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Uniqueness of (dilatonic) charged black holes and black *p*-branes in higher dimensions

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We prove the uniqueness of higher dimensional (dilatonic) charged black holes in static and asymptotically flat spacetimes for an arbitrary vector-dilaton coupling constant. An application to the uniqueness of a wide class of black *p*-branes is also given.

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I. INTRODUCTION

The aim of this paper is to extend our recent work in which we generalized the four-dimensional Israel's theorems on the uniqueness of static vacuum and electrically charged black holes [1] to higher dimensions [2,3]. We also gave a uniqueness theorem for a certain class of charged dilatonic black holes in higher dimensions [3]. There is also earlier work by Hwang [4] on the vacuum case. The motivation for treating charged dilatonic black holes comes from string theory where gauge fields often play an essential role. It is clearly important to be able deal with the most general vector-dilaton coupling constant and that is what we shall do in this present paper.

Another important motivation for our work is to continue our program of proving the uniqueness of static black *p*-brane solutions. Such solutions are $(n+p)$-dimensional spacetimes invariant under the action of a *p*-dimensional Abelian translation group. Reduction to n spacetime dimensions produces a black hole solution of gravity coupled to one or more scalars and an electric 2-form or dual magnetic $(n-2)$-form field strength. A uniqueness theorem for a black hole solution in n spacetime dimensions which asserts the necessity of $SO(n-1)$ invariance acting on S^n orbits is equivalent to asserting the invariance of the static black *p*-brane under the action of $SO(n-1)$ on the $n-1$ space transverse to the *p*-brane. As in the case of 4-dimensional black holes so with higher dimensional charged black holes and branes although we expect that a uniqueness theorem holds in the extreme case we cannot expect such a theorem to assert $SO(n-1)$-invariance because of the existence of multi-p-brane solutions in static equilibrium. In this paper we will only treat the non-extreme case, leaving the extreme case for a later date. We assume only electrically charged black holes coupled to a single 2-form and with a single scalar field. By duality we could also consider magnetically charged black hole coupled to an $(n-2)$ form. In previous work we were limited to the vacuum case [2] or to special values of the the vector-dilaton couplings constant [3], which meant that our results only held for the cases $(n,p)=(5,3)$ or $(6,2)$. The results of this paper will remove this restriction: rather than generalizing to higher dimensions the argument given in [5–7] as was done in our earlier paper [3], we shall generalize the method introduced in four dimensions in [8].

As in the case of higher dimensional vacuum black holes, so in this case, it is essential to assume strict asymptotic flatness because analogues of the Bohm black holes [2,3] also exist in the electrically charged case. In addition, just as in $3+1$ dimensions, we also have to assume that the surface gravity is non-zero (non-extreme), otherwise one has multi-black holes solutions which generalize the Majumdar-Papapetrou (MP) solutions [9]. See Ref. [10] for the discussion the uniqueness of the MP solution under some additional assumptions. We shall also prove the uniqueness of Gibbons-Maeda solution [11] of the Einstein-Maxwell-dilaton system with the general dilaton coupling to the Maxwell field. This amounts to a non-trivial generalization of the earlier papers [2,3,7,8].

The rest of this paper is organized as follows. In Sec. II, we will prove the uniqueness of the higher dimensional Reissner-Nordstrom solution [12] among static black holes in the Einstein-Maxwell system. Then, in Sec. III, a black hole uniqueness for the most general Einstein-Maxwell-dilaton system is given following the four-dimensional proof in [8]. In Sec. IV, we address the uniqueness of black *p*-branes. Finally in Sec. V we provide a summary of the results of the paper.

II. CHARGED BLACK HOLES

In this section, we consider the n-dimensional Einstein-Maxwell system given by the Lagrangian

$$L={}^{n}R-F^2, \tag{1}$$

where F is the Maxwell field, and prove the uniqueness of the static and electrically charged black holes in higher dimensions.

