Perturbative computation of the gluonic effective action via Polyakov's world-line path integral

S. D. Avramis,* A. I. Karanikas,† and C. N. Ktorides‡
Physics Department, Nuclear & Particle Physics Section, University of Athens, GR-15771 Athens, Greece
(Received 8 May 2002; published 19 August 2002)

The Polyakov world-line path integral describing the propagation of gluon field quanta is constructed by employing the background gauge fixing method and is subsequently applied to analytically compute the divergent terms of the one (gluonic) loop effective action to fourth order in perturbation theory. The merits of the proposed approach are that, to a given order, it reduces to performing two integrations, one over a set of Grassmann and one over a set of Feynman-type parameters, through which one manages to accommodate all Feynman diagrams entering the computation at once.

DOI: 10.1103/PhysRevD.66.045017 PACS number(s): 12.38.Bx

I. INTRODUCTION

Improved methods, in comparison to the Feynman diagrammatic ones, for expediting perturbative calculations in QCD have emerged, within the past decade or so, through the adoption of first-quantization-based approaches. The latter involve either string or world-line agents through which one describes the field theoretical system. The original efforts in this direction were string-inspired and were based on realizations, made in the late 1980s [1–5], regarding the relation between string and non-Abelian gauge field theories in the limit of an infinite string tension. Following their own involvement in such studies, Bern and Kosower [6] established a set of rules expediting efficient one loop computations in non-Abelian gauge theories. Through them one could encompass contributions of a host of Feynman diagrams at once. Further extensions of the string-inspired approach were subsequently carried out in Refs. [7–12].

World-line based methodologies aiming at the same goals soon followed through the work of Strassler [13] who proposed suitably defined (one-dimensional) path integrals for the various quantum field systems he considered. Extensive use was made of supersymmetric one-dimensional particle coordinates [14–16] ''living'' on the paths. Focusing on the computation of perturbative contributions to the effective action at the one loop level, the results of Ref. [13] achieved the reproduction of Bern-Kosower type rules at the level of one particle irreducible (1PI) diagrams. Strassler's approach was further pursued in Refs. [17–19], where computations pertaining to multiloop configurations in QED as well as effective actions involving constant, external (chromo)electric and (chromo)magnetic fields were undertaken.

Now, the world-line casting of relativistic quantum systems is an old story, which goes back to Fock [20], Feynman [21] and Schwinger [22]. Notably, relevant contributions followed by several authors [23–26], the latter of which sparked our original interest in the subject [27,28]. What particularly attracted our attention was the geometrical setting underlying the construction of Polyakov's (world-line) path integral. Within this context we pursued the problem of tracing the field theory origins of Polyakov's spin factor, introduced by him to properly account for the propagation of a free, spin-1/2 particlelike entity on a closed (Euclidean) space-time contour. In Ref. [28] we established, via a well-defined procedure, the emergence of the spin factor through the recasting of the spin-1/2 matter field sector of a gauge field theory from a functional to a (world-line) path integral, entering as an appropriate weight to account for spin. At the same time, the field theoretical interaction term [35] $\bar{\psi}\gamma_\mu\psi A^\mu$ is replaced by a ''factorized'' Wilson line (or loop) which accounts for the effect of the gauge field on the world-line paths, equivalently describes its interaction with the matter particles. Our first applications turned in the direction of considering situations when it is justified to set the spin factor to unity—an occurrence which facilitates a factorization of the infra-red sector of the gauge field theory, in its perturbative version [29].

More recently, we have tested the possible merits stemming from the aformentioned disentanglement between spin factor and Wilson line (loop), inherent in the Polyakov (world-line) path integral for spin-1/2 particlelike entities, as far as the task of facilitating effective, perturbative computations in QCD is concerned [30,31]. In the first of these papers the emphasis was placed on extending the world-line methodology to *open* fermionic lines. At the same time we established a procedure by which Strassler's path integral expression for spin-1/2 matter particles, entering an (interacting) gauge field theory and which contains the term $\sigma\cdot F$ in the action, can be recast into the Polyakov form which carries, in its place, the spin factor. Our subsequent manipulations were expedited by the presence of the spin factor and produced, as a bonus, the following physical picture: Nontrivial spin-factor contributions come precisely at those points where a gauge field quantum is emitted or absorbed by the fermionic world-line path. Moreover, each such occurrence signifies the presence of a derivative discontinuity on the path as a four-momentum k_μ is locally injected. This is, indeed, a nice intuitive picture as it connects the mathematical fact pertaining to the dominance of non-differentiable paths on the one hand, with the physical occurence of emission and absorption of quanta on the other. In the second paper we focused our attention on the more pragmatic goal of developing algorithms, always for spin-1/2 par-

*Email address: savramis@cc.uoa.gr
†Email address: akaranik@cc.uoa.gr
‡Email address: cktorid@cc.uoa.gr

ticle (open) world lines, which lead to efficient perturbative computations in QCD pertaining to Green's functions and amplitudes. Two different alternatives were arrived at according to whether the execution of the particle path integral precedes or follows the considerations involving the Wilson line (loop): (a) The Feynman diagrammatic logic is directly visible and comprehensively dealt with; (b) a novel organization of the perturbative expansion is achieved which retains the space-time description all the way. Strategy (a) leads to a neat organization of the resulting perturbative expression which reduces the computation to two straightforward steps. The first pertains, basically, to the spin factor and amounts to a simple integration over a set of Grassmann variables, as many in number as the perturbative order considered. The second integration is over a set of proper time parameters inherited from the expansion of the Wilson exponential. Possible practical merits of alternative (b) constitute an open issue.

Encouraged by the fact that the Polyakov path integral for spin-1/2 particle entities leads to computational procedures which, both logistically and intuitively, seem to present advantages of their own, we undertake, in the present paper, the task of extending the application of the relevant methodology to the gluonic sector of QCD. Such an effort entails, among other things, the determination of the spin factor pertaining to the propagation of a spin-1 particlelike entity. This issue is confronted in Sec. II where we consider the pure gauge field sector of a Yang-Mills system and utilize techniques associated with the background gauge fixing procedure. Focusing on effective action terms at the one gluon loop level, hence to 1PI diagrams in the Feynman context, we proceed, in Sec. III, to produce a master expression furnishing the Mth perturbative order contribution. The overall structure of these terms corresponds to a gluonic world-line loop on which "vertex operators," in the form of plane waves, are attached. As in our previous work [31], pertaining to open fermionic world lines, the overall calculation reduces to an integration over a Grassmann set of parameters followed by one over a set of Feynman-type parameters, the number of variables for each of the two sets being fixed by the perturbative order. At the end of the section we briefly discuss the extension of our approach to Green's functions or amplitudes where tree type configurations attached to loops must also be taken into account. Direct applications of our master formulas are worked out in Sec. IV, where we analytically compute the divergent parts of the second, third, and fourth (perturbative) order one-gluon-loop contributions to the effective action. From the specific manipulations it becomes obvious that the world-line configuration accommodates, to a given order, the totality of the contributing Feynman diagrams. Moreover, it can be easily surmised from our master expressions that no divergent terms make their appearance above the fourth order. Finally, in Sec. V we summarize our findings and formulate our conclusions, while in the Appendix we trace the main steps involved in bringing the spin factor to its final, ready to apply form.

Let us close our introductory discussion with the following two comments. First, as far as the finite contributions to the effective action terms are concerned, we are in the process of finalizing tests of numerical procedures that have been devised for their evaluation. Second, concerning one particle reducible diagrams (in the Feynman sense), a systematic way to include them into our scheme is to employ a classical field theory perturbation expansion (tree diagrams) associated with the background gauge field. We intend to report the details on both of these matters in a forthcoming paper [32].

II. POLYAKOV WORLD-LINE PATH INTEGRAL FOR THE GLUON SECTOR OF QCD

The successful transcription of the fermionic sector of a gauge field theory into its Polyakov path integral [36] form utilizes the fact that the corresponding functional integral is of a Gaussian type [28]. For the gluon sector, of course, such is not the case. Following Refs. [13] and [19], we proceed by employing the background gauge fixing procedure according to which the gauge field A_μ splits into a dynamical component, to be denoted by α_μ, and a background field B_μ. Given that we shall restrict, ourselves, in the present paper, to the computation of effective action terms, the background field will be considered as classical. Let us finally mention that we shall keep our formalism Euclidean throughout our analysis. Transcription of our final results to Minkowski space-time will be made in the end. In this respect, characterizations such as "Lorentz generators," "Lorentz trace," etc. will be employed by abuse of language.

