

Energy density in the Casimir effect

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We compute the expectations of the squares of the electric and magnetic fields in the vacuum region outside a half-space filled with a uniform dispersive dielectric. We find a positive energy density of the electromagnetic field which diverges at the interface despite the inclusion of dispersion in the calculation. We also investigate the mean squared fields and the energy density in the vacuum region between two parallel half-spaces. Of particular interest is the sign of the energy density. We find that the energy density is described by two terms: a negative position independent (Casimir) term, and a positive position dependent term with a minimum value at the center of the vacuum region. We argue that in some cases, including physically realizable ones, the negative term can dominate in a given region between the two half-spaces, so the overall energy density can be negative in this region.

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I. INTRODUCTION

In 1948 Casimir made the remarkable prediction that there is an attractive force between a pair of uncharged parallel plane perfect conductors [1]. Furthermore, he argued that this force arises solely from a shift in the energy of the vacuum state of the quantized electromagnetic field. An early attempt by Sparnaay [2] to observe this force was inconclusive, but in recent years several new experiments [3,4,5,6,7] have been performed which seem to give good agreement with Casimir's prediction. (To be more precise, most of these experiments actually measure the force between a plate and a sphere and incorporate a theoretical correction to compare it to Casimir's result. Of the recent experiments, only that of Bressi *et al.* [7] uses two parallel plates.)

If the energy of the vacuum state is zero in the limit of infinite plate separation, then the attractive force found by Casimir would seem to imply a negative vacuum energy at finite separation. In fact, Brown and Maclay [8] showed that for perfectly conducting plates one has a constant negative vacuum energy density. This conclusion is of great theoretical interest, because negative energy density has the potential to cause some rather bizarre effects in gravity theory. (See, for example, Ref. [9] and references therein.) However, questions have been raised as to whether the negative energy density will still arise in a more realistic treatment in which the plates are not perfect conductors [10,11]. In particular, Helfer and Lang [10] calculated the energy density outside a single half-space filled with a nondispersive dielectric material and obtained a positive result. They interpreted this as a positive self-energy density associated with a single plate which would add to the negative interaction energy density between a pair of plates. Helfer and Lang conjecture that the net Casimir energy density might be positive when the self-energy is accounted for. If this conjecture is correct, then the situation would be analogous to that of the energy density in classical electrostatics. A pair of oppositely charged particles

have a negative interaction energy, but the net energy density, which is proportional to the square of the electric field, is always positive.

However, the Helfer and Lang calculation does not include dispersion, which is essential in a realistic treatment. Pfenning [12] has studied a scalar model with dispersion and found that the energy density can be negative. In the electromagnetic case, numerous authors, beginning with Lifshitz [13], have studied the effects of dispersion upon Casimir forces. However, these authors have been concerned with the force or the total energy, and not the local energy density. The purpose of this paper is to present a calculation of the Casimir energy density in a model in which dispersion is included. For this purpose, we will use the methods of source theory developed by Schwinger and co-workers [14,15]. This is a method based upon the calculation of Green's functions which is especially well suited to dissipative materials, and was used by Schwinger *et al.* [14] to rederive the results of Lifshitz. Milonni and Shih [16] used conventional quantum electrodynamics to reproduce some of the results of source theory. There has also been considerable interest in recent years in quantization of the electromagnetic field inside dissipative materials using operator methods [17,18,19]. The relation between the results of the latter set of authors and those of Schwinger *et al.* has not yet been clarified.

The outline of this paper is as follows. In Sec. II we review the source theory approach as applied to parallel interfaces of dielectric media. In Sec. III we compute the expectation values of the squares of the electric and magnetic fields in the vacuum region outside a half-space filled with a uniform dispersive dielectric. We extend this calculation to the case of two parallel dielectric half-spaces and also discuss the energy density in Sec. IV. Conclusions are given in Sec. V.

II. GREEN'S FUNCTION APPROACH FOR MULTILAYER DIELECTRICS

This section is a review of the formalism of Schwinger *et al.* [14]. One begins by writing the Maxwell equations for the macroscopic electromagnetic fields produced by an ex-

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ternal polarization source \mathbf{P} , which formally describes the zero point fluctuations of the fields¹

$$\begin{aligned}\mathbf{B} &= \nabla \times \mathbf{A}, \\ \mathbf{E} &= -\dot{\mathbf{A}} - \nabla \phi, \\ \nabla \times \mathbf{B} &= \epsilon \dot{\mathbf{E}} + \dot{\mathbf{P}}, \\ \nabla \cdot (\epsilon \mathbf{E} + \mathbf{P}) &= 0,\end{aligned}\quad (1)$$

where ϵ is the dielectric constant of the medium. The wave equation for the electric field resulting from the Maxwell equations is

$$-\nabla \times (\nabla \times \mathbf{E}) = \epsilon \ddot{\mathbf{E}} = \ddot{\mathbf{P}}. \quad (2)$$

By assuming a linear relation between sources and fields, the electric field can be written as a spacetime integral

$$\mathbf{E}(x) = \int d^4x' \vec{\Gamma}(x, x') \mathbf{P}(x'), \quad (3)$$

where $x = (t, \mathbf{r})$, $x' = (t', \mathbf{r}')$, and $\vec{\Gamma}$ is a Green's dyadic, which satisfies Eq. (2) with a δ -function source. Let

$$\vec{\Gamma}(\mathbf{r}, \mathbf{r}', \omega) = \int_{-\infty}^{\infty} d\tau e^{i\omega\tau} \vec{\Gamma}(x, x'), \quad (4)$$

where $\tau = t - t'$. From Eqs. (2) and (3), it follows that $\vec{\Gamma}(\mathbf{r}, \mathbf{r}', \omega)$ satisfies the following equation:

$$-\nabla \times (\nabla \times \vec{\Gamma}) + \omega^2 \epsilon \vec{\Gamma} = -\omega^2 \vec{\mathbf{I}} \delta(\mathbf{r} - \mathbf{r}'). \quad (5)$$

