Generalization of the Einstein-Straus model to anisotropic settings

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We study the possibility of generalizing the Einstein-Straus model to anisotropic settings by considering the matching of locally cylindrically symmetric static regions to the set of G_4 on S_3 locally rotationally symmetric (LRS) spacetimes. We show that such matchings preserving the symmetry are only possible for a restricted subset of the LRS models in which there is no evolution in one spacelike direction. These results are applied to spatially homogeneous (Bianchi) exteriors where the static part represents a finite bounded interior region without holes. We find that it is impossible to embed finite static strings or other locally cylindrically symmetric static objects (such as bottle or coin-shaped objects) in reasonable Bianchi cosmological models, irrespective of the matter content. Furthermore, we find that if the exterior spacetime is assumed to have a perfect fluid source satisfying the dominant energy condition, then only a very particular family of LRS stiff fluid solutions are compatible with this model. Finally, given the interior-exterior duality in the matching procedure, our results have the interesting consequence that the Oppenheimer-Snyder model of collapse cannot be generalized to such anisotropic cases.

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I. INTRODUCTION

An important long standing question in cosmology concerns the way large scale dynamics of the universe influences the behavior on smaller scales. In particular, given the observed large scale expansion of the universe the question is to what extent does this expansion influence the behavior on astrophysical scales, and more precisely what are its effects on, e.g., the planetary orbits, galaxies and clusters of galaxies. Among the earliest works on this question are those by McVittie [1] and Einstein and Straus [2] (see [3] for more historical references). Historically it was McVittie who first found a perfect fluid spherically symmetric solution to Einstein's field equations which could be interpreted as describing a point particle embedded in an expanding Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime [1]. This interpretation has been questioned by some authors subsequently (e.g., Sussman [4], Gautreau [5] and Nolan [6]). The generally accepted ansatz to model the problem is due to Einstein and Straus [2], who proposed a matching between two spacetimes, instead of trying to use a single solution. They successfully matched the spherically symmetric vacuum Schwarzschild solution to an expanding *dust*[1] FLRW exterior across a hypersurface preserving the symmetry. They showed that such a matching was possible across any comoving 2-sphere, as long as the total mass contained inside the 2-sphere was equal to the Schwarzschild mass contained in it. In this way they concluded that there was no influence from the global expansion of the universe on the vacuum region surrounding the Schwarzschild mass. Two objections have been raised against this model: the first by Krasiński [3] who suggested that the Einstein-Straus model is unstable against radial perturbations, and the second by Bonnor [8,9] who pointed out that there are severe restrictions on the scales on which the model is applicable and that it is not suitable for studies within our solar system or even the galaxy.

Another important observation regarding the result of Einstein and Straus is that it involves a number of idealizations, including the fact that the universe is assumed to be representable by a spatially homogeneous and isotropic dust FLRW model. The question then arises as to whether this result is robust with respect to various plausible generalizations. These could involve changes in the symmetry properties of the model as well as the nature of the interior source field, which was originally taken to be vacuum.

A number of interesting attempts have been made in this direction. Among them are models which keep the spherical symmetry but generalize the interior source fields by considering for example Vaidya (see [10] and references therein[2]) or Lemaître-Tolman spacetimes (see [3] for references concerning the latter in connection with formation of voids). There have also been attempts concerning the relaxation of the spherical symmetry assumption, including generalizations to locally cylindrically symmetric spacetimes. These include the example of the embedding of dust FLRW into a nonstatic vacuum exterior across a hypersurface of a constant radius [11] which, due to the freedom in interpreting the two parts being matched as interior or exterior [10] (which we shall refer to as *interior-exterior duality*), is equivalent to an embedding of a nonstatic vacuum region into a dust FLRW. Similarly, the impossibility of the embedding of typical cosmic strings (i.e., Minkowski with deficit angle) as well as some special nonstatic cylindrically symmetric vacuo into *flat* FLRW was shown in [12]. This problem has been further studied by Senovilla and Vera [13] who have shown in full generality that the embedding of a locally orthogonally tran-

[1]This is a consequence of the FLRW model having been matched to a vacuum spacetime (see, e.g. [7]).

