Quasinormal modes of a small Schwarzschild–anti-de Sitter black hole

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We compute the quasinormal modes associated with the decay of the massless scalar field around a small Schwarzschild–anti-de Sitter black hole. The computations show that when the horizon radius is much less than the anti–de Sitter radius, the imaginary part of the frequency goes to zero as r_+^{d-2} while the real part of ω decreases to its minimum and then goes to $d-1$. Thus the quasinormal modes approach the usual AdS modes in the limit $r_+ \to 0$. This agrees with suggestions of Horowitz and Hubeny [Phys. Rev. D **62**, 024027 (2000)].

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The original interest in quasinormal modes (QNMs) of black holes arose since they are the characteristics of black holes (BHs) which do not depend on initial perturbations and are functions of the black hole parameters only. At present, this interest has been renewed since the QN frequencies are in the suggested region of the gravitational wave detectors which are under construction. In general, QNMs are important in black hole dynamics and appear in such processes as the collisions of two black holes and decay of different fields in a BH background.

All this motivated the investigation of the QNM of black holes in asymptotically flat space-time (see [1] for a recent review). The quasinormal modes of asymptotically de Sitter black holes were studied in [2,3]. Recently an unexpected application of quasinormal modes has appeared due to the AdS conformal field theory (CFT) correspondence [4]: it proved that a large black hole in AdS space corresponds to an approximately thermal state in the CFT, and, thereby, the perturbation of the black hole corresponds to the perturbation of the above thermal state, while the decay of the perturbation can be associated with the return to thermal equilibrium. Thus the QN frequencies give us the thermalization time scale which is very hard to compute directly. Nevertheless, recently exact agreement between the QN frequencies for the three-dimensional Bañados-Teitelboim-Zanelli (BTZ) black hole and the location of the poles of the retarded correlation function of a perturbation in the dual conformal field theory has been found [5]. Quasinormal modes for different types of perturbations of black holes in AdS space have been studied also in a lot of papers [6–15].

Black holes are considered to be small (large), when its horizon radius is much smaller (larger) then the anti–de Sitter radius. When computing the QN frequencies associated with the decay of massless scalar field in the background of small Schwarzschild–anti-de Sitter (SAdS) BH a striking conjecture with the black hole critical phenomena was found: $\omega_{\rm Im}$ is proportional to BH radius r_+ to high accuracy, and the slope of the line $\omega_{\rm Im}$ to the r_+ axis, 2.66, turned out to be very close to the special frequency $\lambda = 2.67$ which corresponds to the growing mode $\exp \lambda t$ describing the late time behavior of the critical solution [16]. Yet, the quasinormal frequencies of the SAdS black hole have been computed only for black holes with horizon radius $r_+ \geqslant 0.4R$, where R is the anti–de Sitter radius (except for one mode corresponding to $r_+ = 0.2R$ for which the behavior of the wave function was numerically reproduced). This is not sufficient to study the small black hole limit, but several suggestions were made. In [15] it was stated that both real and imaginary parts of the QN modes for small black holes are very large and proportional to the surface gravity, however, later, it was shown by numerical integration of the wave equation, that at least for $r_+ \geqslant 0.4R$, in agreement with the first study [6,7], both the real and imaginary parts of ω are decreasing, and stated that the behavior $\omega_{\rm Re} \to \text{const}, \quad \omega_{\rm Im} \to 0$ at $r_+ \to 0$ is expected. Note that by considering an approximate symmetry of the SAdS metric in the limit $r_+ \to 0$ it was supposed that $\omega_{\rm Im} \to 0$ as r_+^2 [6]. Here we try to compute the quasinormal modes of black holes with the horizon radius smaller than $0.4R$, to approach the small black holes regime ($r_+ \ll R$) as much as possible, and to obtain more definite hints of very small black hole behavior. It proves that computations of quasinormal modes for very small black holes are quite reliable within the method proposed in [7] provided one avoids accumulating numerical error (see the Appendix).

The d-dimensional Schwarzschild–anti-de Sitter metric is

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 d\Omega_{d-2}^2, \tag{1}$$

where

$$f(r) = 1 - \frac{r_0^{d-3}}{r} + \frac{r^2}{R^2}. \tag{2}$$

Here r_0 is related to the black hole mass

$$M = \frac{(d-2)A_{d-2} r_0^{d-3}}{16\pi G_d},$$

and A_{d-2} is the area of a unit $(d-2)$ sphere.

