Classical black hole production in high-energy collisions

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We investigate the classical formation of a D-dimensional black hole in a high-energy collision of two particles. The existence of an apparent horizon is related to the solution of an unusual boundary-value problem for Poisson's equation in flat space. For a sufficiently small impact parameter, we construct solutions giving such apparent horizons in $D=4$. These supply improved estimates of the classical cross section for black hole production, and of the mass of the resulting black holes. We also argue that a horizon can be found in a region of weak curvature, suggesting that these solutions are valid starting points for a semiclassical analysis of quantum black hole formation.

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I. INTRODUCTION

The proposal that the fundamental Planck mass could be as low as a TeV has excited new interest in the problem of black hole formation in ultrarelativistic collisions. TeV-scale gravity scenarios offer a completely new perspective on the hierarchy problem, and arise via either large extra dimensions [1,2] or compact dimensions with a large warp factor; a model for the latter appears in [3] and string solutions in [4]. It has long been believed that high-energy collisions where the center of mass energy substantially exceeds the Planck mass would produce black holes; this statement can be thought of as a simple extrapolation of Thorne's hoop conjecture [5] and has been more recently discussed in [6]. Lowering the Planck scale to $\mathcal{O}(\text{TeV})$ thus raises the exciting prospect that black holes can be produced at future accelerators, perhaps even at the CERN Large Hadron Collider (LHC) [7,8].[1]

Clearly we would like to better understand this process. One important problem is to estimate the cross section for black hole production. It has been argued that at high energies black hole production has a good semiclassical description [6,7], since in such cases a horizon should form in a region where the curvature is weak and quantum gravity effects are small. This leads to the naive estimate that the cross section for black hole production is roughly given by

$$\sigma \sim \pi r_h^2(\sqrt{s}) \tag{1}$$

where r_h denotes the horizon radius corresponding to c.m. energy $\sqrt{s}$. One would like to make this estimate more precise to improve experimental predictions, and in particular to derive a differential cross section depending on the mass and spin of the black hole produced. Furthermore, the validity of the estimate (1) has been challenged [10,11]. In particular, while Penrose [12] and D'Eath and Payne [13–15] (for a comprehensive treatment see [16]) have studied the problem of a classical collision with a zero impact parameter and shown that a closed trapped surface forms, [11] argues that such collisions will not form black holes at nonzero impact parameter. Finally, collisions of cosmic rays with our atmosphere have energy reach beyond that of the LHC, and reasonable conjectures about neutrino fluxes suggest the possibility that their black hole products might be seen with present or future cosmic ray observatories [17–20,9] or absence of their observation could place improved bounds on the fundamental Planck scale [21]. However, unlike the collider setting, where production above threshhold is copious, conclusive statements here depend sensitively on the exact factors in Eq. (1). This calls for improved estimates.

This paper will reconsider the classical problem of black hole production in high-energy collisions. In particular, understanding the case of a nonzero impact parameter is crucial to improving the estimate (1). We will use the methods of Penrose [12] and of D'Eath and Payne [13–15], where each incoming particle is modeled as a point particle accompanied by a plane-fronted gravitational shock wave, this wave being the Lorentz-contracted longitudinal gravitational field of the particle. At the instant of collision the two shock waves pass through one another, and nonlinearly interact by shearing and focusing. Penrose [12] and D'Eath and Payne [13–15] studied the case of zero impact parameter b, and by finding a closed trapped surface, derived a rigorous lower bound of $M > \sqrt{s/2}$ and improved estimate $M \approx 0.84\sqrt{s}$ for the mass of the resulting black hole. This paper extends this analysis to $b \neq 0$. In the next section we review the basic approximations involved, and generalize the construction of the incoming waves to arbitrary dimensions. We then reduce the problem of finding a closed-trapped surface to a rather unusual boundary-value problem for solutions of Poisson's equation; this problem has a close analogue in the physics of soap bubbles. Using conformal methods which apply only in four dimensions, Sec. III gives an explicit solution to this problem in that case and gives a lower bound

$$\sigma_{\text{BH production}} > 32.5(G^2 s/4) \tag{2}$$

on the classical black hole cross section, where G is Newton's constant. Section IV discusses issues in extending this analysis to higher dimensions, and in better justifying the semiclassical approximation; we follow with conclusions.

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[1]For a review, see [9].

