Time without time: A stochastic clock model

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We study a classical reparametrization-invariant system, in which "time" is not *a priori* defined. It consists of a nonrelativistic particle moving in five dimensions, two of which are compactified to form a torus. There, assuming a suitable potential, the internal motion is ergodic or more strongly irregular. We consider quasilocal observables which measure the system's "change" in a coarse-grained way. Based on this, we construct a statistical timelike parameter, particularly with the help of maximum entropy method and Fisher-Rao information metric. The emergent reparametrization-invariant "time" does not run smoothly but is simply related to the proper time on the average. For sufficiently low energy, the external motion is then described by a unitary quantum mechanical evolution in accordance with the Schrödinger equation.

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I. INTRODUCTION

Motivated by attempts to quantize gravity, based on the classical theory of general relativity, there has recently been interest in the quantization of "timeless" reparametrization-invariant systems, see, for example, Refs. [1–4] and further references therein.

Presently, we begin with the study of a classical system and address the question of whether local observables can be found which allow us to characterize the evolution in a gauge invariant way. In previous work it has always been assumed that the global features of the trajectories are accessible to the observer, which makes it possible, in principle, to express the evolution of an arbitrarily selected degree of freedom "relationally" in terms of others [5,6]. Thereby the Hamiltonian and possibly additional constraints have been eliminated in favor of Rovelli's "evolving constants of motion" [1].

In distinction, we presently attempt to characterize the evolution by invariant quasilocal statistical properties of the ergodic internal "clock" motion. Heuristically, we assume "time is change" and try to quantify the former in terms of measurements of the latter. Our results indicate that a "deparametrized" time evolution can be constructed based on coarse-graining localized observations in a classical reparametrization-invariant system which is ergodic.

We remark that certain forms of globally incomplete statistical knowledge about a classical system lead to its effective quantization locally [7]. This points towards a deterministic origin of quantization and certainly raises further interesting questions about the relation to the problem at hand. Indeed we shall find that the external motion of the particle is described by a (discrete-time) Schrödinger evolution.

Let us consider a five dimensional model of a "timeless" nonrelativistic particle with the action

$$S=\int dtL, \tag{1}$$

where the Lagrangian is defined by

$$L\equiv\frac{1}{2\lambda}[(\partial_t\vec{q})^2+r^2(\partial_t\phi_1)^2+r^2(\partial_t\phi_2)^2]+\frac{\lambda}{2}[r^2\omega^2(\tilde{\phi}_1^2+\tilde{\phi}_2^2)+2r^2\Omega^2\tilde{\phi}_1\tilde{\phi}_2-E]. \tag{2}$$

Here λ stands for an arbitrary "lapse" function of the parameter t, $\vec{q}\in\mathbf{R}^3$ denotes an ordinary vector, and ϕ_1 and ϕ_2 denote the angular variables corresponding to the toroidally compactified dimensions with radius r, respectively; ω^2,Ω^2 are angular velocity squared coupling parameters. The parameter E fixes the total energy of the "external" and the compactified "internal" degrees of freedom.

Suitably redefining the units of length and energy, we set $r\equiv1$ henceforth. The notation $\tilde{\phi}$ indicates that the corresponding terms in L are periodically continued:

$$\tilde{\phi}\equiv\phi-n,\quad \phi\in[n,n+1[, \tag{3}$$

for any integer n. Thus, a ratchet type potential results in the $\phi_{1,2}$ plane. Alternatively, we may consider the angular variables to be normalized to the square $[0,1[\times[0,1[$, of which the opposite boundaries are identified, thus describing the surface of a torus with main radii $1/2\pi$.

We remark that the kinetic energy terms in Eq. (2) come with identical signs, signifying that all three coordinates are spacelike. It is possible to change the metric such that the angular variables are timelike, which recalls one of the characteristic relative signs between "kinetic" terms in simplified cosmological models, as discussed, for example, in Refs. [3,4] and references therein. Since this, however, would not change any of our present results, we consistently consider a nonrelativistic model here. Furthermore, we choose the potential in the compactified dimensions to be unstable, in order to generate a chaotic internal motion which is ergodic and possibly more strongly irregular.

Varying the action with respect to $\vec{q}$, ϕ_1, and ϕ_2, we obtain the equations of motion

$$\frac{1}{\lambda}\partial_t\left(\frac{1}{\lambda}\partial_t\vec{q}\right)=0, \tag{4}$$

$$\frac{1}{\lambda}\partial_t\left(\frac{1}{\lambda}\partial_t\phi_1\right)=(\omega^2\tilde{\phi}_1+\Omega^2\tilde{\phi}_2)\left(1-\sum_n \delta(\phi_1-n)\right), \tag{5}$$

$$\frac{1}{\lambda}\partial_t\left(\frac{1}{\lambda}\partial_t\phi_2\right)=(\omega^2\tilde{\phi}_2+\Omega^2\tilde{\phi}_1)\left(1-\sum_n \delta(\phi_2-n)\right), \tag{6}$$

where the singular terms arise due to the discontinuities of the potential. Variation of S with respect to λ yields the constraint

$$\frac{1}{2\lambda^2}[(\partial_t\vec{q})^2+(\partial_t\phi_1)^2+(\partial_t\phi_2)^2]-\frac{1}{2}[\omega^2(\tilde{\phi}_1^2+\tilde{\phi}_2^2)+2\Omega^2\tilde{\phi}_1\tilde{\phi}_2-E]=0, \tag{7}$$

which will be recognized as a constraint on the Hamiltonian momentarily.

The canonical momenta are defined as usual:

$$\vec{p}\equiv\frac{\partial L}{\partial(\partial_t\vec{q})}=\frac{1}{\lambda}\partial_t\vec{q}, \tag{8}$$

$$\pi_1\equiv\frac{\partial L}{\partial(\partial_t\phi_1)}=\frac{1}{\lambda}\partial_t\phi_1, \tag{9}$$

$$\pi_2\equiv\frac{\partial L}{\partial(\partial_t\phi_2)}=\frac{1}{\lambda}\partial_t\phi_2. \tag{10}$$

Incorporating these, we obtain the Hamiltonian

$$H=\vec{p}\cdot\partial_t\vec{q}+\pi_1\partial_t\phi_1+\pi_1\partial_t\phi_2-L \tag{11}$$

$$=\frac{\lambda}{2}[\vec{p}^2+\pi_1^2+\pi_2^2-\omega^2(\tilde{\phi}_1^2+\tilde{\phi}_2^2)-2\Omega^2\tilde{\phi}_1\tilde{\phi}_2+E]\equiv\lambda C. \tag{12}$$

In terms of C, Eq. (7) implies the primary constraint

$$C=0, \tag{13}$$

which is a weak equality in the sense of Dirac's formalism of constraint systems [8]. Consequently, the Hamiltonian presents a weak constraint, $H=0$.

Finally, in Hamiltonian form, the equations of motion (4)–(6) read

$$\partial_t\vec{p}=0, \tag{14}$$

$$\partial_t\pi_1=\lambda(\omega^2\tilde{\phi}_1+\Omega^2\tilde{\phi}_2)\left(1-\sum_n \delta(\phi_1-n)\right), \tag{15}$$

$$\partial_t\pi_2=\lambda(\omega^2\tilde{\phi}_2+\Omega^2\tilde{\phi}_1)\left(1-\sum_n \delta(\phi_2-n)\right). \tag{16}$$

Employing these equations and the definition of the constraint C in Eq. (13), it follows by explicit calculation that it does not evolve. Consistently, this is also obtained by

$$\partial_t C=\{C,H\}=\lambda^{-1}\{H,H\}=0, \tag{17}$$

employing the Poisson bracket notation, $\{A,B\}\equiv\Sigma(\partial_Q A\partial_P B-\partial_P A\partial_Q B)$, where a sum over all coordinates and canonical momenta, represented by Q and P, is understood. Therefore, no secondary constraints exist in our model.