Quasinormal modes of black holes in asymptotically anti–de Sitter space-time are governed by the wave equation

$$\left(\frac{d^2}{dr_*^2} + \omega^2\right)\Psi(r) = U\Psi(r), \tag{3}$$

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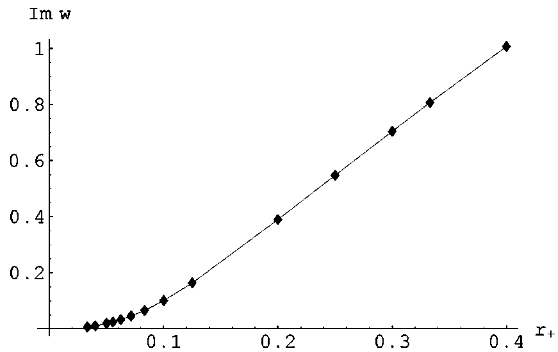

FIG. 1. Imaginary part of ω for $d=4$ black hole, $l=0$, $n=0$.

where the potential U is given by

$$U(r)=f(r)\left(\frac{l(l+d-3)}{r^2}-\frac{(2-d)(d-4)}{4r^2}f(r)+\frac{2-d}{2r}f'(r)\right), \tag{4}$$

and we take $\omega=\omega_{\mathrm{Re}}-\imath\omega_{\mathrm{Im}}$. The tortoise coordinate is $dr^*=f^{-1}(r)dr$, l is the angular harmonic index. By rescaling of r we can put $R=1$. It is essential that the effective potential is infinite at spatial infinity. Thus the wave function vanishes at infinity and satisfies the purely in-going wave condition at the black hole horizon.

We managed to compute the quasinormal modes for the $d=4$ black hole with the radius up to $r_+=1/30R$ (see Figs. 1 and 2). This reasonably approximates behavior in a small black hole regime. It proved that the oscillation frequency falls down to some minimum and then begins to grow approaching $d-1$ when $r_+\to 0$ (see Fig. 2). This minimum of the Re ω equals

$$\min(\omega_{\mathrm{Re}})\approx 2.362\,868 \quad \text{at} \quad r_+=0.395R, \quad d=4,$$

$$\min(\omega_{\mathrm{Re}})\approx 3.705\,140 \quad \text{at} \quad r_+=0.341R, \quad d=5,$$

and (according to our preliminary results) continues to increase for higher dimensions. Here with the more dimensional AdS space is the less black hole radius r_+ at which the real oscillation frequency attains its minimum. Upon thorough consideration of Fig. 1 of [10] one can learn that among the real oscillation frequencies corresponding to $r_+=0.2$, 0.4, 0.6, $0.8R$, ω_{Re} at $r_+=0.4R$ is the least. This agrees with our computations showing the minimum of ω_{Re} at $r_+\approx 0.395R$ for a four-dimensional black hole. The imaginary part of ω falls down to zero, and the closer r_+ is to zero the better the corresponding plot can be fit by the function Ar_+^2. For $d=4$ the best fit for the last five points (from $r_+=1/16$ to $r_+=1/30$) in Table I is $\omega_{\mathrm{Im}}=8.066\,53r_+^2$. The higher the dimensions, the less ω_{Im} of small black holes is, i.e., the more the damping time of a perturbation.

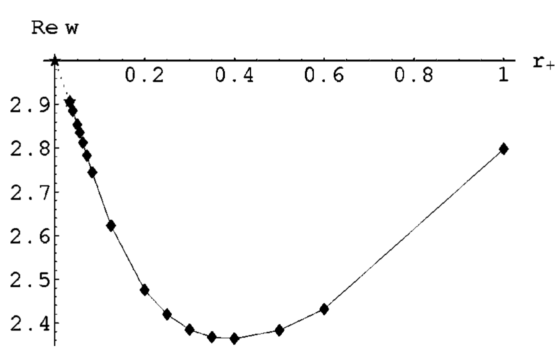

FIG. 2. Real part of ω for $d=4$ black hole, $l=0$, $n=0$.