In general, the metric of an n-dimensional static spacetime has the form

$$ds^2=-V^2dt^2+g_{ij}dx^idx^j, \tag{2}$$

where V and g_{ij} are independent of t and they are regarded as quantities on the $t=$const hypersurface Σ. The event horizon H is a Killing horizon located at the level set $V=0$, which is assumed to be non-degenerate. Then the static field equations become

$$\nabla^2 V=\frac{C^2}{V}(\nabla\psi)^2, \tag{3}$$

$$\nabla^2\psi=\frac{\nabla\psi\cdot\nabla V}{V}, \tag{4}$$

and

$$R_{ij}=\frac{\nabla_i\nabla_j V}{V}-\frac{2}{V^2}\nabla_i\psi\nabla_j\psi+\frac{2(\nabla\psi)^2}{(n-2)V^2}g_{ij}, \tag{5}$$

where $C=[2(n-2)/(n-3)]^{1/2}$, ∇ and R_{ij} denote covariant derivative and the Ricci tensor defined on (Σ,g_{ij}), respectively, and ψ is the electrostatic potential such that $F=d\psi\wedge dt$.

In asymptotically flat space-times, one can find an appropriate coordinate system in which the metric and electrostatic potential have asymptotic expansions of the form

$$V=1-\frac{\mu}{r^{n-3}}+O(1/r^{n-2}), \tag{6}$$

$$g_{ij}=\left(1+\frac{2}{n-3}\frac{\mu}{r^{n-3}}\right)\delta_{ij}+O(1/r^{n-2}), \tag{7}$$

$$\psi=\frac{Q/C}{r^{n-3}}+O(1/r^{n-2}), \tag{8}$$

respectively, where μ, $Q=$const represent the Arnowitt-Deser-Misner (ADM) mass and the electric charge (up to constant factors), respectively, and $r:=\sqrt{\Sigma_i(x^i)^2}$. We assume the non-extremal condition $\mu>|Q|$.

Consider the following two conformal transformations:

$$\hat{g}^{\pm}_{ij}=\Omega^2_{\pm}g_{ij}, \tag{9}$$

where

$$\Omega^2_{\pm}=\left[\left(\frac{1\pm V}{2}\right)^2-\frac{C^2}{4}\psi^2\right]^{1/2}. \tag{10}$$

Then we have two manifolds $(\Sigma^{\pm},\hat{g}^{\pm}_{ij})$. On Σ^+, the asymptotic behavior of the metric becomes

$$\hat{g}^{+}_{ij}=\delta_{ij}+O(1/r^{n-2}). \tag{11}$$

On Σ^-, we have

$$\hat{g}^{-}_{ij}=\frac{[(\mu^2-Q^2)/4]^{2/(n-3)}}{r^4}\delta_{ij}+O(1/r^5)$$
$$=[(\mu^2-Q^2)/4]^{2/(n-3)}(d\varrho^2+\varrho^2d\Omega^2_{n-2}),+O(\varrho^5), \tag{12}$$

where $d\Omega^2_{n-2}$ denotes the round sphere metric and $\varrho:=1/r$ has been defined. Pasting $(\Sigma^{\pm},\hat{g}^{\pm}_{ij})$ across the level set $V=0$ and adding a point $\{p\}$ at $\varrho=0$, we can construct a complete regular surface $\hat{\Sigma}=\Sigma^+\cup\Sigma^-\cup\{p\}$. The Ricci curvature $\hat{R}$ on $\Sigma^{\pm}$ becomes

$$\Omega^2_{\pm}\hat{R}=\frac{\hat{\nabla}^2V}{V}+\frac{2}{n-2}\frac{(\hat{\nabla}\psi)^2}{V^2}-2(n-2)\frac{\hat{\nabla}^2\Omega_{\pm}}{\Omega_{\pm}}$$
$$-(n-2)(n-5)\frac{(\hat{\nabla}\Omega_{\pm})^2}{\Omega^2_{\pm}}$$
$$=\frac{1}{8V^2\Omega^{2(n-3)}_{\pm}}|2V\psi\hat{\nabla}V-(V^2-1+C^2\psi^2)\hat{\nabla}\psi|^2, \tag{13}$$

where we have used the identities

$$\hat{\nabla}^2(V\pm C\psi)=\pm\frac{C}{V}\hat{\nabla}\psi\cdot\hat{\nabla}(V\pm C\psi). \tag{14}$$