So far, the discussion has been purely classical. At this point, Schwinger *et al.* [14] use source theory to identify the Green's dyadic $\vec{\Gamma}$ with an "effective product of electric fields"

$$\frac{i}{\hbar} \langle E_j(\mathbf{r}) E_k(\mathbf{r}') \rangle = \Gamma_{jk}(\mathbf{r}, \mathbf{r}', \omega). \quad (6)$$

We can interpret this as the Fourier transform of the electric field correlation function. From the Maxwell equation $\nabla \times \mathbf{E} = -\dot{\mathbf{B}}$, one finds the corresponding expression for the magnetic field:

$$\frac{i}{\hbar} \langle B_j(\mathbf{r}) B_k(\mathbf{r}') \rangle = \epsilon_{jlm} \epsilon_{knp} (\nabla_l \nabla'_n / \omega^2) \Gamma_{mp}(\mathbf{r}, \mathbf{r}', \omega). \quad (7)$$

Note that \hbar makes its first appearance in these expressions. These expressions can be identified with the vacuum expectation values of products of field operators, which appear in the more conventional field theory approach to quantization of the electromagnetic field. From now onward, we revert to units in which $\hbar = 1$. In order to calculate the field correlation

functions, one needs to find the Green's function Γ occurring in Eq. (5). This amounts to solving a classical boundary value problem.

The interfaces between the media are chosen to be perpendicular to the z direction, so for now it will matter if the dielectric constant changes in the z direction only. Therefore, it is convenient to introduce a transverse spatial Fourier transform

$$\vec{\Gamma}(\mathbf{r}, \mathbf{r}', \omega) = \int d\mathbf{k}_\perp \frac{1}{(2\pi)^2} e^{i\mathbf{k}_\perp \cdot (\mathbf{r} - \mathbf{r}')_\perp} \vec{\Gamma}(z, z', \mathbf{k}_\perp, \omega), \quad (8)$$

where the vector \mathbf{k}_\perp can be chosen to point along the $+x$ axis ($k = |\mathbf{k}_\perp|$).

Some components of $\vec{\Gamma}$ are found to be [14]

$$\Gamma_{xx} = -\frac{1}{\epsilon} \delta(z - z') + \frac{1}{\epsilon} \frac{\partial}{\partial z} \frac{1}{\epsilon'} \frac{\partial}{\partial z'} g^B, \quad (9a)$$

$$\Gamma_{yy} = \omega^2 g^E, \quad (9b)$$

$$\Gamma_{zz} = -\frac{1}{\epsilon} \delta(z - z') + \frac{k^2}{\epsilon \epsilon'} g^B, \quad (9c)$$

$$\Gamma_{xz} = i \frac{k}{\epsilon \epsilon'} \frac{\partial}{\partial z} g^B, \quad (9d)$$

$$\Gamma_{zx} = -i \frac{k}{\epsilon \epsilon'} \frac{\partial}{\partial z'} g^B, \quad (9e)$$

where $\epsilon' = \epsilon(z')$, and g^E , the "transverse electric," and g^B , the "transverse magnetic," Green's functions satisfy

$$\left[-\frac{\partial^2}{\partial z^2} + k^2 - \omega^2 \epsilon \right] g^E(z, z') = \delta(z - z'), \quad (10a)$$

$$\left[-\frac{\partial}{\partial z} \frac{1}{\epsilon} \frac{\partial}{\partial z} + \frac{k^2}{\epsilon} - \omega^2 \right] g^B(z, z') = \delta(z - z'). \quad (10b)$$

By introducing the quantity

$$\kappa^2 = k^2 - \omega^2 \epsilon, \quad (11)$$

Eq. (10) can be written as

$$\left[-\frac{\partial^2}{\partial z^2} + \kappa^2 \right] g^E(z, z') = \delta(z - z'), \quad (12a)$$

$$\left[-\frac{\partial}{\partial z} \frac{1}{\epsilon} \frac{\partial}{\partial z} + \frac{\kappa^2}{\epsilon} \right] g^B(z, z') = \delta(z - z'). \quad (12b)$$

So, in order to find the field correlation functions as defined in Eqs. (6) and (7) in a given situation, one needs to solve these equations with the appropriate boundary conditions. We consider here two cases.

¹Heaviside-Lorentz units with $c = \hbar = 1$ will be used in this paper. Also, it is assumed that the magnetic permeability is unity.

III. ONE-INTERFACE CASE

We now specialize the above discussion to a situation in which the inhomogeneity of the dielectric constant is due to a plane interface separating a dielectric substance from a vacuum:

$$\begin{aligned} z > 0: \quad \epsilon(z) &= 1, \\ z < 0: \quad \epsilon(z) &\equiv \epsilon_d. \end{aligned} \quad (13)$$

Here ϵ_d is a function of frequency, but not of position.

