

Dynamics of spherically symmetric spacetimes: Hydrodynamics and radiation

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Using the 3+1 formalism of general relativity the equations governing the dynamics of spherically symmetric spacetimes with arbitrary sources are obtained. The case of a perfect fluid accompanied by a flow of interacting massless or massive particles (e.g., neutrinos) which are described in terms of relativistic transport theory is then specialized. I focus on three types of gauges: (1) the isotropic-maximal gauge, (2) radial-polar gauge, and (3) isotropic-polar gauge.

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I. INTRODUCTION

One of the most fascinating phenomena in gravitational physics is that of gravitational collapse. Notably, the gravitational collapse of astrophysical bodies culminating in the formation of black holes.

The historical controversy on the final fate of gravitational collapse of compact objects such as white dwarfs and neutron stars raised by the discovery of maximum mass limits and the subsequent stability analysis led to finding, at least for the most ideal configurations, definite answers and concrete predictions depending on the initial conditions and the equation of state of matter (see Refs. [1,2] for a review). The simplest situation describing the gravitational collapse ending in black hole formation is that of a spherical ball of pressureless and homogeneous fluid (the well known Oppenheimer-Snyder dust collapse [3]). The solution shows the appearance of an *event horizon* revealing thus the formation of a black hole after a finite proper time.

Since that pioneering investigation, much has been done for more complicated and realistic initial configurations. On one hand, the accelerated development in the area of computation and the recent advances on the numerical analysis of Einstein's equations have made possible computing the dynamics of rather complex spacetimes faster and for longer term evolutions. On the other hand, the advances in particle and nuclear physics have led to better knowledge of the conditions of matter at high densities, providing then more realistic models for matter in cores and neutron stars.

Most of the recent analyses of gravitational collapse leading to black hole formation have been done in spherical symmetry and following a more "modern" point of view in light of the 3+1 formulation of general relativity and other formulations better adapted to numerical stability (see Refs. [4,5] and references therein). Among these investigations, there is the one of Shapiro and Teukolsky [6] who studied the collapse of polytropes by imposing isotropic (spatial) coordinates and maximal slicing, with initial conditions provided by the Tolman-Oppenheimer-Volkoff equation of hydrostatic equilibrium. Later, a similar study was performed by Schinder, Bludman, and Piran [7] in comoving coordinates with a polar slicing, and byourgoulhon [8,9] in radial co-

ordinates and polar slicing, this latter improving previous analysis by the incorporation of realistic equations of state for the nuclear matter.

The effects of different gauges on the time evolution have been exhibited for two of the most popular choices (isotropic or radial coordinates with maximal or polar slicings) by comparing with the analytical solution of Oppenheimer-Snyder [9–11]. For the time being, the discussion of gauges is postponed to Sec. VII.

The analysis of gravitational collapse of compact stars and iron cores would not be complete if the influence of neutrinos were not taken into account. This has been recognized by a long list of authors since the precursor investigations of Colgate and White [12], and May and White [13], who analyzed the effect of neutrinos in supernova explosions. Later Wilson [14] performed full general relativity computations with a neutrino flow described in terms of the relativistic Boltzmann equation. Wilson's analysis included electron and muon massless neutrinos assuming that the corresponding antineutrinos contributed in the same basis. The interaction of neutrinos with the star's fluid was described by an opacity function. Unlike previous studies, Wilson's found that the heat conduction by neutrinos is not sufficient to eject any material from a collapsing star. In all those studies black hole formation was not analyzed but only evolution configurations terminating in stable states corresponding to white dwarfs or neutron stars.

An updated analysis was carried out by Mayle, Wilson, and Schramm [15] using a Boltzmann code and for a large set of mass configurations. Neutrino signals from various species were analyzed within time scales of ~ 1 s after the supernova explosion. Previously Saenz and Shapiro [16] had computed a nonspherical quasi-Newtonian collapse accompanied by neutrino and gravitational radiation.

Burrows and coworkers have also analyzed during the last 20 years the mechanism of type II supernova (SNII) and the role of neutrinos (see [17,18] and references therein). Among these investigations, one finds an interesting model of long term neutrino emission from the hot proton-neutron star phase to the final outcome of a stable cold neutron star [19].

Many other recent investigations have confirmed and improved in several aspects previous findings on SNII (see [20–25] and references therein). For the case of a core collapse leading to a black hole, two scenarios are recognized [22]. One called *early* black hole formation which is generi-

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cally associated with accreting protoneutron stars which form from the collapse of degenerate cores of massive stars [25,26]. The accretion of some tenths to one solar mass can last a second, and the exceeding of the maximum mass drives the protoneutron star into a black hole collapse in a typical time scale of ~ 0.5 ms. In this scenario the neutrino signal is abruptly cut off after the black hole forms, and the typical neutrino luminosities prior to the cutoff are $\sim 10^{52}$ erg/s per flavor.

The second scenario called *late* black hole formation typically arises by a softening of the high-density equation of state (EOS) of the protoneutron star [27–30]. The phase transition from the neutron star matter to a more exotic state which includes kaon condensates [31,32] or hyperon condensates [33] can lower the maximum allowable mass to $\sim 1.5M_\odot$ [28,30], driving thus a stable protoneutron star to an unstable regime and finally to a collapse into a black hole. This kind of core collapse can last ~ 10 s before the cutoff and the luminosity of neutrinos is ten times lower than the luminosity of the early case.

It is encouraging to note that for a SNII at a distance of ~ 10 kpc which explodes within the early scenario, SuperKamiokande can probe $\bar{\nu}_e$ masses down to 1.8 eV by comparing the arrival times of high and low energy neutrinos within the reaction $\bar{\nu}_e + p \rightarrow e^+ + n$ in a Cherenkov detector (see Ref. [22] for details).

In fact the very recent announcement on the measurements of solar neutrinos from the decay of ^8B by the Sudbury Neutrino Observatory (SNO) [34] via charged current interactions and by the elastic scattering of electrons reveals that neutrinos could be changing flavor as they travel from the sources to the Earth. This discovery if confirmed could corroborate the oscillating behavior of neutrinos and therefore their massive nature. The fluxes measured of the different flavors are in close agreement with the predictions of the solar models. The SNO experiment then implies that the upper limit of the mass squared difference between the ν_e and the ν_τ or ν_μ is less than 10^{-3} eV² [34]. This result when combined with the current bounds on m_{ν_e} of 2.8 eV and $\Delta m_{\nu_\mu\nu_\tau}^2$ (assuming neutrino oscillations) provides a limit for the sum of the masses of the three neutrino species in the range [0.05,8.4] eV [34].

One proposal to measure the ν_τ and ν_μ masses indirectly and that can corroborate the SNO findings is the one which uses a time-of-flight technique [22] for neutrinos emitted in the early black hole formation scenario discussed above. The point is that if neutrinos are massive then there is a delay (relative to a massless neutrino) in the cutoff of the neutrino signal as measured on Earth after the black hole forms, and it is given by $\Delta t \sim (m_\nu/E_\nu)^2$ for distances of ~ 10 kpc. This delay can affect the event rate measured in a detector. The conclusion is that assuming luminosities $L \sim 10^{52}$ erg/s per flavor at the cutoff time, SuperKamiokande can probe e -neutrino masses as small as 1.8 eV for $T_{\nu_e} \sim 3.5$ MeV, whereas the OMNIS [35] or SNO detector can detect m_{ν_μ, ν_τ} masses as small as 6 eV for $T_{\nu_\mu, \nu_\tau} \sim 8$ MeV.

A collapse scenario which is rather different from the

above consists in the accretion of matter by an old neutron star near the maximum mass limit. Gourgoulhon and Haensel [36] have analyzed the neutrino emission during the collapse to a black hole within this scenario via nonequilibrium β processes, assuming that the nuclear matter is transparent for neutrinos (i.e., opacities were neglected). Instead of using neutrino transport, a *regularized* geometrical-optics model adapted to massless neutrinos was adopted. This model was thus intended to provide upper bounds in the neutrino burst. The collapse lasts typically a millisecond (the time that the black hole formation takes place), and in the most favorable conditions the total energy of $\bar{\nu}_e$ and $\bar{\nu}_\mu$ antineutrinos is $\sim 10^{51}$ erg, while the energy of the corresponding neutrinos is several orders of magnitude lower. This is even lower for the ν_τ and $\bar{\nu}_\tau$ neutrinos. The main conclusion is that a collapse of this kind at a distance of ~ 10 kpc would be undetectable by the current neutrino detectors.

Finally, another scenario which has been analyzed in the past is the dynamics of collisionless gas of particles which mimic spherical star clusters, and the possibility of a cluster collapse into a supermassive black hole [37–42]. The motivation was to provide a theoretical description for the formation of supermassive black holes that could exist in the centers of galaxies. Recently, a similar study which included gamma-ray bursts, is the one analyzed by Linke *et al.* [43], where the collapse of supermassive stars ($M \sim 10^5 M_\odot - 10^9 M_\odot$) with emission of thermal neutrinos is considered. In that work, the spacetime is foliated by outgoing null hypersurfaces rather than using a 3+1 foliation of spacetime.

In view of the different scenarios of gravitational collapse available today and the miscellaneous predictions within each of them it is worth pursuing the investigations along these lines. Only in this way there will be at hand a large set of models which the forthcoming (e.g., [35]) and recent observations [34] will validate or rule out.

Although the paper is written in the same spirit of various papers which deal with the system of equations Einstein-hydrodynamics-Boltzmann, several aspects distinguish it from them. For instance, most formalisms treat neutrinos as massless particles, except perhaps the one of Harleston and collaborators [44,45]. Here massive particles are considered from the onset and previous equations are recovered in the massless limit. The relativistic Boltzmann equation is written in terms of 3+1 variables for generic spacetimes. This had been done in the past only in spherical symmetry. Therefore, the relativistic Boltzmann equation presented here is coupled from the onset to the 3+1 Einstein's equations. That is, the curvature effects appear in terms of the lapse, the shift, the extrinsic curvature, and the three-metric. The hydrodynamic equations are derived also in the context of the 3+1 formalism and they couple to the neutrinos via collision integrals. In particular the equation for the velocity field of the fluid is written in several forms each of which is useful whether one uses different numerical methods. In this regard, a general relativistic Euler equation is presented using the tetrad formalism. Its quasi-Newtonian form allows an easy interpretation of several terms, and reduces to well known equations in various limits. Such a Euler equation turns to be better

adapted for spectral methods than the equation for the momentum current density [8,9]. The system of equations are then specialized for spherical symmetry using three different gauges: (1) isotropic-maximal gauge, (2) radial-polar gauge, and (3) isotropic-polar gauge, where isotropic and radial stand for the spatial coordinate choice, and maximal and polar refer to the slicing condition.

This is the first of a series of papers where the gravitational collapse of various kinds of matter will be analyzed.

The paper is organized as follows. Section II presents succinctly the 3+1 formalism of general relativity rather more to fix the notation than to give a detailed description. In Sec. III the case of two interacting sources of matter is specialized: a perfect fluid and a flow of relativistic particles described in terms of relativistic transport theory. Section IV treats the relativistic transport theory. Section V deals with thermodynamics. In Sec. VI spherical symmetry is considered and in Sec. VII three gauge conditions are analyzed and discussed in light of the previous studies. Finally some remarks and the plans for the forthcoming investigations along this line are found at the end.

II. THE 3+1 FORMALISM OF GENERAL RELATIVITY

One of the most popular reformulations of general relativity when tackling numerical problems is the 3+1 formulation, a particular form of which is the Arnowitt-Deser-Misner (ADM) formalism [46]. We shall not enter into the details of the derivation of the equations (see Refs. [47–50]) but rather discuss the general idea in order to fix the notations.

The main idea is as follows: under general assumptions (see [47,49] and references therein for details) a globally hyperbolic spacetime $(M^4, g_{\mu\nu})$ can be foliated by a family of space-like hypersurfaces Σ_t (Cauchy surfaces). Each hypersurface represents a Riemannian sub-manifold (M^3, h_{ij}) endowed by an *induced metric* h_{ij} (the three-metric). It is then assumed a local coordinate system (t, x^i) for the spacetime, the spatial part (x^i) represents a local coordinate system for Σ_t , while t is a global time function that parametrizes Σ_t . The embedding of Σ_t in spacetime is completed by the *extrinsic curvature* of Σ_t . This is defined by

$$K_{\mu\nu} := -\frac{1}{2}\mathcal{L}_n h_{\mu\nu}, \quad (1)$$

where \mathcal{L}_n stands for the Lie derivative along the normal n^μ to Σ_t and $h_{\mu\nu} := g_{\mu\nu} + n_\mu n_\nu$. The vector field n^μ is time-like ($n^\mu n_\mu = -1$) and the convention used for its components with respect to the coordinate base adapted to the spacetime foliation is as follows:

$$n^\mu = (N, N^i). \quad (2)$$

This convention means that n^μ points towards the *future*. Since n^μ is a unit time-like vector, it is customary to interpret n^μ as the four-velocity of the so-called Eulerian observer \mathcal{O}_E . The *scalar* quantity N (the lapse function) represents thus the rate at which \mathcal{O}_E sees the flow of its proper-time as compared with the intervals between two

neighboring hypersurfaces Σ_t and Σ_{t+dt} . The three-vector N^i (the *shift-vector*), represents the coordinate three-velocity at which the Eulerian observer moves with respect to the coordinates (t, x^i) . In this way, the four-metric reads

$$ds^2 = -(N^2 - N^i N_i) dt^2 - 2N_i dt dx^i + h_{ij} dx^i dx^j, \quad (3)$$

noting that in other studies the shift vector is taken with the opposite sign.

A useful formula for K_{ij} obtained from Eq. (1) is

$$\begin{aligned} K_{ij} &= -\nabla_i n_j = -N \Gamma_{ij}^l \\ &= -\frac{1}{2N} \left(\frac{\partial h_{ij}}{\partial t} + {}^3\nabla_j N_i + {}^3\nabla_i N_j \right), \end{aligned} \quad (4)$$

where ${}^3\nabla_j$ stands for the covariant derivative compatible with h_{ij} . This is to be regarded as an evolution equation for the three-metric h_{ij} .

The trace of the extrinsic curvature is simply given by

$$K := -\nabla_\alpha n^\alpha. \quad (5)$$

Another useful quantity is the acceleration of \mathcal{O}_E given by

$$a^\mu := n^\nu \nabla_\nu n^\mu = {}^3\nabla^\mu [\ln N], \quad (6)$$

which allows for the lapse interpretation of the acceleration potential for \mathcal{O}_E [48].

The orthogonal decomposition of the energy-momentum tensor in components tangent and orthogonal to Σ_t leads to [48]

$$T^{\mu\nu} = S^{\mu\nu} + J^\mu n^\nu + n^\mu J^\nu + E n^\mu n^\nu. \quad (7)$$

The tensor $S^{\mu\nu}$ is symmetric and corresponds to the *three-energy-momentum tensor*; J^μ is the *three-momentum density vector*, and E is the *total energy density* measured by the Eulerian observer \mathcal{O}_E . Both $S^{\mu\nu}$ and J^μ are orthogonal to n^μ . For the specific applications considered here, $T^{\mu\nu}$ will be the total energy-momentum tensor of matter which can be composed by the contribution of different types of sources:

$$T^{\mu\nu} = \sum_i T_i^{\mu\nu}. \quad (8)$$

This means that

$$E = \sum_i E_i, \quad J^\mu = \sum_i J_i^\mu, \quad S^{\mu\nu} = \sum_i S_i^{\mu\nu}. \quad (9)$$

The projection of Einstein equations $R_{\mu\nu} = 4\pi G_0(2T_{\mu\nu} - T_{\alpha\beta} g_{\mu\nu})$ in the directions tangent and orthogonal to Σ_t followed by the use of the Gauss-Codazzi-Mainardi equations leads to the 3+1 form of Einstein equations:

$${}^3R + K^2 - K_{ij} K^{ij} = 16\pi G_0 E, \quad (10)$$

known as the Hamiltonian constraint,

$${}^3\nabla_l K_i^l - {}^3\nabla_i K = 8\pi G_0 J_i, \quad (11)$$

known as the *momentum constraint* equations.

Finally, the *dynamic* Einstein equations read

$$\begin{aligned} & \partial_t K_j^i + N^l \partial_l K_j^i + K_l^i \partial_j N^l - K_j^l \partial_l N^i + {}^3\nabla^i {}^3\nabla_j N \\ & - {}^3R_j^i N - N K K_j^i \\ & = 4\pi G_0 N [(S - E) \delta_j^i - 2S_j^i], \end{aligned} \quad (12)$$

where $S = S_l^l$ is the trace of $S^{\mu\nu}$, and all the quantities written with a “3” index refer to those computed with the three-metric h_{ij} . Moreover, under the 3+1 formalism tensor quantities tangent to Σ_t use the three-metric to raise and lower their spatial indices. Equations (4) and (12) are the set of the Cauchy-initial-data evolution equations for the gravitational field subject to the constraints Eqs. (10) and (11).

An evolution equation for the trace K is obtained by taking the trace in Eq. (12):

$$\begin{aligned} & \partial_t K + N^l \partial_l K + {}^3\Delta N - N({}^3R + K^2) \\ & = 4\pi G_0 N [S - 3E] \end{aligned} \quad (13)$$

where ${}^3\Delta$ stands for the Laplacian operator compatible with h_{ij} .

This can be simplified by using Eq. (10) to give

$$\partial_t K + N^l \partial_l K + {}^3\Delta N - N K_{ij} K^{ij} = 4\pi G_0 N [S + E]. \quad (14)$$

In addition to the gravitational field equations, there are the matter equations

$$\nabla_\mu T^{\mu\nu} = 0, \quad (15)$$

which according to Eq. (8), yields

$$\nabla_\mu T_\psi^{\mu\nu} = -\mathcal{F}^\nu, \quad (16)$$

where $T_\psi^{\mu\nu}$ is the energy-momentum tensor of certain fields that are collectively labeled by ψ and $\mathcal{F}_\nu := \nabla_\mu T_{\text{ext}}^{\mu\nu}$ are the “forces” exerted by the external fields (fields other than ψ). For instance, when considering a total energy-momentum tensor given by the combination of a perfect-fluid and a radiated flow of particles:

$$T^{\mu\nu} = T_{\text{PF}}^{\mu\nu} + T_{\text{R}}^{\mu\nu} \quad (17)$$

then $T_\psi^{\mu\nu} = T_{\text{PF}}^{\mu\nu}$ and $T_{\text{ext}}^{\mu\nu} = T_{\text{R}}^{\mu\nu}$; thus the energy-momentum tensor of the perfect-fluid alone will not conserve by separate; \mathcal{F}_ν will represent the “forces” of the radiated flow acting on the perfect fluid in form of *collisions*.

The energy-momentum conservation equations (16) can be written in 3+1 form as well. The projection of Eq. (16) along n^μ leads to the energy conservation equation,

$$\begin{aligned} & \partial_t E_\psi + N^l \partial_l E_\psi + \frac{1}{N} {}^3\nabla_l (N^2 J_\psi^l) \\ & = N (S_\psi^{ij} K_{ij} + E_\psi K) + N n_\nu \mathcal{F}^\nu. \end{aligned} \quad (18)$$

Explicitly $n_\nu \mathcal{F}^\nu = -N\mathcal{F}^t$.