We conclude that the system has four physical degrees of freedom. Its extended phase space, cf. Ref. [1], is ten dimensional, corresponding to the Lagrangian variables in Eq. (2) and the associated canonical momenta. It is, however, reduced to a nine dimensional surface by the constraint. Since the physical phase space is eight dimensional, there must be one "gauge" degree of freedom, which is related to the reparametrization invariance. The study of this gauge symmetry and its consequences will be performed in Sec. II, while the gauge invariant description of the evolution will be developed in Sec. III. In Sec. IV we demonstrate that under certain conditions the external motion of the particle can be mapped onto an evolution according to the Schrödinger equation. We conclude with a brief discussion.

II. GAUGE INVARIANCE AND OBSERVABLES

We observe indeed that the action is invariant under the set of gauge transformations:

$$t\equiv f(t'),\quad x(t)\equiv x'(t'),\quad \lambda(t)\frac{dt}{dt'}\equiv\lambda'(t'), \tag{18}$$

where $x\in\{\vec{q},\phi_1,\phi_2\}$. Corresponding infinitesimal transformations are generated by

$$\delta t\equiv t-t'=\epsilon(t'), \tag{19}$$

where ϵ is infinitesimal. This yields immediately

$$\delta x\equiv x(t')-x'(t')=-\epsilon(t')\partial_{t'}x(t'), \tag{20}$$

$$\delta\lambda\equiv\lambda(t')-\lambda'(t')=-\partial_{t'}[\epsilon(t')\lambda(t')]. \tag{21}$$

Employing the definitions of the canonical momenta, Eqs. (8)–(10), we obtain, from Eq. (20),

$$\delta\vec{q}=-\epsilon\lambda\vec{p}, \tag{22}$$

$$\delta\phi_{1,2}=-\epsilon\lambda\pi_{1,2}. \tag{23}$$

Similarly, with the help of Eqs. (14)–(16), we obtain

$$\delta\vec{p}=0, \tag{24}$$

$$\delta\pi_1=-\epsilon\lambda(\omega^2\tilde{\phi}_1+\Omega^2\tilde{\phi}_2)\left(1-\sum_n\ \delta(\phi_1-n)\right), \tag{25}$$

$$\delta\pi_2=-\epsilon\lambda(\omega^2\tilde{\phi}_2+\Omega^2\tilde{\phi}_1)\left(1-\sum_n\ \delta(\phi_2-n)\right). \tag{26}$$

Comparing with the equations of motion, we see that the evolution of the coordinates and momenta is generated by the gauge transformations.

Obviously, the three-momentum $\vec{p}$ is conserved, in accordance with translation invariance. Its components present three gauge invariant observables which may serve as coordinates of the physical phase space. Another invariant is provided by the internal contribution to the constraint:

$$C_{int}\equiv\vec{p}^{\,2}+E-2C, \tag{27}$$

cf. Eqs. (12), (13); since the constraint does not evolve, i.e. is invariant, as well as $\vec{p}^{\,2}$, this also holds for C_{int}. However, being related via the constraint, C_{int} is not independent of $\vec{p}$. The angular momentum suggests itself as a further observable,

$$\vec{J}\equiv\vec{q}\times\vec{p}, \tag{28}$$

due to rotational invariance. Clearly, $\delta\vec{J}=\delta\vec{q}\times\vec{p}+\vec{q}\times\delta\vec{p}=0$.

As expected, the external motion plays a rather passive role in our system, since it can be almost completely described in terms of the conserved linear and angular momenta. It only contributes with its kinetic energy to the constraint C. However, note that $\vec{p}$ and $\vec{J}$ together determine only the two constant components of $\vec{q}$ transverse to $\vec{p}$, i.e. $\vec{q}_\perp$. Therefore, in order to predict the evolution of the coordinate $\vec{q}$, we need to construct a reference "clock," such that the longitudinal component of Eq. (8) is well determined, when integrated:

$$\vec{q}_\parallel(t)=\vec{q}_\parallel^{(0)}+\vec{p}\int_0^t dt'\lambda(t'). \tag{29}$$

This is one of the objectives of the following section.

Furthermore, we remark that we so far obtained five gauge invariant coordinates ($\vec{p}$, $\vec{q}_\perp$) for the eight dimensional physical phase space plus one constraint (C, or C_{int}), which allows us to eliminate one more of the remaining four internal variables ($\phi_{1,2},\pi_{1,2}$). Because of the intrinsic nonlinearity of the gauge tranformations, Eqs. (22)–(26), we are unable to find further invariants in the general, interacting case. This motivates our attempt to construct other statistical observables, in order to obtain an invariant, even if coarse-grained, description of the physical phase space.

III. ERGODICITY AND COARSE-GRAINED "TIMING"

In order to proceed, we make the crucial assumption that our model forms an ergodic system [9]. We also restrict the allowed lapse functions λ to be (strictly) positive, thus avoiding trajectories which trace themselves backwards (or stall) [2].

In the following, we will consider quantities $N[\phi_{1,2}](c_i)$ which are functionals only of the trajectory determined by the coordinates $\phi_{1,2}$ and possibly depend parametrically on further constraints involving only them. While some explicit examples will be studied in detail, generally speaking, we have quantities in mind which reflect properties of Poincaré sections of the full trajectories. Naturally, we require the constraints c_i to transform as the coordinates under the gauge transformations (18), i.e. $c_i'=c_i$. It follows that such quantities are gauge invariant, since they depend only on geometrical properties of a path and related constraints. Being independent of the momenta, they do not depend on how the trajectory is parametrized: $N[\phi_{1,2}'](c_i')=N[\phi_{1,2}](c_i)$. Thus, they qualify as coarse-grained observables characterizing the internal motion.

Our aim is to construct a timelike variable based on such observables. In the following first subsection we do this based on the idea that the geometric path length covered by the system evolving from an initial to a final state is an invariant measure of the "time" that passed. The crucial point is that this measure can be inferred in an ergodic system approximately from coarse-graining localized observations, provided we understand the dynamics sufficiently well.

In the second subsection, however, considering the interacting nonlinear system, we generalize this approach, in order to extract a "time" from quasilocal measures of the "change" occurring while the system evolves. In particular, we will employ a maximum-entropy method together with the Fisher-Rao information metric, in order to characterize the distance, i.e. the "time" passed, between evolving probability distributions. We point out that our approach is somehow orthogonal to the one of Ref. [10], although we make use of its formalism. There the author launches the ambitious project to derive dynamics from rules of inference and the maximum-entropy principle in particular. We instead assume the reparametrization-invariant dynamics to be given and construct a pertinent notion of "time."

A. Free internal motion on the torus

Illustrating our approach, we begin with the noninteracting case, i.e., with $\omega^2=\Omega^2=0$. Even without interactions, the internal motion is ergodic for almost all initial conditions. In particular, if the ratio of the two independent angular velocities is not a rational number, then, for sufficiently late times t, the trajectory will come arbitrarily close to any point on the surface of the torus. This is easily seen in its parametric representation, Eq. (31) below. In Fig. 1 we show a typical example.