II. TRAPPED SURFACES IN SHOCK-WAVE GEOMETRIES

In the TeV gravity scenarios in question, we will assume that the particles of the standard model propagate on a brane, but that gravity propagates in a higher-dimensional space with either large volume or large warping.[2] This lowers the fundamental Planck scale M_p, perhaps to $\mathcal{O}$(TeV), while maintaining the observed value of the four-dimensional Planck mass, $M_4 \sim 10^{19}$ GeV. We expect to be able to create black holes in scattering processes with center of mass energy $E > M_p$. In general such black hole solutions will have complicated dependence on both the gravitational field of the brane and the geometry of the extra dimensions. However, there are two useful approximations [7] that may be used for a wide class of solutions. First, the brane is expected to have tension given by roughly the Planck or string scale, and so for black holes substantially heavier than the Planck scale the brane's field should be a negligible effect. Secondly, if the geometrical scales of the extra dimensions (radii, curvature radii, variational scale of the warp factor) are all large as compared to $1/M_p$, then there is a large regime where the geometry of the extra dimensions plays no essential role. Therefore it is often a good approximation to consider high-energy collisions in D-dimensional flat space.

In the center of mass frame, each of the high-energy particles has energy

$$\mu = \sqrt{s}/2. \tag{3}$$

We use a coordinate system $(\bar{u}, \bar{v}, \bar{x}^i)$ where retarded and advanced times $(\bar{u}, \bar{v})$ are $(t - z, t + z)$ in terms of Minkowski coordinates, $\bar{z}$ being the direction of motion, and $\bar{x}^i$, $i = 1, \ldots, D-2$, denotes transverse coordinates. The impact parameter is b and the particles are initially incoming along $(\bar{x}^i) = (\pm b/2, 0, \ldots, 0)$.

The gravitational solution for each of the incoming particles can be found by boosting the rest-frame solution. The gravitational field outside a particle is well approximated by the Schwarzschild solution. Since we will be concerned with long-distance phenomena such as formation of large horizons, short-range modifications of the solution should not be relevant. The D-dimensional Schwarzschild solution with mass M is

$$ds^2 = -\left(1 - \frac{16\pi G M}{(D-2)\Omega_{D-2}} \frac{1}{r^{D-3}}\right) dt^2 + \left(1 - \frac{16\pi G M}{(D-2)\Omega_{D-2}} \frac{1}{r^{D-3}}\right)^{-1} dr^2 + r^2 d\Omega_{D-2}^2, \tag{4}$$

where $d\Omega_{D-2}^2$ and Ω_{D-2} are the line element and volume of the unit $(D-2)$-sphere. The Aichelburg-Sexl solution [22] is found by boosting this, taking the limit of large boost and small mass, with fixed total energy μ. The result for a particle moving in the $+z$ direction is the metric

$$ds^2 = -d\bar{u}d\bar{v} + d\bar{x}^{i2} + \Phi(\bar{x}^i)\delta(\bar{u})d\bar{u}^2. \tag{5}$$

Here Φ depends only on the transverse radius $\bar{\rho} = \sqrt{\bar{x}^i \bar{x}_i}$, and takes the form

$$\Phi = \begin{cases} -8G\mu \ln(\bar{\rho}), \; D=4, & (6) \\ \dfrac{16\pi G\mu}{\Omega_{D-3}(D-4)\bar{\rho}^{D-4}}, \; D>4. & (7) \end{cases}$$

Note that Φ satisfies Poisson's equation

$$\nabla^2 \Phi = -16\pi G \mu \delta^{D-2}(\bar{x}^i) \tag{8}$$

in the transverse dimensions [here ∇ is the $(D-2)$-dimensional flat-space derivative in the $(\bar{x}^i)$]. This spacetime solution is manifestly flat except in the null plane $\bar{u}=0$ of the shock wave. If we consider an identical shock wave travelling along $\bar{v}=0$ in the $-z$ direction, by causality these will not be able to influence each other until the shocks collide. This means that we can superpose the two solutions of the form (5) to give the exact geometry outside the future light cone of the collision of the shocks.

The coordinates $\bar{u}, \bar{v}, \bar{x}^i$ suffer the drawback that geodesics and their tangent vectors appear discontinuous across the shock. This can be remedied by going to a new coordinate system defined by

$$\bar{u} = u, \tag{9}$$

$$\bar{v} = v + \Phi\,\theta(u) + \frac{u\,\theta(u)(\nabla\Phi)^2}{4}, \tag{10}$$

$$\bar{x}^i = x^i + \frac{u}{2}\nabla_i \Phi(x)\,\theta(u) \tag{11}$$

(where θ is the Heaviside step function), in which both geodesics and their tangents are continuous across the shock at $u=0$. In these coordinates, the metric of the combined shock waves becomes

