Magnetic catalysis and anisotropic confinement in QCD

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The expressions for dynamical masses of quarks in the chiral limit in QCD in a strong magnetic field are obtained. A low energy effective action for the corresponding Nambu-Goldstone bosons is derived and the values of their decay constants as well as the velocities are calculated. The existence of a threshold value of the number of colors N_c^{thr} , dividing the theories with essentially different dynamics, is established. For the number of colors $N_c \ll N_c^{thr}$, an anisotropic dynamics of confinement with the confinement scale much less than Λ_{QCD} and a rich spectrum of light glueballs is realized. For N_c of order N_c^{thr} or larger, a conventional confinement dynamics takes place. It is found that the threshold value N_c^{thr} grows rapidly with the magnetic field $[N_c^{thr}] \approx 100$ for $|eB| \approx (1 \text{ GeV})^2]$. In contrast with QCD with a nonzero baryon density, there are no principal obstacles for examining these results and predictions in lattice computer simulations.

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I. INTRODUCTION

Since the dynamics of QCD is extremely rich and complicated, it is important to study this theory under external conditions which provide a controllable dynamics. On the one hand, this allows one to understand better the vacuum structure and Green's functions of QCD, and, on the other hand, there can exist interesting applications of such models in themselves. The well known examples are hot QCD (for a review see Ref. [1]) and QCD with a large baryon density (for a review see Ref. [2]).

Studies of QCD in external electromagnetic fields had started long ago [3,4]. A particularly interesting case is an external magnetic field. Using the Nambu-Jona-Lasinio (NJL) model as a low energy effective theory for QCD, it was shown that a magnetic field enhances the spontaneous chiral symmetry breakdown. The understanding of this phenomenon had remained obscure until the universal role of a magnetic field as a catalyst of chiral symmetry breaking was established in Refs. [5,6]. The general result states that a constant magnetic field leads to the generation of a fermion dynamical mass (i.e., a gap in the one-particle energy spectrum) even at the weakest attractive interaction between fermions. For this reason, this phenomenon was called the magnetic catalysis. The essence of the effect is the dimensional reduction $D \rightarrow D - 2$ in the dynamics of fermion pairing in a magnetic field. In the particular case of weak coupling, this dynamics is dominated by the lowest Landau level (LLL) which is essentially (D-2)-dimensional [5,6]. The applications of this effect have been considered both in condensed matter physics [7,8] and cosmology (for reviews see Ref. [9]).

The phenomenon of the magnetic catalysis was studied in gauge theories, in particular, in QED [10-15] and in QCD

[16–18]. In recent work [18], it has been suggested that the dynamics underlying the magnetic catalysis in QCD is weakly coupled at sufficiently large magnetic fields. In this paper, we study this dynamical problem rigorously, from first principles. In fact, we show that, at sufficiently strong magnetic fields, $|eB| \ge \Lambda_{QCD}^2$, there exists a consistent truncation of the Schwinger-Dyson (gap) equation which leads to a reliable asymptotic expression for the quark mass m_q . Its explicit form reads

$$m_q^2 \approx 2C_1 |e_q B| (c_q \alpha_s)^{2/3} \\ \times \exp\left[-\frac{4N_c \pi}{\alpha_s (N_c^2 - 1) \ln(C_2 / c_q \alpha_s)}\right], \qquad (1)$$

where e_q is the electric charge of the *q*-th quark and N_c is the number of colors. The numerical factors C_1 and C_2 equal 1 in the leading approximation that we use. Their value, however, can change beyond this approximation and we can only say that they are of order 1. The constant c_q is defined as follows:

$$c_q = \frac{1}{6\pi} (2N_u + N_d) \left| \frac{e}{e_q} \right|,\tag{2}$$

where N_u and N_d are the numbers of up and down quark flavors, respectively. The total number of quark flavors is $N_f = N_u + N_d$. The strong coupling α_s in the last equation is related to the scale $\sqrt{|eB|}$, i.e.,

$$\frac{1}{\alpha_s} \approx b \ln \frac{|eB|}{\Lambda_{OCD}^2}, \quad \text{where} \quad b = \frac{11N_c - 2N_f}{12\pi}.$$
 (3)

We should note that in the leading approximation the energy scale $\sqrt{|eB|}$ in Eq. (3) is fixed only up to a factor of order 1.

As we discuss below, because of the running of α_s , the value of the dynamical mass (1) grows very slowly with

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increasing the value of the background magnetic field. Moreover, there may exist an intermediate region of fields where the mass decreases with increasing the magnetic field. Another, rather unexpected, consequence is that a strong external magnetic field suppresses the chiral vacuum fluctuations leading to the generation of the usual dynamical mass of quarks $m_{dyn}^{(0)} \approx 300$ MeV in QCD without a magnetic field. In fact, in a wide range of strong magnetic fields $\Lambda^2 \leq B$ $\leq (10 \text{ TeV})^2$ (where Λ is the characteristic gap in QCD without the magnetic field; it can be estimated to be a few times larger than Λ_{OCD}), the dynamical mass (1) remains smaller than $m_{dyn}^{(0)}$. As it will be shown in Sec. IV, this point is intimately connected with another one: in a strong magnetic field, the confinement scale, λ_{OCD} , is much less than the confinement scale Λ_{OCD} in QCD without a magnetic field.