The equations of motion immediately yield the solutions

$$\phi_{1,2}(t)=\phi_{1,2}^{(0)}+\pi_{1,2}^{(0)}\int_0^t dt'\lambda(t'), \tag{30}$$

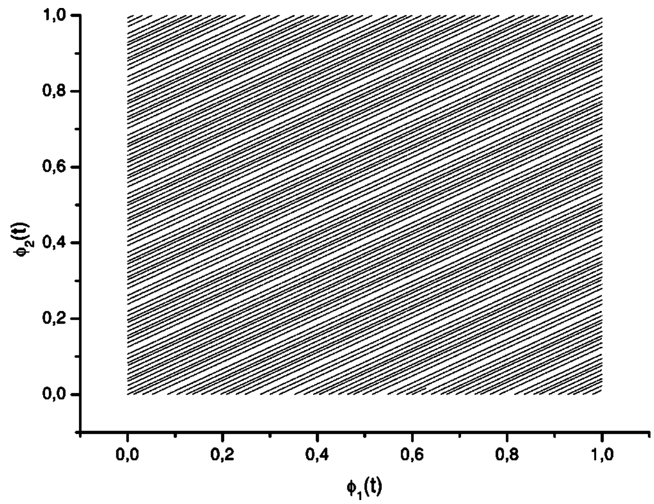

FIG. 1. A typical trajectory. Initial conditions for the free motion on the torus are $\phi_{1,2}^{(0)}=0$ and $\pi_1^{(0)}=1$, $\pi_2^{(0)}=\sqrt{3}$ (arbitrary units). Here $t=\tau$ (proper time); the final time is $\tau=50$.

where $\phi_{1,2}^{(0)}$ and $\pi_{1,2}^{(0)}$ denote the initial coordinates and momenta, respectively; of course, $\pi_{1,2}(t)=\pi_{1,2}^{(0)}$. Eliminating the integral of the lapse function, we obtain, for example, ϕ_2 in terms of ϕ_1:

$$\phi_2(\phi_1)=\phi_2^{(0)}-\frac{\pi_2^{(0)}}{\pi_1^{(0)}}(\phi_1^{(0)}-\phi_1). \tag{31}$$

Following Refs. [1,5], this is a gauge invariant ''relational'' description of the motion: $\phi_2(s)$ gives the value of coordinate ϕ_2, when (''the time is such that'') ϕ_1 has the value s. In this way, ϕ_1 may serve as a time variable, even if not a unique one.

However, similarly as in the models studied previously, there is additional information about the full path, which is necessary to complete this description. In the present case, the solution (31) presents a line in the $\phi_{1,2}$ plane, unwrapping the motion which multiply covers the torus. Thus, when folding it back onto the torus, one has to keep track of which unit square in the plane a respective piece is coming from. This can be labeled by two integers $n_{1,2}$, which may be interpreted as the winding numbers characterizing additionally a given point $(\tilde{\phi}_1,\tilde{\phi}_2)$ on the path. Obviously, this presents highly nonlocal (topological) information, which will be unavailable for a local observer under more realistic circumstances, such as in the presence of nonlinear interactions.

Therefore, we turn to statistical measures of the motion on the torus. Consider the ''incident number'' $I_1^\delta(L)$ which counts the number of times that a given trajectory $[\tilde{\phi}_1(t),\tilde{\phi}_2(t)]$, cuts the ϕ_1 axis in the neighborhood of the observer, i.e., with $\tilde{\phi}_1\in[0,\delta<1]$ and $\tilde{\phi}_2=0$, subject to the constraint that the total path length equals L, taking into account the wrapping around the torus. Because of ergodicity and assumed nonnegative lapse functions, I_1 is a stepwise increasing function of t, while $\tilde{\phi}_{1,2}(t)\in[0,1[$ are of sawtooth type, with details depending on the particular parametrization. The reparametrization-invariant value of I_1, however, only depends on the path length; thus, it implicitly involves nonlocal information.

The incident number I_1 presents a very simple example of the coarse-grained observables discussed before. There exists an unlimited number of different such counting observables. The more of them we introduce and measure, the more detailed will be our reparametrization-invariant description of the internal motion. We will also use the incident number I_2 which is defined like I_1, however, with the roles of coordinates ''1'' and ''2'' exchanged.

Furthermore, we may consider one incident number as a parametric function of the other:

$$I_{21}(k)\equiv \max I_2 \quad \text{with} \quad I_1^\delta(L)=k, \tag{32}$$

where δ is fixed, while the total path length L is the implicit common variable. Note that I_{21} is unique, since we take the respective maximum of I_2, which may increase while I_1 stays constant temporarily.

However, the constraint on a given path length is irrelevant for the actual values assigned to I_1 and I_{21}. One operationally determines them by counting the localized incidents, with $\tilde{\phi}_{1,2}(t)\in[0,\delta]$, and recording one incident number as a function of the other; no knowledge of the path length is required.

Nevertheless, these reparametrization-invariant numbers can be used to determine the corresponding path length by a statistical consideration, since we have sufficient knowledge of the dynamics of the system in the present (integrable) case.

In the absence of interactions the path is composed of straight line segments. We denote the average length of these segments by $\langle l\rangle$. It can be calculated easily due to ergodicity:

$$\langle l\rangle=\frac{[(\pi_1^{(0)})^2+(\pi_2^{(0)})^2]^{1/2}}{\pi_1^{(0)}+\pi_2^{(0)}}\equiv\frac{[2E_{int}]^{1/2}}{\pi_1^{(0)}+\pi_2^{(0)}}, \tag{33}$$

where E_{int} denotes the conserved internal energy, cf. Eqs. (12) and (27). Here we employed Eq. (31) and averaged the straight-line paths with the asymptotically uniform density over the square $[0,1[\times[0,1[$, choosing coordinates such that $\pi_{1,2}^{(0)}>0$.

Now, each increment of I_1 or I_{21} (i.e. I_2) by one unit corresponds to the completion of one line segment. Due to the finite window size $\delta<1$, however, only a corresponding fraction of all incidents happening on both coordinate axes will be recorded on the average. Correcting for this, invoking ergodicity as before, the path length L which leads to the measured incident numbers is simply obtained by

$$L=\langle l\rangle\int_0^{I_1}\frac{ds}{\delta}=\langle l\rangle\frac{I_1+I_{21}(I_1)}{\delta}, \tag{34}$$

where ds is *along* the curve of the function $I_{21}(s)$. Practically, what appears on the right-hand side is the total registered incident number ($I\equiv I_1+I_2$) at a certain instant.

In this way, we obtain a measure of the ''time'' interval T during which N incidents occurred:

$$T \equiv L/[2E_{int}]^{1/2}, \tag{35}$$

i.e., dividing the path length by the constant velocity.

The point of this rather trivial example is that nowhere do we make use of the time parametrizing the evolution nor of the generally unknown path length. Rather, we derive T from reparametrization-invariant localized measurements.

Not surprisingly, the resulting "time" will not run smoothly, due to the coarse-grained description of the internal motion: as if we were reading an analog "clock" under a stroboscopic light. This is precisely the role of a Poincaré section with respect to the increasing invariant path length of an evolving trajectory.

In order to illustrate the behavior of the "time" T, it is convenient to introduce the fictitious proper time (function):

$$\tau \equiv \int_0^t dt' \lambda(t'). \tag{36}$$

Then, keeping the notation as simple as possible, with $x(\tau) \equiv x(t)$ for $x \in \{\vec{q}, \vec{p}, \phi_{1,2}, \pi_{1,2}\}$, we obtain $\partial_t x(t) = \lambda(t) \partial_\tau x(\tau)$. Applying this transformation to the definition of the canonical momenta and the equations of motion, the lapse function can be eliminated; this replaces t by τ and λ by 1 in Eqs. (8)–(10) and Eqs. (14)–(16), respectively. The resulting equations are reparametrization-invariant. Solutions of these equations are to be interpreted "physically" by introducing the inverse function $t(\tau)$ and $x(t) = x(\tau(t))$ [1].

Integrating the free internal motion with respect to the proper time, which replaces the integral in Eq. (30) by τ, we show T as a function of τ in Fig. 2a and Fig. 2b. For the two runs differing in the total (computer) time τ, we find that after a short while, i.e. already at low incident numbers, the constructed physical "time" T approximates qualitatively well the proper time τ. The fluctuations on top of the observed linear dependence naturally reflect the stochastic internal motion. The fact that the slopes are consistently smaller than one can be attributed to the bias towards longer-than-average pieces of trajectory, which is introduced by measuring the incidents close to the origin and extrapolating from there; see Eqs. (33) and (34).

Finally, employing the reparametrization-invariant "time," which runs approximately parallel to the unphysical proper time, $T \approx \kappa\tau$ (κ const), we succeed in describing the external motion with respect to the physical internal "clock" constructed here. Recalling that the external motion is given by Eq. (29), we obtain $\vec{q}_\parallel(\tau) = \vec{q}_\parallel^{(0)} + \tau\vec{p} \approx \vec{q}_\parallel^{(0)} + \kappa^{-1} T \vec{p}$.