On the other hand, the projection of Eq. (16) on Σ_t leads to the momentum conservation equation,

$$\begin{aligned} & \partial_t J_i^\psi + N^l \partial_l J_i^\psi + J_i^\psi \partial_t N^l + N ({}^3\nabla_l \psi S_i^l) \\ & = N K J_i^\psi - ({}_\psi S_i^l + E_\psi \delta_i^l) {}^3\nabla_l N - {}^3\mathcal{F}_i N, \end{aligned} \quad (19)$$

where

$${}^3\mathcal{F}_i = h_{i\mu} \mathcal{F}^\mu = -N_i \mathcal{F}^t + h_{ij} \mathcal{F}^j. \quad (20)$$

Tetrads

In many applications the use of tetrad components of tensors (hereafter physical components) are better adapted to a problem than the coordinate components. This will be the case when writing the equations of relativistic transport for the radiated flow and the equations of motion for the matter.

In the context of the 3+1 formalism the tetrad used here is the local tetrad of the Eulerian observer which is given by $\{n^\mu, e_{(i)}^j\}$, that is, by the time-like vector normal to Σ_t and by a triad on Σ_t . The best choice for a triad $e_{(i)}^{(l)}$ on Σ_t will depend on the particular coordinates used on the hypersurface Σ_t (see Sec. VI for the case of spherical symmetry).

In covariant notation a tetrad is given by

$$e_{(\mu)} = q_{(\mu)}^\nu \frac{\partial}{\partial x^\nu}, \quad (21)$$

where $\partial/\partial x^\nu$ denotes the coordinate basis of the spacetime, and $q_{(\mu)}^\nu$ are the *tetrad coefficients* that allow the normalization. For instance, it turns that $e_{(t)}^\mu \equiv n^\mu$.

The inverse relationship of Eq. (21) is given by

$$\frac{\partial}{\partial x^\mu} = e_{(\nu)}^{(\mu)} e_{(\nu)}, \quad (22)$$

where the coefficients $e_{(\nu)}^{(\mu)}$ are related to $q_{(\mu)}^\nu$ by the completeness relations $e_{(\nu)}^{(\alpha)} q_{(\beta)}^\nu = \delta_{(\beta)}^{(\alpha)}$, and $e_{(\nu)}^{(\alpha)} q_{(\alpha)}^\mu = \delta_\nu^\mu$.

A tetrad is not uniquely defined. The Lorentz invariance $\text{SO}(3,1)$ leaves the freedom on the choice of the six parameters that rotate and boost frames. As mentioned, a convenient choice is as follows:

$$e_{(t)}^{(i)} := -e_{(j)}^{(i)} N^j, \quad (23)$$

$$e_{(i)}^{(t)} := 0, \quad (24)$$

$$e_{(i)}^{(j)} = N, \quad (25)$$

$$h_{ij} = e_{(i)}^{(l)} e_{(j)}^{(l)}. \quad (26)$$

The inverse relations are

$$q_{(t)}^i = \frac{N^i}{N}, \quad (27)$$

$$q_{(i)}^t = 0, \quad (28)$$

$$q_{(i)}^l = N^{-1}, \quad (29)$$

$$h^{ij} = q_{(i)}^l q_{(l)}^j, \quad (30)$$

here $e_i^{(l)} q_{(l)}^j = \delta_j^i e_l^{(i)} q_{(j)}^l = \delta_{(j)}^{(i)}$.

This choice of a tetrad is compatible with the 3+1 decomposition of the four-metric (3), so that

$$ds^2 = e_\mu^{(\alpha)} e_\nu^{(\beta)} \eta_{(\alpha)(\beta)} dx^\mu dx^\nu, \quad (31)$$

with $e_\nu^{(\alpha)}$ given by Eqs. (23)–(26) and $\eta_{(\alpha)(\beta)}$ stands for the Minkowski metric.

Trivial Lorentz transformations of the chosen tetrad relates the Eulerian observer with other possible frames.

Finally, the transformation law for the components of tensors tangent to Σ_t from the coordinate base to the triad is as follows:

$$N^{(i)} = e_l^{(i)} N^l, \quad (32)$$

$$J_{(i)} = q_{(i)}^l J_l, \quad (33)$$

$$K_{(j)}^{(i)} = e_l^{(i)} q_{(i)}^m K_m^l, \quad (34)$$

$$S_{(j)}^{(i)} = e_l^{(i)} q_{(i)}^m S_m^l. \quad (35)$$

The inverse relationships are obtained from above in the obvious way. The triad indices [i.e., spatial indices within “()”] are raised and lowered with $\delta_{(j)}^{(i)}$ (i.e., the triad-covariant and triad-contravariant components of three-tensors are identical to each other). Four-tensor components transform in a similar way using the four-dimensional tetrad coefficients and $\eta_{(\mu)(\nu)}$ ($\eta^{(\mu)(\nu)}$) to lower (raise) indices.

The use of the tetrad formalism will be useful to recast the 3+1 matter equations as well as the relativistic Boltzmann equation (see Sec. IV) in a useful manner. For instance, Eq. (18) reads

$$\begin{aligned} \partial_{(t)} E_\psi + {}^3\nabla_{(i)} J_\psi^{(i)} - E_\psi K + 2J_\psi^{(i)} {}^3\nabla_{(i)} \nu \\ - {}_\psi S^{(i)(j)} K_{(i)(j)} = -\mathcal{F}^{(t)}, \end{aligned} \quad (36)$$

where $\partial_{(t)} = n^\mu \partial_\mu$.

On the other hand, the momentum conservation equation (19) can be written as

$$\begin{aligned} \partial_{(t)} J_\psi^{(i)} + {}^3\nabla_{(j)} S_\psi^{(i)(j)} + [S_\psi^{(i)(j)} + E_\psi \delta^{(i)(j)}] {}^3\nabla_{(j)} \nu - J_\psi^{(i)} K \\ + J_\psi^{(l)} (\mathcal{O}_{(i)(l)}^{(i)} - K_{(i)(l)}^{(i)}) = -{}^3\mathcal{F}^{(i)}. \end{aligned} \quad (37)$$

I remind that the covariant-derivative components with respect to a tetrad employs the four-Ricci rotation coefficients (RRC) as a connection. RRC are defined as follows:

$$\mathcal{O}_{(\beta)(\gamma)}^{(\alpha)} := e_\mu^{(\alpha)} q_{(\beta)}^\nu \nabla_\nu q_{(\gamma)}^\mu = q_{(\beta)}^\sigma e_\mu^{(\alpha)} (\partial_\sigma q_{(\gamma)}^\mu + q_{(\gamma)}^\lambda \Gamma_{\lambda\sigma}^\mu). \quad (38)$$

The three-RRC have an identical expression by restricting the above definition to pure spatial indices and using the three-covariant derivative. While the above definition re-

quires the Christoffel symbols, these can be avoided by using the representation of the RRC in terms of the *structure constants* [50–52].

Although the 3+1 Einstein equations can be written following a tetrad approach [50], for my purposes this will not be necessary and thus I shall not pursue the issue here.

III. PERFECT FLUIDS WITH SOURCES

For the specific applications I have in mind, a combination of a perfect-fluid and a radiated flow will be considered. Then in Eq. (17) I assume

$$T_{\text{PF}}^{\mu\nu} = (\rho + p) u^\mu u^\nu + p g^{\mu\nu}. \quad (39)$$

For the moment the form of the radiating part $T_{\text{R}}^{\mu\nu}$ is not specified. This will be treated in detail in Sec. IV.

The corresponding 3+1 matter variables of the fluid are

$$E_{\text{PF}} = (\rho + p) \Gamma - p, \quad (40)$$

$$J_{\text{PF}}^{(i)} = (E_{\text{PF}} + p) {}^3U^{(i)}, \quad (41)$$

$$S_{\text{PF}}^{(i)(j)} = (E_{\text{PF}} + p) {}^3U^{(i)} {}^3U^{(j)} + \delta^{(i)(j)} p, \quad (42)$$

$$S_{\text{PF}} = (E_{\text{PF}} + p) ({}^3U^{(i)})^2 + 3p, \quad (43)$$

$$\Gamma := -n_\mu u^\mu = u^{(t)} = [1 - ({}^3U^{(i)})^2]^{-1/2}, \quad (44)$$

where

$${}^3U^{(i)} := \frac{u^{(i)}}{\Gamma} = \frac{e_l^{(i)}}{N} (V^l - N^l) = \frac{1}{N} (V^{(l)} - N^{(l)}), \quad (45)$$

$$V^l := u^l / u^t. \quad (46)$$

The equation for conservation of energy (36) applied to a perfect fluid with sources then reads [50,52–54]

$$\begin{aligned} \partial_{(t)} E + {}^3\nabla_{(i)} [(E + p) {}^3U^{(i)}] + (E + p) [2 {}^3U^{(j)} a_{(j)} \\ - {}^3U^{(l)} {}^3U^{(j)} K_{(l)(j)} - K] \\ = -\mathcal{F}_{\text{R}}^{(t)}. \end{aligned} \quad (47)$$

On the other hand, the momentum conservation equation (37) applied to a perfect-fluid with sources can be written as a Euler equation for the velocity field. This reads [50,52–54],

$$\begin{aligned} \partial_{(t)} {}^3U^{(i)} + {}^3U^{(j)} {}^3\nabla_{(j)} {}^3U^{(i)} \\ = -\frac{1}{E_{\text{PF}} + p} [{}^3\partial^{(i)} p + {}^3U^{(i)} \partial_{(i)} p] \\ - a_{(j)} + {}^3U^{(i)} {}^3U^{(l)} (a_{(l)} - {}^3U^{(j)} K_{(l)(j)}) \\ - {}^3U^{(l)} (\mathcal{O}_{(i)(l)}^{(i)} - K_{(i)(l)}^{(i)}) + \frac{1}{E_{\text{PF}} + p} ({}^3U^{(i)} \mathcal{F}_{\text{R}}^{(i)} - \mathcal{F}_{\text{R}}^{(i)}), \end{aligned} \quad (48)$$

where

$$a_{(i)} = {}^3\nabla_{(i)}[\ln N] \quad (49)$$

are the physical components of Eq. (6). This Euler equation is the version in physical components (with the extra dissipative terms $\mathcal{F}_R^{(\mu)}$) of Eq. (2.29) of Ref. [54] (see also Ref. [55]).

Other formulations of general relativistic hydrodynamics that exploit the hyperbolic character of the equations have been analyzed recently (see Ref. [56] for a review). Some of them allow one to handle discontinuous solutions by the use of schemes based on high resolution shock capturing instead of introducing an artificial viscosity. These formulations are specially useful in the study of phenomena that lead to large density gradients like in type II supernova.

IV. RELATIVISTIC TRANSPORT THEORY

The aim is to study the phenomenon of relativistic transport of massive and massless particles (hereafter radiation) within a dense medium. The formalism developed by Lindquist [57] is followed closely, but the details will not be repeated here. The radiated particles of the specie ‘‘R’’ will be treated classically as point particles except when interacting with the dense medium. The interactions and its quantum mechanical effects will be ultimately translated as emission rates and opacity functions. The particles will be characterized thus by a four-momentum

$$p^\mu = \frac{dx^\mu}{d\lambda}, \quad (50)$$

where $d\lambda$ corresponds to an affine parameter (for massless particles) or to the proper-time per mass unit (for massive particles).

According to this formalism, one postulates the existence of a scalar function $F_R(x^\mu, p^\mu)$ (the *invariant* distribution function for the specie R) which is a function from the phase space coordinates (x^μ, p^μ) to the reals. Actually, since we will be interested in particles satisfying the mass shell condition

$$g_{\mu\nu}p^\mu p^\nu = -\tilde{m}, \quad (51)$$

where $\tilde{m} = 0, 1$ for massless or massive particles, respectively, F_R will be a function defined on the *reduced* phase-space. For instance, the set $\mathcal{P} := \{(x^\mu, p^\mu) : x^\mu \in M^4, p^\mu \in T_x^{M^4}, g_{\mu\nu}p^\mu p^\nu \leq 0, p^\mu \text{ future directed}\}$ is the one-particle phase-space (or *future-directed* tangent bundle \mathcal{B}) for particles of arbitrary (real-valued) masses m , and the reduced phase-space (or sphere bundle) \mathcal{P}_m (the mass shell) corresponds to the subspace of \mathcal{P} with tangent vectors p^μ of fixed length (it has dimension seven).

The invariant distribution function so introduced will be such that the number density four-vector and the energy-momentum tensor of the particles are, respectively [57],

$$j_R^\mu = \int p^\mu F_R(x^\mu, p^\mu) dP, \quad (52)$$

$$T_R^{\mu\nu} = \int p^\mu p^\nu F_R(x^\mu, p^\mu) dP, \quad (53)$$

where dP is the invariant volume of the momenta space on shell [57]:

$$dP = -N\sqrt{h} \frac{d^3p}{p_t}, \quad (54)$$

where $d^3p := dp^1 dp^2 dp^3$ represents a coordinate three-volume element and h is the determinant of the three-metric.

The distribution function F_R is related to the dimensionless distribution function \bar{F}_R by

$$F_R = \frac{g_R}{8\pi^3 \hbar^3} \bar{F}_R, \quad (55)$$

where g_R is the statistical weight of the particles of the specie R (e.g., $g_R = 1, 2$ for neutrinos and photons, respectively).

As in the case of the perfect fluid, it is useful to refer the components of four-momenta to a tetrad. In the context of the 3+1 formalism we have

$$e := p^{(t)} = Np^t, \quad (56)$$

$$p^{(i)} = e_i^{(i)}(p^i - p^t N^i), \quad (57)$$

with the inverse relationships given by

$$p^t = e/N, \quad (58)$$

$$p^i = e \frac{N^i}{N} + q_{(i)}^i p^{(i)}, \quad (59)$$

where $p^{(i)}$ are the physical spatial components of the four-momentum (i.e., the spatial physical components of the projection of p^μ onto Σ_t : ${}^3p^\mu := h_t^\mu p^\nu$). The ratio $p^{(i)}/e$ corresponds to the local velocity of particles measured by \mathcal{O}_E .

Introducing the magnitude of the three-momentum as

$$p^2 := p_{(i)} p^{(i)}, \quad (60)$$

it is easy to see that Eq. (51) simply becomes

$$e^2 = p^2 + \tilde{m}^2. \quad (61)$$

Here e is the energy (per mass-unit in the case of massive particles) as measured by the Eulerian observer. Therefore, from Eq. (54), one obtains,

$$dP = \sqrt{h} \frac{d^3p}{e \left(1 + \frac{p_{(i)} e_i^{(i)} N^i}{EN} \right)}, \quad (62)$$

where we used $p_t = e_i^{(i)} p_{(i)} = N p_{(t)} - p_{(i)} e_i^{(i)} N^i = -Ne - p_{(i)} e_i^{(i)} N^i$.

When changing variables in momentum space by using Eqs. (59) and (61) a straightforward manipulation leads to

$$d^3p = \frac{dp^{(1)}dp^{(2)}dp^{(3)}}{\sqrt{h}} \left(1 + \frac{P_{(l)}e_i^{(l)}N^i}{eN} \right), \quad (63)$$

where the relationship $\det[q_{(l)}^i] = 1/\sqrt{h}$ was used. Finally

$$dP = \frac{dp^{(1)}dp^{(2)}dp^{(3)}}{e}, \quad (64)$$

which has exactly the same form as in Minkowski spacetime.

The use of physical spherical variables in momentum space leads to

$$dP = p^2 dp d\Omega_p / e = p de d\Omega_p. \quad (65)$$

Indeed, this is a useful expression when dealing with spherical symmetry.

The use of tetrad components allows us to write Eqs. (52) and (53) as follows:

$$j_R^{(\mu)} = \int p^{(\mu)} F_R(x^\lambda, p^{(\lambda)}) dP, \quad (66)$$

$$T_R^{(\mu)(\nu)} = \int p^{(\mu)} p^{(\nu)} F_R(x^\lambda, p^{(\lambda)}) dP. \quad (67)$$

Therefore, the corresponding 3+1 matter variables are

$$E_R = \int e^2 F_R(x^\lambda, p^{(\lambda)}) dP, \quad (68)$$

$$J_R^{(i)} = \int e p^{(i)} F_R(x^\lambda, p^{(\lambda)}) dP, \quad (69)$$

$$S_R^{(i)(j)} = \int p^{(i)} p^{(j)} F_R(x^\lambda, p^{(\lambda)}) dP, \quad (70)$$

$$S_R = \int p^2 F_R(x^\lambda, p^{(\lambda)}) dP. \quad (71)$$

According to Eq. (9) the total 3+1 matter variables are

$$E = E_{\text{PF}} + E_R, \quad (72)$$

$$J^{(i)} = J_{\text{PF}}^{(i)} + J_R^{(i)}, \quad (73)$$

$$S^{(i)(j)} = S_{\text{PF}}^{(i)(j)} + S_R^{(i)(j)}, \quad (74)$$

$$S = S_{\text{PF}} + S_R, \quad (75)$$

where I remind that the quantities labeled with ‘‘PF’’ are given by Eqs. (40)–(43). It is understood in these expressions that the sum of the different quantities extends to all the species R considered.

A. The Boltzmann equation in curved spacetimes

The ‘‘macroscopic’’ four-current density number of particles of the specie R was defined by Eq. (52). The number density of particles as measured in the local frame of an observer \mathcal{O}_p (with four-velocity v^μ) is

$$n_p^R := -v_\mu j_R^\mu = - \int v_\mu p^\mu F_R(x^\lambda, p^{(\lambda)}) dP. \quad (76)$$

Therefore, the number of particles dN_R with momenta between p^μ and $p^\mu + dp^\mu$ crossing the volume element dV of the space-like hypersurfaces orthogonal to v^μ and which is centered at some point x^μ of spacetime is

$$dN_R = F_R(x^\lambda, p^{(\lambda)}) (-v_\mu p^\mu) dV dP. \quad (77)$$

The quantity $dW = (-v_\mu p^\mu) dV d\lambda$ represents the four-volume spanned by the flow of particles (world lines) crossing dV , which is given by the element of hypersurface dV with normal v^μ and the particle’s infinitesimal displacement orthogonal to dV given by $d\lambda = v_\mu p^\mu d\lambda$ [57]. The quantity $(-v_\mu p^\mu) dV$ is in fact the correct Lorentz invariant four-volume element. From the relativistic form of Liouville’s theorem (see Ref. [57] for the details) $dW dP$ remains invariant along a given set of trajectories. Therefore, the change in the number of world lines within $dW dP$ is proportional to the change in F_R ,

$$\begin{aligned} \delta(dN_R) &= \left[\frac{\partial F_R}{\partial x^\alpha} dx^\alpha + \frac{\partial F_R}{\partial p^\alpha} dp^\alpha \right] (-v_\mu p^\mu) dV dP \\ &= \left[\frac{\partial F_R}{\partial x^\alpha} p^\alpha + \frac{dp^\alpha}{d\lambda} \frac{\partial F_R}{\partial p^\alpha} \right] dW dP. \end{aligned} \quad (78)$$

The evolution for p^μ will be thus governed by the equations of motion of individual particles:

$$\frac{dp^\mu}{d\lambda} = -p^\sigma p^\nu \Gamma_{\sigma\nu}^\mu + \mathcal{F}_{\text{fields}}^\mu + \mathcal{F}_{\text{coll}}^\mu. \quad (79)$$

This equation shows that the trajectory of the particles depends on (1) the spacetime curvature, (2) the forces arising from the interaction of the particles with fundamental fields other than the gravitational one, and (3) the interaction with other particles that can be represented by ‘‘collisions.’’ For my purposes, I will consider that $\mathcal{F}_{\text{fields}}^\mu = 0$. That is to say, the only fundamental field I consider is the gravitational one. All other interactions like the weak ones (in the case of neutrons) will be treated phenomenologically as collision terms, and therefore the set of equations will not include gauge-fields, but rather involve macroscopic quantities that characterize the medium and which are obtained from field theory in a similar fashion as one obtains the equation of state of matter.