The central dynamical issue underlying this dynamics is the effect of screening of the gluon interactions in a magnetic field in the region of momenta relevant for the chiral symmetry breaking dynamics, $m_q^2 \ll |k^2| \ll |eB|$. In this region, gluons acquire a mass M_g of order $\sqrt{N_f \alpha_s} |e_q B|$. This allows to separate the dynamics of the magnetic catalysis from that of confinement. More rigorously, M_g is the mass of a quarkantiquark composite state coupled to the gluon field. The appearance of such mass resembles the pseudo-Higgs effect in the (1+1)-dimensional massive QED (massive Schwinger model) [19] (see below).

Since the background magnetic field breaks explicitly the global chiral symmetry that interchanges the up and down quark flavors, the chiral symmetry in this problem is $SU(N_u)_L \times SU(N_u)_R \times SU(N_d)_L \times SU(N_d)_R \times U^{(-)}(1)_A$. The $U^{(-)}(1)_A$ is connected with the current which is an anomaly-free linear combination of the $U^{(d)}(1)_A$ and $U^{(u)}(1)_A$ currents. [The $U^{(-)}(1)_A$ symmetry is of course absent if either N_d or N_u is equal to zero.] The generation of quark masses breaks this symmetry spontaneously down to $SU(N_u)_V \times SU(N_d)_V$ and, as a result, $N_u^2 + N_d^2 - 1$ gapless Nambu-Goldstone (NG) bosons occur. In Sec. III, we derive the effective action for the NG bosons and calculate their decay constants and velocities.

The present analysis is heavily based on the analysis of the magnetic catalysis in QED done by Gusynin, Miransky, and Shovkovy [11]. A crucial difference is of course the property of asymptotic freedom and confinement in QCD. In connection with that, our second major result is the derivation of the low energy effective action for gluons in QCD in a strong magnetic field [see Eq. (18) below]. The characteristic feature of this action is its anisotropic dynamics. In particular, the strength of static (Coulomb-like) forces along the direction parallel to the magnetic field is much larger than that in the transverse directions. Also, the confinement scale in this theory is much less than that in QCD without a magnetic field. These features imply a rich and unusual spectrum of light glueballs in this theory.

A special and interesting case is QCD with a large number of colors, in particular, with $N_c \rightarrow \infty$ (the 't Hooft limit). In this limit, the mass of gluons goes to zero and the expression for the quark mass becomes essentially different [see Eq. (25) in Sec. V]. In fact, it will be shown that, for any value of an external magnetic field, there exists a threshold value N_c^{thr} , rapidly growing with |eB| [e.g., $N_c^{thr} \ge 100$ for $|eB| \ge (1 \text{ GeV})^2$]. For N_c of the order N_c^{thr} or larger, the gluon mass becomes small and irrelevant for the dynamics of the generation of a quark mass. As a result, expression (25) for m_q takes place for such large N_c . The confinement scale in this case is close to Λ_{QCD} . Still, as is shown in Sec. V, the dynamics of chiral symmetry breaking is under control in this limit if the magnetic field is sufficiently strong.

It is important that, unlike the case of QCD with a nonzero baryon density, there are no principal obstacles for checking all these results and predictions in lattice computer simulations of QCD in a magnetic field.

II. MAGNETIC CATALYSIS IN QCD

We begin by considering the Schwinger-Dyson (gap) equation for the quark propagator. It has the following form:

$$G^{-1}(x,y) = S^{-1}(x,y) + 4 \pi \alpha_s \gamma^{\mu} \\ \times \int G(x,z) \Gamma^{\nu}(z,y,z') \mathcal{D}_{\nu\mu}(z',x) d^4 z d^4 z',$$
(4)

where S(x,y) and G(x,y) are the bare and full fermion propagators in an external magnetic field, $\mathcal{D}_{\nu\mu}(x,y)$ is the full gluon propagator and $\Gamma^{\nu}(x,y,z)$ is the full amputated vertex function. Since the coupling α_s related to the scale |eB| is small, one might think that the rainbow (ladder) approximation is reliable in this problem. However, this is not the case. Because of the (1+1)-dimensional form of the fermion propagator in the LLL approximation, there are relevant higher order contributions [10,11]. Fortunately one can solve this problem. First of all, an important feature of the quark-antiquark pairing dynamics in QCD in a strong magnetic field is that this dynamics is essentially Abelian. This feature is provided by the form of the polarization operator of gluons in this theory. The point is that the dynamics of the quark-antiquark pairing is mainly induced in the region of momenta k much less than $\sqrt{|eB|}$. This implies that the magnetic field yields a dynamical ultraviolet cutoff in this problem. On the other hand, while the contribution of (electrically neutral) gluons and ghosts in the polarization operator is proportional to k^2 , the fermion contribution is proportional to $|e_{a}B|$ [11]. As a result, the fermion contribution dominates in the relevant region with $k^2 \ll |eB|$.

This observation implies that there are three, dynamically very different, scale regions in this problem. The first one is the region with the energy scale above the magnetic scale $\sqrt{|eB|}$. In that region, the dynamics is essentially the same as in QCD without a magnetic field. In particular, the running coupling decreases logarithmically with increasing the energy scale there. The second region is that with the energy scale below the magnetic scale but much larger than the dynamical mass m_q . In this region, the dynamics is Abelianlike and, therefore, the dynamics of the magnetic catalysis is similar to that in QED in a magnetic field. At last, the third region is the region with the energy scale less than the gap. In this region, quarks decouple and a confinement dynamics for gluons is realized.