Interestingly, the coarse-grained "jumpiness" of T introduces a corresponding stochastic component into the external motion. This would be recognized if global relational data or results of increasingly fine-grained local measurements were available with which to compare the change of $\vec{q}_\parallel$. We will further study the consequences of this stroboscopic effect in Sec. IV.

B. Quasilocal analysis of "time is change" with interactions

Now we consider the case of the internal motion on the torus with the interactions ($\propto \omega^2, \Omega^2$) turned on. The interactions additionally mix the trajectories in phase space, since the potentials are of inverted oscillator type. Slightly simplifying the ensuing calculations, we set $\omega^2 = \Omega^2$ henceforth. This leaves the motion parallel to the diagonal $\tilde{\phi}_1 = \tilde{\phi}_2$ exponentially unstable, while the orthogonal motion is free. A typical trajectory for the case of still relatively weak interaction is shown in Fig. 3.

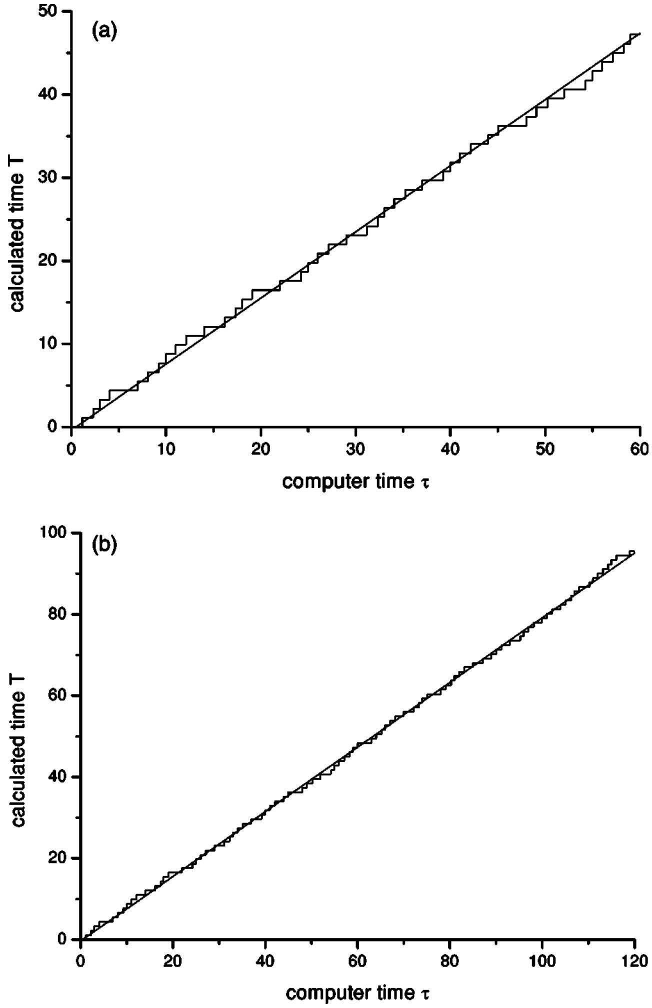

FIG. 2. The reparametrization invariant "time" T, Eq. (35), based on localized incident counting, as a function of (computer) proper time τ for two different proper time intervals, cases (a) and (b); initial conditions as in Fig. 1; window parameter: $\delta = 1/3$. The lines result from linear fits.

The microstates of the internal part of the system can be described by the phase space coordinates $x \equiv (\phi_1, \phi_2; \pi_1, \pi_2)$. However, in general, we will not be able to follow the deterministic evolution through sequences of microstates for any sufficiently complex nonlinear system. Therefore, we develop a coarse-graining statistical approach based on probability distributions. Let $P(x)dx$ denote the expected number of microstates in the volume element dx at x. Then, we may characterize macrostates of the system by giving their coordinates $Q^i, i = 1, \ldots, n$ in the n-dimensional state space:

$$Q^i \equiv \langle \mathcal{Q}^i \rangle \equiv \int dx P(x) \mathcal{Q}^i(x), \tag{37}$$

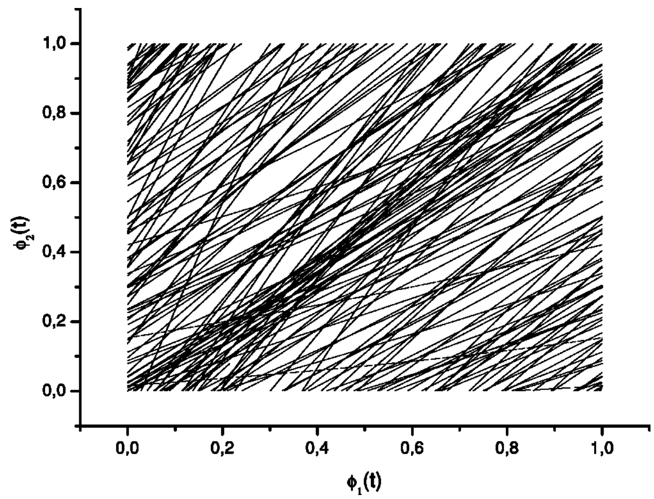

FIG. 3. A typical trajectory with interaction parameters $\omega=\Omega=3$. The initial conditions are $\phi_{1,2}^{(0)}=0$ and $\pi_1^{(0)}=3$, $\pi_2^{(0)}=\sqrt{5}$; the final proper time $(t=\tau)$ is $\tau=30$.

with $\{\mathcal{Q}^i\}$ denoting the relevant set of observables. It is assumed that all information necessary to answer our particular questions about the system is encoded in these observables [10].

The conserved internal energy, see Eq. (27), presents an important observable:

$$\mathcal{H}_{int}(x)\equiv\pi_1^2+\pi_2^2-\omega^2(\tilde{\phi}_1+\tilde{\phi}_2)^2, \tag{38}$$

where we absorbed an inessential factor 2 into the definition for convenience.

Then, the "prior" distribution $P_c(x|H_{int})$ which best reflects our mostly lacking information about the state of the system, given the conserved energy, is obtained by maximizing the entropy

$$S[P]\equiv-\int dxP(x)\ln P(x), \tag{39}$$

subject to the constraint $\langle\mathcal{H}_{int}\rangle=H_{int}$, with this constant being fixed by the initial conditions. The result is the canonical distribution

$$P_c(x|H_{int})=Z_c^{-1}e^{-\beta\mathcal{H}_{int}(x)}, \tag{40}$$

with the partition function Z_c and Lagrange multiplier β, respectively, to be calculated from

$$Z_c\equiv\int dxe^{-\beta\mathcal{H}_{int}(x)},\quad H_{int}=-\frac{\partial}{\partial\beta}\ln Z_c, \tag{41}$$

where $\int dx\equiv\int_{-\infty}^{\infty}d\pi_1\int_{-\infty}^{\infty}d\pi_2\int_0^1 d\phi_1\int_0^1 d\phi_1$. Thus we recover expressions which are familiar from statistical mechanics.