$$ds^2 = -du\,dv + [H_{ik}^{(1)}H_{jk}^{(1)} + H_{ik}^{(2)}H_{jk}^{(2)} - \delta_{ij}]dx^i dx^j \tag{12}$$

where

$$H_{ij}^{(1)} = \delta_{ij} + \frac{1}{2}\nabla_i\nabla_j\Phi(\mathbf{x} - \mathbf{x}_1)u\,\theta(u), \tag{13}$$

$$H_{ij}^{(2)} = \delta_{ij} + \frac{1}{2}\nabla_i\nabla_j\Phi(\mathbf{x} - \mathbf{x}_2)v\,\theta(v) \tag{14}$$

with Φ given by Eqs. (6) and (7), and with

$$\mathbf{x}_1 = (+b/2, 0, \ldots, 0), \quad \mathbf{x}_2 = (-b/2, 0, \ldots, 0). \tag{15}$$

[2] For a brief unified review of these scenarios, see [9].

Here $\mathbf{x}\equiv(x^i)$ in the transverse flat $(D-2)$-space.

A marginally trapped surface $\mathcal{S}$ is a closed spacelike $(D-2)$-surface, the outer null normals of which have zero convergence [23]. For the case of $D=4$ and $b=0$, Penrose [12] found such a surface in the union of the two shock waves. This consisted of two flat disks with radii ρ_c at $\bar{t}=-4G\mu\ln\rho_c$, $\bar{z}=\pm4G\mu\ln\rho_c$. Matching their normals across the boundary, which lies in the collision surface $u=v=0$, then determined $\rho_c=4G\mu=r_h$. This construction immediately generalizes to the case $D>4$, $b=0$ where

$$\rho_c=\left(\frac{8\pi G\mu}{\Omega_{D-3}}\right)^{1/(D-3)}. \tag{16}$$

Generalizing to $b\neq0$ and arbitrary dimensions, we attempt to construct $\mathcal{S}$ in the union of the incoming null hypersurfaces $v\leqslant0=u$ and $u\leqslant0=v$. These hypersurfaces intersect each other in the $(D-2)$-dimensional surface $u=0=v$, and $\mathcal{S}$ will intersect this $(D-2)$-surface in a closed $(D-3)$-dimensional surface $\mathcal{C}$, to be determined. In the first incoming null surface $v\leqslant0=u$, $\mathcal{S}$ will be defined by

$$v=-\Psi_1(\mathbf{x})$$

with

$$\Psi_1>0 \quad \text{interior to } \mathcal{C}, \Psi_1=0 \quad \text{on } \mathcal{C}, \tag{17}$$

and one may straightforwardly check that the outer null normals will have zero convergence for $v<0$ as long as

$$\nabla^2(\Psi_1-\Phi_1)=0 \quad \text{interior to } \mathcal{C}. \tag{18}$$

Similarly, in the second incoming null surface $u\leqslant0=v$, $\mathcal{S}$ will be defined by

$$u=-\Psi_2(\mathbf{x})$$

with

$$\Psi_2>0 \quad \text{interior to } \mathcal{C}, \Psi_2=0 \quad \text{on } \mathcal{C}, \tag{19}$$

and the outer null normals will have zero convergence for $u<0$ as long as

$$\nabla^2(\Psi_2-\Phi_2)=0 \quad \text{interior to } \mathcal{C}. \tag{20}$$

Finally, the outer null normal to $\mathcal{S}$ must be continuous at the intersection $u=0=v$; if not there would be a δ function in the convergence. A necessary condition for continuity is

$$\boldsymbol{\nabla}\Psi_1\cdot\boldsymbol{\nabla}\Psi_2=4 \quad \text{on } \mathcal{C}, \tag{21}$$

since Ψ_α, $\alpha=1,2$, vanish on $\mathcal{C}$, $\boldsymbol{\nabla}_i\Psi_\alpha$ is normal to $\mathcal{C}$ and this condition is also sufficient.

Finding a marginally trapped surface therefore reduces to a simple mathematical problem. Specifically, note that Eqs. (8), (18) imply that Ψ_α satisfies Poisson's equation with sources at $\mathbf{x}_\alpha$. Define the rescaled functions

$$g(\mathbf{x},\mathbf{x}_\alpha;\mathcal{C})=\frac{\Omega_{D-3}}{16\pi G\mu}\Psi_\alpha \tag{22}$$

satisfying

$$\nabla^2_{\mathbf{x}}g(\mathbf{x},\mathbf{x}_\alpha;\mathcal{C})=-\Omega_{D-3}\delta^{D-2}(\mathbf{x}-\mathbf{x}_\alpha), \tag{23}$$

$$g(\mathbf{x},\mathbf{x}_\alpha;\mathcal{C})=0 \quad \text{for } \mathbf{x} \text{ on } \mathcal{C}. \tag{24}$$

These are thus the Dirichlet Green's functions for sources at $\mathbf{x}_1,\mathbf{x}_2$ and with boundary $\mathcal{C}$. The problem of finding the marginally trapped surface is equivalent to the following.