Then the Ricci curvature on $\hat{\Sigma}$ is non-negative. Furthermore, Eq. (11) implies that the total mass also vanishes on $\hat{\Sigma}$. As a consequence of the positive mass theorem [13,14], such a surface $\hat{\Sigma}$ must be flat and

$$2V\psi\hat{\nabla}V=(V^2-1+C^2\psi^2)\hat{\nabla}\psi \tag{15}$$

holds, which implies that the level surfaces of V and ψ coincide. In other words, the physical Cauchy surface Σ is conformally flat. We shall now demonstrate that the conformally transformed event horizon $\hat{H}$ is a geometric sphere in $\hat{\Sigma}$. We choose V as a local coordinate in a neighborhood $U\subset\Sigma$. Let $\{x^A\}$ be coordinates on level sets of V such that their trajectories are orthogonal to each level set. Then, the metric on Σ can be written in the form

$$g=\rho^2dV^2+h_{AB}dx^Adx^B, \tag{16}$$

where $\rho^2:=(\nabla V)^2$. Since Σ is conformally flat, the Riemann invariant has a simple expression in this coordinate system:

$${}^nR_{IJKL}{}^nR^{IJKL}=R_{ijkl}R^{ijkl}+4R_{0i0j}R^{0i0j}$$
$$=\frac{4(n-2)}{(n-3)V^2\rho^2}[k_{AB}k^{AB}+k^2+2\mathcal{D}_A\rho\mathcal{D}^A\rho], \tag{17}$$

where $\mathcal{D}_A$ denotes the covariant derivative on each level set of V, and k_{AB} is the second fundamental form of the level set.

The requirement that the event horizon H is a regular surface leads to the condition

$$k_{AB}|_H=0, \tag{18}$$

$$\mathcal{D}_A\rho|_H=0. \tag{19}$$

In particular, H is a totally geodesic surface in Σ.

Let us consider how the event horizon is embedded into the base space $(\hat{\Sigma},\delta_{ij})$. Define the smooth function

$$v:=(1+V-C\psi)^{-1}, \tag{20}$$

which is the harmonic function on $(\hat{\Sigma},\delta_{ij})$: $\nabla_0^2 v=0$. In terms of this, we can adopt the following local expression for the flat space

$$\delta_{ij}dx^i dx^j=\hat{\rho}^2 dv^2+\hat{h}_{AB}dx^A dx^B. \tag{21}$$

The event horizon is located at some $v=$const surface $\hat{H}$. The extrinsic curvature $\hat{k}_{AB}$ of the level set $v=$const can be expressed as

$$\hat{k}_{AB}=\Omega_+ k_{AB}+\frac{1}{\hat{\rho}}\frac{\partial\Omega_+}{\partial v}h_{AB}. \tag{22}$$

Thus we obtain

$$\hat{k}_{AB}=\frac{1}{\hat{\rho}}\frac{\partial\Omega_+}{\partial v}\bigg|_H \hat{h}_{AB} \tag{23}$$

on $\hat{H}$. In other words, the embedding of $\hat{H}$ into the Euclidean $(n-1)$-space is totally umbilical. It is known that such a embedding must be hyperspherical [15], namely each connected component of $\hat{H}$ is a geometric sphere with a certain radius. The embedding of a hypersphere into the Euclidean space is known to be rigid [16], which means that we can always locate one connected component of $\hat{H}$ at the $r=r_0$ surface of $\tilde{\Sigma}$ without loss of generality. If there is only a single horizon, we have a boundary value problem for the Laplace equation $\nabla_0^2 v=0$ on the base space $\Omega:=E^{n-1}\backslash B^{n-1}$ with the Dirichlet boundary conditions. Such a solution must be spherically symmetric, so that the Birkhoff theorem implies that it is given by the Reissner-Nordstrom solutions.