The relativistic Boltzmann equation (RBE) then reads,

$$\hat{\mathbf{L}} F_R = \left(\frac{dF_R}{d\lambda} \right)_{\text{coll}}, \quad (80)$$

where

$$\hat{\mathbf{L}} := p^\mu \frac{\partial}{\partial x^\mu} - p^\nu p^\sigma \Gamma_{\nu\sigma}^\mu \frac{\partial}{\partial p^\mu}, \quad (81)$$

is the relativistic Liouville operator often written by the fuzzy notation $p^\alpha D/dx^\alpha$ (the directional derivative of F_R along the phase flow), and $(dF_R/d\lambda)_{\text{coll}} = -\mathcal{F}_{R\text{ coll}}^\alpha(\partial F_R/\partial p^\alpha)$ represents collectively the scattering, absorption, and emission processes between the particles of the specie R and the medium. In the absence of collisions, the distribution function remain constant along the particle's path (i.e., along the particle's geodesics). In the language of differential geometry, the operator $D/dx^\alpha = \partial/\partial x^\alpha - p^\lambda \Gamma_{\alpha\lambda}^\mu(\partial/\partial p^\mu)$ corresponds to a coordinate basis vector of the horizontal part of the tangent bundle over the spacetime M^4 [57]. As mentioned before, the distribution function will be defined only on the reduced phase space \mathcal{P}_m (the mass-shell), that is, it will be defined only on the *sphere bundle* (the subbundle of tangent vectors of fixed length). One can incorporate this restriction in the Liouville operator by treating the spatial part p^i of the four-momenta as independent components. Then

$$(D/dx^\alpha)_{\text{ms}} = \frac{\partial F_R}{\partial x^\alpha} - p^\lambda \Gamma_{\alpha\lambda}^i \frac{\partial F_R}{\partial p^i}. \quad (82)$$

B. Tetrad representation and the 3+1 RBE

It will be convenient to use tetrad components to rewrite the RBE. By employing the tetrad formalism it is easy to show that the RBE reads

$$\begin{aligned} & \left(p^{(\alpha)} q_{(\alpha)}^\mu \frac{\partial}{\partial x^\mu} - p^{(\beta)} p^{(\alpha)} \mathcal{O}_{(\beta)(\alpha)}^{(\delta)} \frac{\partial}{\partial p^{(\delta)}} \right) F_R(x^\alpha, p^{(\sigma)}) \\ &= \left(\frac{dF_R}{d\lambda} \right)_{\text{coll}}. \end{aligned} \quad (83)$$

Let us consider Eq. (83), and split it in terms of temporal and spatial contributions. First we define,

$$\begin{aligned} L_p &:= p^{(b)} p^{(a)} \mathcal{O}_{(b)(a)}^{(d)} \frac{\partial}{\partial p^{(d)}} = p^{(b)} p^{(a)} \mathcal{O}_{(b)(a)}^{(t)} \frac{\partial}{\partial e} \\ &+ p^{(b)} p^{(a)} \mathcal{O}_{(b)(a)}^{(i)} \frac{\partial}{\partial p^{(i)}}, \end{aligned} \quad (84)$$

where the notation $e = p^{(t)}$ was used. The properties of the RRC and some straightforward calculations show a useful relationship between the four-RRC and the physical components of 3+1 variables. For instance:

$$\begin{aligned} \mathcal{O}_{(t)(i)}^{(i)} &= \mathcal{O}_{(i)(t)}^{(i)} = a_{(i)}, \\ \mathcal{O}_{(i)(j)}^{(t)} &= \mathcal{O}_{(j)(i)}^{(t)} = -K_{(i)(j)}, \\ \mathcal{O}_{(t)(\mu)}^{(\mu)} &= \mathcal{O}_{(i)(\mu)}^{(\mu)} = 0 \quad (\text{no summation on } \mu), \end{aligned} \quad (85)$$

$$\begin{aligned} \mathcal{O}_{(i)(j)}^{(i)} &= -\frac{1}{2} \left(-\frac{\partial_{(i)} N^{(j)}}{N} + q_{(m)}^l \frac{N^{(m)}}{N} \partial_{(i)} e_l^{(j)} \right. \\ &\quad \left. + q_{(i)}^l \partial_{(t)} e_l^{(j)} - (i \leftrightarrow j) \right), \end{aligned}$$

$$\mathcal{O}_{(i)(j)}^{(l)} = {}^3\mathcal{O}_{(i)(j)}^{(l)}.$$

Here ${}^3\mathcal{O}_{(i)(j)}^{(l)}$ are the 3-Ricci rotation coefficients, i.e., the RRC associated to the local basis frame on Σ_t , and $\partial_{(t)} = (1/N)(\partial/\partial t) + (N^{(i)}/N)\partial_{(i)}$.

In this way we obtain

$$\begin{aligned} L_p &= (ep^{(i)} a_{(i)} - p^{(i)} p^{(j)} K_{(i)(j)}) \frac{\partial}{\partial e} \\ &+ (e^2 a^{(i)} + ep^{(i)} \mathcal{O}_{(i)(j)}^{(i)} - ep^{(j)} K_{(j)}^{(i)}) \\ &+ p^{(l)} p^{(j)} {}^3\mathcal{O}_{(i)(j)}^{(l)} \frac{\partial}{\partial p^{(i)}}. \end{aligned} \quad (86)$$

Finally, the 3+1 decomposition of Eq. (83) is

$$\begin{aligned} & \left[e \partial_{(t)} + p^{(i)} \partial_{(i)} - (ep^{(i)} a_{(i)} - p^{(i)} p^{(j)} K_{(i)(j)}) \frac{\partial}{\partial e} \right. \\ & \quad - (e^2 a^{(l)} + ep^{(i)} \mathcal{O}_{(i)(j)}^{(i)} - ep^{(j)} K_{(j)}^{(i)}) \\ & \quad \left. + p^{(i)} p^{(j)} {}^3\mathcal{O}_{(i)(j)}^{(l)} \frac{\partial}{\partial p^{(l)}} \right] F_R(x^\alpha, p^{(c)}) = \left(\frac{dF_R}{d\lambda} \right)_{\text{coll}}. \end{aligned} \quad (87)$$

Moreover, the properties of RRC imply ${}^3\mathcal{O}_{(i)(l)}^{(l)} \equiv 0$ (no sum convention). Therefore the above equation can be further simplified.

Equation (87) is the 3+1 version of the RBE, here written in terms of physical components. The mass shell condition $e^2 = p^{(i)} p_{(i)} + \tilde{m}^2$ can be imposed on the RBE by considering, for instance, $p^{(i)}$ as independent variables. In that case F_R is to be considered as though it does not depend explicitly on e , i.e., $\partial F_R/\partial e = 0$. Alternatively, the use of spherical-like variables in momentum space (see Sec. VI C) will allow us to consider e as the independent variable and $p^2 = p^{(i)} p_{(i)}$ as the dependent one.

C. Collisions

As I stressed before, particles may be submitted to collisional forces arising from the interacting medium (i.e., in the case of neutrinos these forces come from the weak interaction with baryons). Let me remind that the collision integral as conceived originally by Boltzmann assumes that the interacting medium has a known distribution function. That is, the distribution function of the medium is a data of the problem.

In the present case, the interacting medium is to be considered not as particles but rather as a fluid field, namely a perfect fluid. Thus, for my purpose, it will be more conve-

nient to characterize the collision integral in terms of scalar functions as it is usual in transport theory. In this way collisions will be represented macroscopically by the so-called *invariant opacity* $o(x^\mu, p^\mu)$ and the invariant emissivity $Y(x^\mu, p^\mu)$.

I then assume that the collision integral takes the following form:

$$\left(\frac{dF_R}{d\lambda}\right)_{\text{coll}} = Y - oF_R. \quad (88)$$

In terms of quantities measured in the same frame one can write

$$Y = \frac{\eta(x^\mu, p^\mu)}{e^2} = \frac{\eta'(x'^\mu, p'^\mu)}{e'^2}, \quad (89)$$

here η and e are the matter emissivity and the particle energy, respectively, both measured in the same frame. In the same way, one can introduce the matter opacity as

$$o = e\chi(x^\mu, p^\mu) = e'\chi'(x'^\mu, p'^\mu), \quad (90)$$

where $\chi \sim 1/l$, l being the mean free path of the particle in the corresponding frame.

The collision term takes then the useful form

$$\left(\frac{dF_R}{d\lambda}\right)_{\text{coll}} = e\left(\frac{\eta}{e^3} - \chi F_R\right). \quad (91)$$

The opacity χ and the absorption coefficient κ are related by

$$\chi = \kappa n, \quad (92)$$

where n is the proper number density of particles that composes the medium (e.g., the baryon density), such that $n := -j^\mu u_\mu$ where j^μ is the four-current of baryons.

In this way, an alternative form of the collision integral is

$$\left(\frac{dF_R}{d\lambda}\right)_{\text{coll}} = \kappa_e n(S - F_R), \quad (93)$$

where $k_e = e\kappa$, and $S = Y/o$ is usually referred to as the *effective source function*.

It is to be emphasized that quantities measured in different frames are related to each other via the invariant quantities and Lorentz transformations, for instance, the relationship between the opacities measured in the Eulerian frame and those of the proper frame of the fluid are given, according to Eq. (90), by

$$e\chi = e_p\chi_p, \quad (94)$$

where quantities in the left-hand side (lhs) refer to the Eulerian frame, while the quantities in the right-hand side (rhs) refer to the proper frame of the fluid. Since physical components of four vectors in both frames are related by a Lorentz transformation, for instance,

$$p_p^{(\mu)} = \Lambda_{(\nu)}^{(\mu)p} p^{(\nu)}, \quad (95)$$

where

$$\Lambda_{(\nu)}^{(\mu)p} = \begin{pmatrix} \Gamma & -{}^3U^{(i)}\Gamma \\ -{}^3U^{(i)}\Gamma & \delta_{(j)}^{(i)} + {}^3U^{(i)}{}^3U^{(j)}\frac{\Gamma^2}{\Gamma+1} \end{pmatrix}, \quad (96)$$

with ${}^3U^{(i)}$ given by Eq. (45) and Γ by Eq. (44), then for the time components,

$$\begin{aligned} e_p &= \Gamma(e - {}^3U^{(i)}p^{(i)}) = e\Gamma(1 - {}^3U^{(i)}v^{(i)}) \\ &= e\Gamma(1 - ||{}^3U^{(i)}|| |v^{(i)}| \cos(\theta_R)), \end{aligned} \quad (97)$$

where $v^{(i)} := p^{(i)}/e$ represents the velocity of the particles with respect to the Eulerian frame. In the last formula one recognizes the well known formula for the energy shift due to the relative motion of observers. The type of shift (red or blue) will depend on the angle θ_R between the propagation vector of the particles $v^{(i)}$ and the velocity of the fluid ${}^3U^{(i)}$ (i.e., blueshift or redshift if the fluid is approaching or receding, respectively, from the Eulerian observer). Therefore the transformation formula between opacities yields

$$\chi = \chi_p \Gamma(1 - ||{}^3U^{(i)}|| |v^{(i)}| \cos(\theta_R)). \quad (98)$$

In the case of massless particles $|v^{(i)}| = 1$. In a similar way one can obtain the transformation formulas for the absorption coefficients and the emissivities.

D. Conservation equations for the radiated flow

The particle number current and the energy-momentum tensor of the radiated particles were introduced by Eqs. (52) and (53), respectively. For instance, in the case of perfect quantum gases in thermal equilibrium (i.e., Fermi and Bose gases) the above definitions allow one to recover the usual macroscopic expressions for the energy density, density number, and pressure parametrized by the temperature, particle mass, and chemical potential of the species as measured in the local frame.

When collisions are present, both the particle number and the energy-momentum tensor of the particles will not conserve alone since there will be an exchange of energy and momentum with the interacting dense fluid. Thus we can expect that the conservation equations derived from Eqs. (52) and (53) will have sources arising from the collision integral:

$$\nabla_\nu j_R^\nu = \int \left(\frac{dF_R}{d\lambda}\right)_{\text{coll}} dP \quad (99)$$

and

$$\nabla_\nu T_R^{\mu\nu} = \int p^\mu \left(\frac{dF_R}{d\lambda}\right)_{\text{coll}} dP. \quad (100)$$

Then we write

$$\nabla_\nu j_R^\nu = \mathcal{R}_R, \quad (101)$$

$$\nabla_\nu T_R^{\mu\nu} = -\mathcal{F}_R^\mu = n w_R^\mu, \quad (102)$$

where

$$\mathcal{R}_R := \int \kappa(x^\mu, p^\mu) [\mathcal{S}(x^\mu, p^\mu) - F_R(x^\mu, p^\mu)] dP, \quad (103)$$

$$w_R^\mu := \int p^\mu \kappa(x^\mu, p^\mu) [\mathcal{S}(x^\mu, p^\mu) - F_R(x^\mu, p^\mu)] dP. \quad (104)$$

One can define the *mean* emissivity [energy/(volume \times time)] in the fluid frame as

$$\mathcal{D} := -u_\mu \nabla_\nu T_R^{\mu\nu} = -n u_\mu w_R^\mu. \quad (105)$$

Since I have been using quantities measured in the Eulerian frame, in the above expressions one has to use the corresponding quantities with respect to the same observer. Namely, the particle number density n_E as measured by the Eulerian observer is related to the proper number density of the perfect fluid n by $n_E = n\Gamma$. The same considerations apply for the remaining collision variables. For instance, the emissivity measured in the Eulerian frame $\mathcal{D}_E = -n_\mu n w^\mu = n w^{(t)}$ is related to the proper emissivity $\mathcal{D} = -u_\mu n w^\mu$ by $\mathcal{D} = \Gamma \mathcal{D}_E (1 - {}^3U^{(i)} {}^3W^{(i)})$ where ${}^3W^{(i)} := w^{(i)}/w^{(t)}$.

V. THERMODYNAMICS

In this section the thermodynamic description of the dense matter with which the radiative particles interact will be presented taking into account the mean quantities introduced in the previous section. Such a description is performed in the proper frame of the fluid. I then assume that the EOS of the dense matter (i.e., the perfect fluid) is given in parametrized form as follows:

$$\rho = \rho(s, n_1, \dots, n_m), \quad (106)$$

$$p = p(s, n_1, \dots, n_m), \quad (107)$$

where s is the entropy density and n_M ($1 \leq M \leq m$) is the number density of particles of the specie M (e.g., baryons and the different lepton flavors), all of them measured in the fluid frame. For instance, in the case of a dense matter in hydrostatic equilibrium composed by a mixture of neutrons, protons, and electrons, the electron density is obtained directly from the proper baryon density n by demanding charge neutrality and chemical equilibrium $\mu_n = \mu_p + \mu_e$. This last condition arising from the equilibrium of the nuclear reactions: $n \rightleftharpoons p + e^-$ [9]. In that case the equation of state depends on n solely.

Now, Eqs. (106) and (107) are not independent of each other but linked through the first principle of thermodynamics

$$dU = \theta ds - p dV + \mu^R dN^R, \quad (108)$$

where Θ and μ^R are the temperature and the chemical potential of the specie R of the particles composing the fluid, respectively, defined by

$$\Theta = \left(\frac{\partial \rho}{\partial s} \right)_{n_R}, \quad (109)$$

$$\mu^R = \left(\frac{\partial \rho}{\partial n_R} \right)_{s, n_B \neq n_R}. \quad (110)$$

Using these definitions, Eq. (108) takes the following form in terms of densitized quantities [9]:

$$p = \Theta s + \mu^R n_R - \rho. \quad (111)$$

This equation is often referred to as the compatibility thermodynamic condition between Eqs. (106) and (107) [9].

The conservation equation for the baryon number reads

$$\nabla_\mu j^\mu = 0, \quad (112)$$

where I remind that $j^\mu = n u_\mu$ is the density current of baryons and n the proper baryon number density.

This equation can be written explicitly as an evolution equation for the number density $n_E := -n_\mu j^\mu = n\Gamma$ measured by the Eulerian observer as follows:

$$\partial_t(\sqrt{h} n_E) + \partial_i[\sqrt{h} n_E V^i] = 0, \quad (113)$$

where Γ and V^i are given by Eqs. (44) and (46), respectively. Introducing the physical components of the fluid velocity field given by Eq. (45), I have the alternative expression,

$$\partial_t(\sqrt{h} n_E) + \partial_i[\sqrt{h} n_E (N^i + N q_{(j)}^i {}^3U^{(j)})] = 0. \quad (114)$$

The equation of conservation of the baryon number leads to the conserved total baryon number given by

$$\mathcal{N} = \int_{\Sigma_t} -j^\mu n_\mu \sqrt{h} dx^1 dx^2 dx^3 \quad (115)$$

$$= \int_{\Sigma_t} n \Gamma \sqrt{h} dx^1 dx^2 dx^3. \quad (116)$$

The integral has compact support corresponding to the volume enveloped by the star surface.

In a similar way, the equation for entropy conservation is obtained from

$$u_\nu \nabla_\mu T^{\mu\nu} = u_\nu \nabla_\mu (T_{\text{PF}}^{\mu\nu} + T_R^{\mu\nu}) = 0. \quad (117)$$

The use of Eq. (39) followed by the definitions Eqs. (109), (110), and (111), yields

$$\nabla_\mu(su^\mu) = -\frac{1}{\Theta}(\mu^R\mathcal{R}_R + \mathcal{D}), \quad (118)$$

where

$$\mathcal{R}_R := \nabla_\mu(n_R u^\mu) \quad (119)$$

is the rate of particle production and I remind that

$$\mathcal{D} := -u_\nu \nabla_\mu T_R^{\mu\nu} = u_\nu \nabla_\mu T_{\text{PF}}^{\mu\nu} \quad (120)$$

is the particle's mean emissivity in the fluid frame. Therefore the source for entropy generation in a perfect fluid is from the particle production (e.g., neutrinos). Equation (119) is completely equivalent to Eq. (101) which is given in terms of the distribution function.

One can define the entropy per baryon $\sigma = s/n$ and use Eqs. (112) and (118) to obtain

$$n^\mu \nabla_\mu \sigma = -\frac{1}{n\Theta}(\mu^R\mathcal{R}_R + \mathcal{D}). \quad (121)$$

Explicitly this provides an evolution equation for σ :

$$\partial_i \sigma + V^i \partial_i \sigma = -\frac{N}{n\Gamma}(\mu^R\mathcal{R}_R + \mathcal{D}). \quad (122)$$

In the case of a neutron-star collapse with matter out of β -equilibrium an evolution equation for the temperature Θ can be obtained from Eq. (122) [36].