The quadratic part of the (pure) gauge field action reads (in the Feynman gauge)

$$S_2=\frac{1}{2}\alpha_\mu^a[-(D^2)^{ab}\delta_{\mu\nu}-[D_\mu,D_\nu]^{ab}-igF_{\mu\nu}^{ab}]\alpha_\nu^b + \overline{c^a}[(D^2)^{ab}]c^b, \tag{1}$$

where $D_\mu^{ab}=D_\mu^{ab}(B)=\partial_\mu\delta^{ab}+gf^{abc}B_\mu^c$ is the covariant field derivative in the adjoint representation, while c, $\overline{c}$ are the ghost fields. Obviously, $F_{\mu\nu}$ entering Eq. (1) is the Maxwell tensor for the background gauge field, i.e.

$$F_{\mu\nu}^{ab}=F_{\mu\nu}^{ab}(B)=-if^{abc}F_{\mu\nu}^c(B) =-if^{abc}(\partial_\mu B_\nu^c-\partial_\nu B_\mu^c-gf^{cde}B_\mu^d B_\nu^e). \tag{2}$$

Introducing the Lorentz generators under which four-vectors transform, namely

$$(J_{\rho\sigma})_{\mu\nu}=i(\delta_{\rho\mu}\delta_{\sigma\nu}-\delta_{\rho\nu}\delta_{\sigma\mu}), \tag{3}$$

we rewrite Eq. (1) as follows:

$$S_2=\frac{1}{2}\alpha_\mu^a[-(D^2)^{ab}\delta_{\mu\nu}-g(J\cdot F)_{\mu\nu}^{ab}]\alpha_\nu^b+\overline{c^a}[(D^2)^{ab}]c^b. \tag{4}$$

In the one loop approximation, to which we shall restrict our considerations in this work, the effective action, as a functional of the background field, is given by

$$\Gamma_1[B]=\frac{1}{2}\mathrm{Tr}\ln(-D^2-gJ\cdot F)-\mathrm{Tr}\ln(-D^2)$$
$$\equiv\Gamma_{1,gluons}[B]+\Gamma_{ghosts}[B]. \tag{5}$$

In what follows it suffices to work with $\Gamma_{1,gluons}[B]$ as $\Gamma_{ghosts}[B]$ is simply given by the first of the two terms entering the gluon contribution to the effective action, multiplied by (-2).

Employing Schwinger's parametrization formula [22], we write (trace on "Lorentz" and color indices)

$$\Gamma_{1,gluons}[B]=-\frac{1}{2}\int_0^\infty\frac{dT}{T}\int d^Dx\,\mathrm{Tr}\,K(x,x;T), \tag{6}$$

where

$$K(y,x;T)^{ab}_{\mu\nu}\equiv\langle y|e^{-T(-D^2-gJ\cdot F)}|x\rangle^{ab}_{\mu\nu} \tag{7}$$

corresponds to the (dynamical) gauge field propagator kernel in the background field.

The world-line path integral for $K(y,x;T)^{ab}_{\mu\nu}$ results through standard procedures (see, e.g., Ref. [30]) and reads

$$K(y,x;T)^{ab}_{\mu\nu}=\int_{x(0)=x}^{x(T)=y}\mathcal{D}x(t)\exp\left[-\frac{1}{4}\int_0^T dt\dot{x}^2(t)\right]\times\mathcal{P}\exp\left[ig\int_0^T dt\dot{x}\cdot B+g\int_0^T dtJ\cdot F\right]^{ab}_{\mu\nu}. \tag{8}$$

As already established in Refs. [30,31], the Polyakov path integral results once we apply the "area derivative" operator [33,34] given by

$$\frac{\delta}{\delta s_{\mu\nu}(t)}\equiv\lim_{\varepsilon\to 0}\int_{-\varepsilon}^{\varepsilon}ds\,s\frac{\delta^2}{\delta x_\mu\left(t+\frac{s}{2}\right)\delta x_\nu\left(t-\frac{s}{2}\right)} \tag{9}$$

and use, at the same time, the identities

$$\frac{\delta}{\delta s_{\mu\nu}(t)}\mathcal{P}\exp\left(ig\int_0^T dt\dot{x}\cdot B\right)^{ab}=\mathcal{P}\exp\left(ig\int_t^T dt\dot{x}\cdot B\right)^{aa_1}_{\mu\rho}(-igF[x(t)])^{a_1a_2}_{\rho\sigma}\times\mathcal{P}\exp\left(ig\int_0^t dt\dot{x}\cdot B\right)^{a_2b}_{\sigma\nu} \tag{10}$$

and

$$\int_0^T dt\frac{\delta}{\delta s_{\mu\nu}(t)}\exp\left(-\frac{1}{4}\int_0^T dt\dot{x}^2\right)=\frac{1}{2}\int_0^T dt\,\omega_{\mu\nu}[\dot{x}(t)]\exp\left(-\frac{1}{4}\int_0^T dt\dot{x}^2\right), \tag{11}$$

where $\omega_{\mu\nu}$ expresses the rotation of the vector tangent to the trajectory [26] and, for paths described by differentiable functions, assumes the form

$$\omega_{\mu\nu}[\dot{x}]=\frac{T}{2}(\ddot{x}_\mu\dot{x}_\nu-\dot{x}_\mu\ddot{x}_\nu). \tag{12}$$

A more careful discussion pertaining to the spin factor is conducted in the Appendix.

Once performing a partial integration, Eq. (8) assumes its Polyakov path-integral form which reads

$$K(y,x;T)^{ab}_{\mu\nu}=\int_{x(0)=x}^{x(T)=y}\mathcal{D}x(t)\exp\left(-\frac{1}{4}\int_0^T dt\dot{x}^2\right)\times\mathcal{P}\exp\left(\frac{i}{2}\int_0^T dtJ\cdot\omega\right)_{\mu\nu}\times\mathcal{P}\exp\left(ig\int_0^T dt\dot{x}\cdot B\right)^{ab}. \tag{13}$$

In turn, the corresponding expression for the effective action, including the contribution from the ghost term, becomes

$$\Gamma_1[B]=-\frac{1}{2}\int_0^\infty\frac{dT}{T}\int_P\mathcal{D}x(t)\exp\left(-\frac{1}{4}\int_0^T dt\dot{x}^2\right)\times\{\mathrm{Tr}_L\Phi^{[1]}[\dot{x}]-2\}\mathrm{Tr}_c\mathcal{P}\exp\left(ig\int_0^T dt\dot{x}\cdot B\right), \tag{14}$$

where the subscript P denotes the periodic boundary conditions, $x(0)=x(T)$, imposed on the path integration, while the indices on the traces stand for "Lorentz" (L) and "color" (c). Furthermore, we have introduced the spin-factor expression [37]

$$\Phi^{[1]}[\dot{x}]_{\mu\nu}\equiv\mathcal{P}\exp\left[\frac{i}{2}\int_0^T dtJ\cdot\omega[\dot{x}(t)]\right]_{\mu\nu} \tag{15}$$

which is the appropriate weight pertaining to the description of the propagation of a spin-1 particlelike entity (gluon) in (Euclidean) space-time. It is not difficult to see that the spin factor has a restricted dependence on a path's profile. As argued in Ref. [30] and further deliberated on in the Appendix, contributions of the spin factor to the path integral come solely from points where a four-momentum is applied through an emission or absorption of a gauge field quantum. Roughly speaking, this has to do with the fact that the expectation value $\langle\ddot{x}_\mu\dot{x}_\nu\rangle-\langle\dot{x}_\mu\ddot{x}_\nu\rangle$, as computed through the path integral, vanishes unless a four-momentum k_μ is imparted at the point x.

For the sake of comparison we give the corresponding expression for the one fermionic loop expression which contributes to the effective action [28]. It reads (color matrices in the fundamental representation)

$$\Gamma_{1,f}[B]=\frac{1}{2}\int_0^\infty\frac{dT}{T}\int_P\mathcal{D}x(t)\exp\left(-\frac{1}{4}\int_0^T dt\dot{x}^2\right)\times\mathrm{Tr}_L\Phi^{[1/2]}[\dot{x}]\mathrm{Tr}_c P\exp\left(ig\int_0^T dt\dot{x}\cdot B\right) \quad (16)$$

with the spin factor now given by

$$\Phi^{[1/2]}[\dot{x}]\equiv\mathcal{P}\exp\left[\frac{i}{2}\int_0^T dt\,S\cdot\omega[\dot{x}(t)]\right] \quad (17)$$

and where the corresponding Lorentz generators belong to the spinor representation, i.e.

$$S_{\mu\nu}=\frac{1}{2}\sigma_{\mu\nu}=\frac{i}{4}[\gamma_\mu,\gamma_\nu]. \quad (18)$$

Generally put, the Polyakov path integral recasting of a relativistic quantum field theoretical system provides a unified basis for the description of the propagating particlelike entity; one simply has to adjust the weight provided by the spin factor to its particular form. Thus, for a spin-zero particle the relevant weight factor is, simply, unity (note, in this regard, that ghosts fall into this class irrespective of the anticommutation relations, cf. minus sign, they obey) while for spin-1/2 and spin-1 particlelike entities the corresponding weights are provided by Eqs. (17) and (15), respectively.

For completeness let us mention that the path integral expression for the gluonic Green's function, namely

$$iG(y,x)^{ab}_{\mu\nu}=\int_0^\infty dTK(y,x;T)^{ab}_{\mu\nu}, \quad (19)$$

is determined once the substitution from Eq. (13) is made for the propagation kernel.