One may remove the assumption of the single horizon as follows. Consider the evolution of the level surface in Euclidean space. From the Gauss equation in Euclidean space one obtains the evolution equation for the shear $\hat{\sigma}_{AB}:=\hat{k}_{AB}-\hat{k}\hat{h}_{AB}/(n-2)$:

$$\pounds_{\hat{n}}\hat{\sigma}_{AB}=\hat{\sigma}_A{}^C\hat{\sigma}_{CB}+\frac{1}{n-2}\hat{h}_{AB}\hat{\sigma}_{CD}\hat{\sigma}^{CD} -\frac{1}{\hat{\rho}}\left(\hat{\mathcal{D}}_A\hat{\mathcal{D}}_B-\frac{1}{n-2}\hat{h}_{AB}\hat{\mathcal{D}}^2\right)\hat{\rho}, \tag{24}$$

where $\hat{n}$ denotes the unit normal to the level set of v. Using $\nabla_0^2 v=0$, we obtain

$$\pounds_{\hat{n}}\hat{\mathcal{D}}_A\ln\hat{\rho}=\hat{k}\hat{\mathcal{D}}_A\ln\hat{\rho}+\hat{\mathcal{D}}_A\hat{k}, \tag{25}$$

$$\pounds_{\hat{n}}\hat{k}=-\|\hat{\sigma}\|^2-\frac{1}{n-2}k^2-\frac{1}{\hat{\rho}}\hat{\mathcal{D}}^2\hat{\rho}, \tag{26}$$

$$\pounds_{\hat{n}}\mathcal{D}_A\hat{k}=\hat{\mathcal{D}}_A\pounds_{\hat{n}}\hat{k}+(\hat{\mathcal{D}}_A\ln\hat{\rho})(\pounds_{\hat{n}}\hat{k}). \tag{27}$$

From the above equations, it can be seen that

$$\hat{\sigma}_{AB}=0,\quad \hat{\mathcal{D}}_A\hat{\rho}=0,\quad \hat{\mathcal{D}}_A\hat{k}=0, \tag{28}$$

that is, each level surface of v is totally umbilic and hence spherically symmetric, which implies that the metric is isometric to the Reissner-Nordstrom solution.

This is of course local result since we consider only the region containing no saddle points of the harmonic function v. To obtain the global result, we need a further assumption such as analyticity. However, the assumption that there is no saddle point may be justified as follows. At a saddle point $\rho=0$, the level surface of v is multi-sheeted, that is the embedding of the level surfaces is singular there. One can find at least one level surface such that $k_{AB}\neq 0$ near the saddle point. Then, Eq. (17) implies that the saddle point is singular. If the horizon is not connected, this naked singularity must exist to compensate for the gravitational attraction between black holes.

III. DILATONIC CHARGED BLACK HOLES

We here consider the Einstein-Maxwell-dilaton system

$$L={}^{(n)}R-2(\partial\phi)^2-e^{-\alpha\phi}F^2, \tag{29}$$

for general dilaton coupling constant $\alpha>0$.

Adopting the metric form of Eq. (2), we have the following equations:

$$\nabla^2 V=C^2\frac{e^{-\alpha\phi}}{V}(\nabla\psi)^2, \tag{30}$$

$$\nabla^2\phi=-\frac{\nabla V\cdot\nabla\phi}{V}+\frac{\alpha}{2}\frac{e^{-\alpha\phi}}{V^2}(\nabla\psi)^2, \tag{31}$$

$$\nabla^2\psi=\frac{\nabla V\cdot\nabla\psi}{V}+\alpha\nabla\phi\cdot\nabla\psi, \tag{32}$$

and

$$R_{ij}=\frac{\nabla_i\nabla_j V}{V}+2\nabla_i\phi\nabla_j\phi-2\frac{e^{-\alpha\phi}}{V^2}\nabla_i\psi\nabla_j\psi +\frac{2}{n-2}\frac{e^{-\alpha\phi}(\nabla\psi)^2}{V^2}g_{ij}. \tag{33}$$