Moreover, using Eqs. (44), (45), and the tetrad approach of Sec. II, Eq. (121) takes the alternative form:

$$n^\mu \partial_\mu \sigma + {}^3U^{(i)} \partial_{(i)} \sigma = -\frac{1}{n\Gamma\Theta}(\mu^R\mathcal{R}_R + \mathcal{D}). \quad (123)$$

In the same way, one can introduce the particle number per baryon $x_R = n_R/n$ and write Eq. (119) as

$$u^\mu \nabla_\mu x_R = \frac{1}{n} \mathcal{R}_R, \quad (124)$$

which provides an evolution equation for x_R :

$$\partial_t x_R + V^i \partial_i x_R = \frac{N}{n\Gamma} \mathcal{R}_R, \quad (125)$$

or alternatively,

$$n^\mu \partial_\mu x_R + {}^3U^{(i)} \partial_{(i)} x_R = \frac{1}{n\Gamma} \mathcal{R}_R. \quad (126)$$

In order to close the whole system of equations presented so far one needs the input of particle physics. That is, the EOS for nuclear matter, the rate of particle production, and the opacities (see Ref. [2] for a review). In the case of neutrinos emitted by nuclear matter out of beta equilibrium via direct and inverse β processes during neutron-star collapse, the rate of particle production, the emissivities, and the opacities can be given in terms of rather simple formulas [36,58] (see also Ref. [59] for neutrino emissivities from quark matter in β -equilibrium within neutron stars and Ref. [60] for neutrino emission from hot and dense atmospheres). For the case of reaction rates and opacities in Type II supernovae see, for instance, Ref. [15].

In summary, we can briefly list the variables to be evolved in time: (a) Gravitational variables: h_{ij}, K_j^i are to be evolved from Eqs. (4) and (12), respectively, subject to the initial-value constraints Eqs. (10) and (11); (b) Perfect-fluid variables: E_{PF} and ${}^3U^{(i)}$ (or $J_{\text{PF}}^{(i)}$) are evolved from Eqs. (47) and (48) [or from Eq. (37) for the perfect-fluid case]; (c) Radiation variables: $F_R(t, x^i, E, p^{(i)})$ is evolved from Eq. (87) with the collisional term given, for instance, by Eq. (93). The remaining observables are computed by integration from Eqs. (68)–(70). Another alternative is to evolve the mean radiative variables (together with a closure relationship between them: e.g., the diffusion approximation) instead of the distribution function itself [see Eqs. (192) and (193) for the spherically symmetric case]; (d) Thermodynamic variables: n_E, σ , and x_R are to be evolved from Eqs. (114), (123), and (126), respectively.

The particle-physics input for the above system of evolution equations is the EOS given by Eqs. (106) and (107) the rate of particle production \mathcal{R}_R [see Eq. (119)], and the opacity and source function [see Eq. (93)]. The particular form for these depends on the physical problem to be treated.

VI. SPHERICAL SYMMETRY

In this and the following sections I will focus on spherically symmetric spacetimes. The most general line-element for such spacetimes according to the 3+1 decomposition of the metric Eq. (3) is

$$ds^2 = -(N^2 - N^r N_r) dt^2 - 2N_r dt dr + A^2 dr^2 + B^2 (r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2), \quad (127)$$

where all the metric potentials are functions of the coordinates r and t solely. The three-metric h_{ij} is easily read-off from Eq. (127).

On the other hand, the triad coefficients are

$$e_j^{(i)} = \text{diag}[A(t, r), rB(t, r), r \sin \theta B(t, r)]. \quad (128)$$

The inverse coefficients $q_{(j)}^i$ can be obtained trivially from Eq. (128).

The extrinsic curvature can be computed from Eq. (4). I find [61]

$$(K_{ij}) = \begin{pmatrix} -\frac{A}{N}(\partial_t A + \partial_r N^{(r)}) & 0 & 0 \\ 0 & -\frac{rB^2}{N}\left(\frac{N^{(r)}}{A} + \frac{r}{B}\partial_t B + \frac{rN^{(r)}\partial_r B}{AB}\right) & 0 \\ 0 & 0 & -\frac{rB^2\sin^2(\theta)}{N}\left(\frac{N^{(r)}}{A} + \frac{r}{B}\partial_t B + \frac{rN^{(r)}\partial_r B}{AB}\right) \end{pmatrix}, \quad (129)$$

$$(K_j^i) = (K_{(j)}^{(i)}) = \begin{pmatrix} -\frac{1}{N}\left(\frac{\partial_t A}{A} + \frac{N^r\partial_r A}{A} + \partial_r N^r\right) & 0 & 0 \\ 0 & -\frac{1}{N}\left(\frac{\partial_t B}{B} + \frac{N^r\partial_r B}{B} + \frac{N^r}{r}\right) & 0 \\ 0 & 0 & -\frac{1}{N}\left(\frac{\partial_t B}{B} + \frac{N^r\partial_r B}{B} + \frac{N^r}{r}\right) \end{pmatrix},$$

$$= \begin{pmatrix} -\frac{1}{N}\left(\frac{\partial_t A}{A} + \frac{\partial_r N^{(r)}}{A}\right) & 0 & 0 \\ 0 & -\frac{1}{N}\left(\frac{\partial_t B}{B} + \frac{N^{(r)}\partial_r B}{AB} + \frac{N^{(r)}}{rA}\right) & 0 \\ 0 & 0 & -\frac{1}{N}\left(\frac{\partial_t B}{B} + \frac{N^{(r)}\partial_r B}{AB} + \frac{N^{(r)}}{rA}\right) \end{pmatrix}, \quad (130)$$

where the index of Eq. (130) was raised with h^{ij} from Eq. (129).

The three-scalar of curvature is given by

$${}^3R = \frac{2}{r^2 A^2} \left(\frac{A^2}{B^2} - 1 \right) + \frac{2}{A^2} \left(\frac{2\partial_r A}{rA} + \frac{2(\partial_r B)(\partial_r A)}{BA} - \frac{(\partial_r B)^2}{B^2} - \frac{6\partial_r B}{rB} - \frac{2\partial_{rr}^2 B}{B} \right). \quad (131)$$

A. The 3+1 Einstein equations

It is useful to introduce the new variables,

$$\nu = \ln[N], \quad (132)$$

$$\alpha = \ln[A], \quad (133)$$

$$\beta = \ln[B]. \quad (134)$$

The Hamiltonian constraint Eq. (10) reads

$$\frac{1}{r^2} \left(\frac{A^2}{B^2} - 1 \right) + A^2 [2K_{(r)}^{(r)}K_{(\theta)}^{(\theta)} + (K_{(\theta)}^{(\theta)})^2] + \frac{2\partial_r \alpha}{r} - \frac{6\partial_r \beta}{r} + 2(\partial_r \alpha)(\partial_r \beta) - 3(\partial_r \beta)^2 - 2\partial_{rr}^2 \beta = 8\pi G_0 E A^2, \quad (135)$$

where I have used the fact that the nondiagonal components of K_j^i are null and that $K_{(\theta)}^{(\theta)} = K_{(\phi)}^{(\phi)}$ [cf. Eq. (130)].

The radial component of the momentum constraints Eq. (11), reads

$$(K_{(r)}^{(r)} - K_{(\theta)}^{(\theta)}) \left(\frac{1}{r} + \partial_r \beta \right) - \partial_r K_{(\theta)}^{(\theta)} = 4\pi G_0 A J_{(r)}. \quad (136)$$

Or in terms of the trace K ,

$$2(K_{(r)}^{(r)} - K_{(\theta)}^{(\theta)}) \left(\frac{1}{r} + \partial_r \beta \right) - \partial_r K + \partial_r K_{(r)}^{(r)} = 8\pi G_0 A J_{(r)}. \quad (137)$$

The angular components of Eq. (11) and the spherical symmetry lead to the conditions $J_\theta = 0 = J_\phi$ which implies the absence of ‘‘angular currents.’’

The dynamical equations for the nondiagonal components of Eq. (12), i.e., for $\partial_t K_{(\theta)}^{(r)}$, $\partial_t K_{(\phi)}^{(r)}$, $\partial_t K_{(\phi)}^{(\theta)}$ with the fact that $K_{(\theta)}^{(r)} = K_{(\phi)}^{(r)} = K_{(\phi)}^{(\theta)} = 0$ [cf. Eq. (130)], leads, respectively, to the conditions that $S_{(\theta)}^{(r)} = S_{(\phi)}^{(r)} = S_{(\phi)}^{(\theta)} = 0$. Moreover, taking into account the fact that $K_{(\theta)}^{(\theta)} = K_{(\phi)}^{(\phi)}$, the dynamical equations for $\partial_t K_{(\theta)}^{(\theta)}$ and $\partial_t K_{(\phi)}^{(\phi)}$ [see Eq. (12)], lead to the condition $S_{(\theta)}^{(\theta)} = S_{(\phi)}^{(\phi)}$ corresponding to an ‘‘isotropic’’ energy-momentum tensor which is compatible with the hypothesis of a space-time with spherical symmetry. In this way, the only two non-trivial dynamical equations are

$$\begin{aligned}
& \partial_t K_{(r)}^{(r)} + \frac{N^{(r)} \partial_r K_{(r)}^{(r)}}{A} - N K K_{(r)}^{(r)} - \frac{N}{A^2} \left(\frac{2 \partial_r \alpha}{r} - \frac{4 \partial_r \beta}{r} \right. \\
& \left. + 2(\partial_r \alpha)(\partial_r \beta) - 2(\partial_r \beta)^2 - 2 \partial_{rr}^2 \beta \right) \\
& + \frac{N}{A^2} [\partial_{rr}^2 \nu + (\partial_r \nu)^2 - (\partial_r \alpha)(\partial_r \nu)] \\
& = 4 \pi G_0 N (-S_{(r)}^{(r)} + 2S_{(\theta)}^{(\theta)} - E), \quad (138) \\
& \partial_t K_{(\theta)}^{(\theta)} + \frac{N^{(r)} \partial_r K_{(\theta)}^{(\theta)}}{A} - N K K_{(\theta)}^{(\theta)} \\
& - \frac{N}{A^2} \left[\frac{1}{r^2} \left(\frac{A^2}{B^2} - 1 \right) + \frac{\partial_r \alpha}{r} - \frac{4 \partial_r \beta}{r} + (\partial_r \alpha)(\partial_r \beta) \right. \\
& \left. - 2(\partial_r \beta)^2 - \partial_{rr}^2 \beta \right] + \frac{N}{A^2} \left[(\partial_r \beta)(\partial_r \nu) + \frac{\partial_r \nu}{r} \right] \\
& = 4 \pi G_0 N (S_{(r)}^{(r)} - E). \quad (139)
\end{aligned}$$

From Eq. (14) the evolution equation for the trace of K_{ij} is

$$\begin{aligned}
& \partial_t K + \frac{N^{(r)} \partial_r K}{A} - N [(K_{(r)}^{(r)})^2 + 2(K_{(\theta)}^{(\theta)})^2] + \frac{N}{A^2} \left[\partial_{rr}^2 \nu \right. \\
& \left. + (\partial_r \nu)^2 - (\partial_r \alpha)(\partial_r \nu) + 2(\partial_r \beta)(\partial_r \nu) + \frac{2 \partial_r \nu}{r} \right] \\
& = 4 \pi G_0 N (S + E), \quad (140)
\end{aligned}$$

where we recognize the three-covariant Laplace operator in spherical coordinates of the slices Σ_t [see Eq. (127)]:

$${}^3 \Delta \nu = \frac{1}{A^2} \left[\partial_{rr}^2 \nu + \frac{2 \partial_r \nu}{r} - (\partial_r \alpha)(\partial_r \nu) + 2(\partial_r \beta)(\partial_r \nu) \right]. \quad (141)$$

B. Matter equations

In the present case of a perfect fluid in spherical symmetry we have

$$E_{\text{PF}} = (\rho + p)\Gamma - p, \quad (142)$$

$${}_{\text{PF}} S_{(r)}^{(r)} = (E_{\text{PF}} + p)({}^3 U^{(r)})^2 + p, \quad (143)$$

$${}_{\text{PF}} S_{(\theta)}^{(\theta)} = S_{(\phi)}^{(\phi)} = p, \quad (144)$$

$$J_{(r)}^{\text{PF}} = (E_{\text{PF}} + p) {}^3 U^{(r)}, \quad (145)$$

$$\Gamma = [1 - ({}^3 U^{(r)})^2]^{-1/2}, \quad (146)$$

where

$${}^3 U^{(r)} = \frac{A}{N} (V^r - N^r) = \frac{1}{N} (V^{(r)} - N^{(r)}), \quad (147)$$

with $V^r := u^r / u^t$. The spherical symmetry implies $u^\theta = 0 = u^\phi$ and therefore $U^{(\theta)} = 0 = U^{(\phi)}$.

Note that

$${}_{\text{PF}} S_{(r)}^{(r)} = {}^3 U^{(r)} J_{(r)}^{\text{PF}} + p. \quad (148)$$

Equation (47) reads

$$\begin{aligned}
& \partial_t E_{\text{PF}} + N^r \partial_r E_{\text{PF}} + \frac{N}{A B^2 r^2} \partial_r (A B^2 r^2 J_{\text{PF}}^r) \\
& = N ({}_{\text{PF}} S_{(r)}^{(r)} K_{(r)}^{(r)} + 2 {}_{\text{PF}} S_{(\theta)}^{(\theta)} K_{(\theta)}^{(\theta)} + E_{\text{PF}} K) \\
& - 2 J_{\text{PF}}^r \partial_r N - N^2 \mathcal{F}_R^t. \quad (149)
\end{aligned}$$

Or in terms of $J_{(r)} = A J^r$,

$$\begin{aligned}
& \partial_t E_{\text{PF}} + N^r \partial_r E_{\text{PF}} + \frac{N}{A B^2 r^2} \partial_r (B^2 r^2 J_{(r)}^{\text{PF}}) \\
& = N ({}_{\text{PF}} S_{(r)}^{(r)} K_{(r)}^{(r)} + 2 {}_{\text{PF}} S_{(\theta)}^{(\theta)} K_{(\theta)}^{(\theta)} + E_{\text{PF}} K) \\
& - \frac{2}{A} J_{(r)}^{\text{PF}} \partial_r N - N^2 \mathcal{F}_R^t. \quad (150)
\end{aligned}$$

Using Eqs. (144), (147), and (148) the latter equation can be written in conservative form as

$$\begin{aligned}
& \partial_t E_{\text{PF}} + \frac{1}{r^2} \partial_r (r^2 V^r E_{\text{PF}}) \\
& = E_{\text{PF}} \left(N K + \frac{2 N^r}{r} + \partial_r N^r \right) - \frac{1}{r^2} \partial_r \left(r^2 \frac{N}{A} {}^3 U^{(r)} p \right) \\
& - \frac{N J_{(r)}^{\text{PF}}}{A} [\partial_r \nu + \partial_r \alpha + 2 \partial_r \beta - {}^3 U^{(r)} A K_{(r)}^{(r)}] \\
& + N K p - N^2 \mathcal{F}_R^t. \quad (151)
\end{aligned}$$

The momentum conservation Eq. (19) reads

$$\begin{aligned}
& \partial_t J_r^{\text{PF}} + N^r \partial_r J_r^{\text{PF}} + J_r^{\text{PF}} \partial_r N^r + N ({}^3 \nabla_l {}_{\text{PF}} S_r^l) \\
& = N K J_r^{\text{PF}} - ({}_{\text{PF}} S_r^r + E_{\text{PF}}) \partial_r N - {}^3 \mathcal{F}_r^R N. \quad (152)
\end{aligned}$$

Explicitly

$$\begin{aligned}
& \partial_t J_r^{\text{PF}} + N^r \partial_r J_r^{\text{PF}} + J_r^{\text{PF}} \partial_r N^r + N \partial_r {}_{\text{PF}} S_{(r)}^{(r)} \\
& + 2N ({}_{\text{PF}} S_{(r)}^{(r)} - {}_{\text{PF}} S_{(\theta)}^{(\theta)}) \left(\frac{1}{r} + \frac{\partial_r B}{B} \right) \\
& = - ({}_{\text{PF}} S_{(r)}^{(r)} + E_{\text{PF}}) \partial_r N + N J_r^{\text{PF}} (K_{(r)}^{(r)} + 2 K_{(\theta)}^{(\theta)}) \\
& - {}^3 \mathcal{F}_r^R N. \quad (153)
\end{aligned}$$

Or in terms of $J_{(r)}$, one has

$$\begin{aligned}
& \partial_t J_{(r)}^{\text{PF}} + J_{(r)}^{\text{PF}} \frac{\partial_t A}{A} + \frac{N^{(r)}}{A} \partial_r J_{(r)}^{\text{PF}} + \frac{J_{(r)}^{\text{PF}}}{A} \partial_r N^{(r)} \\
& + \frac{N}{A} \partial_r {}_{\text{PF}}S_{(r)}^{(r)} + \frac{2N}{A} ({}_{\text{PF}}S_{(r)}^{(r)} - {}_{\text{PF}}S_{(\theta)}^{(\theta)}) \left(\frac{1}{r} + \frac{\partial_r B}{B} \right) \\
& = - ({}_{\text{PF}}S_{(r)}^{(r)} + E_{\text{PF}}) \frac{\partial_r N}{A} + N J_{(r)}^{\text{PF}} (K_{(r)}^{(r)} + 2K_{(\theta)}^{(\theta)}) \\
& - {}^3\mathcal{F}_{(r)}^{\text{R}} N. \tag{154}
\end{aligned}$$

When using Eq. (130) to replace the time derivative of A I find

$$\begin{aligned}
& \partial_t J_{(r)}^{\text{PF}} + \frac{N^{(r)}}{A} \partial_r J_{(r)}^{\text{PF}} + \frac{N}{A} \partial_r {}_{\text{PF}}S_{(r)}^{(r)} \\
& + \frac{2N}{A} ({}_{\text{PF}}S_{(r)}^{(r)} - {}_{\text{PF}}S_{(\theta)}^{(\theta)}) \left(\frac{1}{r} + \frac{\partial_r B}{B} \right) \\
& = - ({}_{\text{PF}}S_{(r)}^{(r)} + E_{\text{PF}}) \frac{\partial_r N}{A} + 2N J_{(r)}^{\text{PF}} (K_{(r)}^{(r)} + K_{(\theta)}^{(\theta)}) \\
& - {}^3\mathcal{F}_{(r)}^{\text{R}} N. \tag{155}
\end{aligned}$$

Using Eqs. (147) and (148), one obtains an equation for $J_{(r)}^{\text{PF}}$ in conservative form

$$\begin{aligned}
& \partial_r J_{(r)}^{\text{PF}} + \frac{1}{r^2} \partial_r (r^2 V^r J_{(r)}^{\text{PF}}) \\
& = \frac{N}{A} \left\{ J_{(r)}^{\text{PF}} \left[2A (K_{(r)}^{(r)} + K_{(\theta)}^{(\theta)}) - {}^3U^{(r)} (\partial_r \alpha + 2\partial_r \beta) \right. \right. \\
& \left. \left. - \frac{A}{N} \left(\frac{2}{r} N^r + \partial_r N^r \right) \right] - (E_{\text{PF}} + p) \partial_r \nu - \partial_r p - {}^3\mathcal{F}_{(r)}^{\text{R}} A \right\}, \tag{156}
\end{aligned}$$

with the alternative form for the rhs,

$$\begin{aligned}
& \partial_r J_{(r)}^{\text{PF}} + \frac{1}{r^2} \partial_r (r^2 V^r J_{(r)}^{\text{PF}}) \\
& = J_{(r)}^{\text{PF}} \left[2N (K_{(r)}^{(r)} + K_{(\theta)}^{(\theta)}) - V^r (\partial_r \alpha + 2\partial_r \beta) \right. \\
& \left. + N^r \left(\partial_r \alpha + 2\partial_r \beta - \frac{2}{r} \right) - \partial_r N^r \right] \\
& - \frac{N}{A} [(E_{\text{PF}} + p) \partial_r \nu + \partial_r p + {}^3\mathcal{F}_{(r)}^{\text{R}} A]. \tag{157}
\end{aligned}$$