A. Boundary conditions

In solving Eqs. (12a) and (12b), we use the following boundary conditions. At $z = z'$, g is continuous but the derivative is discontinuous at this point [20]:

$$\left. \frac{\partial g}{\partial z} \right|_{z \rightarrow z'_+} - \left. \frac{\partial g}{\partial z} \right|_{z \rightarrow z'_-} = -1. \quad (14)$$

At the boundary ($z = 0$) we use the conditions for continuity of E_x , E_y , ϵE_z , and B_i . The first three, as seen from Eq. (6), imply the continuity of Γ_{xx} , Γ_{yy} , and $\epsilon \Gamma_{zz}$ and subsequently, from Eq. (9), the continuity of g^E , g^B , and

$$\frac{1}{\epsilon} \frac{\partial}{\partial z} \frac{1}{\epsilon'} \frac{\partial}{\partial z'} g^B.$$

The continuity of B_x implies that of $\nabla_z \nabla_{z'} \Gamma_{yy}$, as seen from Eq. (24), which is given below. From this, using Eqs. (6) and (9b), we deduce the continuity of $\partial g^E / \partial z$.

The solutions g^E and g^B in the vacuum region have the form

$$g^E = \frac{e^{-\kappa_0 |z - z'|} + r e^{-\kappa_0(z + z')}}{2\kappa_0}, \quad (15a)$$

$$g^B = \frac{e^{-\kappa_0 |z - z'|} + r' e^{-\kappa_0(z + z')}}{2\kappa_0}, \quad (15b)$$

where

$$r \equiv \frac{\kappa_0 - \kappa_1}{\kappa_0 + \kappa_1}, \quad (16a)$$

$$r' \equiv \frac{\kappa_0 \epsilon_d - \kappa_1}{\kappa_0 \epsilon_d + \kappa_1}. \quad (16b)$$

Here κ_0 and κ_1 represent the quantity κ as defined in Eq. (11) for the vacuum region ($\epsilon = 1$), and for the dielectric half-space region ($\epsilon = \epsilon_d$), respectively, and r and r' can be identified as reflection coefficients for two polarization states \perp and \parallel , respectively, corresponding to the electric field vector being perpendicular or parallel to the plane of incidence of a linearly polarized electromagnetic wave [20].

B. The electric field

Using Eq. (6), we write the formal expectation value of the square of the electric field at coincident points as

$$\begin{aligned} \langle E^2 \rangle_f &= -i \int_{-\infty}^{\infty} d\omega \frac{1}{2\pi} \int_0^{\infty} dk k \frac{1}{2\pi} \Gamma_{kk} \\ &= -\frac{i}{2\pi^2} \int_0^{\infty} d\omega \int_0^{\infty} dk k \Gamma_{kk}. \end{aligned} \quad (17)$$

In the second step, we assumed that the integrand is an even function of ω . By complex rotation ($\omega \rightarrow i\zeta$), this becomes

$$\langle E^2 \rangle_f = \frac{1}{2\pi^2} \int_0^{\infty} d\zeta \int_0^{\infty} dk k \Gamma_{kk}. \quad (18)$$

Note from Eq. (11) that $\kappa^2 > 0$ when ω is imaginary. By means of Eq. (9), all of the components of $\vec{\Gamma}$ in a given region can be written in terms of Γ_{xx} and Γ_{yy} :

$$\begin{aligned} \Gamma_{xz}(z, z') &= \frac{ik}{\kappa^2} \frac{\partial}{\partial z'} \Gamma_{xx}(z, z'), \\ \Gamma_{zx}(z, z') &= -\frac{ik}{\kappa^2} \frac{\partial}{\partial z} \Gamma_{xx}(z, z'), \\ \Gamma_{zz}(z, z') &= \frac{k^2}{(\kappa^2)^2} \frac{\partial}{\partial z} \frac{\partial}{\partial z'} \Gamma_{xx}(z, z') \\ &\quad + \frac{\omega^2}{\kappa^2} \delta(z - z'). \end{aligned} \quad (19)$$

By taking the limit $z \rightarrow z'$, and thus omitting the delta function, Γ_{kk} becomes

$$\Gamma_{kk} = \Gamma_{xx} + \Gamma_{yy} + \frac{k^2}{(\kappa^2)^2} \nabla_z \nabla_{z'} \Gamma_{xx}, \quad (20)$$

or by Eq. (9), using $\epsilon = 1$,

$$\begin{aligned} \Gamma_{kk} &= \omega^2 g^E + \nabla_z \nabla_{z'} g^B + \frac{k^2}{(\kappa^2)^2} \nabla_z \nabla_{z'} (\nabla_z \nabla_{z'} g^B) \\ &= \omega^2 g^E + (k^2 + \nabla_z \nabla_{z'}) g^B. \end{aligned} \quad (21)$$

Using Eqs. (11) and (15), this becomes

$$\Gamma_{kk} = \frac{\omega^2}{\kappa} + \frac{1}{2\kappa} [\omega^2 r + (2k^2 - \omega^2) r'] e^{-2\kappa z}. \quad (22)$$

Equation (18) gives a formal expectation value only, because the integral is divergent. However, the divergence comes only from the ω^2/κ term in Γ_{kk} and is independent of z . It is the usual empty space vacuum divergence. We will henceforth drop this term and denote the resulting finite expectation value by $\langle E^2 \rangle$. The renormalization results in a quantity which vanishes at large distances from the interface, $\langle E^2 \rangle \rightarrow 0$ as $z \rightarrow \infty$, which amounts to finding the difference in $\langle E^2 \rangle$ with the boundary and without it. Thus we find

$$\langle E^2 \rangle = \frac{1}{4\pi^2} \int_0^\infty d\xi \int_0^\infty dk \frac{k}{\kappa} [-\xi^2 r + (2k^2 + \xi^2)r'] e^{-2\kappa z}. \quad (23)$$