[2]The aim of these studies was, in fact, to generalize to nondust FLRW models.

sitive (OT)[3] cylindrically symmetric *static* cavity in an expanding FLRW is not possible, irrespective of the matter content of the cavity.[4] This result has been further generalized by Mars [16] to the case of axial symmetry. Mars [17] has finally been able to prove that in order to embed *any* static cavity in a FLRW universe then this cavity must be ''almost spherically symmetric;'' more precisely, the boundary as seen from the FLRW exterior is required to be a 2-sphere in space. Furthermore, for standard interior source fields such as vacuum, electrovacuum or perfect fluids, the interior itself must be spherically symmetric, with the boundary comoving with the cosmological flow [16,17]. This implies therefore that static objects which can be embedded in FLRW models must be spherical, and as a result the Einstein-Straus model is, in this sense, not robust.

This result once again raises the question of the possibility of embedding general static cavities in more general universe models (which we refer to as the *generalized Einstein-Straus problem*). Also, since realistic cosmological models cannot be expected to be exactly homogeneous and isotropic, the question arises as to what happens if these symmetry assumptions concerning the exterior metric are further relaxed.

There are two different ways to study departures from FLRW: either perturbatively (see [18] for a perturbed generalization of Einstein-Straus with a small rotation) or using exact solutions. Given that a precise formalism for a perturbed matching of two spacetimes is not fully developed, we shall proceed in the second way. A step in this direction was taken by Bonnor [9], who considered the embedding of a Schwarzschild region in an expanding spherically symmetric inhomogeneous Lemaître-Tolman exterior. He found that such matching is possible in general, and it allows the mass and radius for the Schwarzschild cavity to be chosen independently of the exterior LT density.

An interesting question is whether similar results would hold for cases with nonspherically symmetric interiors. As a step in this direction, we shall first of all study the *local* matching between static OT cylindrically symmetric spacetimes and the class of locally rotationally symmetric (LRS) spacetimes admitting a G_4 on S_3 group of isometries, which constitute an anisotropic generalization of the FLRW models. To make the matching global one expects to have further restrictions. In the particular case of an interior that describes a bounded object without holes, we were able to show that this is in fact the case. Thus if the exterior is assumed to be a spatially homogeneous expanding Bianchi spacetime, then, in order to preserve the symmetry, it has to be locally rotationally symmetric, admitting a G_4 on S_3. We note that the results obtained here for the LRS spacetimes also hold for models representing static cavities embedded in Bianchi spacetimes.

Our main result is then that no locally OT cylindrically symmetric static cavities can be embedded into reasonable evolving anisotropic (Bianchi) spacetimes.

We also note that given the interior-exterior duality in the matching procedure, all our results apply equally to the case where it is the interior that is taken to have a spatially homogeneous geometry, embedded into a locally cylindrically symmetric static background. This would allow our results to be applied to other settings, such as the study of the generalization of the Oppenheimer-Snyder [7] model for collapsing objects, which could be viewed as the ''dual'' to the Einstein-Straus model, in the interior-exterior sense defined here.

The plan of the paper is as follows. In Sec. II we review the matching procedure and the definition of matching preserving the symmetry. In Sec. III we present a compact form of the line element for the G_4 on S_3 LRS spacetimes in coordinates adapted to the axial Killing vector field. This will prove useful in Secs. IV and V, where we calculate the matching conditions for the matching preserving the symmetry between a static OT cylindrically symmetric spacetime and a LRS homogeneous spacetime. In Sec. VI we study the restrictions on the subset of LRS spacetimes that are allowed by the matching conditions. We show that the only perfect fluid solutions in this subset correspond to a particular family which has a stiff fluid equation of state. In Sec. VII we extend our results to the case of the spatially homogeneous exteriors. Finally, Sec. VIII gives our discussions and conclusions.

II. MATCHING PROCEDURE

In this section we shall briefly recall the matching procedure across general hypersurfaces (see [19] for more details). As is well known, the matching of two spacetimes requires two sets of (matching or junction) conditions at the matching hypersurface. The first set of these junction conditions will ensure the continuity of the metrics across the matching hypersurface; while the second is equivalent to a nonsingular Riemann tensor distribution in order to prevent infinite discontinuities of matter and curvature across the matching hypersurface.