Let us first consider the intermediate region relevant for the magnetic catalysis. As was indicated above, the important ingredient of this dynamics is a large contribution of fermions to the polarization operator. It is large because of an (essentially) (1+1)-dimensional form of the fermion propagator in a strong magnetic field. Its explicit form can be obtained by modifying appropriately the expression for the polarization operator in QED in a magnetic field [11]:

$$\mathcal{P}^{AB,\mu\nu} \simeq \frac{\alpha_s}{6\pi} \delta^{AB} (k_{\parallel}^{\mu} k_{\parallel}^{\nu} - k_{\parallel}^2 g_{\parallel}^{\mu\nu}) \sum_{q=1}^{N_f} \frac{|e_q B|}{m_q^2}$$

for $|k_{\parallel}^2| \ll m_q^2$, (5)

$$\mathcal{P}^{AB,\mu\nu} \simeq -\frac{\alpha_s}{\pi} \,\delta^{AB}(k_{\parallel}^{\mu}k_{\parallel}^{\nu} - k_{\parallel}^2 g_{\parallel}^{\mu\nu}) \sum_{q=1}^{N_f} \frac{|e_q B|}{k_{\parallel}^2}$$
for $m_q^2 \ll |k_{\parallel}^2| \ll |eB|,$ (6)

where $g_{\parallel}^{\mu\nu} \equiv \text{diag}(1,0,0,-1)$ is the projector onto the longitudinal subspace, and $k_{\parallel}^{\mu} \equiv g_{\parallel}^{\mu\nu} k_{\nu}$ (the magnetic field is in the x^3 direction). Similarly, we introduce the orthogonal projector $g_{\perp}^{\mu\nu} \equiv g^{\mu\nu} - g_{\parallel}^{\mu\nu} = \text{diag}(0,-1,-1,0)$ and $k_{\perp}^{\mu} \equiv g_{\perp}^{\mu\nu} k_{\nu}$ that we shall use below. Notice that quarks in a strong magnetic field do not couple to the transverse subspace spanned by $g_{\perp}^{\mu\nu}$ and k_{\perp}^{μ} . This is because in a strong magnetic field only the quarks from the LLL matter and they couple only to the longitudinal components of the gluon field. The latter property follows from the fact that spins of the LLL quarks are polarized along the magnetic field [10].

The expressions (5) and (6) coincide with those for the polarization operator in the massive Schwinger model if the parameter $\alpha_s |e_q B|/2$ here is replaced by the dimensional coupling α_1 of QED_{1+1} . As in the Schwinger model, Eq. (6) implies that there is a massive resonance in the $k_{\parallel}^{\mu}k_{\parallel}^{\nu} - k_{\parallel}^{2}g_{\parallel}^{\mu\nu}$ component of the gluon propagator. Its mass is

$$M_g^2 = \sum_{q=1}^{N_f} \frac{\alpha_s}{\pi} |e_q B| = (2N_u + N_d) \frac{\alpha_s}{3\pi} |eB|.$$
(7)

This is reminiscent of the pseudo-Higgs effect in the (1 + 1)-dimensional massive QED. It is not the genuine Higgs effect because there is no complete screening of the color charge in the infrared region with $|k_{\parallel}^2| \ll m_q^2$. This can be seen clearly from Eq. (5). Nevertheless, the pseudo-Higgs effect is manifested in creating a massive resonance and this resonance provides the dominant forces leading to chiral symmetry breaking.

Now, after the Abelian-like structure of the dynamics in this problem is established, we can use the results of the analysis in QED in a magnetic field [11] by introducing appropriate modifications. The main points of the analysis are (i) the so called improved rainbow approximation is reliable in this problem provided a special nonlocal gauge is used in the analysis, and (ii) for a small coupling α_s (α in QED), the



FIG. 1. The dynamical masses of quarks as functions of $\ln(|eB|/\Lambda_{QCD}^2)$ for $N_c = 3$ and two different values of $N_f = N_u + N_d$: (i) masses of *u*-quark (solid line) and *d*-quark (dash-dotted line) for $N_u = 1$ and $N_d = 2$; (ii) masses of *u*-quark (dashed line) and *d*-quark (dotted line) for $N_u = 2$ and $N_d = 2$. The result may not be reliable in the weak magnetic field region (shaded) where some of the approximations break. The values of masses are given in units of $\Lambda_{QCD} = 250$ MeV.

relevant region of momenta in this problem is $m_q^2 \ll |k^2|$ $\ll |eB|$. We recall that in the improved rainbow approximation the vertex $\Gamma^{\nu}(x,y,z)$ is taken to be bare and the gluon propagator is taken in the one-loop approximation. Moreover, as we argued above, in this intermediate region of momenta, only the contribution of quarks to the gluon polarization tensor (6) matters. (It is appropriate to call this approximation the "strong-field-loop improved rainbow approximation." It is an analog of the hard-dense-loop improved rainbow approximation in QCD with a nonzero baryon density [20].) As to the modifications, they are purely kinematic: the overall coupling constant in the gap equation α and the dimensionless combination $M_{\gamma}^2/|eB|$ in QED have to be replaced by $\alpha_s (N_c^2 - 1)/2N_c$ and $M_g^2/|e_q B|$, respectively. This leads us to the expression (1) for the dynamical gap.

After expressing the magnetic field in terms of the running coupling, the result for the dynamical mass takes the following convenient form:

$$m_q^2 \approx 2C_1 \left| \frac{e_q}{e} \right| \Lambda_{QCD}^2 (c_q \alpha_s)^{2/3} \\ \times \exp\left[\frac{1}{b \alpha_s} - \frac{4N_c \pi}{\alpha_s (N_c^2 - 1) \ln(C_2 / c_q \alpha_s)} \right].$$
(8)

As is easy to check, the dynamical mass of the *u*-quark is considerably larger than that of the *d*-quark. It is also noticeable that the values of the *u*-quark dynamical mass become comparable to the vacuum value $m_{dyn}^{(0)} \approx 300$ MeV only when the coupling constant gets as small as 0.05.