III. THE ONE GLUON LOOP, *M*-POINT EFFECTIVE ACTION

In this section we shall perform a number of manipulations through which we shall arrive at ready-to-apply master expressions for the computation of one-loop effective action terms. Let us commence our efforts by giving to the, classical, background field B the plane wave form, i.e. we set [38] $B_\mu(x)=t_G^{a_n}\varepsilon_\mu^{a_n}e^{ip_n\cdot x}$, where the index n tracks the various gauge fields entering the Mth order term in the expansion of the Wilson exponential in Eq. (14). We obtain

$$\Gamma_1^{(M)}(p_1,\ldots,p_M)=-\frac{1}{2}(ig)^M\mathrm{Tr}_C(t_G^{a_M}\ldots t_G^{a_1})\int_0^\infty\frac{dT}{T}\left[\prod_{n=M}^{1}\int_0^T dt_n\right]\theta(t_M,\ldots,t_1)\int_P\mathcal{D}x(t)\prod_{n=M}^{1}\varepsilon^n\cdot\dot{x}(t_n)\times\{\mathrm{Tr}_L\Phi^{[1]}[\dot{x}]-2\}\exp\left[-\frac{1}{4}\int_0^T dt\dot{x}^2+i\sum_{n=1}^M p_n\cdot x(t_n)\right]+\text{permutations}, \quad (20)$$

where $\theta(t_M,\ldots,t_1)=\Pi_{n=M-1}^1\theta(t_{n+1}-t_n)$ and where the indication "permutations" refers to all possible rearrangements of the t_n and the $t_G^{a_n}$ associated with the, M in number, background gauge fields.

Our computational strategy for confronting the above quantity coincides with the one employed in Ref. [31]. It relies on a move to recast the spin-factor expression into an explicitly path-independent form. Once this is done the path integration can be immediately performed given that the "action functional" is a simple Gaussian (with a linear term). Subsequently, we shall deal with the spin factor.

Following the procedure employed in the aformentioned reference we introduce the Grassmann variables $\overline{\xi_n}$ and ξ_n through which the $\varepsilon^n\cdot\dot{x}(t_n)$ factors in Eq. (20) are elevated into exponentials according to [13]

$$i\varepsilon^n\cdot\dot{x}(t_n)=\int d\overline{\xi_n}d\xi_n\exp[i\overline{\xi_n}\xi_n\varepsilon^n\cdot\dot{x}(t_n)]. \quad (21)$$

After substituting in Eq. (20) we obtain

$$\Gamma_1^{(M)}(p_1,\ldots,p_M)=-\frac{1}{2}g^M\mathrm{Tr}_C(t_G^{a_M}\ldots t_G^{a_1})\int_0^\infty\frac{dT}{T}\left[\prod_{n=M}^{1}\int_0^T dt_n\right]\theta(t_M,\ldots,t_1)\left[\prod_{n=M}^{1}\int d\xi_n d\overline{\xi_n}\right]\times\int_P\mathcal{D}x(t)\{\mathrm{Tr}_L\Phi^{[1]}[\dot{x}]-2\}\exp\left[-\frac{1}{4}\int_0^T dt\dot{x}^2+i\sum_{n=1}^M\hat{k}(t_n)\cdot x(t_n)\right]+\text{permutations}, \quad (22)$$

having set

$$\hat{k}_\mu(t_n)\equiv p_{n,\mu}+\overline{\xi_n}\xi_n\varepsilon_\mu^n\frac{\partial}{\partial t_n}. \quad (23)$$

Recalling the original specification of the spin factor, which utilizes the employment of the area derivative, let us rewrite Eq. (22) in the specific form which takes into account the fact that the gauge potentials entering the expansion of the Wilson exponential are plane waves. For the Mth order term we write

$$\mathrm{Tr}_L\Phi^{[1]}[\hat{k}]\equiv\exp\left[-i\sum_{n=1}^{M}\hat{k}(t_n)\cdot x(t_n)\right]\left[\mathrm{Tr}_L\mathcal{P}\exp\left(-i\int_0^T dtJ\cdot\frac{\delta}{\delta s}\right)\right]\exp\left[i\sum_{n=1}^{M}\hat{k}(t_n)\cdot x(t_n)\right]. \tag{24}$$

The above expression illustrates in an immediate, albeit formal, manner the path independence of the spin factor: One observes that the area derivative acting on the exponential will produce delta functions entering the parametric integration, cf. Eq. (9), entering the definition of the area derivative operator. A well defined argument leading to this result is provided in the Appendix.

Returning to the case at hand, we write

$$\Gamma_1^{(M)}(p_1,\ldots,p_M)=-\frac{1}{2}g^M\mathrm{Tr}_C(t_G^{a_M}\ldots t_G^{a_1})\int_0^\infty\frac{dT}{T}\left[\prod_{n=M}^{1}\int_0^T dt_n\right]\theta(t_M,\ldots,t_1)\left[\prod_{n=M}^{1}\int d\xi_n d\bar{\xi}_n\right]$$
$$\times\{\mathrm{Tr}_L\Phi^{[1]}[\hat{k}]-2\}\int_P\mathcal{D}x(t)\exp\left[-\frac{1}{4}\int_0^T dt\dot{x}^2+i\sum_{n=1}^{M}\hat{k}(t_n)\cdot x(t_n)\right]+\text{permutations.} \tag{25}$$

The first task we shall carry out is to perform the, basically Gaussian, path integral. Straightforward manipulations, partly displayed in the Appendix, lead to the result

$$\int_{x(0)=x(T)}\mathcal{D}x(t)\exp\left[-\frac{1}{4}\int_0^T dt\dot{x}^2(t)+i\sum_{n=1}^{M}\hat{k}(t_n)\cdot x(t_n)\right]$$
$$=(2\pi)^D\delta^{(D)}\left(\sum_{n=1}^{M}p_n\right)\frac{1}{(4\pi T)^{D/2}}\exp\left[\sum_{n<m}p_n\cdot p_m G(t_n,t_m)+\sum_{n\neq m}\bar{\xi}_n\xi_n\varepsilon^n\cdot p_m\partial_n G(t_n,t_m)\right.$$
$$\left.+\frac{1}{2}\sum_{n\neq m}\bar{\xi}_n\xi_n\bar{\xi}_m\xi_m\varepsilon^n\cdot\varepsilon^m\partial_n\partial_m G(t_n,t_m)\right]. \tag{26}$$

One notes contributions pertaining solely to the points of attachment of external gauge field on the loop contour. In the above expression the following Green's function [13] has been employed:

$$G(t,t')=|t-t'|\left[1-\frac{|t-t'|}{T}\right]. \tag{27}$$

It corresponds [39] to the motion of a one-dimensional particle moving on a closed contour, i.e.

$$\frac{1}{2}\frac{\partial^2}{\partial t^2}G(t,t')=-\delta(t-t')+\frac{1}{T} \tag{28}$$

and obeys the boundary conditions $G(0,t')=G(T,t')$ and $\dot{G}(0,t')=\dot{G}(T,t')$.

Introducing the dimensionless parameters u_i according to $t_i=Tu_i, i=1,\ldots,n$, the interim result for $\Gamma_1^{(M)}(p_1,\ldots,p_M)$ reads

$$\Gamma_1^{(M)}(p_1,\ldots,p_M)=-\frac{1}{2}g^M(2\pi)^D\delta^{(D)}\left(\sum_{n=1}^{M}p_n\right)\mathrm{Tr}_C(t_G^{a_M}\cdots t_G^{a_1})\frac{1}{(4\pi)^{D/2}}\int_0^\infty dT\,T^{M-D/2-1}\left[\prod_{n=M}^{1}\int_0^1 du_n\right]$$
$$\times\theta(u_M,\ldots,u_1)F^{(M)}(u_1,\ldots,u_M;T)\exp\left[T\sum_{n<m}p_n\cdot p_m G(u_n,u_m)\right]+\text{permutations,} \tag{29}$$

where $G(u_n,u_m)=|u_n-u_m|[1-|u_n-u_m|]$ satisfies the additional properties

$$\partial_n G(u_n,u_m)\equiv\dot{G}(u_n,u_m)=\mathrm{sgn}(u_n-u_m)-2(u_n-u_m)=-\dot{G}(u_m,u_n) \tag{30}$$

and

$$-\partial_n\partial_m G(u_n,u_m)=\partial_n^2 G(u_n,u_m)\equiv\ddot{G}(u_n,u_m)=2[\delta(u_n-u_m)-1]. \tag{31}$$

Finally, in Eq. (29) we have set

$$F^{(M)}(u_1,\ldots,u_M;T)=\left[\prod_{n=M}^{1}\int d\xi_n d\overline{\xi_n}\right](\mathrm{Tr}_L\Phi^{[1]}[\hat{k}]-2)\exp\left[\sum_{n\neq m}\overline{\xi_n}\xi_n\varepsilon^n\cdot p_m\partial_n G(u_n,u_m)+\frac{1}{2T}\sum_{n\neq m}\overline{\xi_n}\xi_n\overline{\xi_m}\xi_m\varepsilon^n\cdot\varepsilon^m\partial_n\partial_m G(u_n,u_m)\right]. \tag{32}$$

The spin factor can now be brought into a ready-to-apply form through a series of manipulations that are outlined in the Appendix. The following result is arrived at:

$$\Phi^{[1]}_{\mu\nu}[\hat{k}]=\mathcal{P}\exp\left[\frac{i}{2}\sum_{n=1}^{M}J\cdot\phi(n)\right]_{\mu\nu}=\delta_{\mu\nu}+\frac{i}{2}(J_{\rho\sigma})_{\mu\nu}\sum_{n=1}^{M}\phi_{\rho\sigma}(n)+\left(\frac{i}{2}\right)^2(J_{\rho_2\sigma_2})_{\mu\lambda}(J_{\rho_1\sigma_1})_{\lambda\nu}\times\sum_{n_2=1}^{M}\sum_{n_1=1}^{n_2-1}\phi_{\rho_2\sigma_2}(n_2)\phi_{\rho_1\sigma_1}(n_1)+\cdots \tag{33}$$

with

$$\phi_{\mu\nu}(n)=2\overline{\xi_n}\xi_n(\varepsilon^n_\mu p_{n,\nu}-\varepsilon^n_\nu p_{n,\mu})+\frac{4}{T}\overline{\xi}_{n+1}\xi_{n+1}\overline{\xi_n}\xi_n(\varepsilon^{n+1}_\mu\varepsilon^n_\nu-\varepsilon^{n+1}_\nu\varepsilon^n_\mu)\times\delta(u_{n+1}-u_n) \tag{34}$$

and where we have designated that $\overline{\xi}_{M+1}=\xi_{M+1}=0$.