Another possibility which has turned out to be very useful in some numerical studies [8,9,36] is the use of the Euler equation for the fluid instead of the equation for $J_{(r)}^{\text{PF}}$. The only nontrivial component of the Euler equation (48) in spherical symmetry reads

$$\begin{aligned}
& \partial_{(t)} {}^3U^{(r)} + {}^3U^{(r)} {}^3\partial_{(r)} {}^3U^{(r)} \\
& = - \frac{1}{E_{\text{PF}} + p} [{}^3\partial^{(r)} p + {}^3U^{(r)} \partial_{(t)} p] \\
& + \frac{1}{\Gamma^2} ({}^3U^{(r)} K_{(r)}^{(r)} - a_{(r)}) \\
& + \frac{1}{E_{\text{PF}} + p} ({}^3U^{(r)} \mathcal{F}_{\text{R}}^{(t)} - \mathcal{F}_{\text{R}}^{(r)}), \tag{158}
\end{aligned}$$

where I used the fact that the RRC are antisymmetric so that the terms $\mathcal{O}_{(t)(r)}^{(r)} = 0 = \mathcal{O}_{(r)(r)}^{(r)}$, and I remind the explicit expressions

$$\partial_{(t)} = \frac{1}{N} \partial_t + \frac{N^r}{N} \partial_r, \tag{159}$$

$${}^3\partial_{(r)} = \frac{1}{A} \partial_r, \tag{160}$$

$$a_{(r)} = {}^3\partial_{(r)} \nu. \tag{161}$$

C. The RBE in spherical symmetry

The most general four-metric for a spherically symmetric spacetime is given by Eq. (127). Then, it is easy to see that the simplest tetrad choice $e_{(\mu)}$ associated with Eq. (127) and which corresponds to the local tetrad of the Eulerian observer, reads

$$e_{(t)} = \frac{1}{N} \frac{\partial}{\partial t} + \frac{N^r}{N} \frac{\partial}{\partial r}, \tag{162}$$

$$e_{(r)} = \frac{1}{A} \frac{\partial}{\partial r}, \tag{163}$$

$$e_{(\theta)} = \frac{1}{rB} \frac{\partial}{\partial \theta}, \tag{164}$$

$$e_{(\phi)} = \frac{1}{rB \sin \theta} \frac{\partial}{\partial \phi}, \tag{165}$$

where $\partial/\partial x^\mu$ [with $x^\mu = (t, r, \theta, \phi)$] denote the coordinate basis.

In the Eulerian frame, the spherical symmetry of configuration space (i.e., spacetime) induces a symmetry on momentum space that can be exploited to simplify the computations. It is convenient to define spherical variables on momentum space as follows: take a unit vector $e_{(r)}$ as the polar axis, i.e., as the symmetry axis on momentum space. Then it is useful to introduce new variables e, ψ, γ in the Eulerian frame as

$$\begin{aligned}
p^{(t)} &= e, & p^{(r)} &= p \cos \psi, & p^{(\theta)} &= p \sin \psi \cos \gamma, \\
p^{(\phi)} &= p \sin \psi \sin \gamma. \tag{166}
\end{aligned}$$

As mentioned earlier in Sec. IV, e is the energy of the radiated particles (e.g., neutrinos) in the Eulerian frame and p^2

$=p^{(i)}p_{(i)}$; ψ is the angle between the polar axis and the neutrino propagation three-vector $p^{(i)}$, and γ is the angle of rotation around $e_{(r)}$. The above variables are consistent with the mass shell condition (61). For massless particles, then $p \equiv e$. Under these new variables the RBE (83) reads

$$\left(p^{(a)}(e, \psi, \gamma) q_{(a)}^\mu \frac{\partial}{\partial x^\mu} - p^{(b)}(e, \psi, \gamma) p^{(a)}(e, \psi, \gamma) \right. \\ \left. \times \mathcal{O}_{(b)(a)}^{(d)} \frac{\partial \tilde{p}^{(c)}}{\partial p^{(d)}} \frac{\partial}{\partial \tilde{p}^{(c)}} \right) F_{\mathbb{R}}(\tilde{\mathbf{p}}) = \left(\frac{dF_{\mathbb{R}}}{d\tau} \right)_{\text{coll}}, \quad (167)$$

where I explicitly stress the dependence of momenta with respect to the new momenta spherical-like variables (e, p, ψ, γ) represented collectively by $\tilde{\mathbf{p}}$.

Indeed, the spherical symmetry and the mass shell condition will be reflected in the RBE by the fact that the distribution function $F_{\mathbb{R}}$ depends only on four phase-space coordinates (t, r, e, μ) or (t, r, p, μ) instead of the original eight (x^μ, p^μ) [57,62], where $\mu := \cos \psi$.

We can now proceed to calculate explicitly the RBE. The only non-null Ricci coefficients are [61]

$$\mathcal{O}_{(t)(r)}^{(t)} = \mathcal{O}_{(t)(t)}^{(r)} = D_r \nu,$$

$$\mathcal{O}_{(r)(r)}^{(t)} = \mathcal{O}_{(r)(t)}^{(r)} = D_t \alpha + \frac{1}{N} D_r N^{(r)},$$

$$\mathcal{O}_{(\theta)(\theta)}^{(t)} = \mathcal{O}_{(\phi)(\phi)}^{(t)} = \mathcal{O}_{(\theta)(t)}^{(\theta)} = \mathcal{O}_{(\phi)(t)}^{(\phi)} \\ = \frac{N^{(r)}}{rAN} + D_t \beta + \frac{N^{(r)}}{N} D_r \beta,$$

$$\mathcal{O}_{(\theta)(\theta)}^{(r)} = \mathcal{O}_{(\phi)(\phi)}^{(r)} = -\mathcal{O}_{(\theta)(r)}^{(\theta)} = -\mathcal{O}_{(\phi)(r)}^{(\phi)} \\ = -r^{-1} A^{-1} - D_r \beta,$$

$$\mathcal{O}_{(\phi)(\phi)}^{(\theta)} = -\mathcal{O}_{(\phi)(\theta)}^{(\phi)} = -\frac{1}{rB} \cot \theta,$$

where $D_t := (1/N) (\partial/\partial t)$, $D_r := (1/A) (\partial/\partial r) \equiv {}^3\partial_{(r)}$. After imposing the mass shell condition, the RBE (167) in terms of the spherical variables reads explicitly [61],

$$e D_t F_{\mathbb{R}}(t, r, e, \mu) + \left(\frac{p\mu}{e} + \frac{N^{(r)}}{N} \right) e D_r F_{\mathbb{R}}(t, r, e, \mu) \\ - p^2 \left[(1 - \mu^2) \frac{N^{(r)}}{N} \left(\frac{1}{rA} + D_r \beta \right) + \mu^2 D_t \alpha + (1 - \mu^2) D_t \beta + \frac{\mu e}{p} D_r \nu + \frac{\mu^2}{N} D_r N^{(r)} \right] \partial_e F_{\mathbb{R}}(t, r, e, \mu) \\ + e(1 - \mu^2) \left[\left(\frac{p}{e} + \frac{\mu N^{(r)}}{N} \right) \left(\frac{1}{rA} + D_r \beta \right) + \mu(D_t \beta - D_t \alpha) - \frac{e}{p} D_r \nu - \frac{\mu}{N} D_r N^{(r)} \right] \partial_\mu F_{\mathbb{R}}(t, r, e, \mu) = \left(\frac{dF_{\mathbb{R}}}{d\lambda} \right)_{\text{coll}}. \quad (168)$$

The alternative 3+1 form of the RBE can be computed from Eq. (87) when changing to the spherical variables $\tilde{p}^{(\mu)}$ in the momentum space:

$$e D_t F_{\mathbb{R}}(t, r, e, \mu) + \left(\frac{p\mu}{e} + \frac{N^{(r)}}{N} \right) e D_r F_{\mathbb{R}}(t, r, e, \mu) - p^2 \left[-(1 - \mu^2) K_{(\theta)}^{(\theta)} - \mu^2 K_{(r)}^{(r)} + \frac{\mu e}{p} D_r \nu \right] \partial_e F_{\mathbb{R}}(t, r, e, \mu) \\ + e(1 - \mu^2) \left[\frac{p}{e} \left(\frac{1}{rA} + D_r \beta \right) + \mu(K_{(r)}^{(r)} - K_{(\theta)}^{(\theta)}) - \frac{e}{p} D_r \nu \right] \partial_\mu F_{\mathbb{R}}(t, r, e, \mu) = \left(\frac{dF_{\mathbb{R}}}{d\lambda} \right)_{\text{coll}}. \quad (169)$$

In normal coordinates where $N^r = 0$, and for massless particles, Eq. (167) reduces to the relativistic Boltzmann equation derived by Lindquist [57] with the choice $B = R(r)/r$. Following Harleston and Vishniac [44], Eq. (169) can be written in ‘‘conservative’’ form which is specially suited for numerical solutions [45]:

$$\partial_t \bar{F}_{\mathbb{R}} + \frac{1}{r^2} \partial_r (r^2 V_p^r \bar{F}_{\mathbb{R}}) + \frac{1}{pe} \partial_e (p^3 \mathcal{H}_e \bar{F}_{\mathbb{R}}) + \partial_\mu (\mathcal{H}_\mu \bar{F}_{\mathbb{R}}) \\ = \frac{NAB^2}{e} \left(\frac{dF_{\mathbb{R}}}{d\lambda} \right)_{\text{coll}}, \quad (170)$$

where

$$\tilde{F}_R = AB^2 F_R(t, r, e, \mu), \quad (171)$$

$$V_p^r = \frac{p^r}{p^t} = \frac{N}{A} \left(\frac{p\mu}{e} + \frac{N^{(r)}}{N} \right), \quad (172)$$

$$\mathcal{H}_e = \frac{N}{p^2} \hat{\mathcal{L}}(e) = -N \left[-(1 - \mu^2) K_{(\theta)}^{(\theta)} - \mu^2 K_{(r)}^{(r)} + \frac{\mu e}{p} D_r \nu \right], \quad (173)$$

$$\mathcal{H}_\mu = \frac{N}{e} \hat{\mathcal{L}}(\mu) = N(1 - \mu^2) \left[\frac{p}{e} \left(\frac{1}{rA} + D_r \beta \right) + \mu (K_{(r)}^{(r)} - K_{(\theta)}^{(\theta)}) - \frac{e}{p} D_r \nu \right], \quad (174)$$

and $\hat{\mathcal{L}}$ stands for the Liouville operator as it appears in the lhs of Eq. (169). Alternatively, the above equation can be written as

$$\begin{aligned} \partial_t \tilde{F}_R + \frac{1}{r^2} \partial_r (r^2 V_p^r \tilde{F}_R) + \frac{1}{p^2} \partial_p (p^3 \mathcal{H}_p \tilde{F}_R) + \partial_\mu (\mathcal{H}_\mu \tilde{F}_R) \\ = \frac{NAB^2}{e} \left(\frac{dF_R}{d\lambda} \right)_{\text{coll}}, \end{aligned} \quad (175)$$

where now $\tilde{F}_R = \tilde{F}_R(t, r, p, \mu)$ is to be regarded as a function of p instead of e and

$$\mathcal{H}_p = \frac{N}{pe} \hat{\mathcal{L}}(p) = -N \left[-(1 - \mu^2) K_{(\theta)}^{(\theta)} - \mu^2 K_{(r)}^{(r)} + \frac{\mu e}{p} D_r \nu \right]. \quad (176)$$

A particular gauge choice will only affect the form of the quantities given by Eqs. (171)–(174) and (176), and the rhs of Eqs. (170) and (175).

Finally, I emphasize that all the momentum variables which appear in the various forms of the RBE in spherical symmetry are components with respect to the orthonormal tetrad carried by the Eulerian observer. Therefore the corresponding quantities in the collision integral are to be referred to the same observer.

D. Mean radiative variables

In Sec. IV the energy-momentum tensor of particles was defined in terms of their microscopic four-momenta. However, one can introduce mean radiative variables which are to be interpreted as their counterparts of the continuum case. Such variables are called the *moments of the distribution function*. In order to write them explicitly, I shall use the physical components of the energy-momentum tensor of particles Eq. (67) and the spherical variables in momentum

space in addition to the invariant volume element of momentum space given by Eq. (65). I have then

$$\begin{aligned} T_R^{(\mu)(\nu)} &= \int_0^\infty \int_{-1}^1 \int_0^{2\pi} p^{(\mu)} p^{(\nu)} F_R(t, r, p, \mu) \frac{p^2}{e} dp d\mu d\gamma \\ &= \int_{\tilde{m}}^\infty \int_{-1}^1 \int_0^{2\pi} p^{(\mu)} p^{(\nu)} F_R(t, r, e, \mu) \\ &\quad \times \sqrt{e^2 - \tilde{m}^2} de d\mu d\gamma. \end{aligned} \quad (177)$$

The only non-null moments are

$$\begin{aligned} E_R &:= T_R^{(t)(t)} \\ &= 2\pi \int_0^\infty p^2 \sqrt{p^2 + \tilde{m}^2} dp \int_{-1}^1 F_R(t, r, p, \mu) d\mu \\ &= 2\pi \int_{\tilde{m}}^\infty e^2 \sqrt{e^2 - \tilde{m}^2} de \int_{-1}^1 F_R(t, r, e, \mu) d\mu, \end{aligned} \quad (178)$$

$$\begin{aligned} H_R &:= T_R^{(t)(r)} = 2\pi \int_0^\infty p^3 dp \int_{-1}^1 \mu F_R(t, r, e, \mu) d\mu \\ &= 2\pi \int_{\tilde{m}}^\infty e (e^2 - \tilde{m}^2) de \int_{-1}^1 \mu F_R(t, r, e, \mu) d\mu, \end{aligned} \quad (179)$$

$$\begin{aligned} p_R &:= T_R^{(r)(r)} \\ &= 2\pi \int_0^\infty \frac{p^4}{\sqrt{p^2 + \tilde{m}^2}} dp \int_{-1}^1 \mu^2 F_R(t, r, e, \mu) d\mu \\ &= 2\pi \int_{\tilde{m}}^\infty (e^2 - \tilde{m}^2)^{3/2} de \int_{-1}^1 \mu^2 F_R(t, r, e, \mu) d\mu. \end{aligned} \quad (180)$$

The tangential pressures are given by

$$\begin{aligned} p_R^T &:= T_R^{(\theta)(\theta)} = T_R^{(\phi)(\phi)} = \pi \int_0^\infty \frac{p^4}{\sqrt{p^2 + \tilde{m}^2}} dp \\ &\quad \times \int_{-1}^1 (1 - \mu^2) F_R(t, r, e, \mu) d\mu \\ &= \pi \int_{\tilde{m}}^\infty (e^2 - \tilde{m}^2)^{3/2} de \\ &\quad \times \int_{-1}^1 (1 - \mu^2) F_R(t, r, e, \mu) d\mu. \end{aligned} \quad (181)$$

I note that in the massless case,

$$p_R^T = \frac{1}{2} (E_R - p_R). \quad (182)$$

The corresponding 3+1 variables are

$$J_{\mathbf{R}}^{(r)} = H_{\mathbf{R}}, \quad (183)$$

$$S_{\mathbf{R}}^{(r)(r)} = p_{\mathbf{R}}, \quad (184)$$

$$S_{\mathbf{R}}^{(\theta)(\theta)} = S_{\mathbf{R}}^{(\phi)(\phi)} = p_{\mathbf{R}}^T, \quad (185)$$

$$S_{\mathbf{R}} = {}_{\mathbf{R}}S_{(i)}^{(i)} = p_{\mathbf{R}} + 2p_{\mathbf{R}}^T. \quad (186)$$

The effective pressure of radiation which can be defined as $p_{\text{eff}}^{\mathbf{R}} = S_{\mathbf{R}}/3$, turns out to be in the case of massless particles $p_{\text{eff}}^{\mathbf{R}} = E_{\mathbf{R}}/3$, which corresponds precisely to the equation of state of an ultrarelativistic gas. In Eq. (179), $H_{\mathbf{R}}$ is the mean radiative flux of energy in the radial direction.

It is usual to introduce the so-called variable Eddington factor

$$\Xi = \frac{p_{\mathbf{R}}}{E_{\mathbf{R}}} \quad (187)$$

used to measure the degree of ‘‘anisotropy’’ in the particle flow. In the case of massless particles if $\Xi = 1/3$ then $p_{\mathbf{R}}^T = p_{\mathbf{R}} = E_{\mathbf{R}}/3$ which corresponds to a fully isotropic flow. Moreover, in the free streaming approximation we have

$$E_{\mathbf{R}} = J_{\mathbf{R}}^{(r)} = S_{\mathbf{R}}^{(r)(r)}, \quad (188)$$

and so $\Xi = 1$, which is the case of a highly anisotropic flow ($p_{\mathbf{R}}^T = 0$) with a purely radial flux of radiation.

In the same way, the macroscopic particle number density current measured in the Eulerian frame is given by Eq. (66) and it in terms of the spherical variables of momentum space gives

$$\begin{aligned} j_{\mathbf{R}}^{(\mu)} &= \int_0^\infty \int_{-1}^1 \int_0^{2\pi} p^{(\mu)} F_{\mathbf{R}}(t, r, p, \mu) \frac{p^2}{e} dp d\mu d\gamma \\ &= \int_{\tilde{m}}^\infty \int_{-1}^1 \int_0^{2\pi} p^{(\mu)} F_{\mathbf{R}}(t, r, e, \mu) \sqrt{e^2 - \tilde{m}^2} de d\mu d\gamma. \end{aligned} \quad (189)$$

In particular, the mean number density and flux of particles measured by the Eulerian observer are given, respectively, by

$$\begin{aligned} n_{\mathbf{E}}^{\mathbf{R}} &:= -n_{\mu} j_{\mathbf{R}}^{\mu} = j_{\mathbf{R}}^{(t)} = 2\pi \int_0^\infty \int_{-1}^1 F_{\mathbf{R}}(t, r, p, \mu) p^2 dp d\mu \\ &= 2\pi \int_{\tilde{m}}^\infty \int_{-1}^1 e F_{\mathbf{R}}(t, r, e, \mu) \sqrt{e^2 - \tilde{m}^2} de d\mu, \end{aligned} \quad (190)$$

$$\begin{aligned} j_{\mathbf{R}}^{(r)} &= 2\pi \int_0^\infty \frac{p^3}{\sqrt{p^2 + \tilde{m}^2}} dp \int_{-1}^1 \mu F_{\mathbf{R}}(t, r, e, \mu) d\mu \\ &= 2\pi \int_{\tilde{m}}^\infty (e^2 - \tilde{m}^2) de \int_{-1}^1 \mu F_{\mathbf{R}}(t, r, e, \mu) d\mu, \end{aligned} \quad (191)$$

and $j_{\mathbf{R}}^{(\theta)} = 0 = j_{\mathbf{R}}^{(\phi)}$.