Problem C. Given two points $\mathbf{x}_1$ and $\mathbf{x}_2$ in Euclidean $(D-2)$-space, and a constant $B>0$. Let $g(\mathbf{x},\mathbf{x}_\alpha;\mathcal{C})$ be the Dirichlet Green's functions satisfying Eqs. (23),(24). Find a closed $(D-3)$-surface $\mathcal{C}$ enclosing the points with the following property:

$$\boldsymbol{\nabla}_{\mathbf{x}}g(\mathbf{x},\mathbf{x}_1;\mathcal{C})\cdot\boldsymbol{\nabla}_{\mathbf{x}}g(\mathbf{x},\mathbf{x}_2;\mathcal{C})=B^2 \tag{25}$$

for all points $\mathbf{x}$ on $\mathcal{C}$.

As a trivial example, if $\mathbf{x}_1=\mathbf{x}_2$, then the unit $(D-3)$-sphere about $\mathbf{x}_1$ is a solution $\mathcal{C}$ to problem C for $B=1$. This reproduces Penrose's trapped surface in $D=4$ with suitable scaling, and gives its generalization, Eq. (16), for $D>4$. Given general $\mathbf{x}_1$ and $\mathbf{x}_2$, does a solution $\mathcal{C}$ always exist? Clearly not if the points are too distant from each other, because a collision at large enough impact parameter cannot produce a black hole. Given general $\mathbf{x}_1$ and $\mathbf{x}_2$, is the solution $\mathcal{C}$ unique? We shall see that it is not. We also remark that solutions for different B are related by simple scale transformations.

One way to understand problem C is via another physical problem that serves as a simple analogue. Consider a ring of wire with shape $\mathcal{C}$ in the x,y plane in three dimensions, and suppose that this ring is spanned by a soap film. If we apply pressure to the soap film, then in the limit of small displacement its vertical displacement $z(x,y)$ satisfies the equation

$$\nabla^2z(x,y)=-\frac{p(x,y)}{\sigma}, \tag{26}$$

where σ is the film's tension. Generalize to the case of two films, held apart by pressures in the $+z$ and $-z$ directions. If the pressure is exerted at points $\mathbf{x}_\alpha$, then the solutions to Eq. (26) are the above Dirichlet Green's functions. If the horizontal positions of the pressure points are the same, and the ring $\mathcal{C}$ is circular, then the angles θ_1,θ_2 of the soap films with the x,y plane are constant around $\mathcal{C}$. Now separate the pressure points slightly in the x direction—this will change these angles, and they will be functions of position along $\mathcal{C}$. Problem C is that of finding the curve $\mathcal{C}$ for which

$$B^2=\tan\theta_1\tan\theta_2 \tag{27}$$

is constant over $\mathcal{C}$, and equal to the value for zero x separation of the pressure points. This problem can also be generalized to a higher-dimensional analogue. One can argue for the plausibility of a solution, at least for small enough x displacement, by noting that deforming a point on $\mathcal{C}$ toward the center increases the angles and thus the local value of B, and deforming away decreases B. This suggests that $\mathcal{C}$ can be adjusted point by point to make B equal to the given constant value over the ring. The next section will solve this problem explicitly in the special case $D=4$.

III. EXPLICIT CONSTRUCTION FOR $D=4$

In $D=4$, problem C may be readily solved, at least for sufficiently close points, by a trial-and-correction construction, in two steps. First, choose trial points $\tilde{\mathbf{x}}_1$ and $\tilde{\mathbf{x}}_2$, and a trial curve $\mathcal{T}$; then construct $g(\tilde{\mathbf{x}},\tilde{\mathbf{x}}';\mathcal{T})$, and evaluate

$$f(\tilde{\mathbf{x}};\mathcal{T})\equiv \boldsymbol{\nabla}_{\tilde{\mathbf{x}}} g(\tilde{\mathbf{x}},\tilde{\mathbf{x}}_1;\mathcal{T})\cdot \boldsymbol{\nabla}_{\tilde{\mathbf{x}}} g(\tilde{\mathbf{x}},\tilde{\mathbf{x}}_2;\mathcal{T}), \quad \tilde{\mathbf{x}} \text{ on } \mathcal{T}. \tag{28}$$