Two observations of practical interest can be made in connection with the above expression for the spin factor. First, it is clear that the number of terms in the expansion of the exponential in Eq. (33) terminates at M as the saturation point of the Grassmann variables is by then reached. Second, the delta-function-containing term in Eq. (33) implies that for a given ordering there is a contribution from coinciding points, u_n and u_{n+1} in this case. This occurrence signifies the presence of a ‘‘four-gluon vertex’’ which is automatically included in a given perturbative calculation, along with the (derivative-dependent) ‘‘three-gluon vertices’’ represented by the first term. One thereby concludes that the Mth order perturbative contributions to the effective action are classified, via the spin factor, exclusively by the number of the points of gluon (single or pairwise) attachments on the closed worldline contour, in all possible permutations. Accordingly, the computation of the M-point effective action term will collect all Mth order, 1PI Feynman diagrams.

We mention in passing that fermionic loop contributions to the effective action easily follow by referring to Eqs. (16)–(18). One simply has to make the substitution $\phi_{\rho\sigma}(J_{\rho\sigma})_{\mu\nu}\rightarrow S_{\mu\nu}\phi_{\mu\nu}$ in Eqs. (33) and (34) and use the fundamental representation of the group.

Given the expressions we have arrived at, what remains to be carried out, as far as 1PI configurations are concerned, are the integrations over the Grassmann variables as well as the parametric integrations entering Eq. (29). Numerical methods have been developed for this purpose whose report is forthcoming [32]. For the rest of this paper we restrict ourselves to the computation of the divergent part of the effective action. In this regard, let us observe, by looking at Eq. (32), that ultraviolet divergences will occur only for terms of order $M=2,3,4$. Specifically, by focusing on the terms that have the minimum number of $p_{n,\mu}$ factors one determines, through dimensional considerations, that they should carry the compensating factor T^{2-M} for $M=2,3,4$. The latter combines with $T^{M-1-D/2}$ in Eq. (29) to produce divergent terms $\sim\Gamma(2-D/2)$. Further inspection shows that no such terms arise for $M\geqslant 5$, a fact that directly complies with the renormalizability of the theory.

Concerning our earlier statement that the classical, background field B takes, for the present purposes, the form of plane waves we offer the following comment. Suppose we become completely general and set

$$B_\mu(x)=t^a_G\int\frac{d^Dq}{(2\pi)^D}e^{iq\cdot x}\tilde{\varepsilon}^a_\mu(q). \tag{35}$$

Then, the only resulting difference is that our master expression, given by Eq. (29), would include integrations over the momenta. The plane-wave representation employed in the present paper has been made entirely for reasons of economy, coupled with the fact that we shall be focusing on the divergent contributions to the effective action. Referring to Eq. (35), we have chosen $\tilde{\varepsilon}^a_\mu(q)=(2\pi)^D\delta(q-p)\varepsilon^a_\mu$, with $p\cdot\varepsilon=0$. Let us also point out that in our subsequent computations we shall, for IR protection purposes, set the gluon four-momenta to $p_n^2=\lambda^2$, $n=1,2,\ldots$, with $\lambda^2>\Lambda^2_{QCD}$.

Let us finally remark that the master formula can also be employed for the purpose of computing Green's functions or amplitudes where one particle reducible diagrams, in the language of the Feynman organization of perturbation theory,

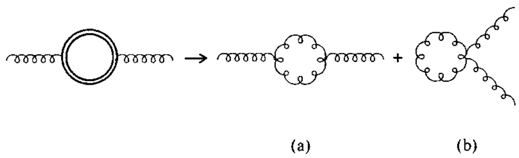

FIG. 1. Second order, one-loop gluon contribution to the effective action. Depicted to the left is the ''world-line diagram'' representing the $M=2$ master formula. To the right, the classes of Feynman diagrams, accommodated by the world-line one, are displayed. The ghost loop Feynman diagram has not been drawn.

also enter. To this end it is sufficient to represent the classical background field in the form it would assume had we solved the equation

$$D_\mu F_{\mu\nu}(B)=0 \tag{36}$$

via the classical field perturbation theory. We would, then, write

$$\tilde{\varepsilon}^a_\mu=(2\pi)^D\delta(q-p_1)\varepsilon^a_\mu+g(2\pi)^D\delta(q-p_1-p_2)\times(t^a_G)_{a_1a_2}\frac{\varepsilon^1\cdot p_2\varepsilon^2_\mu}{(p_1+p_2)^2}+\mathcal{O}(g^2), \tag{37}$$

which furnishes an attachment of a tree diagram with a three gluon vertex. One could similarly proceed to determine terms corresponding to attachments of higher order. With the help of the series implied in Eq. (37) and employing the expression given by Eq. (35), we can use the master formula given by Eq. (29) to compute (one-loop) amplitude terms. As mentioned earlier, these matters will be dealt with in a forthcoming paper [32].

IV. COMPUTATION OF DIVERGENT ONE-LOOP EFFECTIVE ACTION TERMS TO FOURTH ORDER

In this section we shall apply our comprehensive formulas given by Eqs. (29) and (32)–(34) towards the computation of the divergent contributions to the $M=2,3,4$ terms in the expansion of the effective action—in fact, the only terms which exhibit ultra-violet divergences. We leave the task of computing finite contributions, to the same order, to a future paper where numerical methods will be applied.

A. The two gluon contribution $(M=2)$ to the effective action

The present calculation pertains to the classes of Feynman diagrams displayed in Fig. 1 as (a) and (b). Our master expression accommodates the two depicted classes plus the ghost contribution. From Eqs. (33) and (34) we determine, for $M=2$,

$$\mathrm{Tr}_L\Phi^{[1]}=D-8\bar{\xi}_1\xi_1\bar{\xi}_2\xi_2\varepsilon^1\cdot\varepsilon^2 p_1\cdot p_2. \tag{38}$$

Upon substituting in Eq. (32) and performing the Grassmann integrations we obtain

$$F^{(M=2)}(u_1,u_2;T)=-\frac{1}{T}(D-2)\varepsilon^1\cdot\varepsilon^2\ddot{G}(u_1,u_2)-8\varepsilon^1\cdot\varepsilon^2 p_1\cdot p_2. \tag{39}$$

One notes that the delta function entering the specification of $\ddot{G}(u_1,u_2)$ accommodates the contribution coming from the class of Feynman diagrams wherein the two (truncated) external gluons attach themselves to the loop through a four-point vertex.

The above result when substituted in Eq. (32) gives, after an integration by parts which results in the replacement $\ddot{G}(u_1,u_2)\to -Tp_1\cdot p_2\dot{G}^2(u_1,u_2)$,

$$\Gamma_1^{(M=2)}(p_1,p_2)=-\frac{1}{2}(2\pi)^4\delta^{(4)}(p_1+p_2)\mathrm{Tr}_C(t^{a_2}_G t^{a_1}_G)\times\frac{g^2}{(4\pi)^{D/2}}\varepsilon^1\cdot\varepsilon^2 p_1\cdot p_2\times\int_0^\infty dT T^{1-D/2}\int_0^1 du_2\int_0^{u_2}du_1\times[(D-2)\dot{G}^2(u_1,u_2)-8]\times\exp[-T\lambda^2 G(u_1,u_2)]+\text{permutations}, \tag{40}$$

where the infra-red cutoff λ has been introduced by going off shell. The integrations in the last equation can be easily performed and lead to the final result

$$\Gamma_1^{(M=2)}(p_1,p_2)=-\frac{1}{2}(2\pi)^4\delta^{(4)}(p_1+p_2)N\delta^{a_2a_1}\frac{g^2}{(4\pi)^2}\times\left(4\pi\frac{\mu^2}{\lambda^2}\right)^{2-D/2}\varepsilon^1\cdot\varepsilon^2 p_1\cdot p_2\times\Gamma\left(2-\frac{D}{2}\right)\frac{11-7(2-D/2)}{3-2(2-D/2)}\times B\left(\frac{D}{2}-1,\frac{D}{2}-1\right), \tag{41}$$

where the adjustment $g^2\to g_D^2=g^2\mu^{4-D}$ was made in order to restore dimensional consistency. The term ''permutations'' in Eq. (29) has been duly taken care of by taking into account all the rearrangements of indices (1,2) and dividing by 2! in order to comply with boson non-distinguishability.