Straightforward calculation yields the partition function

$$Z_c=\frac{\pi}{\beta}Z_{c,conf}=\frac{\pi}{\beta}J(1,1), \tag{42}$$

where $Z_{c,conf}$ denotes its configurational factor. It is given here in terms of the integral

$$J(a,b)\equiv\frac{1}{\beta\omega^2}\int_0^{a\sqrt{\beta\omega^2}}ds_1\int_0^{b\sqrt{\beta\omega^2}}ds_2e^{(s_1+s_2)^2} \tag{43}$$

$$=\frac{1}{2}\sqrt{\frac{\pi}{\beta\omega^2}}\big((a+b)\mathrm{erfi}[(a+b)\sqrt{\beta\omega^2}]-a\,\mathrm{erfi}(a\sqrt{\beta\omega^2})-b\,\mathrm{erfi}(b\sqrt{\beta\omega^2})\big)-\frac{1}{2\beta\omega^2}\big(e^{(a+b)^2\beta\omega^2}-e^{a^2\beta\omega^2}-e^{b^2\beta\omega^2}+1\big) \tag{44}$$

$$=ab\left(1+\beta\omega^2\left[\frac{1}{3}a^2+\frac{1}{2}ab+\frac{1}{3}b^2\right]+\frac{1}{2}(\beta\omega^2)^2\left[\frac{1}{5}a^4+\frac{1}{2}a^3b+\frac{2}{3}a^2b^2+\frac{1}{2}ab^3+\frac{1}{5}b^4\right]+\mathrm{O}[(\beta\omega^2)^3]\right), \tag{45}$$

involving the imaginary error function, $\mathrm{erfi}(x)\equiv2\pi^{-1/2}\int_0^x ds\,\exp(s^2)$. We will make more use of this integral in the following.

For example, incorporating the small-β expansion, we calculate the relation between H_{int} and β:

$$\frac{H_{int}}{\omega^2}=\frac{1}{\beta\omega^2}-\frac{7}{6}-\frac{127}{180}\beta\omega^2+\mathrm{O}[(\beta\omega^2)^2], \tag{46}$$

using Eqs. (41), (42), and (45). Here the error in comparison with the exact result rapidly decreases with energy and is less than 5% for $H_{int}/\omega^2>1$. We are presently interested in the positive energy regime, since only there the trajectories can explore all of the torus surface, cf. Eq. (38).

In order to improve the prior distribution P_c with the help of quasilocal measurements, we again consider simple observables for illustration. We define two "window functions:"

$$\mathcal{I}_1(x)\equiv\Theta(\phi_1-\epsilon)\Theta(\delta-\phi_1)\Theta(\phi_2)\Theta(\epsilon-\phi_2), \tag{47}$$

with $0<\epsilon\ll\delta<1$; $\mathcal{I}_2$ is defined by the analogous expression with ϕ_1 and ϕ_2 exchanged. Thus $\mathcal{I}_1$ and $\mathcal{I}_2$ project out small rectangles along the ϕ_1 and ϕ_2 axis, respectively, which do not overlap. Here we explicitly introduced the small but finite window width ϵ, which in any case is necessary for a correct counting of incidents in numerical simulations with a finite resolution. In the following we will adapt to the present case the measurement of incident numbers.

We decompose the prior distribution with respect to the windows determined by $\mathcal{I}_1$ and $\mathcal{I}_2$:

$$P_c(x|H_{int})=[\mathcal{I}_1(x)+\mathcal{I}_2(x)]P_c(x|H_{int})+[1-\mathcal{I}_1(x)-\mathcal{I}_2(x)]P_c(x|H_{int})\equiv P_w'(x|H_{int})+\overline{P}_w(x|H_{int}). \tag{48}$$

While $\overline{P_w}$ describes their complement, the distribution P'_w is the one which is related to measurements within the windows. Its normalized counterpart P_w is

$$P_w(x|H_{int}) \equiv \frac{[\mathcal{I}_1(x)+\mathcal{I}_2(x)]P_c(x|H_{int})}{\int dx[\mathcal{I}_1(x)+\mathcal{I}_2(x)]P_c(x|H_{int})} = \frac{\beta}{2\pi}\frac{[\mathcal{I}_1(x)+\mathcal{I}_2(x)]e^{-\beta\mathcal{H}_{int}(x)}}{J(\delta,\epsilon)-J(\epsilon,\epsilon)}, \tag{49}$$

employing the configurational integral of Eq. (43).

Then, the distribution $P_w(x|H_{int};\langle\mathcal{I}_i\rangle)$ which best reflects the information contained in the prior distribution $P_w(x|H_{int})$ and in the data from measurements of the window functions is obtained by maximizing the entropy [10]

$$S[P] \equiv -\int dx[\mathcal{I}_1(x)+\mathcal{I}_2(x)]P(x)\ln\frac{P(x)}{P_w(x|H_{int})}, \tag{50}$$

subject to the constraints

$$\langle\mathcal{I}_i\rangle = \frac{I_i}{I_1+I_2}\bigg|_{i=1,2}. \tag{51}$$

Here we determine the average of the incident functions, i.e. the (total) probabilities of observing an incident in the respective windows, in terms of the measured incident numbers. This obviously presents a crude coarse-graining. An improved description is obtained, for example, by binning the incident numbers with respect to the main axis of each window. Many more detailed measurements can be envisaged, but the simplest ones will suffice here.

It is straightforward to show that this procedure yields a grand-canonical distribution:

$$P_w(x|H_{int};\langle\mathcal{I}_i\rangle) = Z^{-1}[\mathcal{I}_1(x)+\mathcal{I}_2(x)] \times e^{-\beta\mathcal{H}_{int}(x)-\alpha_1\mathcal{I}_1(x)-\alpha_2\mathcal{I}_2(x)}, \tag{52}$$

where, in this case, the partition function and Lagrange multipliers are determined by

$$Z \equiv \int dx[\mathcal{I}_1(x)+\mathcal{I}_2(x)] \times e^{-\beta\mathcal{H}_{int}(x)-\alpha_1\mathcal{I}_1(x)-\alpha_2\mathcal{I}_2(x)} = \int dx[\mathcal{I}_1(x)\lambda_1+\mathcal{I}_2(x)\lambda_2]e^{-\beta\mathcal{H}_{int}(x)}, \tag{53}$$

introducing the fugacities $\lambda_i \equiv \exp(-\alpha_i)|_{i=1,2}$, and

$$\langle\mathcal{I}_i\rangle = -\frac{\partial}{\partial\alpha_i}\ln Z = \lambda_i\frac{\partial}{\partial\lambda_i}\ln Z\bigg|_{i=1,2}, \tag{54}$$

together with the constraints (51). Here β is considered to be a known feature of the prior distribution P_c, previously determined in Eq. (46).

The present situation differs in an important way from the usual one in statistical mechanics, where H_{int} and $\langle\mathcal{I}_i\rangle$ would *all* correspond to conserved quantities and be treated on an equal footing. Presently, the (window functions related to the) incident numbers are evolving quantities which we measure in order to learn about the change occurring in the system.

The grand-canonical partition function can be calculated directly, resulting in

$$Z = \frac{\pi}{\beta}Z_{conf} = \frac{\pi}{\beta}(\lambda_1+\lambda_2)[J(\delta,\epsilon)-J(\epsilon,\epsilon)], \tag{55}$$

where Z_{conf} is the configurational factor of this partition function. Furthermore, we obtain

$$\langle\mathcal{I}_i\rangle = \frac{\lambda_i}{\lambda_1+\lambda_2}\bigg|_{i=1,2}, \tag{56}$$

which implies $\lambda_1/\lambda_2 = \langle\mathcal{I}_1\rangle/\langle\mathcal{I}_2\rangle$. We set $\lambda_i = C\langle\mathcal{I}_i\rangle|_{i=1,2}$, with a common (undetermined) constant C. Incorporating these results, the distribution follows:

$$P_w(x|H_{int};\langle\mathcal{I}_i\rangle) = \frac{\beta}{\pi}\frac{[\langle\mathcal{I}_1\rangle\mathcal{I}_1(x)+\langle\mathcal{I}_2\rangle\mathcal{I}_2(x)]}{J(\delta,\epsilon)-J(\epsilon,\epsilon)} \times e^{-\beta\mathcal{H}_{int}(x)}, \tag{57}$$

using $\langle\mathcal{I}_1+\mathcal{I}_2\rangle = 1$ and cancelling the common factor C.