Let us define the following quantities:

$$\Phi_{\pm 1}=\frac{1}{2}\left[e^{\alpha\phi/2}V\pm\frac{e^{-\alpha\phi/2}}{V}-C^2(1+\lambda)\frac{e^{-\alpha\phi/2}\psi^2}{V^2}\right], \tag{34}$$

$$\Phi_0=C(1+\lambda)^{1/2}\frac{e^{-\alpha\phi/2}\psi}{V}, \tag{35}$$

and

$$\Psi_{\pm 1}=\frac{1}{2}(e^{-2C^2\phi/\alpha}V\pm e^{2C^2\phi/\alpha}V^{-1}), \tag{36}$$

where $\lambda=\alpha^2/4C^2$ has been defined.

Let us consider the conformal transformation defined by

$$\tilde{g}_{ij}=V^{2/(n-3)}g_{ij}, \tag{37}$$

and introduce the following symmetric tensors defined on this space:

$$\tilde{G}_{ij}=\tilde{\nabla}_i\Phi_{-1}\tilde{\nabla}_j\Phi_{-1}-\tilde{\nabla}_i\Phi_0\tilde{\nabla}_j\Phi_0-\tilde{\nabla}_i\Phi_1\tilde{\nabla}_j\Phi_1 \tag{38}$$

and

$$\tilde{H}_{ij}=\tilde{\nabla}_i\Psi_{-1}\tilde{\nabla}_j\Psi_{-1}-\tilde{\nabla}_i\Psi_1\tilde{\nabla}_j\Psi_1. \tag{39}$$

Then the field equations become

$$\tilde{\nabla}^2\Phi_A=\tilde{K}\Phi_A, \tag{40}$$

$$\tilde{\nabla}^2\Psi_A=\tilde{H}\Psi_A, \tag{41}$$

and

$$\tilde{R}_{ij}=\frac{2}{C^2}(1+\lambda)(\tilde{G}_{ij}+\lambda\tilde{H}_{ij}), \tag{42}$$

where $A=-1,0,1$.

Furthermore, we perform the following conformal transformations:

$${}^{\Phi}g^{\pm}_{ij}={}^{\Phi}\omega_{\pm}^{2/(n-3)}\tilde{g}_{ij} \tag{43}$$

and

$${}^{\Psi}g^{\pm}_{ij}={}^{\Psi}\omega_{\pm}^{2/(n-3)}\tilde{g}_{ij}, \tag{44}$$

where

$${}^{\Phi}\omega_{\pm}=\frac{\Phi_1\pm 1}{2} \tag{45}$$

and

$${}^{\Psi}\omega_{\pm}=\frac{\Psi_1\pm 1}{2}. \tag{46}$$

The extreme case should be excluded to keep the above conformal factors non-negative. In fact, there exist multi-black-hole solutions in the extreme limit [17].

Now we have four manifolds $({}^{\Phi}\Sigma_+,{}^{\Phi}g^+_{ij})$, $({}^{\Phi}\Sigma_-,{}^{\Phi}g^-_{ij})$, $({}^{\Psi}\Sigma_+,{}^{\Psi}g^+_{ij})$, and $({}^{\Psi}\Sigma_-,{}^{\Psi}g^-_{ij})$. Pasting $({}^{\Phi(\Psi)}\Sigma_{\pm},{}^{\Phi(\Psi)}g^{\pm}_{ij})$ across the surface $V=0$, we can construct a complete regular surface ${}^{\Phi(\Psi)}\Sigma={}^{\Phi(\Psi)}\Sigma^+\cup{}^{\Phi(\Psi)}\Sigma^-$. Thus, we have two regular surfaces, ${}^{\Phi}\Sigma$ and ${}^{\Psi}\Sigma$. As in the previous section, we can check that each total gravitational mass on ${}^{\Phi}\Sigma$ and ${}^{\Psi}\Sigma$ vanishes.