Now, by trading the coupling constant for the magnetic field scale |eB|, we get the dependence of the dynamical mass on the value of the external field. The numerical results are presented in Fig. 1 [we used $C_1 = C_2 = 1$ in Eq. (8)].

As one can see in Fig. 1, the value of the quark gap in a wide window of strong magnetic fields, $\Lambda_{QCD}^2 \ll |eB| \approx (10 \text{ TeV})^2$, remains smaller than the dynamical mass of quarks $m_{dyn}^{(0)} \approx 300$ MeV in QCD without a magnetic field. In other words, the chiral condensate is partially *suppressed*

for those values of a magnetic field. The explanation of this, rather unexpected, result is actually simple. The magnetic field leads to the mass M_g (7) for gluons. In a strong enough magnetic field, this mass becomes larger than the characteristic gap Λ in QCD without a magnetic field (Λ , playing the role of a gluon mass, can be estimated as a few times larger than Λ_{QCD}). This, along with the property of the asymptotic freedom (i.e., the fact that α_s decreases with increasing the magnetic field), leads to the suppression of the chiral condensate [21].

This point also explains why our result for the gap is so different from that in the NJL model in a magnetic field [3]. Recall that, in the NJL model, the gap logarithmically (i.e., much faster than in the present case) grows with a magnetic field. This is related to the assumption that both the dimensional coupling constant $G = g/\Lambda^2$ (with Λ playing a role similar to that of the gluon mass in QCD), as well as the scale Λ do not depend on the value of the magnetic field. Therefore, in that model, in a strong enough magnetic field, the value of the chiral condensate is overestimated.

The picture which emerges from this discussion is the following. For values of a magnetic field $|eB| \leq \Lambda^2$ the dynamics in QCD should be qualitatively similar to that in the NJL model. For strong values of the field, however, it is essentially different, as was described above. This in turn suggests that there should exist an intermediate region of fields where the dynamical masses of quarks decrease with increasing the background magnetic field.

III. EFFECTIVE ACTION OF NG BOSONS

The presence of the background magnetic field breaks explicitly the global chiral symmetry that interchanges the up and down quark flavors. This is related to the fact that the electric charges of the two sets of quarks are different. However, the magnetic field does not break the global chiral symmetry of the action completely. In particular, in the model with the N_u up quark flavors and the N_d down quark flavors, the action is invariant under the chiral symmetry $SU(N_u)_L \times SU(N_u)_R \times SU(N_d)_L \times SU(N_d)_R \times U^{(-)}(1)_A$. The $U^{(-)}(1)_A$ is connected with the current which is an anomaly-free linear combination of the $U^{(d)}(1)_A$ and $U^{(u)}(1)_A$ currents. [The $U^{(-)}(1)_A$ symmetry is of course absent if either N_d or N_u is equal to zero.]

The global chiral symmetry of the action is broken spontaneously down to the diagonal subgroup $SU(N_u)_V \times SU(N_d)_V$ when dynamical masses of quarks are generated. In agreement with the Goldstone theorem, this leads to the appearance of an $N_u^2 + N_d^2 - 1$ number of the Nambu-Goldstone (NG) gapless excitations in the lowenergy spectrum of QCD in a strong magnetic field. Notice that there is also a pseudo-NG boson connected with the conventional (anomalous) $U(1)_A$ symmetry which can be rather light in a sufficiently strong magnetic field.

Now, in the chiral limit, the general structure of the low energy action for the NG bosons could be easily established from the symmetry arguments alone. First of all, such an action should be invariant with respect to the space-time symmetry $SO(1,1) \times SO(2)$ which is left unbroken by the

background magnetic field [here the SO(1,1) and the SO(2) are connected with Lorentz boosts in the x_0 - x_3 hyperplane and rotations in the x_1 - x_2 plane, respectively]. Besides that, the low-energy action should respect the original chiral symmetry

$$SU(N_u)_L \times SU(N_u)_R \times SU(N_d)_L \times SU(N_d)_R \times U^{(-)}(1)_A$$
.

These requirements lead to the following general form of the action:

$$\mathcal{L}_{NG} \approx \frac{f_u^2}{4} \operatorname{tr}(g_{\parallel}^{\mu\nu}\partial_{\mu}\Sigma_u\partial_{\nu}\Sigma_u^{\dagger} + v_u^2 g_{\perp}^{\mu\nu}\partial_{\mu}\Sigma_u\partial_{\nu}\Sigma_u^{\dagger}) + \frac{f_d^2}{4} \operatorname{tr}(g_{\parallel}^{\mu\nu}\partial_{\mu}\Sigma_d\partial_{\nu}\Sigma_d^{\dagger} + v_d^2 g_{\perp}^{\mu\nu}\partial_{\mu}\Sigma_d\partial_{\nu}\Sigma_d^{\dagger}) + \frac{\tilde{f}^2}{4} (g_{\parallel}^{\mu\nu}\partial_{\mu}\widetilde{\Sigma}\partial_{\nu}\widetilde{\Sigma}^{\dagger} + \widetilde{v}^2 g_{\perp}^{\mu\nu}\partial_{\mu}\widetilde{\Sigma}\partial_{\nu}\widetilde{\Sigma}^{\dagger}).$$
(9)