C. The magnetic field

Now we compute the expectation value of the magnetic field. Using Eq. (7), we find

$$\begin{aligned} i\langle B_x(\mathbf{r})B_x(\mathbf{r}') \rangle &= \frac{1}{\omega^2} (\nabla_z \nabla_{z'} \Gamma_{yy} - \nabla_y \nabla_{z'} \Gamma_{zy} \\ &\quad - \nabla_z \nabla_{y'} \Gamma_{yz} + \nabla_y \nabla_{y'} \Gamma_{zz}), \\ i\langle B_y(\mathbf{r})B_y(\mathbf{r}') \rangle &= \frac{1}{\omega^2} (\nabla_z \nabla_{z'} \Gamma_{xx} - \nabla_x \nabla_{z'} \Gamma_{zx} \\ &\quad - \nabla_z \nabla_{x'} \Gamma_{xz} + \nabla_x \nabla_{x'} \Gamma_{zz}), \\ i\langle B_z(\mathbf{r})B_z(\mathbf{r}') \rangle &= \frac{1}{\omega^2} (\nabla_y \nabla_{y'} \Gamma_{xx} - \nabla_y \nabla_{x'} \Gamma_{xy} \\ &\quad - \nabla_x \nabla_{y'} \Gamma_{yx} + \nabla_x \nabla_{x'} \Gamma_{yy}). \end{aligned} \quad (24)$$

From the definition of \mathbf{k}_\perp , it follows that all derivatives in y vanish, so we can write the sum of the above terms as

$$\begin{aligned} i\langle B_i(\mathbf{r})B_i(\mathbf{r}') \rangle &= \frac{1}{\omega^2} (\nabla_z \nabla_{z'} \Gamma_{yy} + \nabla_z \nabla_{z'} \Gamma_{xx} - \nabla_x \nabla_{z'} \Gamma_{zx} \\ &\quad - \nabla_z \nabla_{x'} \Gamma_{xz} + \nabla_x \nabla_{x'} \Gamma_{zz} + \nabla_x \nabla_{x'} \Gamma_{yy}). \end{aligned} \quad (25)$$

Using Eq. (19), we have

$$\begin{aligned} \nabla_x \nabla_{x'} \Gamma_{zz}(\mathbf{r}, \mathbf{r}', \omega) &= \int \frac{d\mathbf{k}_\perp}{(2\pi)^2} \frac{(k^2)^2}{(\kappa^2)^2} \nabla_z \nabla_{z'} \Gamma_{xx}, \\ \nabla_z \nabla_{x'} \Gamma_{xz}(\mathbf{r}, \mathbf{r}', \omega) &= \int \frac{d\mathbf{k}_\perp}{(2\pi)^2} \frac{k^2}{\kappa^2} \nabla_z \nabla_{z'} \Gamma_{xx}, \\ \nabla_x \nabla_{z'} \Gamma_{zx}(\mathbf{r}, \mathbf{r}', \omega) &= \int \frac{d\mathbf{k}_\perp}{(2\pi)^2} \frac{k^2}{\kappa^2} \nabla_z \nabla_{z'} \Gamma_{xx}. \end{aligned} \quad (26)$$

This leads to

$$\begin{aligned} i\langle B(x)B(x') \rangle &= \int \frac{d\omega}{2\pi} \int \frac{d\mathbf{k}_\perp}{(2\pi)^2} \left[\frac{1}{\omega^2} (k^2 + \nabla_z \nabla_{z'}) \Gamma_{yy}(z, z') \right. \\ &\quad \left. + \frac{\omega^2}{\kappa^4} \nabla_z \nabla_{z'} \Gamma_{xx}(z, z') \right]. \end{aligned} \quad (27)$$

Using Eq. (9), this becomes

$$\begin{aligned} i\langle B(x)B(x') \rangle &= \int \frac{d\omega}{2\pi} \int \frac{d\mathbf{k}_\perp}{(2\pi)^2} [(k^2 + \nabla_z \nabla_{z'}) g^E(z, z') \\ &\quad + \omega^2 g^B(z, z')]. \end{aligned} \quad (28)$$

Following the same procedure as used above in calculating $\langle E^2 \rangle$, we find the finite mean squared magnetic field to be

$$\langle B^2 \rangle = \frac{1}{4\pi^2} \int_0^\infty d\xi \int_0^\infty dk \frac{k}{\kappa} [(2k^2 + \xi^2)r - \xi^2 r'] e^{-2\kappa z}. \quad (29)$$

Note from Eqs. (23) and (29) that $\langle E^2 \rangle \leftrightarrow \langle B^2 \rangle$ under interchange of r and r' . Now, the mean energy density can be calculated as

$$U = \frac{1}{2} (\langle E^2 \rangle + \langle B^2 \rangle). \quad (30)$$

Using Eqs. (23) and (29), this becomes

$$U = \frac{1}{4\pi^2} \int_0^\infty d\xi \int_0^\infty dk \frac{k^3}{\kappa} (r + r') e^{-2\kappa z}. \quad (31)$$

We can write U in a form more convenient for numerical calculation by introducing polar coordinates u and θ ($\xi = u \cos \theta$, $k = u \sin \theta$):

$$U = \frac{1}{4\pi^2} \int_0^\infty du u^3 \int_0^{\pi/2} d\theta (\sin \theta)^3 (r + r') e^{-2uz}. \quad (32)$$

We use the Drude model for the dielectric function

$$\epsilon_d(\omega) = 1 - \frac{\omega_p^2}{\omega^2}, \quad (33)$$

where ω_p is the plasma frequency. From Eqs. (16a), (16b), and (33), we find

$$r = \frac{u - \sqrt{u^2 + \omega_p^2}}{u + \sqrt{u^2 + \omega_p^2}}, \quad (34a)$$

$$r' = \frac{u^2 (\cos \theta)^2 + \omega_p^2 - u (\cos \theta)^2 \sqrt{u^2 + \omega_p^2}}{u^2 (\cos \theta)^2 + \omega_p^2 + u (\cos \theta)^2 \sqrt{u^2 + \omega_p^2}}. \quad (34b)$$