Finally, re-normalizing $P_w(x|H_{int};\langle\mathcal{I}_i\rangle)$ and using the resulting distribution in place of P'_w, cf. Eq. (48), we obtain the properly normalized distribution for the whole phase space:

$$P(x|H_{int};\langle\mathcal{I}_i\rangle) = 2[\langle\mathcal{I}_1\rangle\mathcal{I}_1(x)+\langle\mathcal{I}_2\rangle\mathcal{I}_2(x)]P_c(x|H_{int}) + [1-\mathcal{I}_1(x)-\mathcal{I}_2(x)]P_c(x|H_{int}), \tag{58}$$

which is updated by measuring the incident numbers $I_{1,2}$ and employing Eq. (51).

With the distribution at hand, we could proceed similarly as in the previous Sec. III A, trying to estimate the average path length related to the increasing incident numbers in particular, in order to gain a measure of the change taking place in the system.

However, in the following we proceed differently, in a way which appears more suitable to further generalization. We introduce the Fisher-Rao information metric for the purpose of quantifying the change due to the chaotic, even if deterministic, motion from one configuration to the next [11,12]. It is the uniquely determined Riemannian metric (except for an overall multiplicative constant) on the space of states which are probability distributions [10]. In our present case the states are simply described by the pair of coordinates $Q^i \equiv \langle\mathcal{I}_i\rangle|_{i=1,2} \in [0,1]$, considering the coordinate H_{int} to be fixed at a constant value. Then, the "distance" ds between the states $P(x|H_{int};Q^k)$ and $P(x|H_{int};Q^k+dQ^k)$ is given by

$$ds^2 = g_{ij}dQ^i dQ^j, \tag{59}$$

with the metric

$$g_{ij}=\int dxP(x|H_{int};Q^k)\frac{\partial \ln P(x|H_{int};Q^k)}{\partial Q^i}\times\frac{\partial \ln P(x|H_{int};Q^k)}{\partial Q^j} \quad (60)$$

$$=2\frac{J(\delta,\epsilon)-J(\epsilon,\epsilon)}{J(1,1)}(Q^i)^{-1}\delta_{ij}, \quad (61)$$

employing Eqs. (40), (42), (43), and particularly Eq. (58) in the explicit calculation. Note that the coordinates, being probabilities, are constrained by $Q^1+Q^2=1$, which further simplifies the result obtained here.

However, we would like to express Eqs. (59)–(61) in terms of the directly measured incident numbers. Thus, with the help of Eqs. (51), we obtain

$$\Delta s=2\frac{J(\delta,\epsilon)-J(\epsilon,\epsilon)}{J(1,1)}\frac{|I_2\Delta I_1-I_1\Delta I_2|}{\sqrt{I_1I_2}(I_1+I_2)} \quad (62)$$

$$=2\frac{J(\delta,\epsilon)-J(\epsilon,\epsilon)}{J(1,1)(I_1+I_2)}\left(\sqrt{\frac{I_2}{I_1}}\Delta I_1+\sqrt{\frac{I_1}{I_2}}\Delta I_2\right), \quad (63)$$

where the second equality is due to the dichotomous character of the pair of increments ΔI_1 and ΔI_2; i.e., we have $\Delta I_k=0,1$, however, the two increments present the results of two mutually exclusive measurements which cannot happen in coincidence.

Several remarks are in order here. First, Eq. (63) is invariant under the similarity transformation $I_k\rightarrow\xi\cdot I_k|_{k=1,2}$ and $\Delta s\rightarrow\Delta s/\xi$. Thus, generally, the amount of ''change per incident'' decreases inversely with the total number of incidents, i.e., loosely speaking, with the age of the evolution going on. This seems natural in a statistical context, but less familiar in the context of dynamics, where we are accustomed to a linearly flowing time.

Second, the dichotomous increments of the incident numbers can be mapped on a binary sequence of zeros and ones, for example, according to $(\Delta I_1=1)\rightarrow 0$ and $(\Delta I_2=1)\rightarrow 1$. Then, the number of zeros and ones in the produced bitstring is simply related to I_1-1 and I_2-1, respectively. Thus, one could compare the behavior of the function Δs with corresponding ones for different dynamical models or with randomly generated sequences.

Third, the statistical factors in the results of Eqs. (61), (62), encoding the dynamics, are nicely factorized from the quantities incorporating the evolving observables I_1 and I_2; an analogous factorization is seen in Eq. (35). We believe that the very simple structure of our results is indeed of a more general kind and similarly will be encountered, whenever one succeeds in identifying pairs of dichotomous observables, which somehow reflect the symmetry of the system.

Furthermore, the generalization involving n-tuples of exclusive observables can be worked out along these lines.

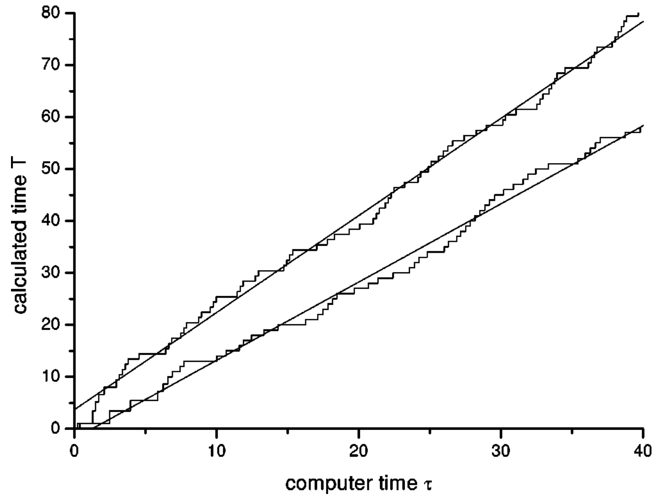

FIG. 4. The reparametrization invariant ''time'' T, from Eq. (64) with constant prefactor set to one, as a function of (computer) proper time τ for two different sets of initial conditions: $\phi_{1,2}^{(0)}=0$ (always), $\pi_1^{(0)}=5$, $\pi_2^{(0)}=\sqrt{17}$ (upper curve) and $\pi_1^{(0)}=3$, $\pi_2^{(0)}=\sqrt{5}$ (lower curve); interaction parameters: $\omega=\Omega=3$; window parameter: $\delta=1/3$. Lines from linear fits.

Coming back to the construction of a reparametrization invariant ''time'' T, we now set

$$\Delta T\equiv\frac{N\Delta s}{\sqrt{H_{int}}}, \quad (64)$$

with $N\equiv I_1+I_2$, i.e. stretching the time with an extra factor N, in order to compensate for the aging of the measure Δs, which we mentioned. Furthermore, we divide out the square-root-of-energy factor, which is determined by the initial conditions. This results in a scaling with energy, similarly as in Sec. III A. At high energy H_{int}, Eq. (38), equals (two times) the kinetic energy, which might seem more appropriate here. However, it is by itself not a reparametrization invariant quantity in the present interacting case.

We calculate the ''time'' passed by accumulating the time steps ΔT. The constant, though energy dependent, overall prefactor contained in the result of Eq. (64) is set equal to one, recalling that the Fisher-Rao metric which enters here, see Eqs. (59)–(61), is unique only up to a constant factor. Figure 4 shows the resulting behavior of the parametrization invariant ''time'' T as a function of the proper time τ, Eq. (36), which is also incorporated in the present numerical simulations. For the two initial conditions, differing in the initial velocities, we observe the expected scaling with energy. Furthermore, while the size of individual fluctuations is comparable to what was found in the noninteracting case of Fig. 2, we notice here characteristic deviations from the average linear dependence over longer proper time scales. Such nonlinear behavior becomes even more prominent in Fig. 5, with long time excursions.