Second, correct the trial solution by finding a conformal transformation $\mathbf{x}=\mathbf{x}(\tilde{\mathbf{x}})$ that sends $f(\tilde{\mathbf{x}};\mathcal{T})$ to B^2; this will send $\tilde{\mathbf{x}}_1$ and $\tilde{\mathbf{x}}_2$ to some points $\mathbf{x}_1$ and $\mathbf{x}_2$, and will send $\mathcal{T}$ to a curve $\mathcal{C}$ obeying Eq. (25). Thus we obtain a solution of problem C. This works because Poisson's equation in dimension 2 is conformally invariant, while f transforms as

$$f(\mathbf{x};\mathcal{C})=\left|\frac{\partial\tilde{\mathbf{x}}}{\partial\mathbf{x}}\right|^2 f(\tilde{\mathbf{x}};\mathcal{T}). \tag{29}$$

Let us now construct some solutions to problem C. Given two trial points $\tilde{\mathbf{x}}_1$ and $\tilde{\mathbf{x}}_2$ such that $(1/2)|\tilde{\mathbf{x}}_1-\tilde{\mathbf{x}}_2|\equiv a<1$, we may take

$$\tilde{\mathbf{x}}_1=(a,0), \quad \tilde{\mathbf{x}}_2=(-a,0). \tag{30}$$

For the trial curve $\mathcal{T}$ choose the unit circle $|\tilde{\mathbf{x}}|=1$. Then the Green's function evaluated for the points $\tilde{\mathbf{x}}_1$ and $\tilde{\mathbf{x}}_2$ is

$$g(\tilde{\mathbf{x}},\tilde{\mathbf{x}}_1;\mathcal{T})=-\frac{1}{2}\ln\left(\frac{(\tilde{x}-a)^2+\tilde{y}^2}{(a\tilde{x}-1)^2+a^2\tilde{y}^2}\right), \tag{31}$$

$$g(\tilde{\mathbf{x}},\tilde{\mathbf{x}}_2;\mathcal{T})=-\frac{1}{2}\ln\left(\frac{(\tilde{x}+a)^2+\tilde{y}^2}{(a\tilde{x}+1)^2+a^2\tilde{y}^2}\right), \tag{32}$$

and

$$f(\tilde{\mathbf{x}};\mathcal{T})=\frac{(1-a^2)^2}{(1-2a\tilde{x}+a^2)(1+2a\tilde{x}+a^2)}. \tag{33}$$

To conformally send $f(\tilde{\mathbf{x}};\mathcal{T})$ to B^2, we use complex analytic variables $z\equiv x+iy$ and $\tilde{z}\equiv\tilde{x}+i\tilde{y}$. We evidently require a transformation $z(\tilde{z})$, analytic on and within the unit circle $\mathcal{T}$ parametrized by $\tilde{z}=\exp(i\tilde{\phi})$, with

$$B^2\left|\frac{dz}{d\tilde{z}}\right|^2=f(\tilde{\mathbf{x}};\mathcal{T})$$

$$=\frac{(1-a^2)^2}{(1-2a\tilde{x}+a^2)(1+2a\tilde{x}+a^2)} \tag{34}$$

$$=\frac{(1-a^2)^2}{[1-a^2\exp(2i\tilde{\phi})][1-a^2\exp(-2i\tilde{\phi})]}. \tag{35}$$

By inspection, the required tranformation satisfies

$$\frac{dz}{d\tilde{z}}=\frac{1-a^2}{B(1-a^2\tilde{z}^2)} \tag{36}$$

which integrates to

$$z(\tilde{z})=\frac{1-a^2}{2Ba}\ln\left(\frac{1+a\tilde{z}}{1-a\tilde{z}}\right). \tag{37}$$

The true points $(\mathbf{x}_1,\mathbf{x}_2)$ are then found to lie at

$$x=\pm\frac{1-a^2}{2Ba}\ln\left(\frac{1+a^2}{1-a^2}\right), \quad y=0. \tag{38}$$