By a substitution [$\cos \theta \rightarrow t$], U becomes

$$U = \frac{1}{4\pi^2} \int_0^\infty du u^3 \left[\int_0^1 dt (1-t^2)(r+r') \right] e^{-2uz}. \quad (35)$$

By the same coordinate transform, Eqs. (23) and (29) become

$$\langle E^2 \rangle = \frac{1}{4\pi^2} \int_0^\infty du u^3 \left\{ \int_0^1 dt [-t^2 r + (2-t^2)r'] \right\} e^{-2uz}, \quad (36a)$$

$$\langle B^2 \rangle = \frac{1}{4\pi^2} \int_0^\infty du u^3 \left\{ \int_0^1 dt [(2-t^2)r - t^2 r'] \right\} e^{-2uz}. \quad (36b)$$

The plot for $\langle E^2 \rangle$ and $\langle B^2 \rangle$, as well as U is shown in Fig. 1. As we can see from the figure, the energy density is positive. Now we consider some limiting cases.

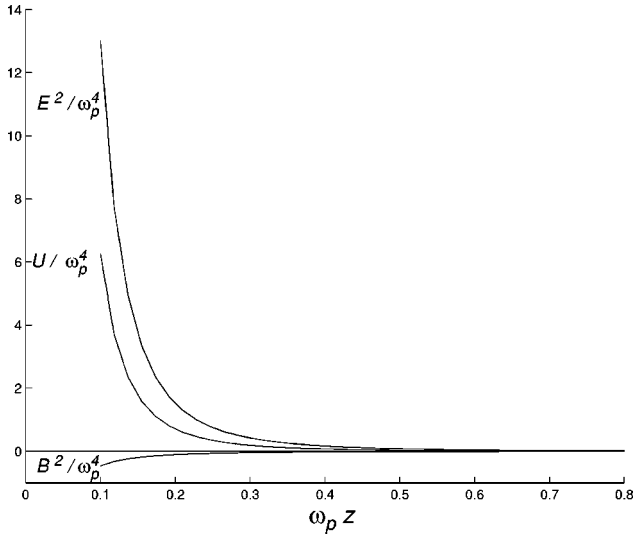


FIG. 1. The expectations of the squares of electric field, magnetic field, and energy density near the dielectric half-space are illustrated.

D. The fields near the interface

To see how U behaves for small z (large u), we first perform the t integration in Eq. (35), which can be done analytically, and then Taylor expand the resulting expression in the square brackets in powers of u^{-1} . That is, we are expanding all of the integrand except for the exponential factor. To the leading order we find

$$U \sim \frac{\sqrt{2}\omega_p}{64\pi} \frac{1}{z^3}. \quad (37)$$

The asymptotic behavior of the mean squared fields (36a), (36b) in this limit is

$$\langle E^2 \rangle \sim \frac{\sqrt{2}\omega_p}{32\pi} \frac{1}{z^3}, \quad (38a)$$

$$\langle B^2 \rangle \sim -\frac{5\omega_p^2}{96\pi} \frac{1}{z^2}. \quad (38b)$$

We see that $\langle E^2 \rangle$ dominates over $\langle B^2 \rangle$, so that $U \approx (1/2)\langle E^2 \rangle$; this is due to the fact that the leading order in the expression in braces in Eq. (36a) is proportional to u^{-1} as compared to u^{-2} in Eq. (36b). If we compare these expressions to ones that would result if dispersion were not included in the calculation, it can be seen from Eq. (36) that in this case $\langle E^2 \rangle \propto z^{-4}$, and the same for $\langle B^2 \rangle$ (see also [10]). As seen from Eq. (38), the inclusion of dispersion in the calculation reduces the power in z up to two orders, but it does not remove the singularity of the results at $z=0$, as might be naively expected.

After more careful consideration, it is not surprising that dispersion alone is insufficient to render the results finite at the interface. The integrals for $\langle E^2 \rangle$ and $\langle B^2 \rangle$ at $z=0$ diverge quartically in a frequency cutoff. However, in general dielectric functions go as

$$\epsilon(\omega) \sim 1 + O(\omega^{-2}) \quad (38c)$$

as $\omega \rightarrow \infty$. Thus the reflection coefficients will go to zero no faster than ω^{-2} , leaving the integrals quadratically divergent. This argument explains why $\langle B^2 \rangle \propto z^{-2}$ for small z , but understanding the behavior of $\langle E^2 \rangle$ requires examining the dependence of the reflection coefficients r and r' upon the transverse momentum k . In fact the contribution of the coefficient r , which describes modes with the polarization vector perpendicular to the plane of incidence, does go as z^{-2} . This coefficient depends only upon frequency, and falls as ω^{-2} for large ω , as can be seen from Eq. (34a). The coefficient r' describes modes with the polarization vector parallel to the plane of incidence, and goes to 1 as $\theta \rightarrow \pi/2$ (corresponding to grazing incidence) for all frequencies. It is this behavior which leads to the z^{-3} singularity in $\langle E^2 \rangle$ and hence in U . (The role of cutoffs for the quantized electromagnetic field in dielectrics has been discussed in more detail by Candelas [21]. Barton [22] has recently emphasized the fact that dispersion alone will not remove all divergences.)

The divergence of U is not considered to be physical, but as resulting from the idealization of the wall as a perfectly smooth surface. One way of removing this singularity is to allow the position of the boundary to fluctuate [23]. It seems plausible that such effects as surface roughness or the atomic nature of matter on small scales can also introduce a physical cutoff that makes the mean squared fields and the energy density finite everywhere. In the scalar models of Pfenning [12], the contributions of high frequencies are at least exponentially suppressed, resulting in a finite energy density on the boundary.