The unitary matrix fields $\Sigma_u \equiv \exp(i\Sigma_{A=1}^{N_u^2-1}\lambda^A \pi_u^A/f_u)$, $\Sigma_d \equiv \exp(i\Sigma_{A=1}^{N_d^2-1}\lambda^A \pi_d^A/f_d)$, and $\widetilde{\Sigma} \equiv \exp(i\sqrt{2} \, \widetilde{\pi}/\widetilde{f})$ describe the NG bosons in the up, down, and $U^{(-)}(1)_A$ sectors of the original theory. The decay constants f_u , f_d , \widetilde{f} and transverse velocities v_u , v_d , \widetilde{v} can be calculated by using the standard field theory formalism (for a review, see for example the book [22]). Let us first consider the $N_u^2 + N_d^2 - 2$ NG bosons in the up and down sectors, assigned to the adjoint representation of the $SU(N_u)_V \times SU(N_d)_V$ symmetry. The basic relation is

$$\delta^{AB} P^{\mu}_{q} f_{q} = -i \int \frac{d^{4}k}{(2\pi)^{4}} \operatorname{tr} \left[\gamma^{\mu} \gamma^{5} \frac{\lambda^{A}}{2} \chi^{B}_{q}(k, P) \right], \quad (10)$$

where $P_q^{\mu} = (P^0, v_q^2 \vec{P}_{\perp}, P^3)$ and $\chi_q^A(k, P)$ is the Bethe-Salpeter (BS) wave function of the NG bosons (*P* is the momentum of their center of mass). In the weakly coupled dynamics at hand, one could use an analogue of the Pagels-Stokar approximation [22,23]. In this approximation, the BS wave function is determined from the Ward identities for axial currents. In fact, the calculation of the decay constants and velocities of NG bosons resembles closely the calculation in the case of a color superconducting dense quark matter [24]. In the LLL approximation, the final result in Euclidean space is

$$f_q^2 = 4N_c \int \frac{d^2 k_{\perp} d^2 k_{\parallel}}{(2\pi)^4} \\ \times \exp\left(-\frac{k_{\perp}^2}{|e_q B|}\right) \frac{m_q^2}{(k_{\parallel}^2 + m_q^2)^2}, \quad v_q = 0.$$
(11)

The evaluation of this integral is straightforward. As a result, we get

$$f_{u}^{2} = \frac{N_{c}}{6\pi^{2}} |eB|, \qquad (12)$$

$$f_d^2 = \frac{N_c}{12\pi^2} |eB|.$$
 (13)

The remarkable fact is that the decay constants are nonzero even in the limit when the dynamical masses of quarks approach zero. The reason for that is the (1+1)-dimensional character of this dynamics: as one can see from expression (11), in the limit $m_q \rightarrow 0$, the infrared singularity in the integral cancels the mass m_q in the numerator. A similar situation takes place in color superconductivity [24,25]: in that case the (1+1)-dimensional character of the dynamics is provided by the Fermi surface.

Notice that the transverse velocities of the NG bosons are equal to zero. This is also a consequence of the (1+1)-dimensional structure of the quark propagator in the LLL approximation. The point is that quarks can move in the transverse directions only by hopping to higher Landau levels. Taking into account higher Landau levels would lead to nonzero velocities suppressed by powers of $|m_q|^2/|eB|$. In fact, the explicit form of the velocities was derived in the weakly coupled NJL model in an external magnetic field [see Eq. (65) in the second paper of Ref. [10]]. It is

$$v_{u,d}^{2} \sim \frac{|m_{u,d}|^{2}}{|eB|} \ln \frac{|eB|}{|m_{u,d}|^{2}} \ll 1.$$
(14)

A similar expression should take place also for the transverse velocities of the NG bosons in QCD.

Now, let us turn to the NG boson connected with the spontaneous breakdown of the $U^{(-)}(1)_A$. It is a $SU(N_u)_V$ $\times SU(N_d)_V$ singlet. Neglecting the anomaly, we would actually get two NG singlets, connected with the up and down sectors, respectively. Their decay constants and velocities would be given by expression (10) in which λ^A has to be replaced by λ^0 . The latter is proportional to the unit matrix and normalized as the λ^A matrices: tr[$(\lambda^0)^2$]=2. It is clear that their decay constants and velocities would be the same as for the NG bosons from the adjoint representation. Taking now into account the anomaly, we find that the anomaly-free $U^{(-)}(1)_A$ current is connected with the traceless matrix $\tilde{\lambda}^0/2 \equiv (\sqrt{N_d/N_f} \lambda_u^0 - \sqrt{N_u/N_f} \lambda_d^0)/2$. Therefore the genuine NG singlet $|1\rangle$ is expressed through those two singlets, $|1,d\rangle$ and $|1,u\rangle$, as $|1\rangle = \sqrt{N_d/N_f} |1,u\rangle - \sqrt{N_u/N_f} |1,d\rangle$. This implies that its decay constant is

$$\tilde{f}^2 = \frac{(N_d f_u + N_u f_d)^2}{N_f^2} = \frac{(\sqrt{2}N_d + N_u)^2 N_c}{12\pi^2 N_f^2} |eB|.$$
(15)

Its transverse velocity is of course zero in the LLL approximation.

IV. ANISOTROPIC CONFINEMENT OF GLUONS

Let us now turn to the infrared region with $|k| \leq m_d$, where all quarks decouple (notice that we take here the smaller mass of *d* quarks). In that region, a pure gluodynamics is realized. However, its dynamics is quite unusual. The point is that although gluons are electrically neutral, their dynamics is strongly influenced by an external magnetic field, as one can see from expression (5) for their polarization operator. In a more formal language, while quarks decouple and do not contribute into the equations of the renormalization group in that infrared region, their dynamics strongly influence the boundary (matching) conditions for those equations at $k \sim m_d$.