More precisely, let $(\mathcal{V}^+,g^+)$ and $(\mathcal{V}^-,g^-)$ be two C^3 spacetimes with oriented boundaries σ^+ and σ^-, respectively, such that σ^+ and σ^- are diffeomorphic. The matched spacetime $(\mathcal{V},g)$ is the disjoint union of $\mathcal{V}^\pm$ with the points in $\sigma^\pm$ identified such that the junction conditions are satisfied (see [19–22]). Since $\sigma^\pm$ are diffeomorphic, one can view these boundaries as diffeomorphic to a 3-dimensional oriented manifold σ which can be embedded in $\mathcal{V}^+$ and $\mathcal{V}^-$. Let $\{\xi^a\}$ $(a=1,2,3)$ and $\{x^{\pm\alpha}\}$ be coordinate systems on σ and $\mathcal{V}^\pm$, respectively. The two boundaries are given by two C^3 maps

$$\begin{aligned}\Phi^\pm\colon \sigma &\rightarrow \mathcal{V}^\pm,\\ \xi^a &\mapsto x^{\alpha\pm}=\Phi^{\alpha\pm}(\xi^a),\end{aligned}\tag{1}$$

such that $\sigma^\pm=\Phi^\pm(\sigma)$. At every point $p\in\sigma$ the natural basis $\{\partial/\partial\xi^a|_p\}$ of the tangent plane $T_p\sigma$ is pushed forward by the

[3]For most matter contents one is interested, this assumption is actually a consequence of having an axis of symmetry [14]; see also below.

[4]We note that nonexpanding exteriors can be matched; see [15].

rank-3 differential maps $d\Phi|^{\pm}_{p}$ into three linearly independent vectors at $\Phi^{\pm}(p)$, denoted by $\vec{e}^{\,\pm}_a|_{\Phi^{\pm}(p)}$, defined only in the corresponding hypersurfaces $\sigma^{\pm}$, as follows:

$$d\Phi^{\pm}\left(\frac{\partial}{\partial\xi^a}\right)=\frac{\partial\Phi^{\pm\mu}}{\partial\xi^a}\,\frac{\partial}{\partial x^{\pm\mu}}\bigg|_{\sigma^{\pm}}\equiv\vec{e}^{\,\pm}_a=e^{\pm\mu}_a\frac{\partial}{\partial x^{\pm\mu}}\bigg|_{\sigma^{\pm}}. \quad (2)$$

Using the pull-backs $\Phi^{\pm *}$, the metrics $g^{\pm}$ at any point $\Phi^{\pm}(p)\in\sigma^{\pm}$ are mapped to the dual space at $p\in\sigma$ providing two symmetric 2-covariant tensors $\overline{g^{+}}$ and $\overline{g^{-}}$, whose components in the natural basis $\{d\xi^a\}$ are $\overline{g}^{\pm}_{ab}=e^{\pm\mu}_a e^{\pm\nu}_b g_{\mu\nu}|_{\sigma^{\pm}}=(\vec{e}^{\,\pm}_a\cdot\vec{e}^{\,\pm}_b)$. These are the first fundamental forms of σ inherited from $(\mathcal{V}^{\pm},g^{\pm})$. Now, as shown in [19,22], the necessary and sufficient condition for the existence of a *continuous* extension g of the metric to the whole manifold $\mathcal{V}$ such that $g|_{\mathcal{V}^+}=g^+$ and $g|_{\mathcal{V}^-}=g^-$ is

$$\overline{g^{+}}=\overline{g^{-}}. \quad (3)$$

These relations, which can also be expressed as $ds^{2+}|_{\sigma^+}\overset{\sigma}{=}ds^{2-}|_{\sigma^-}$ (where $\overset{\sigma}{=}$ implies that both sides of the equality must be evaluated on σ), are the *preliminary junction conditions* [19]. Now, the bases $\{\vec{e}^{\,+}_a\}$ and $\{\vec{e}^{\,-}_a\}$ can be identified,

$$d\Phi^{+}\left(\frac{\partial}{\partial\xi^a}\right)=d\Phi^{-}\left(\frac{\partial}{\partial\xi^a}\right), \quad (4)$$

as can the hypersurfaces $\sigma^{+}=\sigma^{-}$, so henceforth we represent both $\sigma^{\pm}$ by σ. Essentially, we are identifying the abstract manifold σ with its images $\sigma^{+}=\sigma^{-}$ in $(\mathcal{V},g)$.