Restoring physical dimensions,

$$\Psi_1=8G\mu g(\mathbf{x},\mathbf{x}_1;\mathcal{C}), \tag{39}$$

$$\Psi_2=8G\mu g(\mathbf{x},\mathbf{x}_2;\mathcal{C}), \tag{40}$$

$$B=\frac{1}{4G\mu}, \tag{41}$$

$$\mathbf{x}_1=\left(\frac{2G\mu(1-a^2)}{a}\ln\left(\frac{1+a^2}{1-a^2}\right),0\right)=-\mathbf{x}_2, \tag{42}$$

$$b(a)=\frac{4G\mu(1-a^2)}{a}\ln\left(\frac{1+a^2}{1-a^2}\right). \tag{43}$$

Thus we have constructed a marginally trapped surface $\mathcal{S}$ for any value of impact parameter $b(a)$ that can be obtained from the above formula for some a, with $0\leqslant a<1$. The area of $\mathcal{S}$ is found to be

$$\mathrm{Area}(\mathcal{S})=16\pi(G\mu)^2\frac{(1-a^2)^2}{a^2}\ln\left(\frac{1+a^2}{1-a^2}\right). \tag{44}$$

Now $\mathcal{S}$ may or may not be an apparent horizon, which is defined as the *outermost* marginally trapped surface. However, the existence of $\mathcal{S}$ means either that $\mathcal{S}$ is in fact the apparent horizon, or that an apparent horizon exists in the exterior of $\mathcal{S}$. Because $\mathcal{S}$ can be shown to be convex, and because the two-metric is Euclidean, Area($\mathcal{S}$) is a lower bound on the area of the apparent horizon. Modulo technical issues about cosmic censorship, the existence of an apparent horizon means that the collision will produce a black hole (or more than one black hole, although this seems unlikely in the present setup). Moreover, by the area theorem, the mass of the final black hole (assumed single) is bounded below,

$$M_{\text{final bh}} > 2\mu \frac{1-a^2}{2a}\sqrt{\ln\left(\frac{1+a^2}{1-a^2}\right)}, \tag{45}$$

and the fraction of total energy $2\mu=\sqrt{s}$ emitted as gravitational radiation is bounded above,

$$\frac{E_{\text{grav rad}}}{2\mu} < 1 - \frac{1-a^2}{2a}\sqrt{\ln\left(\frac{1+a^2}{1-a^2}\right)}. \tag{46}$$

In fact, $E_{\text{grav rad}}$ may be significantly smaller because the final black hole is expected to be rotating (unless $b=0$), tying up substantial energy.

The function $b(a)$ (for given μ) reaches a maximum value of

$$b_{\max} \approx 3.219 G\mu \tag{47}$$

at

$$a_{\max} \approx 0.6153, \tag{48}$$

so this is the greatest impact parameter for which this construction can demonstrate production of a black hole. The corresponding lower limit on the cross section is

$$\sigma_{\text{BH production}} \geqslant \pi b_{\max}^2 \approx 32.552 (G\mu)^2. \tag{49}$$

Previous estimates for the black hole production cross section used Eq. (1), which gives $\sigma \approx 50.27 (G\mu)^2$, where $r_h = 2G\sqrt{s} = 4G\mu$ is the Schwarzschild radius belonging to the total energy available. Our lower limit is about 65% of this estimate. Another interesting quantity is the mass of the final black hole. Equation (45) together with (43) gives a lower bound on the mass as a function of impact parameter. We find a range from $0.71\sqrt{s}$ for $b=0$ to $0.45\sqrt{s}$ for $b=b_{\max}$. The perturbative analysis of [13–15] raises the former to $M \approx 0.84\sqrt{s}$; we expect a corresponding increase in the latter upon further analysis.

At values $a > a_{\max}$ (but still $a<1$) our construction produces a second, smaller marginally trapped surface for the same b. This shows that solutions to problem C are usually not unique.

It is also interesting to better understand the shape of the curve $\mathcal{C}$ for the $D=4$ solution. This is readily found from Eq. (37), and takes the form

$$(1-a^2)\sinh^2\frac{ax}{4G\mu(1-a^2)} + (1+a^2)\sin^2\frac{ay}{4G\mu(1-a^2)} = a^2. \tag{50}$$

Obviously this approaches a circle of radius $4G\mu$ as $a \to 0$. Figure 1 displays the curve $\mathcal{C}$ in the transverse collision plane for various values of b.