A conventional way to describe this dynamics is the method of the low energy effective action. By taking into account the polarization effects due to the background magnetic field, we arrive at the following quadratic part of the low-energy effective action of gluons:

$$\mathcal{L}_{g,eff}^{(2)} = -\frac{1}{2} \sum_{A=1}^{N_c^2 - 1} A_{\mu}^A(-k) [g^{\mu\nu}k^2 - k^{\mu}k^{\nu} + \kappa (g_{\parallel}^{\mu\nu}k_{\parallel}^2 - k_{\parallel}^{\mu}k_{\parallel}^{\nu})] A_{\nu}^A(k), \qquad (16)$$

where

$$\kappa = \frac{\alpha_s}{6\pi} \sum_{q=1}^{N_f} \frac{|e_q B|}{m_q^2}$$
$$= \frac{1}{12C_1 \pi} \sum_{q=1}^{N_f} \left(\frac{\alpha_s}{c_q^2}\right)^{1/3}$$
$$\times \exp\left(\frac{4N_c \pi}{\alpha_s (N_c^2 - 1) \ln(C_2/c_q \alpha_s)}\right) \ge 1.$$
(17)

By making use of the quadratic part of the action as well as the requirement of the gauge invariance, we could easily restore the whole low-energy effective action (including selfinteractions) as follows:

$$\mathcal{L}_{gl} \approx \frac{1}{2} \sum_{A=1}^{N_c^2 - 1} (\vec{E}_{\perp}^A \cdot \vec{E}_{\perp}^A + \epsilon E_3^A E_3^A) - \vec{B}_{\perp}^A \cdot \vec{B}_{\perp}^A - B_3^A B_3^A), \qquad (18)$$

where the (chromo-) dielectric constant $\epsilon \equiv 1 + \kappa$ was introduced. Also, we introduced the notation for the chromoelectric and chromo-magnetic fields as follows:

$$E_i^A = \partial_0 A_i^A - \partial_i A_0^A + g f^{ABC} A_0^B A_i^C, \qquad (19)$$

$$B_i^A = \frac{1}{2} \varepsilon_{ijk} (\partial_j A_k^A - \partial_k A_j^A + g f^{ABC} A_j^B A_k^C).$$
(20)

This low energy effective action is relevant for momenta $|k| \leq m_d$. Notice the following important feature of the ac-

tion: the coupling g, playing here the role of the "bare" coupling constant related to the scale m_d , coincides with the value of the vacuum QCD coupling related to the scale $\sqrt{|eB|}$ (and not to the scale m_d). This is because g is determined from the matching condition at $|k| \sim m_d$, the lower border of the intermediate region $m_d \leq |k| \leq \sqrt{|eB|}$, where, because of the pseudo-Higgs effect, the running of the coupling is essentially frozen. Therefore the "bare" coupling g indeed coincides with the value of the vacuum QCD coupling related to the scale $\sqrt{|eB|}$: $g = g_s$. Since this value is much less than that of the vacuum QCD coupling related to the scale m_d , this implies that the confinement scale λ_{QCD} of the action (18) should be much less than Λ_{QCD} in QCD without a magnetic field.

Actually, this consideration somewhat simplifies the real situation. Since the LLL quarks couple to the longitudinal components of the polarization operator, only the effective coupling connected with longitudinal gluons is frozen. For transverse gluons, there should be a logarithmic running of their effective coupling. It is clear, however, that this running should be quite different from that in the vacuum QCD. The point is that the timelike gluons are now massive and their contribution in the running in the intermediate region is severely reduced. On the other hand, because of their negative norm, just the timelike gluons are the major players in producing the antiscreening running in QCD (at least in covariant gauges). Since now they effectively decouple, the running of the effective coupling for the transverse gluons should slow down. It is even not inconceivable that the antiscreening running can be transformed into a screening one. In any case, one should expect that the value of the transverse coupling related to the matching scale m_d will be also essentially reduced in comparison with that in the vacuum QCD. Since the consideration in this section is rather qualitative, we adopt the simplest scenario with the value of the transverse coupling at the matching scale m_d also coinciding with g_s .

In order to determine the new confinement scale λ_{QCD} , one should consider the contribution of gluon loops in the perturbative loop expansion connected with the *anisotropic* action (18), a hard problem being outside the scope of this paper. Here we will get an estimate of λ_{QCD} , without studying the loop expansion in detail. Let us start by calculating the interaction potential between two static quarks in this theory. It reads

$$V(x,y,z) \simeq \frac{g_s^2}{4\pi\sqrt{z^2 + \epsilon(x^2 + y^2)}}.$$
 (21)

Because of the dielectric constant, this Coulomb-like interaction is anisotropic in space: it is suppressed by a factor of $\sqrt{\epsilon}$ in the transverse directions compared to the interaction in the direction of the magnetic field. The potential (21) corresponds to the classical, tree, approximation which is good only in the region of distances much smaller than the confinement radius $r_{QCD} \sim \lambda_{OCD}^{-1}$. Deviations from this interaction are described by loop corrections. Let us estimate the value of a fine structure constant connected with the perturbative loop expansion.