In any case, this demonstrates the existence of a monotonous ''time'' function also for the present more strongly irregular system, as compared to the ergodic (but integrable) one of Sec. III A. Presumably, an approximately linear de-

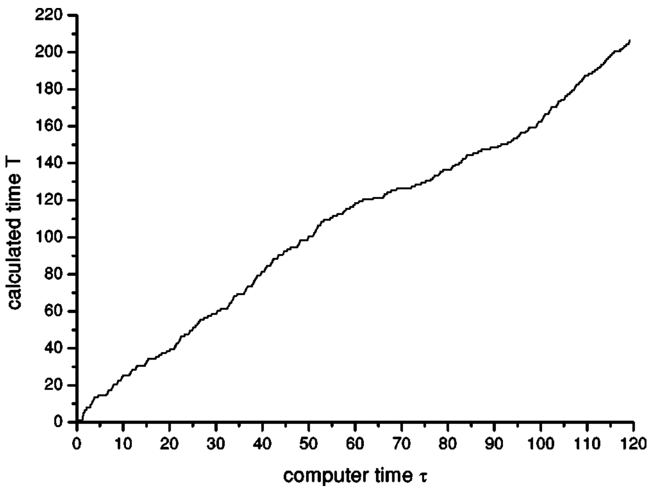

FIG. 5. Same as Fig. 4 (upper curve), illustrating nonlinear long-time behavior.

pendence will only become recognizable for much longer runs, i.e. for larger incident numbers. On the other hand, of course, the statistical construction of T based on quasilocal observables could be refined by considering a higher-dimensional state space from the outset, cf. Eq. (37), i.e. incorporating more observables.

This completes our discussion of two examples of the construction of a coarse-grained parametrization invariant evolution parameter.

IV. STROBOSCOPIC QUANTIZATION?

It seems worthwhile to draw attention to a possible connection between the present subject and the *deterministic classical systems which are effectively quantum mechanical*, as recently discussed by 't Hooft; see Ref. [13] and further references therein.

The total number of incidents obtained from our exclusive observables always increases by one. These are the "ticks of the clock," numbered by $n\in\mathbf{N}$. However, via the constructed "time" sequence $\{T_n\}$, the proper times $\tau(T_n)$ vary irregularly, in general, due to the quasi-periodic internal motion. Therefore, the free external motion, $\vec{q}_\parallel(T)\equiv\vec{q}_\parallel(\tau(T))=\vec{q}_\parallel^{(0)}+\tau(T)\vec{p}$, shows stochastic behavior in terms of T. We emphasize that $\tau(T)$ here is *not* directly related to $\tau(t)$ introduced in Eq. (36).

Let us assume that we could exhaust all possible *invariant quasilocal observables* and cannot further resolve the "time between the incidents" marking T and $T+\delta T$.

That there arises a discreteness in the optimal "time" sequence generated in this way in a generic nonintegrable system [9] seems plausible for two reasons. First, by realistically restricting the observables to be localized near to the observer, we lose information about the full trajectories in coordinate space. Second, by insisting on gauge invariant observables, we lose the possibility to make use of additional momentum space observables (except for a possible limited number of constants of motion), or higher proper time derivatives, in order to reconstruct the full trajectories and a smooth evolution parameter based on them.

Anyhow, for given initial conditions, the incident counts as well as the coordinates $\vec{q}_\parallel(T_n)$ present discrete physical data which are deterministically determined in our model. (The constant orthogonal components are of no interest here.) Then, the following question arises: can we still give an invariant meaning to and predict the future of the free external motion, instead of letting the system evolve and measuring? In general, the answer is "no," since neither the future discrete values of T, see Eq. (64), for example, nor the value of τ at those future instants will be predictable.

However, motivated by the two practical examples of Sec. III, we introduce the notion of a well-behaved "time," which will be sufficient to maintain some predictability indeed.

Let a *well-behaved "time" T* be such that the one-dimensional motion of a free particle, co-ordinated in a reparametrization-invariant way by the sequence $\{q(T_n)\}$, is of limited variation:

$$\begin{aligned}&\{T_n\}\text{ is well-behaved}\\ &\Leftrightarrow\exists\,\overline{\tau},\Delta\tau:\overline{\tau}n-\Delta\tau\leqslant\tau(T_n)\leqslant\overline{\tau}n+\Delta\tau,\quad\forall n\in\mathbf{N}\\ &\Leftrightarrow\exists\,\overline{q},\Delta q:\overline{q}n-\Delta q+q^{(0)}\leqslant q(T_n)\leqslant\overline{q}n+\Delta q\\ &\qquad+q^{(0)},\quad\forall n\in\mathbf{N},\end{aligned}\tag{65}$$

with $\overline{q}\equiv\overline{\tau}p^{(0)}$, $\Delta q\equiv\Delta\tau p^{(0)}$, and where $q^{(0)},p^{(0)}$ denote the initial data. This is the next best we can expect from a reasonable "time," motivated by Newtonian physics here. It is important to realize that there is no hope for continuity or periodicity, since our construction of T is based generally on the ergodicity of a quasi-periodic process and related Poincaré sections.

Assuming a well-behaved "time," we see that the sequence of points $\{q(T_n)\}$ can be mapped into a regular lattice of possibly overlapping cells of size $2\Delta q$ and spacing $\overline{q}$. "As the clock ticks," i.e. $n\rightarrow n+1$, the particle moves from one cell to the next. On the average this takes a proper time $\overline{\tau}$ and physical "time" $\overline{T}$. Here we assume for convenience that $\{T_n\}$ itself is of limited variation, $\exists\,\overline{T},\Delta T:T_n=\overline{T}n\pm\Delta T$, $\forall n\in\mathbf{N}$. In the simplest example, in Sec. III A, we have $\overline{T}=\delta^{-1}(\pi_1^{(0)}+\pi_2^{(0)})^{-1}$ and $\Delta T=0$, showing a dependence on the initial momenta and the scale of localization of the observed incidents.

Thus, for a well-behaved "time" sequence (of limited variation), we have mappings from "clock ticks" to ("time" as well as) space intervals. We identify the space intervals, containing the respective $q(T_n)$, with *primordial states*:

$$|n)\equiv[\overline{q}n-\Delta q+q^{(0)},\overline{q}n+\Delta q+q^{(0)}].\tag{66}$$

In order to handle the subtle limit of an infinite system, we impose reflecting boundary conditions and distinguish left- and right-moving states. Finally, then, the evolution is described by the deterministic rules:

$$n\rightarrow n+1,\quad n\in\mathbf{N},\tag{67}$$

$$|n)\to|n+1), \quad -N+1\le n\le N-1, \tag{68}$$

$$|N)\to|-N+1), \tag{69}$$

with $2N$ states in all, $|-N+1),\ldots,|N)$; states with a negative (positive) label correspond to left- (right-) moving states, according to negative (positive) particle momentum. Reaching the states $|0)$ or $|N)$, the particle changes its direction of motion.

The rules (67)–(69) can altogether be represented by a unitary evolution operator:

$$U(\delta T=\bar{T})\equiv e^{-iH\bar{T}} = e^{-i\pi(N+1/2)/N}\begin{pmatrix} 0 & & & & 1\\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & \ddots & \ddots & \\ & & & 1 & 0\end{pmatrix}, \tag{70}$$

which acts on the $2N$-dimensional vector composed of the primordial states; the overall phase factor is introduced for later convenience. Here $\bar{T}$ is the natural scale for the Hamiltonian H.

In the following analysis we apply 't Hooft's method, who considered the discrete and strictly periodic motion of a classical particle on a circle, introducing a Hilbert space based on the primordial states [13]. The evolution operator turns out to be diagonal with respect to the discrete Fourier transforms of the states $|n)$. We define the basis functions:

$$\langle k|n)\equiv f_k(n)\equiv(2N)^{-1/2}\exp\left(\frac{i\pi kn}{N}\right). \tag{71}$$

They present a complete orthonormal basis, since $\Sigma_{k=-N+1}^{N}(n'|k\rangle\langle k|n)=\delta_{n'n}$, with $(n|k\rangle\equiv\langle k|n)^*$, and noting that $f_k(n)=f_n(k)$. This yields indeed

$$\langle k'|U|k\rangle=\delta_{k'k}\exp\left(-i\pi\left[N+\frac{1}{2}-k\right]/N\right)\Leftrightarrow\langle m'|U|m\rangle = \delta_{m'm}\exp\left(-\frac{2\pi i}{2s+1}\left[s+m+\frac{1}{2}\right]\right), \tag{72}$$

where the equivalence follows from $2s+1\equiv 2N$ and relabeling and replacing the states $|k\rangle$ by states $|m\rangle$, with $m\equiv -k+1/2$ and $-s\le m\le s$. The phase factor of Eq. (70) contributes the additional terms which assure a positive definite (bounded) spectrum similar to the harmonic oscillator case.