From now on we use the conformal positive energy theorem [18] to show that the static slice is conformally flat. See the Appendix for the conformal positive energy theorem in higher dimensions. We consider another conformal transformation

$$\hat{g}^{\pm}_{ij}:=[({}^{\Phi}\omega_{\pm})^2({}^{\Psi}\omega_{\pm})^{2\lambda}]^{1/(n-3)(1+\lambda)}\tilde{g}_{ij}. \tag{47}$$

The Ricci curvature on this space can be shown to become

$$\begin{aligned}(1+\lambda)\hat{R}^{\pm}=&[({}^{\Phi}\omega_{\pm})^2({}^{\Psi}\omega_{\pm})^{2\lambda}]^{-1/(n-3)(1+\lambda)}\\&\times[({}^{\Phi}\omega_{\pm})^{2/(n-3)}({}^{\Phi}R^{\pm})\\&+\lambda({}^{\Psi}\omega_{\pm})^{2/(n-3)}({}^{\Psi}R^{\pm})]\\&+\frac{(n-2)\lambda}{1+\lambda}(\hat{\nabla}\ln{}^{\Psi}\omega_{\pm}-\hat{\nabla}\ln{}^{\Phi}\omega_{\pm})^2.\end{aligned} \tag{48}$$

The first term of the right-hand side (RHS) turns out to be non-negative:

$$\begin{aligned}&({}^{\Phi}\omega_{\pm})^{2/(n-3)}({}^{\Phi}R^{\pm})+\lambda({}^{\Psi}\omega_{\pm})^{2/(n-3)}({}^{\Psi}R^{\pm})\\&=\frac{2}{C^2}\left|\frac{\Phi_0\tilde{\nabla}\Phi_{-1}-\Phi_{-1}\tilde{\nabla}\Phi_0}{\Phi_1\pm 1}\right|^2.\end{aligned} \tag{49}$$

The conformal positive mass theorem implies that

$$\frac{{}^{\Phi}\omega_{\pm}}{{}^{\Psi}\omega_{\pm}}=\text{const}, \tag{50}$$

$$\Phi_0=\text{const}\times\Phi_{-1} \tag{51}$$

and that each $({}^{\Phi}\Sigma,{}^{\Phi}g_{ij})$, $({}^{\Psi}\Sigma,{}^{\Psi}g_{ij})$ and $(\hat{\Sigma},\hat{g}_{ij})$ is flat space. In other words, (Σ,g_{ij}) is conformally flat. We define the function

$$v:=({}^{\Phi}\omega_{\pm}V)^{-1/2}. \tag{52}$$

Noting that

$$\hat{g}_{ij}=v^{4/(n-3)}\,{}^{\Phi}g_{ij}, \tag{53}$$

we have

$$v^{4/(n-3)}\hat{R}={}^{\Phi}R-\frac{4(n-2)}{n-3}\frac{\nabla_0^2 v}{v}. \tag{54}$$

Since we already know that $\hat{R}={}^{\Phi}R=0$, v turns out to be the harmonic function on the flat space:

$$\nabla_0^2 v=0. \tag{55}$$

With the procedure given in the previous section, we can show that the static solution must hyperspherically symmetric, therefore given by the metric given in Ref. [11].