First of all, because of the form of the potential (21), the effective coupling constants connected with the parallel and transverse directions are different: while the former is equal to $g_{eff}^{\parallel} = g_s$, the latter is $g_{eff}^{\perp} = g_s / \epsilon^{1/4}$. On the other hand, the loop expansion parameter (fine structure constant) is $g_{eff}^2/4\pi v_g$, where v_g is the velocity of gluon quanta. Now, as one can notice from Eq. (16), while the velocity of light c = 1, there are gluon quanta with the velocity $v_g^{\perp} = 1/\sqrt{\epsilon}$ in the transverse directions. This seems to suggest that the fine structure coupling may remain the same, or nearly the same, despite the anisotropy: the factor $\sqrt{\epsilon}$ in $(g_{eff}^{\perp})^2$ will be canceled by the same factor in v_g^{\perp} . Therefore the fine structure constant can be estimated as $\alpha_s = g_s^2/4\pi$ [although, as follows from Eq. (16), there are quanta with the velocity $v_g^{\perp} = 1$, their contribution in the perturbative expansion is suppressed by the factor $1/\sqrt{\epsilon}$].

This consideration is of course far from being quantitative. Introducing the magnetic field breaks the Lorentz group SO(3,1) down to $SO(1,1) \times SO(2)$, and it should be somehow manifested in the perturbative expansion. Still, we believe, this consideration suggests that the structure of the perturbative expansion in this theory can be qualitatively similar to that in the vacuum QCD, modulo the important variation: while in the vacuum QCD, α_s is related to the scale |eB|, it is now related to much smaller scale m_d .

By making use of this observation, we will approximate the running in the low-energy region by a vacuum-like running:

$$\frac{1}{\alpha'_{s}(\mu)} = \frac{1}{\alpha_{s}} + b_{0} \ln \frac{\mu^{2}}{m_{d}^{2}}, \quad \text{where} \quad b_{0} = \frac{11N_{c}}{12\pi}, \quad (22)$$

where the following condition was imposed: $\alpha'_s(m_d) = \alpha_s$. From this running law, we estimate the new confinement scale,

$$\lambda_{QCD} \simeq m_d \left(\frac{\Lambda_{QCD}}{\sqrt{|eB|}} \right)^{b/b_0}.$$
 (23)

We emphasize again that expression (23) is just an estimate of the new confinement scale. In particular, both the exponent, taken here to be equal to b/b_0 , and the overall factor in this expression, taken here to be equal to 1, should be considered as being fixed only up to a factor of order 1.

The hierarchy $\lambda_{QCD} \ll \Lambda_{QCD}$ is intimately connected with a somewhat puzzling point that the pairing dynamics decouples from the confinement dynamics despite the fact that it produces quark masses of order Λ_{QCD} or less [for a magnetic field all the way up to the order of (10 TeV)²]. The point is that these masses are heavy in units of the new confinement scale λ_{QCD} and the pairing dynamics is indeed weakly coupled.

V. QCD WITH A LARGE NUMBER OF COLORS

In this section, we will discuss the dynamics in QCD in a magnetic field when the number of colors is large, in particular, we will consider the ('t Hooft) limit $N_c \rightarrow \infty$. Just a look at expression (7) for the gluon mass is enough to recognize that the dynamics in this limit is very different from that considered in the previous sections. Indeed, as is well known, the strong coupling constant α_s is proportional to $1/N_c$ in this limit. More precisely, it rescales as

$$\alpha_s = \frac{\widetilde{\alpha}_s}{N_c},\tag{24}$$

where the new coupling constant $\tilde{\alpha}_s$ remains finite as $N_c \rightarrow \infty$. Then, expression (7) implies that the gluon mass goes to zero in this limit. This in turn implies that the appropriate approximation in this limit is not the improved rainbow approximation but the rainbow approximation itself, when *both* the vertex and the gluon propagator in the SD equation (4) are taken to be bare.

In order to get the expression for the quark in this case, we can use the results of the analysis of the SD equation in the rainbow approximation in QED in a magnetic field [10], with the same simple modifications as in Sec. II. The result is

$$m_q^2 = C |e_q B| \exp\left[-\pi \left(\frac{\pi N_c}{(N_c^2 - 1)\alpha_s}\right)^{1/2}\right],$$
 (25)

where the constant C is of order 1. As $N_c \rightarrow \infty$, one gets

$$m_{q,\infty}^2 = C |e_q B| \exp\left[-\pi \left(\frac{\pi}{\tilde{\alpha}_s}\right)^{1/2}\right].$$
 (26)

It is natural to ask how large N_c should be before the expression (25) becomes reliable. From our discussion above, it is clear that the rainbow approximation may be reliable only when the gluon mass is small; i.e., it is of the order of the quark mass m_q or less. Equating expressions (7) and (25), we derive an estimate for the threshold value of N_c :

$$N_c^{thr} \sim \frac{2N_u + N_d}{\ln|eB|/\Lambda_{QCD}^2} \exp\left[\frac{\pi}{2\sqrt{3}} \left(11\ln\frac{|eB|}{\Lambda_{QCD}}\right)^{1/2}\right].$$
 (27)

Expression (25) for the quark mass is reliable for the values of N_c of the order of N_c^{thr} or larger. Decreasing N_c below N_c^{thr} , one comes to expression (1).

It is quite remarkable that one can get a rather close estimate for N_c^{thr} by equating expressions (1) and (7):

$$N_c^{thr} \sim \frac{2N_u + N_d}{\ln|eB| / \Lambda_{QCD}^2} \exp\left[\left(11 \ln \frac{|eB|}{\Lambda_{QCD}}\right)^{1/2}\right]$$
(28)

[notice that the ratio of the exponents (27) and (28) is equal to 0.91]. The similarity of estimates (27) and (28) implies that, crossing the threshold N_c^{thr} , expression (25) for m_q smoothly transfers into expression (1).