Recalling the algebra of $SU(2)$ generators, and with $S_z|m\rangle=m|m\rangle$ in particular, we obtain the Hamiltonian:

$$H=\frac{2\pi}{(2s+1)\bar{T}}\left(S_z+s+\frac{1}{2}\right), \tag{73}$$

i.e., diagonal with respect to $|m\rangle$ states of the half-integer representations determined by s.

Continuing with standard notation, we have $S^2\equiv S_x^2+S_y^2+S_z^2=s(s+1)$, which suffices to obtain the following identity:

$$H=\frac{2\pi}{(2s+1)^2\bar{T}}\left(S_x^2+S_y^2+\frac{1}{4}\right)+\frac{\bar{T}}{2\pi}H^2. \tag{74}$$

Furthermore, using $S_\pm\equiv S_x\pm iS_y$, we introduce coordinate and conjugate momentum operators:

$$q\equiv\frac{1}{2}(aS_-+a^*S_+), \quad p\equiv\frac{1}{2}(bS_-+b^*S_+), \tag{75}$$

where a and b are complex coefficients. Calculating the basic commutator with the help of $[S_+,S_-]=2S_z$ and using Eq. (73), we obtain

$$[q,p]=i\left(1-\frac{\bar{T}}{\pi}H\right), \tag{76}$$

provided we choose $\Im(a^*b)\equiv -2(2s+1)^{-1}$. Incorporating this choice, we calculate

$$S_x^2+S_y^2=\frac{(2s+1)^2}{4}\left(|a|^2p^2+|b|^2q^2-(\Im a\cdot\Im b+\Re a\cdot\Re b)\times\{q,p\}\right). \tag{77}$$

In order to obtain a reasonably simple Hamiltonian in the infinite system limit ($s\to\infty$), we finally choose

$$a\equiv i\sqrt{\frac{\bar{T}}{\pi}}, \quad b\equiv\frac{2}{2s+1}\sqrt{\frac{\pi}{\bar{T}}}. \tag{78}$$

Then, defining $\omega\equiv 2\pi/(2s+1)\bar{T}$, the previous Eq. (74) becomes

$$H=\frac{1}{2}p^2+\frac{1}{2}\omega^2q^2+\frac{\bar{T}}{2\pi}\left(\frac{1}{4}\omega^2+H^2\right), \tag{79}$$

showing a nonlinearly modified harmonic oscillator Hamiltonian at this stage, which corresponds to the result of Ref. [13].

Following our construction of the reparametrization-invariant "time," we presently keep $\bar{T}$ finite while considering the infinite system limit. Thus, for $s\to\infty$, we have $\omega\to 0$ and obtain

$$H=\frac{\pi}{\bar{T}}\left(1-\left(1-\frac{\bar{T}}{\pi}p^2\right)^{1/2}\right), \tag{80}$$

which has the low-energy limit $H\approx p^2/2$. On the other side, the energy is bounded from above by $\pi/\bar{T}$, since we have

$(\overline{T}/\pi)p^2=4(2s+1)^{-2}S_x^2$ ($\leqslant 1$, for $s\to\infty$, when diagonalized).

Interestingly, towards the upper bound the violation of the basic quantum mechanical commutation relation, as calculated in Eq. (76), becomes maximal, i.e., there q and p commute and behave like classical observables.

We conclude here with calculating the matrix elements of the operators q and p between primordial states. Relating the primordial states $|n)$ to the $|m\rangle$ states with the help of Eq. (71) and using $S_\pm|m\rangle=[s(s+1)-m(m\pm1)]^{1/2}|m\pm1\rangle$, we obtain

$$\begin{aligned} m(n'|S_-\pm S_+|n) &= \frac{1}{2s+1}\left(\exp\left(-\frac{2\pi i}{2s+1}n\right)\pm\exp\left(\frac{2\pi i}{2s+1}n'\right)\right) \\ &\quad\times\sum_{m=-s}^{s}\exp\left(\frac{2\pi i}{2s+1}(n'-n)(m-1/2)\right) \\ &\quad\times[s(s+1)-m(m+1)]^{1/2}. \end{aligned} \tag{81}$$

For large s the summation is replaced by an integral. Then, using Eqs. (75) and (78), the results are ($s\to\infty$)

$$(n'|q|n)=\frac{1}{2}\sqrt{\pi\overline{T}}\,\frac{J_1(\pi(n'-n))}{n'-n}\,\frac{n'+n}{2}, \tag{82}$$

$$(n'|p|n)=\frac{1}{2}\sqrt{\frac{\overline{T}}{\pi}}\,\frac{J_1(\pi(n'-n))}{n'-n}, \tag{83}$$

where J_1 denotes an ordinary Bessel function of the first kind. Thus, neither the position nor the momentum operator is diagonal in the basis of the primordial states.

In any case, we find here once more an emergent quantum model based on a deterministic classical evolution, similar to that in Ref. [13].

The interesting feature presently is that the construction of a reparametrization-invariant "time" based on quasilocal observables naturally induces the particular stochastic features in the behavior of the external particle motion. The remaining predictable aspects of its motion can then most simply be described by a unitary discrete-time quantum mechanical evolution.

V. CONCLUSIONS

Our construction of a reparametrization-invariant "time" is motivated by the observation that "time passes" when there is an observable change, which is localized with the observer. More precisely, necessary are incidents, i.e. observable unit changes, which are recorded, and from which invariant quantities characterizing the change of the evolving system can be derived.

A basic ingredient is the assumption of ergodicity, such that the system explores dynamically the whole allowed energy surface in phase space. Generally, then there will be sufficiently frequent usable incidents next to the observer.

We illustrate this in Sec. III with a simple timeless model of a nonrelativistic particle moving classically in five dimensions, of which two internal ones are compactified to form a torus. Defining suitable window functions, localized observations of incidents correspond here to the trajectory passing by the window. Thus, the incidents reflect properties of the dynamics with respect to (subsets of) Poincaré sections. Roughly, the passing "time" corresponds to the observable change there.

In the first example in Sec. III A the dynamics is very simple and even integrable [9]. We know that the system is ergodic in this case and use this fact in order to calculate statistically, based on quasilocal incident counts, the invariant path length run by the system. This provides an invariant measure of "time." We find that the proper time is linearly related to this, however, subject to stochastic fluctuations.

In the second example, Sec. III B, we have additional interactions which mix the trajectories in a much more complicated way. We assume that the system is ergodic. Our statistical formalism then allows us to measure the change of the phase space distribution which is updated by the information coming from the incident counts. An essential ingredient is the Fisher-Rao information metric [10–12]. This calculation again provides an invariant "time" function, which behaves qualitatively similar to the previous example, however, with characteristic long-time deviations from an average linear relation to proper time.

It will be interesting to generalize the present model to relativistic systems, as well as to generalize the statistical formalism. A related question is as follows: which are the (discrete) limitations of such a construction of "time" based on localized observables, i.e. how close can one get to a linearly flowing Newtonian time, in a generic nonintegrable system?

Finally, we pointed out in Sec. IV a possible connection of the present subject to the study of quantum mechanical systems which have an underlying deterministic classical dynamical model [13]. Indeed, we find that for certain "well-behaved time sequences" the remaining deterministic aspects of the induced stochastic classical motion of the (in our model) external particle can be most simply described by a discrete-time quantum mechanical Schrödinger evolution. This "stroboscopic quantization" may arise under more general circumstances, if one insists on constructing by statistical means an invariant "time" from localized observables.

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