E. Case of a perfect conductor

Now we consider the limit $\epsilon \rightarrow \infty$. In this limit, as seen from Eqs. (11), (16a), and (16b), $r \rightarrow -1$ and $r' \rightarrow 1$. Equation (31) implies that U becomes zero, as expected, and Eqs. (36a) and (36b) give

$$\langle E^2 \rangle \sim \frac{3}{16\pi^2} \frac{1}{z^4}, \quad (39)$$

$$\langle B^2 \rangle \sim -\frac{3}{16\pi^2} \frac{1}{z^4}. \quad (40)$$

These well-known results are consistent with the asymptotic Casimir-Polder potential [24]

$$V_{\text{CP}} \sim -\frac{3}{32\pi^2} \frac{\alpha_0}{z^4} = -\frac{1}{2} \alpha_0 \langle E^2 \rangle, \quad (41)$$

where α_0 is the static polarizability of an atom near the interface.

IV. THE ENERGY DENSITY BETWEEN DIELECTRICS

In this section we calculate the energy density in a vacuum region of width a between two dielectric half-spaces. We define the dielectric constant as

$$\begin{aligned} 0 < z < a. & \quad \epsilon = 1, \\ z < 0 \text{ and } z > a: & \quad \epsilon = \epsilon_d. \end{aligned} \quad (42)$$

In the vacuum region, g^E occurring in Eq. (12a) has the form

$$g^E = -\frac{1}{2\kappa_0} \left\{ \frac{\exp(-\kappa_0|z-z'|) + r^2 \exp(-2\kappa_0 a) \exp(\kappa_0|z-z'|)}{r^2 \exp(-2\kappa_0 a) - 1} + \frac{r \{ \exp[-\kappa_0(z+z')] + \exp(-2\kappa_0 a) \exp[\kappa_0(z+z')] \}}{r^2 \exp(-2\kappa_0 a) - 1} \right\}, \quad (43)$$

and g^B has the same form as g^E , with r and r' interchanged, where r and r' are defined in Eqs. (16a) and (16b). We can now calculate the electric and magnetic fields in this region.

A. The electric field

The first and the second term on the right hand side of Eq. (21) for the present case, in the limit $z \rightarrow z'$, are

$$\omega^2 g^E = \frac{\omega^2}{2\kappa} \frac{1+r^2 e^{-2\kappa a}}{1-r^2 e^{-2\kappa a}} + \frac{\omega^2}{2\kappa} \frac{r(e^{-2\kappa z} + e^{-2\kappa a} e^{2\kappa z})}{1-r^2 e^{-2\kappa a}}, \quad (44)$$

$$(k^2 + \nabla_z \nabla_{z'}) g^B = \frac{\omega^2}{2\kappa} \frac{1+r'^2 e^{-2\kappa a}}{1-r'^2 e^{-2\kappa a}} + \frac{k^2 + \kappa^2}{2\kappa} \times \frac{r'(e^{-2\kappa z} + e^{-2\kappa a} e^{2\kappa z})}{1-r'^2 e^{-2\kappa a}}, \quad (45)$$

so that Γ_{kk} becomes

$$\Gamma_{kk} = \frac{\omega^2}{\kappa} + \frac{\omega^2}{\kappa} \left(\frac{r^2}{e^{2\kappa a} - r^2} + \frac{r'^2}{e^{2\kappa a} - r'^2} \right) + \frac{1}{\kappa} \left[\omega^2 \frac{r}{1-r^2 e^{-2\kappa a}} + (2k^2 - \omega^2) \frac{r'}{1-r'^2 e^{-2\kappa a}} \right] \times e^{-\kappa a} \cosh[\kappa(2z-a)]. \quad (46)$$

We again drop the first term on the right hand side of the above expression. After introducing polar coordinates and using Eqs. (11) and (46), (18) we find

$$\langle E^2 \rangle = \frac{1}{2\pi^2} \int_0^\infty du u^3 \int_0^{\pi/2} d\theta \sin(\theta) \times \left\{ \cos^2(\theta) \left(\frac{r^2}{r^2 - e^{2ua}} + \frac{r'^2}{r'^2 - e^{2ua}} \right) + \left[-\cos^2(\theta) \frac{r}{1-r^2 e^{-2ua}} + [1 + \sin^2(\theta)] \times \frac{r'}{1-r'^2 e^{-2ua}} \right] e^{-ua} \cosh[u(2z-a)] \right\}, \quad (47)$$

or with $\cos(\theta) \rightarrow t$

$$\langle E^2 \rangle = \frac{1}{2\pi^2} \int_0^\infty du u^3 \int_0^1 dt \left\{ t^2 \left(\frac{r^2}{r^2 - e^{2ua}} + \frac{r'^2}{r'^2 - e^{2ua}} \right) + \left[-t^2 \frac{r}{1-r^2 e^{-2ua}} + (2-t^2) \frac{r'}{1-r'^2 e^{-2ua}} \right] \times e^{-ua} \cosh[u(2z-a)] \right\}. \quad (48)$$

B. The magnetic field

In the same way, Eq. (28) leads to

$$\langle B^2 \rangle = \frac{1}{2\pi^2} \int_0^\infty du u^3 \int_0^1 dt \left\{ t^2 \left(\frac{r^2}{r^2 - e^{2ua}} + \frac{r'^2}{r'^2 - e^{2ua}} \right) + \left[(2-t^2) \frac{r}{1-r^2 e^{-2ua}} - t^2 \frac{r'}{1-r'^2 e^{-2ua}} \right] \times e^{-ua} \cosh[u(2z-a)] \right\}. \quad (49)$$

This expression differs from the one for $\langle E^2 \rangle$ only in the z dependent term with $r \leftrightarrow r'$. A plot of $\langle E^2 \rangle$, $\langle B^2 \rangle$, and U is shown in Fig. 2 for $\omega_p a = 200$. It can be seen in this figure that significant but not complete cancellation occurs between $\langle E^2 \rangle$ and $\langle B^2 \rangle$.

