 469 (1979).
[58] H. Osborn, Nucl. Phys. **B159**, 497 (1979).
[59] B. Berg and M. Lüscher, Nucl. Phys. **B160**, 281 (1979).
[60] I. Jack, Nucl. Phys. **B174**, 526 (1980).