From Eq. (41) we verify, once returning to Minkowski space-time, the well known result (which does not take into account the contribution from the fermionic loop)

$$Z_g=1-\frac{1}{2}\frac{g^2}{(4\pi)^2}N\frac{11}{3}\frac{1}{2-D/2}. \tag{42}$$

It is of interest to observe, by referring to Eqs. (13), (19) and as has been explicitly demonstrated in Refs. [30,31], that the corresponding formulas resulting from Polyakov's path integral for open lines have the same basic structure with the ones that have resulted from the present considerations pertaining to loops. It then becomes a straightforward matter to surmise the validity of the Ward identity $Z_B^{1/2}Z_g=1$ which is known to hold in the framework of the background gauge fixing method.

B. The three gluon contribution $(M=3)$ to the effective action

We now turn our attention to $\Gamma_1^{(M=3)}$ which summarizes the contributions from the classes of Feynman diagrams depicted in Fig. 2 (plus ghost ones). Again, our first task is to compute the corresponding expression for the spin factor. Equations (33) and (34) now give

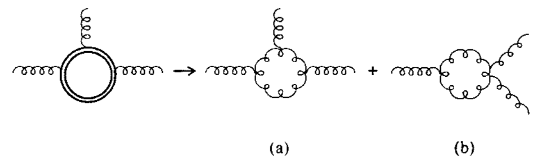

FIG. 2. Same as in Fig. 1 for third order contributions ($M=3$).

$$\mathrm{Tr}_L\Phi^{[1]}=D+8\bar{\xi}_2\xi_2\bar{\xi}_3\xi_3(\varepsilon^2\cdot p_3\varepsilon^3\cdot p_2-\varepsilon^2\cdot\varepsilon^3 p_2\cdot p_3)+8\bar{\xi}_1\xi_1\bar{\xi}_3\xi_3(\varepsilon^1\cdot p_3\varepsilon^3\cdot p_1-\varepsilon^1\cdot\varepsilon^3 p_1\cdot p_3)+8\bar{\xi}_1\xi_1\bar{\xi}_2\xi_2(\varepsilon^1\cdot p_2\varepsilon^2\cdot p_1$$
$$-\varepsilon^1\cdot\varepsilon^2 p_1\cdot p_2)+\frac{16}{T}\bar{\xi}_1\xi_1\bar{\xi}_2\xi_2\bar{\xi}_3\xi_3[(\varepsilon^1\cdot\varepsilon^2\varepsilon^3\cdot p_1-\varepsilon^1\cdot\varepsilon^3\varepsilon^2\cdot p_1)\,\delta(u_3-u_2)+(\varepsilon^1\cdot\varepsilon^3\varepsilon^2\cdot p_3-\varepsilon^2\cdot\varepsilon^3\varepsilon^1\cdot p_3)$$
$$\times\delta(u_2-u_1)+T\varepsilon^1\cdot p_3\varepsilon^2\cdot p_1\varepsilon^3\cdot p_2]. \tag{43}$$

The integration over the Grassmann variables can be systematically performed, yielding the result

$$F_1^{(M=3)}(u_1,u_2,u_3;T)=-\frac{D-2}{T}\{\varepsilon^1\cdot\varepsilon^2[\varepsilon^3\cdot p_1\dot{G}(u_3,u_1)+\varepsilon^3\cdot p_2\dot{G}(u_3,u_2)]\ddot{G}(u_1,u_2)+\varepsilon^1\cdot\varepsilon^3[\varepsilon^2\cdot p_1\dot{G}(u_2,u_1)$$
$$+\varepsilon^2\cdot p_3\dot{G}(u_2,u_3)]\ddot{G}(u_1,u_3)+\varepsilon^2\cdot\varepsilon^3[\varepsilon^1\cdot p_2\dot{G}(u_1,u_2)+\varepsilon^1\cdot p_3\dot{G}(u_1,u_3)]\ddot{G}(u_2,u_3)\}$$
$$+\frac{16}{T}(\varepsilon^1\cdot\varepsilon^2\varepsilon^3\cdot p_1-\varepsilon^1\cdot\varepsilon^3\varepsilon^2\cdot p_1)\,\delta(u_3-u_2)+\frac{16}{T}(\varepsilon^1\cdot\varepsilon^3\varepsilon^2\cdot p_3-\varepsilon^2\cdot\varepsilon^3\varepsilon^1\cdot p_3)\,\delta(u_2-u_1)+f.t., \tag{44}$$

where *f.t.* stands for "terms with finite contribution." Obviously the latter involve terms with T to the 0th power or higher, equivalently, they involve more than one (external) momentum variables. Let us reiterate that the finite terms should be computable through numerical methods that are currently being developed.

Substituting the above result in Eq. (29) we obtain

$$\Gamma_1^{(M=3)}(p_1,p_2,p_3)=-\frac{1}{2}(2\pi)^4\delta^{(4)}\left(\sum_{i=1}^{3}p_i\right)\mathrm{Tr}_c(t_G^{a_3}t_G^{a_2}t_G^{a_1})\frac{g^3}{(4\pi)^{D/2}}\int_0^\infty dT\,T^{1-D/2}\int_0^1 du_3\int_0^{u_3}du_2\int_0^{u_2}du_1\{4(D-2)$$
$$\times[\varepsilon^1\cdot\varepsilon^2\varepsilon^3\cdot p_2(u_2-u_1)+\varepsilon^1\cdot\varepsilon^3\varepsilon^2\cdot p_1(1-(u_3-u_1))]+[\varepsilon^2\cdot\varepsilon^3\varepsilon^1\cdot p_3(u_3-u_2)]-16(\varepsilon^1\cdot\varepsilon^2\varepsilon^3\cdot p_2$$
$$+\varepsilon^1\cdot\varepsilon^3\varepsilon^2\cdot p_1)\,\delta(u_3-u_2)-16(\varepsilon^1\cdot\varepsilon^3\varepsilon^2\cdot p_1+\varepsilon^2\cdot\varepsilon^3\varepsilon^1\cdot p_3)\,\delta(u_2-u_1)+f.t.\}\exp\left\{-\frac{T\lambda^2}{2}[(u_2-u_1)\right.$$
$$\left.\times(1-(u_2-u_1))+(u_3-u_2)(1-(u_3-u_2))+(u_3-u_1)(1-(u_3-u_1))]\right\}+\text{permutations}. \tag{45}$$

It is easy to see that the first term in the curly brackets takes care of the Feynman diagrams represented by Fig. 2(a) while the other two, which carry the delta functions, collect the contributions from diagrams depicted by Fig. 2(b). To further guide the reader let us also mention that use was made of Eqs. (30) and (31). Accordingly, the above expression refers to the specific ordering which enters these equations and underlies the particular integrations over the parameters u_1, u_2, and u_3, an occurrence that will be rectified shortly. Finally, in the exponential factor we have set

$$p_1^2=p_2^2=p_3^2=\lambda^2,\quad 2p_1\cdot p_2=2p_1\cdot p_3=2p_2\cdot p_3=-\lambda^2. \tag{46}$$

Next, we make the variable change $u_2-u_1=x_2$ and $u_3-u_1=x_3$ which casts Eq. (45) into the form

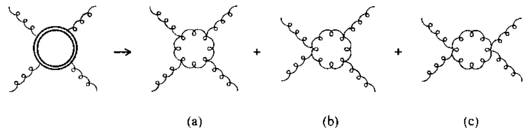

FIG. 3. Same as in Fig. 1 for fourth order contributions $(M=4)$.

$$\Gamma_1^{(M=3)}(p_1,p_2,p_3)=-\frac{1}{2}(2\pi)^4\delta^{(4)}\left(\sum_{i=1}^{3}p_i\right)\mathrm{Tr}_c(t_G^{a_3}t_G^{a_2}t_G^{a_1})g\frac{g^2}{(4\pi)^2}\left(4\pi\frac{\mu^2}{\lambda^2}\right)^{2-D/2}\{-4(D-2)a_D(\varepsilon^1\cdot\varepsilon^2\varepsilon^3\cdot p_2+\varepsilon^1\cdot\varepsilon^3\varepsilon^2\cdot p_1$$
$$+\varepsilon^2\cdot\varepsilon^3\varepsilon^1\cdot p_3)+8b_D(\varepsilon^1\cdot\varepsilon^2\varepsilon^3\cdot p_2+3\varepsilon^1\cdot\varepsilon^3\varepsilon^2\cdot p_1+2\varepsilon^2\cdot\varepsilon^3\varepsilon^1\cdot p_3)\}\Gamma\left(2-\frac{D}{2}\right)+f.t.+\text{permutations}, \tag{47}$$

where we have introduced

$$a_D=2^{2-D/2}\int_0^1 dx_3\int_0^{x_3}dx_2x_2[x_2(1-x_2)+(x_3-x_2)(1-x_3+x_2)+x_3(1-x_3)]^{D/2-2} \tag{48}$$

and

$$b_D=\int_0^1 dx_3(x_3(1-x_3))^{D/2-2}. \tag{49}$$

Obviously $a_4=\frac{1}{6}$ and $b_4=1$.