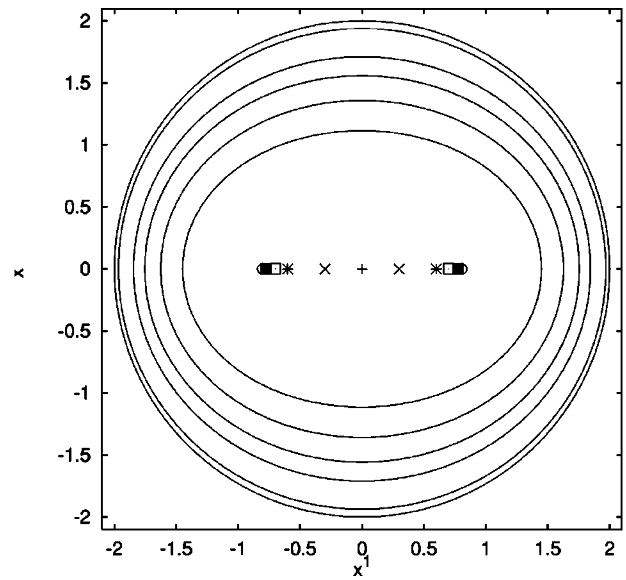

FIG. 1. The intersection curve $\mathcal{C}$ of the marginally trapped surface $\mathcal{S}$ with the collision plane $(u=0=v)$. Several curves $\mathcal{C}$, for various impact parameters b, are superposed; spacetime dimension is $D=4$. Distances are in units of $G\sqrt{s}=2G\mu$; in these units $r_h=2$. Incoming particle pairs appear in the horizontal line $x^2=0$ at pair separation b; *wider* pairs therefore correspond to *smaller* curves $\mathcal{C}$. Values of b are 0 +, 0.6 ×, 1.2 *, 1.4 □, 1.55 ●, and 1.609 ○, the last being maximal.

IV. FURTHER RESULTS AND DIRECTIONS

Clearly it would be desirable to find an explicit solution to problem C in the higher-dimensional case, in order to give more careful estimates of the cross section in the physically interesting situation where the extra dimensions are relevant. Nonetheless, we have given a heuristic argument for the existence of such a solution, which is buttressed by the explicit construction of the trapped surface in the four-dimensional case. This appears to demonstrate classical black hole formation at nonzero impact parameter—answering the criticism of [11].

For head-on collisions, $b=0$, in $D>4$, the apparent horizon is the union of two balls of radius $\rho_c = r_h$, Eq. (16). The corresponding lower limit on black hole mass is displayed in Table I.

Solving problem C in $D>4$ for $b>0$ may require numerical work. Note that by symmetry it still remains a two-dimensional problem of finding a curve $\hat{\mathcal{C}}$ which produces the surface $\mathcal{C}$ as a surface of revolution about the axis connecting the two source points. However, in the $D>4$ case, the relevant Green's function is of the form (7) and no longer transforms nicely under conformal transformations. Of course, even solving this problem is only a starting point—it provides a lower bound on the mass of the black hole, but, as was found in the axially symmetric case in [13–15], the resulting black hole will absorb more of the energy contained in its external fields, and it should be possible to raise this bound by studying this subsequent evolution. Recall that in the axially symmetric case in $D=4$ this resulted in an addi-

TABLE I. Lower limit on M_{BH} in head-on collisions $b=0$, for various dimensions D. Here $A_{\rm Sch}$ is the $(D-2)$-dimensional horizon area of a Schwarzschild black hole of mass $\sqrt{s}$, while $A_{\rm trap}$ is the area of the Penrose marginally trapped surface.

D	$A_{\rm trap}/A_{\rm Sch}$	$M_{\rm BH}/\sqrt{s}$	$1-M_{\rm BH}/\sqrt{s}$
4	0.50000	0.70711	0.29289
5	0.54270	0.66533	0.33467
6	0.55032	0.63894	0.36106
7	0.55080	0.62057	0.37943
8	0.54928	0.60696	0.39304
9	0.54720	0.59642	0.40358
10	0.54502	0.58798	0.41202
11	0.54293	0.58105	0.41895

tional enhancement of approximately 119%. Going beyond this, the ultimate goal of such an analysis—and its quantum extension—is to compute the differential cross section depending on the mass and spin of the resulting black hole.

One might also wonder if a black hole is produced in the high-energy collision of a particle with a purely gravitational shock wave, in any dimension $D\geqslant 4$. The present analysis suggests that the answer is "no." If we attempt to replace $\Phi_2(\mathbf{x})$ by a source-free solution of Laplace's equation, to model a purely gravitational shock wave, then Eqs. (19),(20) have no solution at all, by the maximum principle for elliptic equations. Therefore no apparent horizon exists in the incoming wave front surface. Similarly, the collision of two purely gravitational shock waves [24] seems not to produce a black hole. Moreover, in the collision of two particles at $b>0$, as studied here, one might wonder if additional, smaller marginally trapped surfaces might appear, enclosing one particle but not the other; a similar argument shows not. We have not considered trapped surfaces that might exist to the future of the incoming wave surface, however, so these arguments cannot conclusively rule out black holes.