IV. BRANES

As already mentioned, one motivation for the present work was to establish the uniqueness of static p-brane solutions. In general these take the form

$$ds^2 = e^{2\gamma\Phi}(d\mathbf{y}_p^2) + e^{2\delta\Phi} g_{\mu\nu} dx^\mu dx^\nu. \qquad (56)$$

Dimensional reduction on $\mathbf{E}^p$ will take the Einstein-Hilbert action to the Einstein-Hilbert action if $(n-2)\delta + p\gamma = 0$. If the higher dimensional metric is coupled to an $n-2$ form F_{n-2} with no scalars, we obtain the Lagrangian [19]

$$R - 2(\nabla\phi)^2 - \frac{2}{(n-2)!} e^{-2\alpha\phi} F_{n-2}^2, \qquad (57)$$

with

$$\gamma^2 = \frac{2(n-2)}{p(n+p-2)} \qquad (58)$$

and

$$\alpha^2 = \frac{2p(n-3)^2}{(n-2)(n+p-2)}. \qquad (59)$$

In this way, we obtain a uniqueness theorem for black p-branes described by the metric ansatz of Eq. (56). Note that, by contrast with our previous paper [3], the values of (n,p) are unrestricted.

V. SUMMARY

We presented the proof of the uniqueness theorem for static charged dilatonic black holes in higher dimensions. We excluded the extreme case from consideration. Our theorem also provides a uniqueness theorem for black p-branes.

Since the extreme case is a BPS state, the remaining issue about the extreme case is important. In this paper we considered only electrically charged black holes. However, the generalization to the magnetically charged case is straightforward because of the duality as mentioned in the last section.

Finally we comment on the assumption on asymptotic flatness. If we drop, there will be infinite sequence of solutions constructed from the the Bohm metrics [20] as in our previous work [2,3]. Since the positive energy theorem does not hold in such cases, it is unclear whether proof of the uniqueness presented here can be generalized to cover this case. The stability of Bohm black holes solutions is currently under investigation.

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APPENDIX: THE CONFORMAL POSITIVE ENERGY THEOREM IN HIGHER DIMENSIONS

In this appendix we briefly describe the statement and the proof of the conformal positive energy theorem in higher dimensions. The proof is given by a straightforward extension of that of Simon [18] in four dimensions. Simon in turn was inspired by Masood-ul-Alam's proof of the uniqueness of the Gibbons-Maeda solution in four dimensions.

Theorem: Let $({}^\Phi\Sigma, {}^\Phi g_{ij})$ and $({}^\Psi\Sigma, {}^\Psi g_{ij})$ be asymptotically flat Riemannian $(n-1)$-dimensional manifolds with ${}^\Psi g_{ij} = \Omega^2\, {}^\Phi g_{ij}$. Then ${}^\Phi m + \beta\, {}^\Psi m \geqslant 0$ if ${}^\Phi R + \beta\Omega^2\, {}^\Psi R \geqslant 0$ hold for a positive constant β. The equality holds if and only if $({}^\Phi\Sigma, {}^\Phi g_{ij})$ and $({}^\Psi\Sigma, {}^\Psi g_{ij})$ are flat.

Proof: Let us consider the conformal transformation as $\bar{g}_{ij} = \Omega^{[2\beta/(1+\beta)]}\, {}^\Phi g_{ij}$. It is easy to see that

$$(1+\beta)\bar{R} = \Omega^{-2\beta/(1+\beta)}({}^\Phi R + \beta\Omega^2\, {}^\Psi R) + (n-2)(n-3)\frac{\beta}{1+\beta}\left(\frac{\bar{D}\Omega}{\Omega}\right)^2. \qquad (A1)$$

Like Witten's positive energy theorem [13,14], now, we can prove the positivity of the total gravitational mass on $(\bar{\Sigma}, \bar{g}xs_{ij})$ which is

$$\bar{m} = (1+\beta)^{-1}({}^\Phi m + \beta\, {}^\Psi m) \geqslant 0. \qquad (A2)$$

For the case of $\bar{m}=0$, we see that $(\bar{\Sigma}, \bar{g}_{ij})$ is flat and Ω is constant. They imply that $({}^\Phi\Sigma, {}^\Phi g_{ij})$ and $({}^\Psi\Sigma, {}^\Psi g_{ij})$ are also flat.

It is trivial that $\bar{m}=0$ if $({}^\Phi\Sigma, {}^\Phi g_{ij})$ and $({}^\Psi\Sigma, {}^\Psi g_{ij})$ are flat. Q.E.D.

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