In order to obtain the final result we need to take into account contributions coming from all permutations of the variables u_1, u_2, u_3 and divide by 3! to compensate for boson indistinguishability. The result can be easily obtained using the cyclic properties of the trace. One, finally, obtains

$$\Gamma_1^{(M=3)}(p_1,p_2,p_3)=\frac{1}{2}(2\pi)^4\delta^{(4)}\left(\sum_{i=1}^{3}p_i\right)\mathrm{Tr}_c(t_G^{a_3}t_G^{a_2}t_G^{a_1})g\frac{g^2}{(4\pi)^2}\left(4\pi\frac{\mu^2}{\lambda^2}\right)^{2-D/2}\{4(D-2)a_D(\varepsilon^1\cdot\varepsilon^2\varepsilon^3\cdot p_2+\varepsilon^1\cdot\varepsilon^3\varepsilon^2\cdot p_1$$
$$+\varepsilon^2\cdot\varepsilon^3\varepsilon^1\cdot p_3)-16b_D(\varepsilon^1\cdot\varepsilon^2\varepsilon^3\cdot p_2+\varepsilon^1\cdot\varepsilon^3\varepsilon^2\cdot p_1+\varepsilon^2\cdot\varepsilon^3\varepsilon^1\cdot p_3)\}\Gamma\left(2-\frac{D}{2}\right)+f.t. \tag{50}$$

in agreement with the known result.

C. The four gluon contribution $(M=4)$ to the effective action

The computation in the present subsection pertains to a ‘‘world-line’’ diagram which collectively accommodates all the classes of the contributing Feynman diagrams, i.e. with zero, one and two four-vertices (plus, of course, contributions from ghost diagrams) (see Fig. 3). As our present analytic computations refer to the divergent part, let us isolate the relevant contribution (terms with the factor $1/T^2$) entering the expression for the spin factor. We find

$$\mathrm{Tr}_L\Phi^{[1]}=D+\frac{32}{T^2}\bar{\xi}_4\xi_4\bar{\xi}_3\xi_3\bar{\xi}_2\xi_2\bar{\xi}_1\xi_1(\varepsilon^1\cdot\varepsilon^4\varepsilon^2\cdot\varepsilon^3-\varepsilon^1\cdot\varepsilon^3\varepsilon^2\cdot\varepsilon^4)\delta(u_4-u_3)\delta(u_2-u_1)+f.t. \tag{51}$$

Integration over the Grassmann variables is a straightforward matter and gives

$$F^{(M=4)}(u_4,u_3,u_2,u_1;T)=\frac{D-2}{T^2}[\varepsilon^1\cdot\varepsilon^2\varepsilon^3\cdot\varepsilon^4\ddot{G}(u_1,u_2)\ddot{G}(u_3,u_4)+\varepsilon^1\cdot\varepsilon^3\varepsilon^2\cdot\varepsilon^4\ddot{G}(u_1,u_3)\ddot{G}(u_2,u_4)$$
$$+\varepsilon^1\cdot\varepsilon^4\varepsilon^2\cdot\varepsilon^3\ddot{G}(u_1,u_4)\ddot{G}(u_2,u_3)]+\frac{32}{T^2}(\varepsilon^1\cdot\varepsilon^4\varepsilon^2\cdot\varepsilon^3-\varepsilon^1\cdot\varepsilon^3\varepsilon^2\cdot\varepsilon^4)$$
$$\times\delta(u_4-u_3)\delta(u_2-u_1)+f.t. \tag{52}$$

Substituting the above expression into Eq. (29) we get

$$\Gamma_1^{(M=4)}(p_1,p_2,p_3,p_4)=-\frac{1}{2}(2\pi)^4\delta^{(4)}\left(\sum_{i=1}^{4}p_i\right)\mathrm{Tr}_c(t_G^{a_4}t_G^{a_3}t_G^{a_2}t_G^{a_1})\frac{g_D^4}{(2\pi)^{D/2}}\Gamma\left(2-\frac{D}{2}\right)\int_0^1 du_4\int_0^{u_4}du_3\int_0^{u_3}du_2$$
$$\times\int_0^{u_2}du_1\{4(D-2)(\varepsilon^1\cdot\varepsilon^2\varepsilon^3\cdot\varepsilon^4+\varepsilon^1\cdot\varepsilon^3\varepsilon^2\cdot\varepsilon^4+\varepsilon^1\cdot\varepsilon^4\varepsilon^2\cdot\varepsilon^3)-4(D-2)\varepsilon^1\cdot\varepsilon^2\varepsilon^3\cdot\varepsilon^4[\delta(u_2-u_1)$$
$$+\delta(u_4-u_3)]-4(D-2)\varepsilon^1\cdot\varepsilon^4\varepsilon^2\cdot\varepsilon^3\delta(u_3-u_2)+4(D-2)\varepsilon^1\cdot\varepsilon^2\varepsilon^3\cdot\varepsilon^4\delta(u_2-u_1)\delta(u_4-u_3)$$
$$+32(\varepsilon^1\cdot\varepsilon^4\varepsilon^2\cdot\varepsilon^3-\varepsilon^1\cdot\varepsilon^3\varepsilon^2\cdot\varepsilon^4)\delta(u_2-u_1)\delta(u_4-u_3)\}\left[\sum_{n=1}^{4}\sum_{m=n+1}^{4}p_n\cdot p_m G(u_n,u_m)\right]^{D/2-2}$$
$$+f.t.+\text{permutations}. \tag{53}$$

One can easily verify that the first term inside the curly brackets represents contributions corresponding to the Feynman diagrams with no four-vertices, the next two to those with one and the last to those with two. Of course, the above expression pertains to a particular ordering of the variables u_1, u_2, u_3, u_4 as reflected in the explicit delta functions which make their entrance.

Performing the parametric integrations, in the specific ordering that appears in Eq. (53), one obtains

$$\Gamma_1^{(M=4)}(p_1,p_2,p_3,p_4)=-\frac{1}{2}(2\pi)^4\delta^{(4)}\left(\sum_{i=1}^{4}p_i\right)\mathrm{Tr}_c(t_G^{a_4}t_G^{a_3}t_G^{a_2}t_G^{a_1})\frac{g_D^4}{(2\pi)^{D/2}}\Gamma\left(2-\frac{D}{2}\right)\{4(D-2)A_D(\varepsilon^1\cdot\varepsilon^2\varepsilon^3\cdot\varepsilon^4+\varepsilon^1\cdot\varepsilon^3\varepsilon^2\cdot\varepsilon^4$$
$$+\varepsilon^1\cdot\varepsilon^4\varepsilon^2\cdot\varepsilon^3)-4(D-2)(B_D-C_D)\varepsilon^1\cdot\varepsilon^2\varepsilon^3\cdot\varepsilon^4-4(D-2)D_D\varepsilon^1\cdot\varepsilon^4\varepsilon^2\cdot\varepsilon^3+32C_D(\varepsilon^1\cdot\varepsilon^4\varepsilon^2\cdot\varepsilon^3$$
$$-\varepsilon^1\cdot\varepsilon^3\varepsilon^2\cdot\varepsilon^4)\}+f.t.+\text{permutations}, \tag{54}$$

where we have set

$$A_D\equiv\int_0^1 du_4\int_0^{u_4}du_3\int_0^{u_3}du_2\int_0^{u_2}du_1\mathcal{G}^{D/2-2},$$

$$B_D\equiv\int_0^1 du_4\int_0^{u_4}du_3\int_0^{u_3}du_2\int_0^{u_2}du_1[\delta(u_2-u_1)+\delta(u_4-u_3)]\mathcal{G}^{D/2-2},$$

$$C_D\equiv\int_0^1 du_4\int_0^{u_4}du_3\int_0^{u_3}du_2\int_0^{u_2}du_1\delta(u_2-u_1)\delta(u_4-u_3)\mathcal{G}^{D/2-2},$$

$$D_D\equiv\int_0^1 du_4\int_0^{u_4}du_3\int_0^{u_3}du_2\int_0^{u_2}du_1\delta(u_3-u_2)\mathcal{G}^{D/2-2}, \tag{55}$$

with

$$\mathcal{G}\equiv\sum_{n=1}^{4}\sum_{m=n+1}^{4}p_n\cdot p_m G(u_n,u_m). \tag{56}$$

One trivially finds $A_4=\frac{1}{6}$, $B_4=\frac{3}{4}$, $C_4=\frac{1}{2}$ and $D_4=\frac{1}{4}$.