It is also interesting to compare the estimate (49) to a heuristic argument presented in [21]. Anchordoqui *et al.* argue that one may improve estimates of the cross section by taking into account the angular momentum dependence of the Schwarzschild radius. Specifically, for c.m. energy $\sqrt{s}$ and impact parameter b, the angular momentum is $J=b\sqrt{s}/2$. One expects that the maximum impact parameter occurs near a value of b that equals the corresponding angular momentum dependent radius r_h. This is given by [25]

$$r_h^{D-5}\left(r_h^2+\frac{(D-2)^2J^2}{4M^2}\right)=\frac{16\pi GM}{(D-2)\Omega_{D-2}} \xrightarrow{J\to 0} r_h^{D-3}=\frac{16\pi GM}{(D-2)\Omega_{D-2}}. \tag{51}$$

If we set $b=r_h$ in Eq. (51), that gives

$$r_h^{D-3}=\frac{16\pi GM}{(D-2)\Omega_{D-2}}\left[1+\frac{(D-2)^2}{16}\right]^{-1}. \tag{52}$$

This leads to a cross-section estimate

$$\sigma\approx\pi r_h^2=\left[1+\frac{(D-2)^2}{16}\right]^{-2/(D-3)}\pi r_{h,\ \rm spherical}^2. \tag{53}$$

In $D=4$, this gives a relative correction factor of 64%, closely matching the factor in Eq. (49). The correspondence of these results suggests that this estimation technique may indeed be approximately correct in higher dimensions.

A final point concerns the validity of the semiclassical approximation. We expect that the semiclassical approximation for black hole formation should be justified if a horizon forms at small curvature.[3] Of course the surface $\mathcal{S}$ that we have constructed lies in the plane of the shocks, and thus is in a sense close to a region of large curvature; understanding the importance of this would require regulating the solution taking into account finite size/mass effects. However, we believe that this is not necessary, as it is possible to find a trapped surface outside the planes of the incoming shocks. First, in the case $b=0$, consider Penrose's flat disk $\mathcal{S}$ of radius $\bar{\rho}=\rho_c$ in the incoming null wave front surface $\bar{u}=0$. Construct the null plane $\mathcal{N}$ emanating from $\mathcal{S}$ in the opposite direction $v=-$const; $\mathcal{N}$ has zero convergence because it is a null plane. Deform $\mathcal{S}$ into the future along $\mathcal{N}$, while leaving it fixed in some neighborhood of its boundary $\mathcal{C}$. This deformed $(D-2)$-surface $\mathcal{S}'$ will still have zero convergence along the v-direction everywhere, and thus will be an apparent horizon, now with weak spacetime curvature on it. (In fact, $\mathcal{S}'$ has exactly the same area as $\mathcal{S}$, and so gives exactly the same bound on the black hole mass.)

Second, in the case $b>0$, we can proceed similarly. We can still construct a null surface $\mathcal{N}$ emanating to the future from our original marginally trapped surface $\mathcal{S}$, generated by null geodesics normal to $\mathcal{S}$. However, $\mathcal{N}$ will not be a flat null plane, because the null normals to $\mathcal{S}$ have nonzero shear. We can still deform $\mathcal{S}$ some distance to the future along $\mathcal{N}$ to create a new $(D-2)$-surface $\mathcal{S}'$ but the convergence of the null normals of $\mathcal{S}'$ will go positive, due to the shear (we cannot go too far or we will run into a caustic, i.e., the convergence will go to $+\infty$). Thus $\mathcal{S}'$ will actually be a trapped surface, not a marginally trapped surface. This would mean that a marginally trapped surface must lie somewhere outside it. In any case $\mathcal{S}'$ has weak spacetime curvature everywhere on it, and implies the existence of a black hole.

V. CONCLUSIONS

The existence of a closed trapped surface in the collision geometry of two ultrarelativistic particles clearly demonstrates classical black hole formation. The argument that these surfaces are present in the weak curvature region further suggests that this process can be consistently treated in a semiclassical analysis, and should help lay the foundation for a more rigorous justification of such an analysis. Furthermore, we have found improved estimates on the production

[3] In the context of string theory, the Schwarzschild radius must also be larger than the string length $\sqrt{\alpha'}$ [26,27].

cross section for black holes. While these estimates are not enormously different from the more naive estimates of [7,8], it is important to know their size in improving discussion of the sensitivity of cosmic ray observations to black hole production. Future directions include more complete analysis of the higher-dimensional classical problem, which may require numerical work. The present classical analysis should serve as a starting point for a more complete investigation of the semiclassical approximation to black hole formation.

After completing this work we were kindly informed by R. Penrose that he had found related unpublished results on the existence of a maximal impact parameter in $D=4$, many years ago.

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