C. The energy density

Now, using Eqs. (30), (48), and (49), the energy density in the vacuum region can be calculated as

$$U = \frac{1}{2\pi^2} \int_0^\infty du u^3 \int_0^1 dt \left\{ t^2 \left(\frac{r^2}{r^2 - e^{2ua}} + \frac{r'^2}{r'^2 - e^{2ua}} \right) + (1-t^2) \left[\frac{r}{1-r^2 e^{-2ua}} + \frac{r'}{1-r'^2 e^{-2ua}} \right] \times e^{-ua} \cosh[u(2z-a)] \right\}. \quad (50)$$

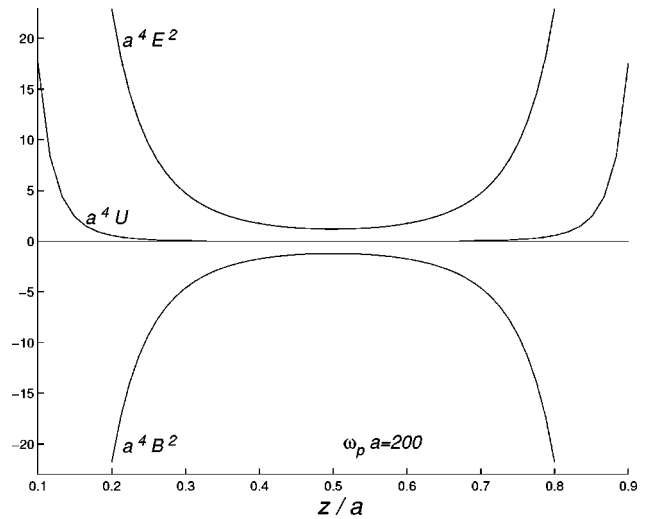


FIG. 2. The expectation values of the squared electric and magnetic fields, as well as the energy density in the vacuum region, are illustrated for $\omega_p a = 200$.

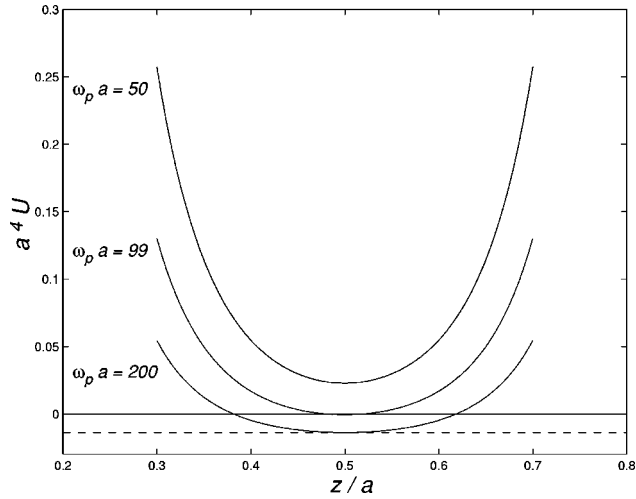


FIG. 3. The energy density in the vacuum region between two dielectric half-spaces is illustrated for three values of the parameter $\omega_p a$. The dashed horizontal line is the energy density for the perfectly conducting limit Eq. (52).

By analyzing this expression we can make some conclusions about the sign of U . First we note that it is position dependent, and we also note that the first term is always negative and the second term is always positive. The overall sign of U depends on the choice of a and ω_p . As $\omega_p a$ grows, U at the midpoint decreases, becoming negative for $\omega_p a \approx 100$, as seen in Figs. 3 and 4.

In Fig. 4 we see how the energy density at the center of the vacuum region changes as the product $\omega_p a$ increases. It can be seen in both Figs. 3 and 4 that U approaches the value given in Eq. (52) as $\omega_p a$ becomes large. The separation at which U becomes negative at the center is

$$a > a_c = \frac{99}{\omega_p} = (1.3 \mu\text{m}) \times \left(\frac{14.8 \text{ eV}}{\omega_p} \right), \quad (51)$$

where 14.8 eV is the plasma frequency of aluminum.

D. A perfect conductor case

In the limit $\omega_p \rightarrow \infty$, $r \rightarrow -1$ and $r' \rightarrow 1$. Then only the first (z independent) term survives in Eq. (50), and we get the familiar result [8]

$$U = -\frac{\pi^2}{720a^4}. \quad (52)$$

It is also interesting to examine the expressions for $\langle E^2 \rangle$ and $\langle B^2 \rangle$ in this limit. After performing the t integration, Eq. (48) can be written as

$$\begin{aligned} \langle E^2 \rangle = & \frac{1}{2\pi^2} \int_0^\infty du u^3 \left[\frac{2}{3} \frac{1}{1-e^{2ua}} \right. \\ & \left. + \left(\frac{e^{-2u(a-z)}}{1-e^{-2ua}} + \frac{e^{-2uz}}{1-e^{-2ua}} \right) \right], \quad (53) \end{aligned}$$