In terms of the above macroscopic variables, the energy-momentum conservation equation in spherical symmetry Eq. (100) reads according to Eqs. (36) and (37) as follows [61]:

$$\begin{aligned} &\partial_{(t)} E_{\mathbf{R}} + {}^3\partial_{(r)} H_{\mathbf{R}} - E_{\mathbf{R}} K \\ &\quad + 2H_{\mathbf{R}} \left[{}^3\partial_{(r)} \nu + \frac{1}{rA} + {}^3\partial_{(r)} \beta \right] \\ &\quad - (p_{\mathbf{R}} K_{(r)}^{(r)} + 2p_{\mathbf{R}}^T K_{(\theta)}^{(\theta)}) = -\mathcal{F}_{\mathbf{R}}^{(t)}, \quad (192) \\ &\partial_{(t)} H_{\mathbf{R}} + {}^3\partial_{(r)} p_{\mathbf{R}} + 2 \left[\frac{1}{rA} + \partial_{(r)} \beta \right] [p_{\mathbf{R}} - p_{\mathbf{R}}^T] \\ &\quad + [p_{\mathbf{R}} + E_{\mathbf{R}}] {}^3\partial_{(r)} \nu - H_{\mathbf{R}} (K + K_{(r)}^{(r)}) = -\mathcal{F}_{\mathbf{R}}^{(r)}. \end{aligned} \quad (193)$$

Depending on the gauge choice some of the terms within these equations can vanish. Actually such evolution equations can be obtained directly from the RBE when multiplying this by the momenta and then integrating in momentum space.

We emphasize that it is more convenient to calculate $E_{\mathbf{R}}$, $H_{\mathbf{R}}$, and $p_{\mathbf{R}}$ directly from their definition once the distribution function has been computed, rather than using the above equations. In any case, such conservation equations can be used to verify the self-consistence of the system. The disadvantage of using the system of Eqs. (192) and (193) for the moments of the distribution instead of solving the RBE is that such a system is undetermined (i.e., there are more variables than equations). Then a closure relation is needed to remove the ambiguity (e.g., the diffusion approximation relating $E_{\mathbf{R}}$ and $p_{\mathbf{R}}$).

Finally, in spherical symmetry the evolution equations (113), (114), (122), (123), (125), and (126) write, respectively,

$$\partial_t (AB^2 n_{\mathbf{E}}) + \frac{1}{r^2} \partial_r (r^2 V^r n_{\mathbf{E}} AB^2) = 0, \quad (194)$$

$$\partial_t (AB^2 n_{\mathbf{E}}) + \frac{1}{r^2} \partial_r \left[AB^2 r^2 n_{\mathbf{E}} \left(N^r + \frac{N}{A} {}^3U^{(r)} \right) \right] = 0, \quad (195)$$

$$\partial_t \sigma + V^r \partial_r \sigma = -\frac{N}{n\Gamma} (\mu^{\mathbf{R}} \mathcal{R}_{\mathbf{R}} + \mathcal{D}), \quad (196)$$

$$\partial_t \sigma + N^r \partial_r \sigma + \frac{N^3 U^{(r)}}{A} \partial_r \sigma = -\frac{N}{n\Gamma} (\mu^{\mathbf{R}} \mathcal{R}_{\mathbf{R}} + \mathcal{D}), \quad (197)$$

$$\partial_t x_{\mathbf{R}} + V^r \partial_r x_{\mathbf{R}} = \frac{N}{n\Gamma} \mathcal{R}_{\mathbf{R}}, \quad (198)$$

$$\partial_t x_{\mathbf{R}} + N^r \partial_r x_{\mathbf{R}} + \frac{N^3 U^{(r)}}{A} \partial_r x_{\mathbf{R}} = \frac{N}{n\Gamma} \mathcal{R}_{\mathbf{R}}. \quad (199)$$

Equation (194) has a conservative form for the quantity AB^2n_E . One can alternatively write an equation in conservative form for n_E as

$$\partial_t n_E + \frac{1}{r^2} \partial_r (r^2 V^r n_E) + n_E [\partial_t \alpha + 2 \partial_r \beta + V^r (\partial_r \alpha + 2 \partial_r \beta)] = 0. \quad (200)$$

In a similar way,

$$\begin{aligned} \partial_t n_E + \frac{1}{r^2} \partial_r \left[r^2 n_E \left(N^r + \frac{N}{A} {}^3U^{(r)} \right) \right] \\ + n_E \left[\partial_t \alpha + 2 \partial_r \beta + \left(N^r + \frac{N}{A} {}^3U^{(r)} \right) (\partial_r \alpha + 2 \partial_r \beta) \right] = 0. \end{aligned} \quad (201)$$

The total contribution of sources [Eqs. (72)–(75)] that appear in the 3+1 Einstein equations write in spherical symmetry as follows:

$$E = E_{\text{PF}} + E_{\text{R}}, \quad (202)$$

$$J^{(r)} = A J^r = (E_{\text{PF}} + p) U^{(r)} + H_{\text{R}}, \quad (203)$$

$$S_{(r)}^{(r)} = (E_{\text{PF}} + p) (U^{(r)})^2 + p + p_{\text{R}}, \quad (204)$$

$$S_{(\theta)}^{(\theta)} = S_{(\phi)}^{(\phi)} = p + p_{\text{R}}^T, \quad (205)$$

$$S = S_{(i)}^{(i)} = (E_{\text{PF}} + p) (U^{(r)})^2 + 3p + p_{\text{R}} + 2p_{\text{R}}^T. \quad (206)$$

In summary, for the specific case of spherical symmetry, the spacetime is completely characterized (as it will be shown in Sec. VII) by the determination of the metric potentials N , N^r , and A in time and space. The metric potential B can be fixed from the shift conditions. Furthermore, these metric potentials can be determined completely from the differential equations that result when combining the gauge conditions and the constraint equations as shown in Secs. VII A–VII C. This means that the metric potentials will evolve through the matter terms and therefore it will not be necessary to use the evolution Einstein equations. On the other hand, the perfect-fluid variables like E_{PF} and ${}^3U^{(r)}$ (or $J_{(r)}^{\text{PF}}$) are evolved from Eqs. (151) and (158) [or Eq. (156)], respectively. The thermodynamic quantities like n_E , σ , and x_{R} are to be evolved from the different alternatives given by Eqs. (194)–(201). Finally the distribution function $F_{\text{R}}(t, r, e, \mu)$ for the species considered is used to determine the radiation observables (e.g., energy-density, fluxes, radiation pressure) and it is evolved using the RBE (170) [or alternatively Eq. (175)]. Another possibility is to use Eqs. (192) and (193) within a closure relation to evolve the radiative quantities E_{R} , H_{R} , and p_{R} , instead of using the RBE. As mentioned at the end of Sec. V, the whole system of differential equations is to be closed by the particle physics input like the EOS of matter, the nuclear-reaction rates, and

the different collision terms. Such information will depend on the specific physical problem to be analyzed.

VII. GAUGE CONDITIONS

Einstein's equations are by construction diffeomorphic invariant. This means that one has the freedom of choosing coordinates or *gauges* without affecting the physics of the spacetime. In the framework of the 3+1 formalism, the gauge choice includes the *time slicing* (lapse condition) and the choice of spatial coordinates (shift condition). In spherical symmetry two of the most popular gauge choices are the *radial* and *isotropic* spatial coordinates with the *maximal* and *polar* time slicings (see Refs. [48,63,64] for a more general discussion on the slicing choices). Moreover, it is in the framework of asymptotically flat spacetimes (a condition usually demanded in astrophysical applications) that they become specially useful; it is in this context that such gauges will be discussed in the following.

The *isotropic coordinates* $A=B$ with the maximal slicing conditions $K=0$, $\partial_t K=0$ (hereafter isotropic-maximal-slicing gauge or IMSG) have been employed by several authors (e.g., see [6,10,20,38,40,42]). In the vacuum and static case, that choice leads to the well known Schwarzschild solution in isotropic coordinates. The maximal slicing has the advantage of freezing the evolution in regions near the formation of space-like singularities while allowing a faster evolution in the outer regions (a feature usually quoted as “singularity avoidance” property). The time slicing leads to an elliptic equation for the lapse, and therefore, for rather general matter conditions (e.g., *strong energy condition*), one can use the maximum-minimum principle to determine the qualitative behavior for the lapse (cf. [48,63]). The lapse function has a minimum at $r=0$ and a maximum at $r \rightarrow \infty$. During the evolution and for the case of black-hole formation, the minimum tends to zero as $t \rightarrow \infty$ (the collapse of the lapse) halting the proper time separation between neighboring slices as the singularity forms. However, far from the origin, $N \rightarrow 1$, which allows one to advance the evolution in the asymptotic regions. The IMSG also has the advantage that the three-metric remains regular at the formation of apparent horizons, which allows one to continue the evolution. The drawback is that eventually the metric potential A grows exponentially at the origin and then the coordinates are “sucked down” to the black hole, which avoids a good description of the evolution outside the event horizon (this is the well known phenomenon of grid or slice *stretching* [e.g., see Ref. [40]).

The *radial coordinates* $B=1$ with the polar slicing condition (hereafter radial-polar-slicing gauge or RPSG) $K=K^r$, have been employed systematically byourgoulhon [8,9,36]. These coordinates are a generalization of the Schwarzschild coordinates to the nonstatic and nonvacuum spacetimes. The field equations turn out to be much more simpler than those of the IMSG since the equations for A and N reduce to first order in r . Furthermore, the shift N^r is zero everywhere on the slices. The RPSG has a central “singularity avoidance” property which is even stronger than that of the IMSG. In fact, the slowing of the evolution is such that it

avoids the formation of apparent horizons, the metric potential A , however, develops a “spike” at the star’s surface as the matter enters the Schwarzschild radius [8,11], leading, unlike the IMSG, to a coordinate crash. Thus these coordinates do not serve to describe the black hole’s interior. Nevertheless, the pathological behavior of the RPSG occurs at large t ’s, and from the astrophysical point of view (e.g., from the point of view of an observer at spatial infinity), these coordinates are good enough to describe the entire evolution of matter outside the black hole; the ingoing matter takes by the way, for an observer at infinity, an infinite time to cross the event horizon (the evolution is thus “frozen”).

The hybrid choice of isotropic coordinates with polar slicings (IPSG) has been less used in the past. However, it seems that they overcome the drawbacks of the above coordinate choices (cf. [11,40]; see also Ref. [64] for an analysis of these coordinates in the context of axisymmetry).

Another popular coordinate choice is the comoving coordinates (Lagrangian coordinates). Those coordinates have been particularly used in the study of supernova collapse with synchronous [13,24,65–67] and polar slicings [7,62]. The comoving coordinates have the advantage that the hydrodynamic equations are simpler since there is no advection. The synchronous slicings are orthogonal in the sense that there is no shift ($N^r=0$). The disadvantage of the latter is that they fail badly when black holes start forming (cf. [7,68]). The asynchronous slicings can remedy this problem. In particular, Schinder and coauthors [7,62] have used polar slicings to avoid the pathologies of the Lagrangian-synchronous gauge. Another modification to the comoving and synchronous coordinates that allows one to handle the formation of black holes is the introduction of an outgoing null coordinate instead of the usual time coordinate [27,69].

Finally, I mention the isotropic coordinates with constant-mean-curvature slicings $K=K(t)$, employed by Harleston and coauthors [44,45]. Such a slicing contains the maximal slicing as a particular case. It also possesses the feature of strong crushing coordinate avoidance. These coordinates generalize (to the nonhomogeneous case) the comoving coordinates of homogeneous and isotropic spacetimes which are relevant in the standard cosmology. Such a choice is thus useful when the spacetime is required to be asymptotically Friedmann-Robertson-Walker [45,44]. The main difference between this choice and the maximal slicing condition is thus the behavior of the hypersurfaces asymptotically: the maximal hypersurfaces reach spatial infinity while the $K(t)$ hypersurfaces reach future or past null infinity whether K is positive or negative [68].

A. Isotropic coordinates and maximal slicing (IMSG)

The isotropic choice $A=B$ ($\alpha=\beta$), implies from Eq. (130) that

$$K_r{}^r = K_{(r)}{}^{(r)} = -\frac{1}{N}(\partial_t\alpha + \partial_r N^r + N^r\partial_r\alpha). \quad (207)$$

Therefore

$$\partial_t\alpha = -NK_r{}^r - \partial_r N^r - N^r\partial_r\alpha. \quad (208)$$

On the other hand, the maximal slicing condition $K=0$ implies that

$$K_r{}^r = -2K_\theta{}^\theta, \quad (209)$$

or equivalently

$$3\partial_t\alpha + 3N^r\partial_r\alpha + \partial_r N^r + \frac{2N^r}{r} = 0. \quad (210)$$

Using Eq. (208) in the previous equation one obtains

$$\partial_r N^r - \frac{N^r}{r} = -\frac{3}{2}NK_r{}^r. \quad (211)$$

This can be written as to give the following differential equation for N^r [cf. Eq. (21) of Shapiro and Teukolsky [6]],

$$\partial_r\left(\frac{N^r}{r}\right) = -\frac{3}{2r}NK_r{}^r. \quad (212)$$

On the other hand, using Eq. (211) in Eq. (208) one obtains an evolution equation for α :

$$\partial_t\alpha = -N^r\left(\partial_r\alpha + \frac{1}{r}\right) + \frac{1}{2}NK_r{}^r. \quad (213)$$

With the above choice and with Eq. (209), the Hamiltonian constraint Eq. (135) reads

$$2\partial_{rr}^2\alpha + (\partial_r\alpha)^2 + \frac{4\partial_r\alpha}{r} = -8\pi G_0EA^2 - \frac{3}{4}A^2(K_r{}^r)^2. \quad (214)$$

Adopting the variable

$$\tilde{\alpha} = \alpha/2 \quad (215)$$

one obtains a second order differential equation for $\tilde{\alpha}$

$$\partial_{rr}^2\tilde{\alpha} + \frac{2\partial_r\tilde{\alpha}}{r} = -A^2\left[2\pi G_0E + \frac{3}{16}(K_r{}^r)^2\right] - (\partial_r\tilde{\alpha})^2, \quad (216)$$

where one recognizes in the lhs the Laplacian operator of a spherically symmetric Euclidean space [cf. Eq. (19) of Shapiro and Teukolsky [6] for a source term E including a perfect fluid alone, Eq. (142)].

The momentum constraint Eq. (137) reads

$$3K_r{}^r\left(\frac{1}{r} + \partial_r\alpha\right) + \partial_r K_r{}^r = 8\pi G_0J_r, \quad (217)$$

where I used $J_r = AJ_{(r)}$. This can be written as a differential equation for $K_r{}^r$ as [cf. Eq. (20) of Shapiro and Teukolsky [6]]

$$\partial_r(A^3r^3K_r{}^r) = 8\pi G_0r^3A^3J_r. \quad (218)$$

Equation (140) with the maximal slicing condition $K = 0$, $\partial_t K = 0$, provides an elliptic equation for N [cf. Eq. (18) of Shapiro and Teukolsky [6] for source terms E and S of a perfect fluid, Eqs. (142) and (143)],

$$\begin{aligned} & \partial_{rr}^2 \nu + (\partial_r \nu)^2 + (\partial_r \alpha)(\partial_r \nu) + \frac{2\partial_r \nu}{r} \\ & = 4\pi G_0 A^2 (S + E) + \frac{3}{2} A^2 (K_r^r)^2. \end{aligned} \quad (219)$$

We have then four differential Eqs. (212), (216), (217), and (219) for N^r , A , K_r^r , and N , respectively. It is to note in those equations that the field variables evolve in time through the matter fields. Although the evolution equation for K_r^r is redundant, for completeness I write it in the IMSG. Equations (138), (214), and (219), lead to

$$\begin{aligned} & \partial_t K_{(r)}^{(r)} + \frac{N^{(r)} \partial_r K_{(r)}^{(r)}}{A} + \frac{3}{4} N (K_{(r)}^{(r)})^2 \\ & - \frac{N}{A^2} \left[(\partial_r \nu + \partial_r \alpha) \left(\frac{2}{r} + \partial_r \alpha \right) + (\partial_r \nu)(\partial_r \alpha) \right] \\ & = -8\pi G_0 N S_{(r)}^{(r)}. \end{aligned} \quad (220)$$

Concerning the energy conservation equation (149), this reads

$$\begin{aligned} \partial_t E_{\text{PF}} + N^r \partial_r E_{\text{PF}} & = -N \partial_r J_{\text{PF}}^r - N J_{\text{PF}}^r \left(2\partial_r \nu + \frac{2}{r} + 3\partial_r \alpha \right) \\ & + N K_{(r)}^{(r)} (\text{PF} S_{(r)}^{(r)} - \text{PF} S_{(\theta)}^{(\theta)}) - N^2 \mathcal{F}_R^t. \end{aligned} \quad (221)$$

Or in terms of $J_r = A^2 J^r$ one obtains

$$\begin{aligned} \partial_t E_{\text{PF}} + N^r \partial_r E_{\text{PF}} & = -\frac{N}{A^2} \partial_r J_r^{\text{PF}} - \frac{N}{A^2} J_r^{\text{PF}} \left(2\partial_r \nu + \frac{2}{r} + \partial_r \alpha \right) \\ & + N K_{(r)}^{(r)} (\text{PF} S_{(r)}^{(r)} - \text{PF} S_{(\theta)}^{(\theta)}) - N^2 \mathcal{F}_R^t. \end{aligned} \quad (222)$$

The conservative form Eq. (151) writes in this gauge as

$$\begin{aligned} & \partial_t E_{\text{PF}} + \frac{1}{r^2} \partial_r (r^2 V^r E_{\text{PF}}) \\ & = E_{\text{PF}} \left(\frac{2N^r}{r} + \partial_r N^r \right) - \frac{1}{r^2} \partial_r \left(r^2 \frac{N}{A} {}^3 U^{(r)} p \right) \\ & - \frac{N J_{(r)}^{\text{PF}}}{A} [\partial_r \nu + 3\partial_r \alpha - {}^3 U^{(r)} A K_{(r)}^{(r)} + N A \mathcal{F}_R^t]. \end{aligned} \quad (223)$$

The momentum conservation equation writes

$$\begin{aligned} \partial_t J_r^{\text{PF}} + N^r \partial_r J_r^{\text{PF}} & = -J_r^{\text{PF}} \partial_r N^r - N \left[\partial_r \text{PF} S_{(r)}^{(r)} \right. \\ & + 2(\text{PF} S_{(r)}^{(r)} - \text{PF} S_{(\theta)}^{(\theta)}) \left(\frac{1}{r} + \partial_r \alpha \right) \\ & \left. + (\text{PF} S_{(r)}^{(r)} + E_{\text{PF}}) \partial_r \nu + {}^3 \mathcal{F}_R^r \right]. \end{aligned} \quad (224)$$

In terms of triad components Eq. (157) reads

$$\begin{aligned} & \partial_r J_{(r)}^{\text{PF}} + \frac{1}{r^2} \partial_r (r^2 V^r J_{(r)}^{\text{PF}}) \\ & = J_{(r)}^{\text{PF}} \left[N K_{(r)}^{(r)} - 3V^r \partial_r \alpha + N^r \left(3\partial_r \alpha - \frac{2}{r} \right) - \partial_r N^r \right] \\ & - \frac{N}{A} [(E_{\text{PF}} + p) \partial_r \nu + \partial_r p + {}^3 \mathcal{F}_R^r A]. \end{aligned} \quad (225)$$

In the IMSG, the Euler equation (158) reads

$$\begin{aligned} & \partial_{(t)} {}^3 U^{(r)} + {}^3 U^{(r)} {}^3 \partial_{(r)} {}^3 U^{(r)} \\ & = -\frac{1}{E_{\text{PF}} + p} [{}^3 \partial^{(r)} p + {}^3 U^{(r)} \partial_{(t)} p] \\ & + \frac{1}{\Gamma^2} ({}^3 U^{(r)} K_{(r)}^{(r)} - {}^3 \partial_{(r)} \nu) \\ & + \frac{1}{E_{\text{PF}} + p} ({}^3 U^{(r)} \mathcal{F}_R^{(t)} - \mathcal{F}_R^{(r)}). \end{aligned} \quad (226)$$

Under the IMSG the evolution equations for the entropy per baryon and the particle number per baryon keep the same form as Eqs. (196) and (198) or the alternative form given by Eqs. (197) and (199), where Eqs. (194) and (195) read

$$\partial_t (A^3 n_E) + \frac{1}{r^2} \partial_r [r^2 V^r n_E A^3] = 0, \quad (227)$$

$$\partial_t (A^3 n_E) + \frac{1}{r^2} \partial_r \left[A^3 r^2 n_E \left(N^r + \frac{N}{A} {}^3 U^{(r)} \right) \right] = 0. \quad (228)$$

Equation (227) has a conservative form for the quantity $n_E A^3$.