TABLE I. The fundamental quasinormal frequencies corresponding to $d=4$ SAdS black hole, $R=1$.

r_+	ω_{Re}	ω_{Im}
0.3	2.38447	0.70413
0.25	2.41945	0.54735
0.2	2.47511	0.3899
0.125	2.62274	0.16392
0.1	2.69282	0.10096
1/12	2.74472	0.06616
1/14	2.78341	0.04578
1/16	2.81289	0.03311
1/18	2.83574	0.02491
1/20	2.8539	0.01932
1/25	2.88584	0.01138
1/30	2.9065	0.0074

Thus even though the boundary conditions at $r=r_+$ do not reduce to regularity at the origin in the limit $r_+\to 0$, the quasinormal modes approach the usual AdS modes [17] in this limit (we checked it out for $d=4,5$), i.e., $\omega_{\mathrm{Re}}\to d-1$ and $\omega_{\mathrm{Im}}\to 0$, as was discussed in [6] (see Figs. 3 and 4).

APPENDIX

When computing the quasinormal mode one has to truncate the sum representing the wave function

$$\psi(x)=\sum_{n=0}^{\infty} a_n(x-x_+)^n, \quad x=\frac{1}{r} \tag{A1}$$

with some large N and find the roots of the equation $|\psi(x)|=0$ at $r=\infty(x=0)$. After simplification the truncated sum (A1) takes the polynomial in ω form, and the necessary roots

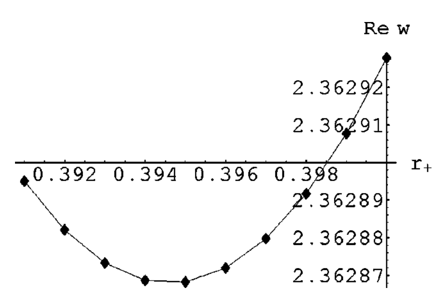

FIG. 3. Real part of ω for $d=4$ black hole near its minimum, $l=0$, $n=0$.

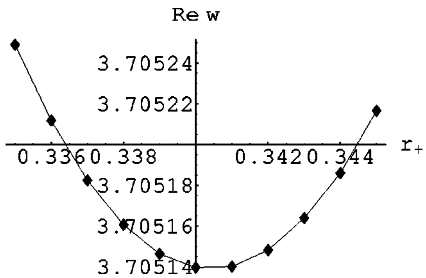

FIG. 4. Real part of ω for $d=5$ black hole near its minimum, $l=0$, $n=0$.

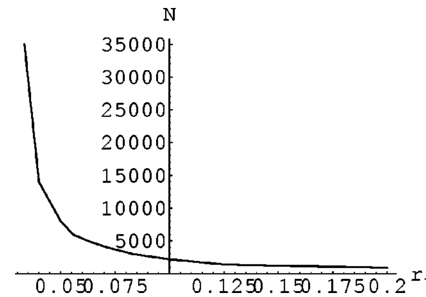

FIG. 5. Convergence plot for Re ω, $d=4$, $l=0$, $n=0$.

can easily be found by MATHEMATICA. However, this reduction to a polynomial form takes a lot of computer time and cannot be performed for a sum of order $N\sim200$ or more. Thus for small black holes we have to use the trial and error method. Herewith there is a danger of missing the fundamental mode we are seeking, and, of ''catching'' another overtone. Fortunately, for reasonably small black holes there are no other overtones close to the fundamental one, and the minimums of $|\psi|$ are sufficiently widely separated. Another difficulty is that for small values of r_+ the initial tiny errors when determining the quantities involved in the sum in Eq. (A1) (namely u_i, t_i, and s_i of the paper [6]) begin to grow when coming to N of order 1000 or greater. Therefore one has to improve the precision of these quantities (with the help of a built-in function of MATHEMATICA) until further increasing of precision will not influence the result. It proved that the 50-digital precision of u_i, t_i, s_i is quite enough in this paper. In addition one must set a higher precision of recurrence relations for coefficients a_n. When approaching the limit $r_+=0$ the number N of the truncated sum (A1) at which an approximate frequency converges grows as is shown in Fig. 5 for $d=4$. The more d, the more the number N is giving good approximation of the frequency. When following all these receptions one can be sure that the convergence plot will be smooth and that at small changing of r_+ the corresponding frequency will not change noisily.

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