and after performing the u integration as

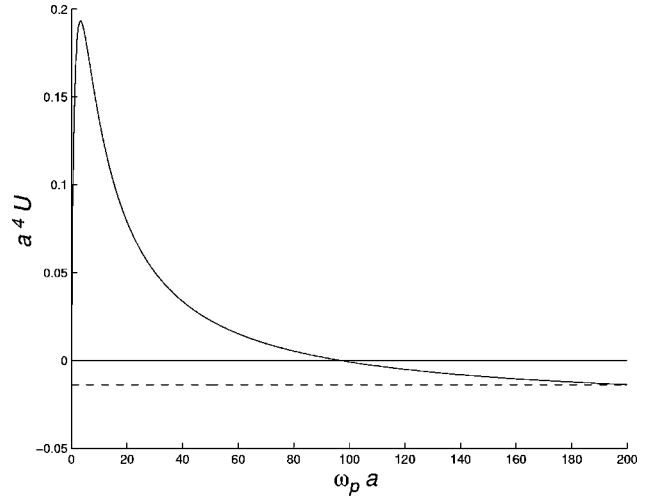


FIG. 4. The graph represents the energy density at the center of the gap between the two dielectric half-spaces as a function of $\omega_p a$. As seen from the graph, the energy density at the center becomes negative when $\omega_p a \approx 99$ or larger. Again the dashed line is the perfectly conducting limit.

$$\langle E^2 \rangle = -\frac{\pi^2}{720a^4} + \frac{1}{32\pi^2 a^4} \left[\psi^{(3)}\left(\frac{z}{a}\right) + \psi^{(3)}\left(1 - \frac{z}{a}\right) \right]. \quad (54)$$

Here $\psi^{(3)} = (d^4/dz^4) \ln \Gamma(z)$ is the polygamma function of order 3. It satisfies the reflection formula [25]

$$\psi^{(3)}\left(\frac{z}{a}\right) + \psi^{(3)}\left(1 - \frac{z}{a}\right) = -\pi \frac{d^3}{d(z/a)^3} \cot\left(\pi \frac{z}{a}\right). \quad (55)$$

This yields

$$\langle E^2 \rangle = -\frac{\pi^2}{720a^4} + \frac{\pi^2}{16a^4} \frac{1 + 2 \cos^2(\pi z/a)}{\sin^4(\pi z/a)}. \quad (56)$$

In the same way, one finds

$$\langle B^2 \rangle = -\frac{\pi^2}{720a^4} - \frac{\pi^2}{16a^4} \frac{1 + 2 \cos^2(\pi z/a)}{\sin^4(\pi z/a)}. \quad (57)$$

These results are in agreement with those given by previous authors [26,27].

E. Energy density near the boundary

To see how U grows near the interface we note that for small z (large u) Eq. (50) becomes

$$U \approx \frac{1}{4\pi^2} \int_0^\infty du u^3 \int_0^1 dt (1-t^2)(r+r') e^{-2uz}, \quad (58)$$

so it reduces to Eq. (35), the solution for the one-interface case. In the $z \rightarrow 0$ limit, this expression reduces to Eq. (37). By the same reasoning, the expressions for $\langle E^2 \rangle$ and $\langle B^2 \rangle$, Eqs. (48) and (49), reduce to Eqs. (36a) and (36b), respectively, in the small z limit.

V. CONCLUSION

In this paper we examined the mean squared fields and the energy density in the region between a pair of half-spaces filled with dispersive media. We found that these quantities diverge at the boundaries of the media, despite the inclusion of dispersion in the calculation. This divergence indicates a breakdown of a continuum description in which the dielectric function changes discontinuously at the boundary. It also shows that there is a positive self-energy density in the region outside a single plate. Nonetheless, we also found that it is possible for the net energy density in the region between the plates to be negative, depending upon the plate separation and the plasma frequency of the material involved. The existence of an attractive Casimir force is not an indicator of whether the energy density at the center of the plates is actually negative or not.

We have found that the energy density at the center becomes negative when $\omega_p a > 100$. Thus for fixed plasma frequency ω_p , the energy density always becomes negative for sufficiently large separation a . Of course, in this limit the magnitude of the energy density is also becoming small. Similarly, for fixed a , the energy density becomes negative for sufficiently large ω_p . In the limit that $\omega_p \rightarrow \infty$ our results approach the constant negative energy density of the perfectly conducting plates. It should not come as a surprise that there is a regime of negative energy density. The calculation assuming perfect conductivity does have a region of validity so long as ω_p is large and one is not too close to one plate. Qualitatively similar behavior has recently been found for the vacuum energy density near a domain wall [28].

In this paper, we assumed a particular form for the dielectric function (33) given by the collisionless Drude model. This is a good model for many metals, especially alkali met-

als, and is the generic form for the dielectric function of all materials at high frequencies. Thus taking a different form for ϵ_d would change the details of our results, especially far from an interface, but should lead to the same limiting forms near an interface. We have also assumed zero temperature throughout this paper. For systems at room temperature, this should be a good approximation when the separations are of the order of a few micrometers. More generally, one can ignore thermal effects at distances small compared to $1/(kT)$.

Contrary to the view expressed by Lamoreaux [11], the appearance of negative energy density in a quantum field theory is very natural. One can easily find quantum states of the free quantized electromagnetic field in empty space which have local negative energy densities. A squeezed vacuum state is an example [29,30]. The energy density of a quantized field has to be defined as a difference between that in empty Minkowski spacetime, and that in a given state and is no longer positive definite, as it was for a classical field. Apart from coupling to gravity, which produces extremely small effects, no clear way has been found to directly observe the local energy density. In certain limits, the negative energy density in a squeezed vacuum state has been shown theoretically [30] to produce an effect on the magnetic moment of a spin system. Whether this effect could ever be observed and whether negative Casimir energy density can produce similar effects is unknown.

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