When using the evolution equation (213), Eqs. (227) and (228) become, respectively,

$$\begin{aligned} & \partial_t n_E + \frac{1}{r^2} \partial_r [r^2 V^r n_E] \\ & + 3n_E \left[(V^r - N^r) \partial_r \alpha - \frac{N^r}{r} + \frac{1}{2} N K_r^r \right] = 0, \end{aligned} \quad (229)$$

$$\begin{aligned} \partial_t n_E + \frac{1}{r^2} \partial_r \left[r^2 n_E \left(N^r + \frac{N}{A} {}^3U^{(r)} \right) \right] \\ + 3n_E \left[\frac{N}{A} {}^3U^{(r)} \partial_r \alpha - \frac{N^r}{r} + \frac{1}{2} N K_r^r \right] = 0. \end{aligned} \quad (230)$$

Equation (229) has a conservative form for the quantity n_E .

B. Radial coordinates and polar slicing (RPSG)

The radial-coordinate condition is achieved by setting $B = 1$ ($\beta = 0$) and the polar slicing $K = K_r^r$ is equivalent to $K_\theta^\theta + K_\phi^\phi = 0$. Since $K_\theta^\theta = K_\phi^\phi$, the RPSG leads to $N^r = 0$.

The Hamiltonian constraint Eq. (135) reads

$$\frac{1}{r^2} (A^2 - 1) + \frac{2}{r} \partial_r \alpha = 8\pi G_0 E A^2. \quad (231)$$

Moreover, by defining

$$A(t, r) := \left(1 - \frac{2G_0 m(t, r)}{r} \right)^{-1/2}, \quad (232)$$

Eq. (231) reads

$$\partial_r \alpha = A^2 G_0 \left(4\pi r E - \frac{m}{r^2} \right) \quad (233)$$

or even [cf. Eq. (18) of Ref. [8] or Eq. (3.29) of Ref. [36] for a perfect fluid alone or for a perfect fluid accompanied by a neutrino flow, respectively],

$$\partial_r m = 4\pi r^2 E. \quad (234)$$

The momentum constraint (136) for the present gauge choice reads

$$\begin{aligned} K_{(r)}^{(r)} &= -\frac{1}{AN} \partial_t A = 4\pi r G_0 J_r = 4\pi r G_0 A^2 J^r \\ &= 4\pi r G_0 A J^{(r)}. \end{aligned} \quad (235)$$

This with Eq. (232) results in an evolution equation for $m(t, r)$ [cf. Eq. (20) of Ref. [8] or Eq. (3.31) of Ref. [36] for a perfect fluid alone or for a perfect fluid accompanied by a neutrino flow, respectively]:

$$\partial_t m = -4\pi r^2 N J^r. \quad (236)$$

The evolution equation (139) gives

$$\frac{\partial_r \nu}{r} - \left[\frac{1}{r^2} (A^2 - 1) + \frac{\partial_r A}{rA} \right] = 4\pi G_0 A^2 (S_{(r)}^{(r)} - E). \quad (237)$$

With Eqs. (232) and (231) this writes [cf. Eq. (22) of Ref. [8] and Eq. (3.32) of Ref. [36] for a source term $S_{(r)}^{(r)}$ of a perfect fluid alone, Eq. (143), or that of a perfect fluid accompanied by a neutrino flow Eq. (204), respectively],

$$\partial_r \nu = G_0 A^2 \left(\frac{m}{r^2} + 4\pi r S_{(r)}^{(r)} \right). \quad (238)$$

Therefore Eqs. (234) and (238) are the field equations for the two variables A and N , respectively. These quantities evolve in time through the matter variables. Therefore, as in the IMSG, the evolution equation for $K_{(r)}^{(r)}$ is also redundant in the RPSG. For completeness I write it using Eq. (138) [cf. Eq. (21) of Ref. [8] for a perfect fluid alone],

$$\begin{aligned} \partial_t K_{(r)}^{(r)} - N (K_{(r)}^{(r)})^2 \\ - \frac{N}{A^2} \left[\frac{2\partial_r \alpha}{r} - \partial_{rr}^2 \nu + (\partial_r \nu)(\partial_r \alpha - \partial_r \nu) \right] \\ = 4\pi G_0 N (-S_{(r)}^{(r)} + 2S_{(\theta)}^{(\theta)} - E). \end{aligned} \quad (239)$$

Concerning the evolution equations for the matter, Eq. (149) reads

$$\begin{aligned} \partial_t E_{\text{PF}} + \frac{N}{Ar^2} \partial_r (Ar^2 J_{\text{PF}}^r) \\ = NK_{(r)}^{(r)} ({}_{\text{PF}}S_{(r)}^{(r)} + E_{\text{PF}}) - 2J_{\text{PF}}^r \partial_r N - N^2 \mathcal{F}_R^t, \end{aligned} \quad (240)$$

which can be written as

$$\begin{aligned} \partial_t E_{\text{PF}} + \frac{1}{r^2} \partial_r (Nr^2 J_{\text{PF}}^r) = NK_{(r)}^{(r)} ({}_{\text{PF}}S_{(r)}^{(r)} + E_{\text{PF}}) \\ - NJ_{\text{PF}}^r (\partial_r \nu + \partial_r \alpha) - N^2 \mathcal{F}_R^t. \end{aligned} \quad (241)$$

The gradients of the metric potentials can be replaced by using Eqs. (231) and (237) which imply

$$\partial_r \nu + \partial_r \alpha = 4\pi r G_0 A^2 (S_{(r)}^{(r)} + E). \quad (242)$$

Then

$$\begin{aligned} \partial_t E_{\text{PF}} + \frac{1}{r^2} \partial_r (Nr^2 J_{\text{PF}}^r) \\ = NK_{(r)}^{(r)} ({}_{\text{PF}}S_{(r)}^{(r)} + E_{\text{PF}}) \\ - 4\pi r G_0 N A^2 J_{\text{PF}}^r (S_{(r)}^{(r)} + E) - N^2 \mathcal{F}_R^t. \end{aligned} \quad (243)$$

Using the momentum constraint (235), one obtains

$$\begin{aligned} \partial_t E_{\text{PF}} + \frac{1}{r^2} \partial_r (Nr^2 J_{\text{PF}}^r) \\ = 4\pi r G_0 N A^2 [J^r ({}_{\text{PF}}S_{(r)}^{(r)} + E_{\text{PF}}) \\ - J_{\text{PF}}^r (S_{(r)}^{(r)} + E)] - N^2 \mathcal{F}_R^t. \end{aligned} \quad (244)$$

Now, since

$$J^r = J_{\text{PF}}^r + J_R^r, \quad (245)$$

$$E = E_{\text{PF}} + E_{\text{R}}, \quad (246)$$

$$S_{(r)}^{(r)} = {}_{\text{PF}}S_{(r)}^{(r)} + {}_{\text{R}}S_{(r)}^{(r)}, \quad (247)$$

I find

$$\begin{aligned} \partial_t E_{\text{PF}} + \frac{1}{r^2} \partial_r (Nr^2 J_{\text{PF}}^r) &= 4\pi r G_0 N A^2 [J_{\text{R}}^r ({}_{\text{PF}}S_{(r)}^{(r)} + E_{\text{PF}}) \\ &\quad - J_{\text{PF}}^r ({}_{\text{R}}S_{(r)}^{(r)} + E_{\text{R}})] - N^2 \mathcal{F}_{\text{R}}^t. \end{aligned} \quad (248)$$

Using that $J_{\text{PF}}^{(r)} = A J_{\text{PF}}^r$, $J_{\text{R}}^{(r)} = A J_{\text{R}}^r$, and Eqs. (143) and (145), the energy conservation Eq. (248) finally reads [cf. Eq. (25) of Ref. [8] or Eq. (3.55) of Ref. [36] for a perfect fluid alone or for a perfect fluid accompanied by a neutrino flow, respectively],

$$\begin{aligned} \partial_t E_{\text{PF}} + \frac{1}{r^2} \partial_r [r^2 (E_{\text{PF}} + p) V^r] \\ = 4\pi r G_0 N A (E_{\text{PF}} + p) [J_{\text{R}}^{(r)} ((U^{(r)})^2 + 1) \\ - U^{(r)} ({}_{\text{R}}S_{(r)}^{(r)} + E_{\text{R}})] - N^2 \mathcal{F}_{\text{R}}^t. \end{aligned} \quad (249)$$

The alternative expression of Eq. (249) in conservative form reads

$$\begin{aligned} \partial_t E_{\text{PF}} + \frac{1}{r^2} \partial_r (r^2 V^r E_{\text{PF}}) \\ = - \frac{1}{r^2} \partial_r (r^2 V^r p) + 4\pi r G_0 N A (E_{\text{PF}} + p) \\ \times [J_{\text{R}}^{(r)} ((U^{(r)})^2 + 1) - U^{(r)} ({}_{\text{R}}S_{(r)}^{(r)} + E_{\text{R}})] - N^2 \mathcal{F}_{\text{R}}^t. \end{aligned} \quad (250)$$

The momentum conservation Eq. (153) reads

$$\begin{aligned} \partial_t J_r^{\text{PF}} + N \partial_r {}_{\text{PF}}S_{(r)}^{(r)} + \frac{2N}{r} {}_{\text{PF}}S_{(r)}^{(r)} - {}_{\text{PF}}S_{(\theta)}^{(\theta)} \\ = - ({}_{\text{PF}}S_{(r)}^{(r)} + E_{\text{PF}}) \partial_r N + N J_r^{\text{PF}} K_{(r)}^{(r)} - {}^3\mathcal{F}_r^{\text{R}} N. \end{aligned} \quad (251)$$

This can be written as

$$\begin{aligned} \partial_t J_r^{\text{PF}} = N \left[- \frac{1}{r^2 N} \partial_r (r^2 N {}_{\text{PF}}S_{(r)}^{(r)}) + \frac{2}{r} {}_{\text{PF}}S_{(\theta)}^{(\theta)} \right. \\ \left. - E_{\text{PF}} \partial_r \nu + J_r^{\text{PF}} K_{(r)}^{(r)} - {}^3\mathcal{F}_r^{\text{R}} \right]. \end{aligned} \quad (252)$$

Using Eq. (157) the conservative form of the latter reads

$$\begin{aligned} \partial_t J_{(r)}^{\text{PF}} + \frac{1}{r^2} \partial_r (r^2 V^r J_{(r)}^{\text{PF}}) \\ = J_{(r)}^{\text{PF}} [2NK_{(r)}^{(r)} - V^r \partial_r \alpha] - \frac{N}{A} [(E_{\text{PF}} + p) \partial_r \nu + \partial_r p \\ + {}^3\mathcal{F}_{(r)}^{\text{R}} A]. \end{aligned} \quad (253)$$

Furthermore, the use of Eqs. (232), (233), (235), and (238) lead to

$$\begin{aligned} \partial_t J_{(r)}^{\text{PF}} + \frac{1}{r^2} \partial_r (r^2 V^r J_{(r)}^{\text{PF}}) \\ = G_0 J_{(r)}^{\text{PF}} \left[8\pi r A N J^{(r)} - A^2 V^r \left(4\pi r E - \frac{m}{r^2} \right) \right] \\ - G_0 N A (E_{\text{PF}} + p) \left(\frac{m}{r^2} + 4\pi r S_{(r)}^{(r)} \right) \\ - \frac{N}{A} (\partial_r p + {}^3\mathcal{F}_{(r)}^{\text{R}} A). \end{aligned} \quad (254)$$

Finally, using Eqs. (245) and (247) and the expressions (143) and (145), I obtain

$$\begin{aligned} \partial_t J_{(r)}^{\text{PF}} + \frac{1}{r^2} \partial_r (r^2 V^r J_{(r)}^{\text{PF}}) \\ = G_0 J_{(r)}^{\text{PF}} \left\{ 4\pi r A N (J_{(r)}^{\text{PF}} + 2J_{\text{R}}^{(r)}) \right. \\ \left. - A^2 V^r \left[4\pi r (E_{\text{PF}} + E_{\text{R}}) - \frac{m}{r^2} \right] \right\} \\ - G_0 N A (E_{\text{PF}} + p) \left[\frac{m}{r^2} + 4\pi r (p + {}_{\text{R}}S_{(r)}^{(r)}) \right] \\ - \frac{N}{A} (\partial_r p + {}^3\mathcal{F}_{(r)}^{\text{R}} A). \end{aligned} \quad (255)$$

The Euler equation (158) and Eqs. (235) and (238) lead to

$$\begin{aligned} \partial_{(t)} {}^3U^{(r)} + {}^3U^{(r)} \partial_{(r)} {}^3U^{(r)} \\ = - \frac{1}{E_{\text{PF}} + p} [{}^3\partial^{(r)} p + {}^3U^{(r)} \partial_{(t)} p] \\ - \frac{G_0 A}{\Gamma^2} \left[\frac{m}{r^2} + 4\pi r (S_{(r)}^{(r)} - {}^3U^{(r)} J_{(r)}) \right] \\ + \frac{1}{E_{\text{PF}} + p} ({}^3U^{(r)} \mathcal{F}_{\text{R}}^{(t)} - {}^3\mathcal{F}_{\text{R}}^{(r)}). \end{aligned} \quad (256)$$

Again, using Eqs. (245) and (247) and the expressions (143) and (145), I obtain

$$\begin{aligned}
& \partial_t^3 U^{(r)} + V^r \partial_r^3 U^{(r)} \\
&= -\frac{1}{(E_{\text{PF}} + p)} \left(\frac{N}{A} \partial_r p + {}^3U^{(r)} \partial_t p \right) \\
&\quad - \frac{ANG_0}{\Gamma^2} \left[\frac{m}{r^2} + 4\pi r(p + {}_R S_{(r)}^{(r)} - {}^3U^{(r)} J_R^{(r)}) \right] \\
&\quad + \frac{1}{(E_{\text{PF}} + p)} ({}^3U^{(r)} \mathcal{F}_R^{(t)} - {}^3\mathcal{F}_R^{(r)} N), \tag{257}
\end{aligned}$$

where I also used that in this gauge $\partial_{(t)} = 1/N \partial_t$, $\partial_{(r)} = 1/A \partial_r$, and $U^{(r)} = (A/N)V^r$.

This is the Euler equation of the fluid in spherical symmetry which includes the forces of the radiation fields acting on the fluid [cf. Eq. (34) of Ref. [8] or Eq. (3.56) of Ref. [36] for a perfect fluid alone or for a perfect fluid accompanied by a neutrino flow, respectively].

A relation that turns out to be useful in this gauge is obtained by combining Eqs. (235) and (242)

$$\frac{1}{AN} \partial_t (A^2) + \partial_r \nu + \partial_r \alpha = 4\pi r G_0 A^2 [S_{(r)}^{(r)} + E - 2J^{(r)}]. \tag{258}$$

For example, in the static case and for perfect fluids, this provides a simple relation between the gradients of the metric potentials and the pressure and the energy density of matter. Moreover, outside the star surface the only contributions to the total matter variables are those of radiated particles. Under the free streaming approximation [cf. Eq. (188)], this implies that the rhs of Eq. (258) vanishes. This situation was investigated analytically in the past using a different gauge [70,71] and corresponds to the external solution.

Finally, the integrability condition $\partial_{r_t}^2 m = \partial_{t_r}^2 m$ imposed in Eqs. (234) and (236) results in the relationship

$$\partial_t E + \frac{1}{r^2} \partial_r (r^2 N J^r) = 0. \tag{259}$$

This equation is in fact compatible with the evolution equation for the total energy density of matter.

Indeed, subtracting Eq. (244) or more specifically Eq. (249) from Eq. (259) one obtains an evolution equation for the energy density of radiation E_R [cf. Eq. (192)].

The evolution equation (113) and the alternative form (195) in the RPSG read, respectively,

$$\partial_t (A n_E) + \frac{1}{r^2} \partial_r [r^2 V^r A n_E] = 0, \tag{260}$$

$$\partial_t (A n_E) + \frac{1}{r^2} \partial_r (r^2 n_E N^3 U^{(r)}) = 0. \tag{261}$$

Note that Eq. (260) has a conservative form for the quantity $A n_E$.

Using Eqs. (235), (233), (245), (145), and (147) in Eq. (260) it becomes

$$\begin{aligned}
& \partial_t n_E + \frac{1}{r^2} \partial_r [r^2 n_E V^r] - G_0 A^2 n_E \left[V^r \left(\frac{m}{r^2} + 4\pi r p \right) \right. \\
& \quad \left. + 4\pi r N J_R^r \right] = 0, \tag{262}
\end{aligned}$$

which provides an equation in conservative form for n_E .