These estimates show that the value of N_c^{thr} rapidly grows with the magnetic field. For example, taking Λ_{QCD} = 250 MeV and N_u =1, N_d =2, we find from Eq. (27) that $N_c^{thr} \sim 10^2$, 10^3 , and 10^4 for $|eB| \sim (1 \text{ GeV})^2$, (10 GeV)², and (100 GeV)², respectively.

As was shown in Sec. IV, in the regime with the number of colors $N_c \ll N_c^{thr}$, the confinement scale λ_{QCD} in QCD in a strong magnetic field is essentially smaller than Λ_{QCD} . What is the value of λ_{QCD} in the regime with N_c being of the order of N_c^{thr} or larger? It is not difficult to see that $\lambda_{QCD} \simeq \Lambda_{QCD}$ in this case. Indeed, now the gluon mass and, therefore, the contribution of quarks in the polarization operator are small (the latter is suppressed by the factor $1/N_c$ with respect to the contribution of gluons). As a result, the β -function in this theory is close to that in QCD without a magnetic field, i.e., $\lambda_{QCD} \simeq \Lambda_{QCD}$.

Expression (25) implies that, for a sufficiently strong magnetic field, the dynamical mass m_q is much larger than the confinement scale Λ_{QCD} . Indeed, expressing the magnetic field in terms of the running coupling, one gets

$$m_q^2 \simeq C \left| \frac{e_q}{e} \right| \Lambda_{QCD}^2 \exp\left[\frac{1}{b \,\alpha_s} - \pi \left(\frac{\pi N_c}{(N_c^2 - 1) \,\alpha_s} \right)^{1/2} \right], \quad (29)$$

and for small values of $b \alpha_s \sim N_c \alpha_s \equiv \tilde{\alpha}_s$ (i.e., for large values of |eB|) the mass m_q is indeed much larger than Λ_{QCD} . This point is important for proving the reliability of the rainbow approximation in this problem. Indeed, the relevant region of momenta in this problem is $m_q^2 \ll |k^2| \ll |eB|$ [10] where, because $m_q^2 \gg \Lambda_{QCD}^2$ for a strong enough field, the running coupling is small. Therefore the rainbow approximation is indeed reliable for sufficiently strong magnetic fields in this case.

VI. CONCLUSION

QCD in a strong magnetic field yields an example of a rich, sophisticated and (that is very important) controllable dynamics. Because of the property of asymptotic freedom, the pairing dynamics, responsible for chiral symmetry breaking in a strong magnetic field, is weakly interacting. The key point why this weakly interacting dynamics manages to produce spontaneous chiral symmetry breaking is the fact that it is essentially (1+1)-dimensional: in the plane orthogonal to the external field the motion of charged quarks is restricted to a region of a typical size of the order of the magnetic length, $l=1/\sqrt{|e_qB|}$. Moreover, such a dynamics almost completely decouples from the dynamics of confinement which develops at very low energy scales in the presence of a strong magnetic field.

While the pairing dynamics decouples from the dynamics of confinement, the latter is strongly modified by the polarization effects due to quarks that have an effective (1+1)-dimensional dynamics in the lowest Landau level. As a result, the confinement scale in QCD in a strong magnetic field λ_{QCD} is much less than the confinement scale Λ_{QCD} in the vacuum QCD. This implies a rich spectrum of light glueballs in this theory.

This picture changes drastically for QCD with a large number of colors ($N_c \gtrsim 100$ for $|eB| \gtrsim 1$ GeV²). In that case a conventional confinement dynamics, with the confinement scale $\lambda_{QCD} \sim \Lambda_{QCD}$, is realized.

The dynamics of chiral symmetry breaking in QCD in a magnetic field has both similarities and important differences with respect to the dynamics of color superconductivity in QCD with a large baryon density [2]. Both dynamics are essentially (1 + 1)-dimensional. However, while the former is anisotropic [the rotational SO(3) symmetry is explicitly broken by a magnetic field], the rotational symmetry is preserved in the latter. This fact is in particular connected with the fact that, while in dense QCD quarks interact both with chromo-electric and chromo-magnetic gluons [20], in the present theory they interact only with the longitudinal components of chromo-electric gluons. This in turn leads to very different expressions for the dynamical masses of quarks in these two theories.

Another important difference is that while the pseudo-Higgs effect takes place in QCD in a magnetic field, the genuine Higgs (Meissner-Higgs) effect is realized in color superconducting dense quark matter. Because of the Higgs effect, the color interactions connected with broken generators are completely screened in infrared in the case of color superconductivity. In particular, in the color-flavor locked phase of dense QCD with three quark flavors, the color symmetry is completely broken and, therefore, the infrared dynamics is under control in that case [26]. As for dense QCD with two quark flavors, the color symmetry is only partially broken down to $SU(2)_c$, and there exists an analog of the pseudo-Higgs effect for the electric modes of gluons connected with the unbroken $SU(2)_c$. As a result, the confinement scale of the gluodynamics of the remaining $SU(2)_c$ group is much less than Λ_{QCD} [27], like in the present case. The essential difference, however, is that, unlike QCD in a magnetic field, the infrared dynamics of a color superconductor is isotropic.

Last but not least, unlike the case of QCD with a nonzero baryon density, there are no principal obstacles for examining all these results and predictions in lattice computer simulations of QCD in a magnetic field.

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from the confinement dynamics may seem to be puzzling. As it will be shown in Sec. IV, the confinement scale λ_{QCD} in QCD in a strong magnetic field is much less than Λ_{QCD} , i.e., the quark masses are large if expressed in units of the new confinement scale. This point implies that the pairing dynamics is indeed weakly coupled.

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