In order to impose the remaining junction conditions we need a one-form, $\mathbf{n}$, normal to the hypersurface, defined through the condition $\mathbf{n}^{\pm}(\vec{e}^{\,\pm}_a)=0$. Since in the final matched manifold $\mathcal{V}$ the normals are to be identified as a single object, both must have the same norm. Also if $\mathbf{n}^{+}$ is to point $\mathcal{V}^{+}$ outwards, then $\mathbf{n}^{-}$ has to point $\mathcal{V}^{-}$ inwards, and conversely. In order to deal with general hypersurfaces, including spacelike and null hypersurfaces, we will also need the rigging vectors $\vec{l}^{\,\pm}$ on $\sigma^{\pm}$ [23], which are defined as vector fields on $\sigma^{\pm}$ and transversal to $\sigma^{\pm}$.[5] The riggings are therefore characterized everywhere on σ by

$$\mathbf{n}^{+}(\vec{l}^{\,+})\overset{\sigma}{=}\mathbf{n}^{-}(\vec{l}^{\,-})\neq 0, \quad (5)$$

so that the vectors $\{\vec{l}^{\,\pm},\vec{e}^{\,\pm}_a\}$ constitute a basis for the tangent spaces to $\mathcal{V}^{\pm}$ at $\sigma^{\pm}$. Given that the preliminary conditions allow us to identify $\{\vec{e}^{\,+}_a\}$ with $\{\vec{e}^{\,-}_a\}$, it only remains to choose the riggings such that the bases $\{\vec{l}^{\,\pm},\vec{e}^{\,\pm}_a\}$ have the same orientation with

$$l^{+}_{\mu}l^{+\mu}\overset{\sigma}{=}l^{-}_{\mu}l^{-\mu}, \quad l^{+}_{\mu}e^{+\mu}_a\overset{\sigma}{=}l^{-}_{\mu}e^{-\mu}_a. \quad (6)$$

[5]Note that in the case of non-null hypersurfaces the normal vector is itself a rigging vector.

In this way, we can identify the whole 4-dimensional tangent spaces of $\mathcal{V}^{\pm}$ at σ, $\{\vec{l}^{\,+},\vec{e}^{\,+}_a\}=\{\vec{l}^{\,-},\vec{e}^{\,-}_a\}\equiv\{\vec{l},\vec{e}_a\}$. We note that if the second equation in Eq. (6) and the preliminary junction conditions hold then the first relation in Eq. (6) is equivalent to Eq. (5) up to a sign.

The remaining junction conditions amount to the equality of the generalized second fundamental forms $H^{\pm}_{ab}$

$$H^{\pm}_{ab}=-l^{\pm}_{\nu}e^{\pm\mu}_a\nabla^{\pm}_{\mu}e^{\pm\nu}_b.$$

In the case of non-null hypersurfaces, choosing $\vec{l}=\vec{n}$, the tensors $H^{\pm}_{ab}$ coincide with the second fundamental forms $K^{\pm}_{ab}=-n^{\pm}_{\nu}e^{\pm\mu}_a\nabla^{\pm}_{\mu}e^{\pm\nu}_b$ inherited by $\sigma^{\pm}$ from $\mathcal{V}^{\pm}$ [19,22,24]. Note that the junction conditions $H^{+}_{ab}\overset{\sigma}{=}H^{-}_{ab}$ do not depend on the specific choice of the rigging vector [19].

When symmetries are present, as in most of the works dealing with spacetime matchings, one is interested in the cases where the matching surface σ inherits a particular symmetry of the two space-times $(\mathcal{V}^{\pm},g^{\pm})$. Such matching is said to *preserve the symmetry*. In practice one demands that the matching hypersurface is tangent to the orbits of the symmetry group to be preserved. A more rigorous definition of matching preserving the symmetry was recently presented in [25]. Thus if $(\mathcal{V}^{+},g^{+})$ and $(\mathcal{V}^{-},g^{-})$ both admit a m-dimensional group of symmetries, the final matched spacetime $(\mathcal{V},g)$ is said to preserve the symmetry G_m if there exist m vectors on σ that are mapped by the push-forwards $d\Phi^{+}$ and $d\Phi^{-}$ to the restrictions of the generators of G_m to σ^{+} and σ^{-}, respectively. Furthermore, if there is an intrinsically distinguishable generator of G_m in $\mathcal{V}^{+}$ and $\mathcal{V}^{-}$, such as an axial Killing vector, then the matching preserving the symmetry must ensure its identification at σ.