The remaining step is to perform all reorderings of the u variables and divide by 4!. In this way one arrives at the final expression

$$\Gamma_1^{(M=4)}(p_1,p_2,p_3,p_4)=-\frac{1}{2}(2\pi)^4\delta^{(4)}\left(\sum_{i=1}^4 p_i\right)\mathrm{Tr}_c(t_G^{a_4}t_G^{a_3}t_G^{a_2}t_G^{a_1})\frac{g_D^4}{(2\pi)^{D/2}}\Gamma\left(2-\frac{D}{2}\right)\{4(D-2)A_D(\varepsilon^1\cdot\varepsilon^2\varepsilon^3\cdot\varepsilon^4+\varepsilon^1\cdot\varepsilon^3\varepsilon^2\cdot\varepsilon^4$$
$$+\varepsilon^1\cdot\varepsilon^4\varepsilon^2\cdot\varepsilon^3)-2(D-2)(B_D-C_D+D_D)(\varepsilon^1\cdot\varepsilon^2\varepsilon^3\cdot\varepsilon^4+\varepsilon^1\cdot\varepsilon^4\varepsilon^2\cdot\varepsilon^3)+16C_D(\varepsilon^1\cdot\varepsilon^2\varepsilon^3\cdot\varepsilon^4$$
$$+\varepsilon^1\cdot\varepsilon^4\varepsilon^2\cdot\varepsilon^3)-32C_D\varepsilon^1\cdot\varepsilon^3\varepsilon^2\cdot\varepsilon^4\}+f.t., \tag{57}$$

which is in full agreement with the known results.

V. CONCLUDING COMMENTS

Given the schemes pioneered by Bern and Kosower [6] and reformulated by Strassler [13] based on string and world-line agents, respectively, and which aim at expediting perturbative computations in QCD both economically and efficiently, it becomes important to assess the relevant merits of yet another competitive proposal advanced in the present paper which utilizes the Polyakov world-line path integral. Directing, to begin with, our comments towards making comparisons with Strassler's approach we could say that the basic difference between the two world-line based schemes is how the disentanglement, between the weight factor pertaining to the spin of the propagating particlelike object on a given path and the dynamical factor represented by the Wilson line (loop), is accomplished. In Strassler's case this task is confronted by using super-particle degrees of freedom (one dimensional) and generates a term in the corresponding Lagrangian of the form $\psi^\mu F_{\mu\nu}\psi^\nu$. In the Polyakov (world-line) version, on the other hand, the issue is addressed via the introduction of the spin factor. We believe that the separation, featured by the latter scheme, between "geometrical" characteristics of paths on the one hand and dynamics—as embodied in the Wilson line (loop) factor—on the other, leads to an organization of the path integral expression which further facilitates the "efficiency factor" for performing perturbative computations. In particular, it offers a unified basis for treating spinors, gauge fields and ghosts; all one has to do is adjust the master formula, which yields the computational rules, to the appropriate spin factor. Moreover, it lends itself to straightforward extensions for applications to processes involving *open* fermionic world lines, as established in Refs. [30,31]. Referring, finally, to the string-based approach of Bern and Kosower, we remark that, modulo the pending [32] detailed demonstration of how one particle reducible connected configurations are handled through the application of the classical perturbative expansion, our master formulas can bypass the aformentioned authors' pinching rules as they lead directly to perturbative calculations in QCD. We expect to further demonstrate the virtues of the Polyakov world-line path integral scheme toward the calculation of two gluon loop contributions to the effective action by generating the corresponding master formulas.

ACKNOWLEDGMENTS

One of us (S.D.A.) acknowledges financial support from the Greek State Scholarships Foundation (I.K.Y.). A.I.K. and C.N.K. acknowledge the support from the General Secretariat of Research and Technology of the University of Athens.

APPENDIX

In this appendix we shall pay closer attention to the spin factor with respect to both carrying out the path integral in Eq. (22) and establishing the result encoded in Eqs. (33) and (34). Looking at the identity given by Eq. (11) we present the proper (regularized) expression for the tensor $\omega_{\mu\nu}$ reads as follows:

$$\int_0^T dt\,\omega_{\mu\nu}[\dot{x}(t)]=\lim_{\varepsilon\to0}\frac{1}{4}\int_{-\varepsilon}^{\varepsilon}ds\int_0^T dt_2\int_0^T dt_1[\ddot{x}_\mu(t_2)\dot{x}_\nu(t_1)-\ddot{x}_\nu(t_2)\dot{x}_\mu(t_1)]\delta(t_2-t_1-s). \tag{A1}$$

Now, if the functions (on the line) $x_\mu(t)$ are infinitely differentiable, then we can, once taking into account that $|t_2-t_1|<\varepsilon$, write $\ddot{x}_\mu(t_2)=\ddot{x}_\mu(t_1)+\mathcal{O}(s)$ as well as $\dot{x}_\nu(t_1)=\dot{x}_\nu(t_2)+\mathcal{O}(s)$ and immediately conclude that

$$\int_0^T dt\,\omega_{\mu\nu}[\dot{x}(t)]=\frac{T}{2}\int_0^T dt[\ddot{x}_\mu(t)\dot{x}_\nu(t)-\ddot{x}_\nu(t)\dot{x}_\mu(t)]. \tag{A2}$$

Otherwise, one should use the limiting expression according to Eq. (A1) when performing manipulations that involve the spin factor.

Let us proceed with the computation of the path integral entering Eq. (25). We set

$$I_{\mu\nu}=\int d^4a\int_{x(0)=x(T)=a}\mathcal{D}x(t)\Phi^{[1]}[\dot{x}(t)]_{\mu\nu}\times\exp\{-S[x]\}, \tag{A3}$$

where the translational zero mode has been explicitly separated and where

$$S[x]=\frac{1}{4}\int_0^T dt\dot{x}^2(t)-i\sum_{n=1}^M\hat{k}(t_n)\cdot x(t_n). \tag{A4}$$

To compute $I_{\mu\nu}$ we make the variable change $x\to x+x^{cl}$, where x^{cl} is a solution of the classical equation of motion resulting from the above action. Specifically, we have

$$\ddot{x}^{cl}_{\mu}(t)=-2i\sum_{n=1}^{M}\hat{k}(t_n)\,\delta(t-t_n)$$
$$\Rightarrow x^{cl}_{\mu}(t)$$
$$=2i\sum_{n=1}^{M}\hat{k}(t_n)\Delta(t,t_n)+a, \tag{A5}$$

where we have employed the Green's function

$$\Delta(t,t')=\frac{t(T-t')}{T}\,\theta(t'-t)+\frac{t'(T-t)}{T}\,\theta(t-t'),$$
$$\Delta(0,t')=\Delta(T,t')=0. \tag{A6}$$

The new action functional is now specified by

$$S[x]\rightarrow\frac{1}{4}\int_0^T dt\dot{x}^2(t)+S[x^{cl}], \tag{A7}$$

where

$$S[x^{cl}]=\sum_{n=1}^{M}\sum_{m=1}^{M}\hat{k}(t_n)\cdot\hat{k}(t_m)\Delta(t_n,t_m)-i\sum_{n=1}^{M}p_n\cdot a. \tag{A8}$$

We immediately observe that integration over a leads to momentum conservation which enters Eq. (26). The rest of the expression for $S[x^{cl}]$ produces the terms entering the exponential factor in the same equation.

Turning our attention to the spin factor we first note that the variable change $x\rightarrow x+x^{cl}$ leads to

$$\int_0^T dt\,\omega_{\mu\nu}[\dot{x}]\rightarrow\int_0^T dt\,\omega_{\mu\nu}[\dot{x}^{cl}]+\frac{T}{2}\int_0^T dt[\ddot{x}_{\mu}(t)\dot{x}_{\nu}(t)$$
$$-\ddot{x}_{\nu}(t)\dot{x}_{\mu}(t)], \tag{A9}$$

having taken into account that the contours $x(t)$ are to be integrated with respect to a quadratic action functional [cf. Eq. (A7)], which implies, cf. Eqs. (A1) and (A5), that mixed terms in x and x^{cl} drop out. Let us finally note that for paths that are infinitely differentiable, in which case Eq. (A2) strictly holds true, the integration of the spin factor with respect to the quadratic action functional yields unity. All this leads to the following result, as far as performing the path integral in Eq. (A3) is concerned:

$$I_{\mu\nu}=(2\pi)^D\delta^{(D)}\left(\sum_{n=1}^{M}p_n\right)\frac{1}{(4\pi T)^{D/2}}\Phi^{[1]}[\dot{x}^{cl}]_{\mu\nu}$$
$$\times\exp\left[\sum_{n<m}p_n\cdot p_m G(t_n,t_m)\right.$$
$$+\sum_{n\neq m}\bar{\xi}_n\xi_n\varepsilon^n\cdot p_m\partial_n G(t_n,t_m)$$
$$\left.+\frac{1}{2}\sum_{n\neq m}\bar{\xi}_n\xi_n\bar{\xi}_m\xi_m\varepsilon^n\cdot\varepsilon^m\partial_n\partial_m G(t_n,t_m)\right]. \tag{A10}$$

The above result explicitly demonstrates our assertion that the overall contribution from the spin factor is exclusively determined by those points on a given path where a momentum is imparted via the action of an external gauge field.