In the same way, using Eqs. (235) and (238) in the alternative Eq. (261) I obtain

$$\begin{aligned}
& \partial_t n_E + \frac{N}{A r^2} \partial_r (r^2 {}^3U^{(r)} n_E) + n_E G_0 A N \left[-4\pi r J^{(r)} \right. \\
& \quad \left. + {}^3U^{(r)} \left(\frac{m}{r^2} + 4\pi r S_{(r)}^{(r)} \right) \right] = 0. \tag{263}
\end{aligned}$$

The evolution Eqs. (196) and (198) do not change in form under the RPSG. However, the alternative form given by Eqs. (197) and (199) in the RPSG write, respectively,

$$\partial_t \sigma + \frac{N^3 U^{(r)}}{A} \partial_r \sigma = -\frac{N}{n \Gamma \Theta} (\mu^R \mathcal{R}_R + \mathcal{D}), \tag{264}$$

$$\partial_t x_R + \frac{N^3 U^{(r)}}{A} \partial_r x_R = \frac{N}{n \Gamma} \mathcal{R}_R. \tag{265}$$

C. Isotropic coordinates and polar slicing (IPSG)

The isotropic choice $A = B$ (i.e., $\alpha = \beta$), and the polar slicing condition $K_\theta^\theta + K_\phi^\phi = 0$, implies due to the spherical symmetry that $K_\theta^\theta = 0$. The latter leads to an evolution equation for α :

$$\partial_t \alpha = -N^r \partial_r \alpha - \frac{N^r}{r}. \tag{266}$$

This and the expression for K^r_r [cf. Eq. (130)] provide an equation for the shift:

$$\partial_r \left(\frac{N^r}{r} \right) = -\frac{N}{r} K_{(r)}^{(r)}. \tag{267}$$

The Hamiltonian constraint (135) leads to

$$\partial_{r_r}^2 \tilde{\alpha} + \frac{2 \partial_r \tilde{\alpha}}{r} = -2\pi G_0 A^2 E - (\partial_r \tilde{\alpha})^2, \tag{268}$$

where

$$\tilde{\alpha} = \alpha/2. \tag{269}$$

The momentum constraint (136) writes

$$K_{(r)}^{(r)} \left(\frac{1}{r} + \partial_r \alpha \right) = 4\pi G_0 A J_{(r)} = 4\pi G_0 J_r. \tag{270}$$

Equation (139) together with Eq. (268) provide an equation for the lapse,

$$\partial_r \nu \left(\frac{1}{r} + 2 \partial_r \tilde{\alpha} \right) = 4 \pi G_0 A^2 S_{(r)}^{(r)} - 2 \partial_r \tilde{\alpha} \left(\frac{1}{r} + \partial_r \tilde{\alpha} \right). \quad (271)$$

In the case of a perfect fluid alone, this corresponds to Eq. (9) of Shapiro and Teukolsky [40], and to Eq. (6) of Schinder *et al.* [7] where the authors use a Lagrangian-polar-slicing gauge. Such a gauge can be easily transformed to the IPSG.

Finally, the evolution equation for $K_{(r)}^{(r)}$ given by Eq. (138) together with Eq. (268) yield

$$\begin{aligned} \partial_t K_{(r)}^{(r)} + \frac{N^{(r)} \partial_r K_{(r)}^{(r)}}{A} - N (K_{(r)}^{(r)})^2 \\ - \frac{N}{A^2} \left[-\partial_{rr}^2 \nu + (\partial_r \nu) (\partial_r \alpha - \partial_r \nu) + (\partial_r \alpha) \left(\frac{2}{r} + 2 \partial_r \alpha \right) \right] \\ = 4 \pi G_0 N (-S_{(r)}^{(r)} + 2S_{(\theta)}^{(\theta)} + E). \end{aligned} \quad (272)$$

The equation of conservation of energy (151) reads in this gauge as follows:

$$\begin{aligned} \partial_t E_{\text{PF}} + \frac{1}{r^2} \partial_r (r^2 V^r E_{\text{PF}}) \\ = E_{\text{PF}} \left(N K_{(r)}^{(r)} + \frac{2N^r}{r} + \partial_r N^r \right) - \frac{1}{r^2} \partial_r \left(r^2 \frac{N}{A} {}^3 U^{(r)} p \right) \\ - \frac{N J_{(r)}^{\text{PF}}}{A} [\partial_r \nu + 3 \partial_r \alpha - {}^3 U^{(r)} A K_{(r)}^{(r)}] \\ + N K_{(r)}^{(r)} p - N^2 \mathcal{F}_R^t. \end{aligned} \quad (273)$$

The equation of conservation of momentum Eq. (157) reads

$$\begin{aligned} \partial_t J_{(r)}^{\text{PF}} + \frac{1}{r^2} \partial_r (r^2 V^r J_{(r)}^{\text{PF}}) \\ = J_{(r)}^{\text{PF}} \left[2N K_{(r)}^{(r)} - 3V^r \partial_r \alpha + N^r \left(3 \partial_r \alpha - \frac{2}{r} \right) \right. \\ \left. - \partial_r N^r \right] - \frac{N}{A} [(E_{\text{PF}} + p) \partial_r \nu + \partial_r p + {}^3 \mathcal{F}_{(r)}^R A]. \end{aligned} \quad (274)$$

The Euler equation (158) in this gauge takes the same form of Eq. (226), and Eqs. (197), (199) and (195) [see Eqs. (227) and (228)] also keep the same form as in the IMSG. Alternatively, one can also use the simpler form of Eqs. (196) and (198). Furthermore, when using the evolution equation (266) in Eqs. (200) and (201) one obtains in the IPSG,

$$\begin{aligned} \partial_t n_E + \frac{1}{r^2} \partial_r [r^2 V^r n_E] \\ + 3n_E \left[(V^r - N^r) \partial_r \alpha - \frac{N^r}{r} \right] = 0, \end{aligned} \quad (275)$$

$$\begin{aligned} \partial_t n_E + \frac{1}{r^2} \partial_r \left[r^2 n_E \left(N^r + \frac{N}{A} {}^3 U^{(r)} \right) \right] \\ + 3n_E \left[\frac{N}{A} {}^3 U^{(r)} \partial_r \alpha - \frac{N^r}{r} \right] = 0. \end{aligned} \quad (276)$$

D. Boundary conditions and initial data

A typical feature of spherically symmetric spacetimes is that the gravitational field variables can evolve in time through the matter fields. So the initial conditions for the matter variables and the boundary conditions fix automatically the initial values for the gravitational field by solving the constraint equations and the gauge condition equations. For spacetimes with less symmetries one is always forced to solve the dynamic Einstein equations to evolve the gravitational field. Let me thus discuss first the boundary conditions.

I call *exterior solution* the solution of field equations outside the perfect-fluid domain (usually a compact support). In the present case, it does not correspond to the Schwarzschild vacuum solution since in general, the radiated matter will extend to spatial infinity. Thus the exterior solution has to be found also numerically. The exterior sources of the field equations will be provided by the energy-momentum tensor of particles [cf. Eq. (177)]. The matter variables of particles will evolve in time through the distribution function. Moreover, outside the star, the radiated particles can interact only with themselves, however, this interaction is rather weak in comparison with the interaction inside the star. In a first approximation one can thus neglect such interactions and consider that the particles will follow geodesics; the distribution function will thus remain constant along them.

Regarding the boundary conditions, these are rather regularity and asymptotic conditions. For instance, the regularity and the asymptotic flatness condition for the shift are, respectively (see Ref. [64] for a more detailed analysis about regularity and boundary conditions),

$$N^r(t, 0) = 0, \quad (277)$$

$$N^r(t, r)_{r \rightarrow \infty} \rightarrow 0. \quad (278)$$

Similar conditions apply for K_r^r . These boundary conditions are enforced from Eqs. (212) and (267):

$$N^r(t, r) = \lambda_{\text{TS}} r \int_r^\infty \frac{N(t, r')}{r'} K_{(r)}^{(r)}(t, r') dr', \quad (279)$$

where $\lambda_{\text{TS}} = 1, 3/2$ for the IGPS and IGMS coordinates, respectively.

The condition for the lapse at the star's center is such that the asymptotic flatness condition $N \rightarrow 1$ is verified. Therefore

$$N(t, 0) = N_c(t), \quad (280)$$

with $N_c(t)$ such that

$$N(t, r)_{r \rightarrow \infty} \rightarrow 1. \quad (281)$$

Since *a priori* this is difficult to enforce, a better strategy consists of rescaling the lapse as $\tilde{N}=N/N_c$ so that $\tilde{N}_c(t) \equiv 1$, then the values of the true lapse can be recovered by using $N_c(t)=1/\tilde{N}(t,r)_{r \rightarrow \infty}$ where the asymptotic value $\tilde{N}_\infty(t):=\tilde{N}(t,r)_{r \rightarrow \infty}$ is found numerically at every time step. This rescaling allows one to integrate the equations spatially in only one cycle. The rescaling will not affect the relevant equations of motion for N or A since only the derivatives of ν appear there [i.e., Eqs. (216), (219), (231), (237), (268), and (271) for N and A in the different gauges are invariant to such a rescaling]. However, this is not true for the shift Eqs. (212) and (267) and for the Boltzmann equation where N appears explicitly. However, this does not pose a problem since a simultaneous rescaling $\tilde{N}^r=N^r/N_c$ leaves all equations invariant as well as the boundary conditions for \tilde{N}^r .

In the case of the RPSG and IPSG one can find an integral expression for the lapse satisfying the boundary conditions. For instance, from Eq. (238)

$$\nu(t,r) = G_0 \int_0^r A^2 \left(\frac{m}{r'^2} + 4\pi r' S_{(r)}^{(r)} \right) dr' + \nu(t,0). \quad (282)$$

The asymptotic flatness condition Eq. (281) leads to

$$\nu(t,r)_{r \rightarrow \infty} \rightarrow 0. \quad (283)$$

Therefore from Eq. (282) and the asymptotic condition one obtains

$$\nu(t,0) = -G_0 \int_0^\infty A^2 \left(\frac{m}{r'^2} + 4\pi r' S_{(r)}^{(r)} \right) dr'. \quad (284)$$

This corresponds precisely to the renormalized value $-\tilde{\nu}_\infty$. So finally,

$$\nu(t,r) = -G_0 \int_r^\infty A^2 \left(\frac{m}{r'^2} + 4\pi r' S_{(r)}^{(r)} \right) dr'. \quad (285)$$

The value for the lapse at the star surface is provided by

$$\nu[t,R(t)] = -G_0 \int_{R(t)}^\infty A^2 \left(\frac{m}{r'^2} + 4\pi r' S_{(r)}^{(r)} \right) dr', \quad (286)$$

where $R(t)$ corresponds to the RPSG- r coordinate at the star surface at time t . I emphasize that, outside the star $S_{(r)}^{(r)} = {}_R S_{(r)}^{(r)}$, that is, the only contribution to $S_{(r)}^{(r)}$ is from the radiated particles. In fact, outside the star I can write,

$$\begin{aligned} \nu(t,r)_{\text{out}} &= -G_0 \int_{r \geq R(t)}^\infty A^2 \left(\frac{m}{r'^2} + 4\pi r' S_{(r)}^{(r)} \right) dr' \\ &= [\ln A]_{r \geq R(t)}^\infty - 4\pi \int_{r \geq R}^\infty A^2 r' ({}_R S_{(r)}^{(r)} + E_R) dr' \end{aligned}$$

$$\begin{aligned} &= \ln A(t,r)^{-1} \Big|_{r \geq R(t)} \\ &\quad - 4\pi \int_{r \geq R(t)}^\infty A^2 r' ({}_R S_{(r)}^{(r)} + E_R) dr' \\ &= \ln \left[1 - \frac{2G_0 m(t,r)}{r} \right]_{r \geq R(t)}^{1/2} \\ &\quad - 4\pi \int_{r \geq R(t)}^\infty A^2 r' ({}_R S_{(r)}^{(r)} + E_R) dr', \quad (287) \end{aligned}$$

where I used Eq. (233) in order to replace the term m/r'^2 and also the asymptotic flatness condition on A [cf. Eq. (290) below]. It is to be stressed that in the absence of matter outside the star surface, the first term of Eq. (287) corresponds to the expression for ν of the Schwarzschild metric [with $m(t,r)|_{r \geq R(t)} = M_*$, M_* being the total mass of the star]. However, in the present case there are contributions (due to the energy-density E_R and pressure ${}_R S_{(r)}^{(r)}$) of particles which fills some part of the space outside the star) which are responsible for the actual metric to deviate from the Schwarzschild one. These contributions arise from the second term of Eq. (287). In some cases (e.g., the free streaming approximation) numerical analysis shows that these deviations are so small they can be neglected (cf. [36]).

Deviations of this kind can be appreciated more easily in the presence of nontrivial scalar fields, for instance in the phenomenon of spontaneous scalarization [72]. Moreover, for $r > R(t)$, the mass function $m(t,r)$ is larger than $m[t,R(t)]$ due to the contribution of E_R to the total energy density [cf. Eq. (234)]. Indeed the mass difference is given by

$$\delta m = 4\pi \int_{R(t)}^{r > R(t)} E_R r'^2 dr'. \quad (288)$$

Another way to appreciate such deviations is by noting that Eq. (258) together with the asymptotic conditions imply, in the case of vacuum, the relationship $AN=1$ which is characteristic of the Schwarzschild solution in the RPSG. However, when matter is present outside the star, then $AN \neq 1$ (e.g., see [72]), except of course at spatial infinity.

The boundary conditions for A are similar to those for N . Therefore

$$A(t,0) = A_c(t), \quad (289)$$

with $A_c(t)$ such that

$$A(t,r)_{r \rightarrow \infty} \rightarrow 1. \quad (290)$$

In the case of the RPSG the reparametrization Eq. (232) imposes the regularity condition

$$m(t,0) = 0. \quad (291)$$

Since near the origin $m \sim r^3$, the reparametrization enforces that $A(t,0) = 1$. Then $\partial_t A(t,0) = 0$. The three-metric is thus locally flat at the origin. Moreover, provided that the energy density of sources falls off at least as fast as $1/r^4$ outside the

star, the same mass parametrization will drive A to the required asymptotic value. Note that this behavior of the metric potential A is compatible with the regularity and asymptotic conditions for K_r^r that I mentioned above. Moreover, these conditions also imply that J^r vanish at the origin as well as asymptotically [cf. Eq. (235)].

In the case of the IMSG and IPSG, one has an elliptic equation for $\tilde{\alpha}$ which is not invariant to a rescaling on A . So unlike the RPSG where $A_c = 1$, the central value A_c should be determined from a shooting method or otherwise in order to satisfy the asymptotic flatness condition. In fact, near the origin $N^r \sim r$, therefore from Eq. (213), it turns out that at the origin

$$\partial_t \alpha(t, 0) = \text{const}, \quad (292)$$

and thus from the definition Eq. (133) and the regularity condition (293) (see below) one concludes that (depending on the sign of the constant) $A(t, 0)$ can grow exponentially (grid stretching phenomenon) [6,10,38,40,42]. The three-metric is thus conformally flat at the origin in the isotropic gauge.

In addition to the previous regularity conditions we have also

$$\partial_r Q|_{r=0} = 0 = Q^r|_{r=0}, \quad (293)$$

where Q represents collectively the metric potentials and the scalar matter field variables (e.g., N, A, m, p, ρ, E , etc.) and Q^r tensor field components (e.g., J^r, K_r^r , etc.).

A convenient way to impose the asymptotic conditions accurately is by compactifying the outer domain with the help of a transformation $u = 1/r$ from the star's surface $r = R$ to spatial infinity. In this way the infinite domain $r \in [R, \infty)$ is mapped to the compact domain $u \in [1/R, 0)$, so the integration can be performed from $1/R$ to "zero" with a high degree of accuracy (cf. [72]). Obviously, the radiation variables as well as the metric are to be matched continuously at R . It is to be emphasized, however, that compactifying the spatial slices in the outer domain for evolution equations can be troublesome since a lack of resolution at the exterior could produce spurious reflections of outgoing pulses which in turn makes the boundaries behave badly.

Regarding the distribution function, the regularity condition at the center on the particle's radial energy flux is $H_R = 0$ [see Eq. (179)] which means that the average of the particles' radial velocity as measured by the Eulerian observer at the origin is zero (local isotropy at $r=0$). The same considerations apply for the radial particle-flux-number $j^{(r)}$ [cf. Eq. (191)]. A sufficient condition for $H_R = 0 = j_R^{(r)}$ at $r = 0$ is

$$F_R(t, 0, e, \mu) = G_R(t, e, \mu) \\ \text{with } G_R(t, e, \mu) = G_R(t, e, -\mu), \quad (294)$$

that is, the distribution function being a pair function of μ at $r=0$ enforces that the integrals given by Eqs. (179) and (191) vanish identically at the origin. This condition ensures in addition that with a suitable form of $G_R(t, e, \mu)$ in the

energy domain the energy density, pressure, and density number of particles in the Eulerian frame are finite at the star's center. Usually the assumption of an ideal quantum gas is adopted as the initial condition for the particles so that the distribution function is isotropic and given by a Fermi-Dirac or Bose-Einstein distribution (whether the particles are fermions or bosons) [20,44,45]. Then the particles' energy density and pressure will be parametrized initially only by the thermodynamic variables like the temperature. In that case $H_R = 0$ all over the initial spacelike hypersurface.

The regularity condition for F_R at $r=0$ is like other scalar quantities,

$$\partial_r F_R|_{r=0} = 0. \quad (295)$$

Another boundary condition is that the inward flux of particles at the star surface is zero. This is imposed by [57],

$$F_R|_{r=R} = 0 \quad \text{for } -1 \leq \mu < 0. \quad (296)$$

Returning to the initial conditions, the form of these will characterize first the type of configuration (e.g., supernova, neutron star, supermassive star, star cluster) and second the dynamics and ultimate fate of the precursor (e.g., proto-neutron star, neutron star, black hole). The goal of future numerical investigation will be to explore a large set of initial conditions and their consequences (cf. [73]).

VIII. CONCLUSIONS

Several astrophysical phenomena involve the dynamics of relativistic objects. Some of the most interesting ones end up in the formation of black holes or neutron stars, like the collapse of cores and supernova explosions. While most of the astrophysical objects are rotating, the role of rotation in relativity can be neglected with regards to the structure of the object when the rotation frequency is low. Aside from fast pulsars, most of the astrophysical objects are endowed with a low rotation frequency. Therefore the spherical symmetry is an assumption that can be very useful in a wide range of applications. On the other hand, the mere existence of fast pulsars reveals that rotation has to be taken into account in a more realistic situation. Moreover, it seems that the deviations from spherical symmetry in supernova explosions is central in the phenomenon [17], and that rotation can influence the cooling mechanism in the early phases of neutron stars [74]. In general relativity this is a difficult task and only a few attempts have been succeeded within an evolutionary code (see [75] and references therein).

One can separate the problem of the dynamics of relativistic bodies in two sets. The first one involves the formulation, the geometry, and the numerics. A convenient coordinate choice and a powerful numerical analysis can allow long term evolutions leading to a better understanding of several physical phenomena. Thus this is a crucial point which has been recognized by almost all the numerical relativists. Investigations of the effect of several gauges are always an important issue. One of the aims of this paper was therefore to derive the fundamental equations under different

gauges and write them in several forms suitable for different numerical schemes.

The second set involves the physical approximations used in the model. In the case of gravitational collapse, I have discussed that neutrinos cannot only play an important role in the dynamics but also that the signal carried by them can be fundamental for a better understanding of the underlying physics and as an invaluable imprint of the ultimate fate of the collapsed object. In particular, if neutrinos turn out to be massive particles [34], mechanisms like the early black hole formation could be tested [22].

Furthermore, the equation of state at high densities can also lead to different time scale processes during the collapse and the accretion phase. Therefore it becomes necessary to

pursue the analysis with the incorporation of the most recent advances in particle and nuclear physics.

While the formalism presented here included only hydrodynamics and transport theory, the equations are quite general as to include other types of matter like scalar fields, which are very useful in the analysis of critical phenomena.

My aim for future investigations is to perform an extensive numerical analysis of several issues discussed here and more generally to analyze the dynamics of spherically symmetric spacetimes with several kinds of matter sources.

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