The final result is obtained once we substitute Eq. (A5) into Eq. (A1). We get

$$\int_0^T dt\,\omega_{\mu\nu}[\dot{x}^{cl}]=-2\sum_{n=1}^{M}\bar{\xi}_n\xi_n(\varepsilon^n_{\mu}p_{n,\nu}-\varepsilon^n_{\nu}p_{n,\mu})$$
$$+\sum_{n=0}^{M}\sum_{m=0}^{M}\bar{\xi}_n\xi_n\bar{\xi}_m\xi_m(\varepsilon^n_{\mu}\varepsilon^m_{\nu}-\varepsilon^n_{\nu}\varepsilon^m_{\mu})$$
$$\times\int_{-\varepsilon}^{\varepsilon}ds\frac{\partial}{\partial t_n}\delta(t_n-t_m-s). \tag{A11}$$

The correct handling of the last term follows once we take into consideration that, first, $m\neq n$ on account of the Grassmann variables and, second, it is to be integrated in consistence with the time ordering implicit in Eq. (20) of the text. Specifically, we have

$$\dots\int_0^T dt_{n+1}\int_0^T dt_n\int_0^T dt_{n-1}\,\theta(t_{n+1}-t_n)\,\theta(t_n-t_{n-1})\int_{-\varepsilon}^{\varepsilon}ds\frac{\partial}{\partial t_n}\delta(t_n-t_m-s)\dots$$
$$=\dots\int_0^T dt_{n+1}\int_0^T dt_n\int_0^T dt_{n-1}\,\theta(t_{n+1}-t_n)\,\theta(t_n-t_{n-1})[2\,\delta_{n+1,m}\delta(t_{n+1}-t_n)-2\,\delta_{m,n-1}\delta(t_n-t_{n-1})]\dots, \tag{A12}$$

which allows us to return to Eq. (A11) and infer that

$$\int_0^T dt\,\omega_{\mu\nu}[\dot{x}^{cl}]=-\frac{i}{2}\int_0^T dt(J\cdot\omega^{cl})_{\mu\nu}=-2\sum_{n=0}^{M}\bar{\xi}_n\xi_n(\varepsilon^n_{\mu}p_{n,\nu}-\varepsilon^n_{\nu}p_{n,\mu})-4\sum_{n=1}^{M}\bar{\xi}_{n+1}\xi_{n+1}\bar{\xi}_n\xi_n(\varepsilon^{n+1}_{\mu}\varepsilon^n_{\nu}-\varepsilon^{n+1}_{\nu}\varepsilon^n_{\mu})\,\delta(t_{n+1}-t_n). \tag{A13}$$

With the above results in place, Eqs. (33) and (34) in the text follow directly.

The careful course of reasoning we have followed in this appendix can be circumvented by the more formal line of procedure adopted in the text. Thus, the validity of the aforementioned equations can be established once we observe that

$$\exp\left[-i\sum_{n=1}^{M}\hat{k}(t_n)\cdot x(t_n)\right]\int_0^T dt\,\frac{\delta}{\delta s_{\mu\nu}(t)}\exp\left[i\sum_{n=1}^{M}\hat{k}(t_n)\cdot x(t_n)\right]$$
$$=-\lim_{\varepsilon\to 0}\int_0^T dt\sum_{n=1}^{M}\sum_{m=1}^{M}\int_{-\varepsilon}^{\varepsilon}ds s\hat{k}_\mu(t_n)\,\delta\left(t_n-t-\frac{s}{2}\right)\hat{k}_\nu(t_m)\,\delta\left(t_m-t+\frac{s}{2}\right)$$
$$=\frac{1}{4}\lim_{\varepsilon\to 0}\int_0^T dt\int_{-\varepsilon}^{\varepsilon}ds s\ddot{x}_\mu^{cl}\left(t+\frac{s}{2}\right)\ddot{x}_\nu^{cl}\left(t-\frac{s}{2}\right)$$
$$=\lim_{\varepsilon\to 0}\frac{1}{8}\int_{-\varepsilon}^{\varepsilon}ds\int_0^T dt_2\int_0^T dt_1[\ddot{x}_\mu(t_2)\dot{x}_\nu(t_1)-\ddot{x}_\nu(t_2)\dot{x}_\mu(t_1)]\,\delta(t_2-t_1-s). \tag{A14}$$

[1] R.R. Metsaev and A.A. Tseytlin, Nucl. Phys. **B298**, 109 (1988).
[2] J.A. Minahan, Nucl. Phys. **B298**, 36 (1988).
[3] T.R. Taylor and G. Veneziano, Phys. Lett. B **212**, 147 (1988).
[4] V.S. Kaplunovsky, Nucl. Phys. **B307**, 145 (1988); **B382**, 436(E) (1988).
[5] Z. Bern and D.A. Kosower, Phys. Rev. D **38**, 1888 (1988).
[6] Z. Bern and D.A. Kosower, Nucl. Phys. **B321**, 605 (1989); **B379**, 451 (1992).
[7] Z. Bern, L.J. Dixon, and D.A. Kosower, Phys. Rev. Lett. **70**, 2677 (1993).
[8] Z. Bern, Phys. Lett. B **296**, 85 (1992).
[9] Z. Bern and D.C. Dunbar, Nucl. Phys. **B379**, 562 (1992).
[10] P. Di Vecchia, A. Lerda, L. Magnea, and R. Marotta, Phys. Lett. B **351**, 445 (1995).
[11] E.S. Fradkin and A.A. Tseytlin, Phys. Lett. **163B**, 123 (1985).
[12] P. Di Vecchia, L. Magnea, A. Lerda, R. Russo, and R. Marotta, Nucl. Phys. **B469**, 235 (1996).
[13] M.J. Strassler, Nucl. Phys. **B385**, 145 (1992).
[14] R. Casalbuoni, J. Gomis, and G. Longhi, Nuovo Cimento Soc. Ital. Fis., A **24**, 249 (1974).
[15] L. Brink, S. Deser, B. Zumino, P. Di Vecchia, and P. Howe, Phys. Lett. **64B**, 435 (1976); L. Brink, P. Di Vecchia, and P. Howe, Nucl. Phys. **B118**, 76 (1977).
[16] A.P. Balachandran, P. Salomonson, B.S. Skagerstam and J.O. Winnberg, Phys. Rev. D **15**, 2308 (1977).
[17] M.G. Schmidt and C. Schubert, Phys. Lett. B **331**, 69 (1994).
[18] M.G. Schmidt and C. Schubert, Phys. Rev. D **53**, 2150 (1996).
[19] M. Reuter, M.G. Schmidt, and C. Schubert, Ann. Phys. (N.Y.) **259**, 313 (1997).
[20] V. A. Fock, Izv. Akad. Nauk. USSR, OMEN, p. 557 (1937).
[21] R.P. Feynman, Phys. Rev. **80**, 440 (1950).
[22] J.S. Schwinger, Phys. Rev. **82**, 664 (1951).
[23] E.S. Fradkin, Nucl. Phys. **B76**, 588 (1966).
[24] M.B. Halpern, A. Jevicki, and P. Senjanovic, Phys. Rev. D **16**, 2476 (1977); M.B. Halpern and W. Siegel, *ibid.* **16**, 2486 (1977).
[25] L. Alvarez-Gaume, Commun. Math. Phys. **90**, 161 (1983).
[26] A. M. Polyakov, in *Fields, Strings And Critical Phenomena*, edited by E. Brezin and J. Zinn-Justin (North-Holland, Amsterdam,1990).
[27] A.I. Karanikas and C.N. Ktorides, Phys. Lett. B **275**, 403 (1992).
[28] A.I. Karanikas and C.N. Ktorides, Phys. Rev. D **52**, 5883 (1995).
[29] G.C. Gellas, A.I. Karanikas, and C.N. Ktorides, Ann. Phys. (N.Y.) **255**, 228 (1997); Phys. Rev. D **55**, 5027 (1997); **57**, 3763 (1998); A.I. Karanikas and C.N. Ktorides, *ibid.* **59**, 016003 (1999); A.I. Karanikas, C.N. Ktorides, N.G. Stefanis, and S.M. Wong, Phys. Lett. B **455**, 291 (1999).
[30] A.I. Karanikas and C.N. Ktorides, J. High Energy Phys. **11**, 033 (1999).
[31] A.I. Karanikas and C.N. Ktorides, Phys. Lett. B **500**, 75 (2001).
[32] A.S. Kapoyannis, A.I. Karanikas, and C.N. Ktorides (in preparation).
[33] A.M. Polyakov, Nucl. Phys. **B164**, 171 (1980).
[34] A.A. Migdal, Phys. Rep. **102**, 199 (1983).
[35] The interaction term is displayed generically, independent as to whether the gauge theory is Abelian or non-Abelian.
[36] For convenience we drop the characterization "world line" from hereon, even though we recognize the fact that we are using a term that has been established for the characterization of the same author's path integral pertaining to string quantization.
[37] Strictly speaking, Polyakov's spin factor is given by the (Lorentz) trace of $\Phi^{[1]}[\dot{x}]_{\mu\nu}$.
[38] Further insight on this point will be provided at the end of this section.
[39] To be exact, the solution of Eq. (27) includes an arbitrary constant which does not appear in Eq. (27) on the account that momentum conservation has been imposed.