In the cases we shall consider below, $(\mathcal{V}^{+},g^{+})$ will correspond to a G_4 on S_3 LRS spacetime, thus admitting a cylindrical symmetry (Abelian G_2 subgroup), and $(\mathcal{V}^{-},g^{-})$ to a static OT cylindrically symmetric spacetime. We shall consider the matching preserving the cylindrical symmetry, which is represented by an Abelian group G_2 [14,26,27].

III. GENERAL METRIC FORMS WITH A G_4 ON S_3 WHICH ARE LRS

In this section we shall write down in explicit cylindrical-like coordinates a general compact form of the metric corresponding to LRS spatially homogeneous spacetimes, which admit a G_4 group of motions on spatial 3-hypersurfaces S_3. We begin by combining the standard metric forms for all possible G_4 on S_3 LRS spaces, with $k=\pm 1,0$, given by [28] (see also [29])

$$ds^2=-dt^2+a^2(t)dx^2+b^2(t)(dy^2+\Sigma^2(y,k)d\omega^2), \quad (7)$$

$$ds^2=-dt^2+a^2(t)\boldsymbol{\sigma}^2_k+b^2(t)(dy^2+\Sigma^2(y,k)dw^2), \quad (8)$$

$$ds^2=-dt^2+a^2(t)dx^2+b^2(t)e^{2x}(dy'^2+dw'^2), \quad (9)$$

where

$$\Sigma(y,k)=\begin{cases}\sin y, & k=+1,\\ y, & k=0,\\ \sinh y, & k=-1,\end{cases}$$

$$\boldsymbol{\sigma}_k=\begin{cases}dx+\cos y\,dw, & k=+1,\\ dx-y^2/2dw, & k=0,\\ dx+\cosh y\,dw, & k=-1.\end{cases} \tag{10}$$

Performing a change to polar coordinates $\{y'=y\sin w, w'=y\cos w\}$ in the line element (9), and following [29] in a first step, the above metrics can be combined into a compact form given by

$$ds^2=-dt^2+a^2(t)\hat{\boldsymbol{\theta}}_{nk}^{\,2}+b^2(t)e^{2\epsilon x}(dy^2+\Sigma^2(y,k)dw^2), \tag{11}$$

where

$$\hat{\boldsymbol{\theta}}_{nk}=dx+nF(y,k)dw, \tag{12}$$

$$F(y,k)=\begin{cases}-\cos y, & k=+1,\\ y^2/2, & k=0,\\ \cosh y, & k=-1\end{cases} \tag{13}$$

and where ϵ and n are such that

$$\epsilon=0,1,\quad n=0,1,\quad \epsilon n=\epsilon k=0. \tag{14}$$

We note that

$$\Sigma=F_{,y},\quad (\Sigma_{,y})^2+k\Sigma^2=1, \tag{15}$$

where here and throughout the comma denotes the partial derivative with respect to the indicated variable. The metrics (7), (8) and (9) are recovered with $\{\epsilon=0,n=0\}$, $\{\epsilon=0,n=1\}$ and $\{\epsilon=1\}$ respectively.[6] The metric (11) with $\epsilon=n=0$ and $k=1$ is the Kantowski-Sachs metric, which admits no simply transitive G_3 subgroup. All the other cases included in Eq. (11) possess a simply transitive G_3 subgroup of symmetries that can be classified according to their Bianchi types: metric (8) corresponds to type II for $k=0$ and to types III, VIII and IX for $k\neq 0$; metric (9) corresponds to types V and VII_h and metric (7) corresponds to types I, III and VII_0. These classifications are summarized in Table I. The cases with $n=0$ admit a multiply transitive G_3 on S_2.

The axial Killing vector associated with the metric (11) is $\vec{\eta}_1=\partial_w+nk\partial_x$, which can be easily shown to define a regular axis at $y=0$, that is

[6]Note that for convenience we have introduced a change in the sign of w in Eq. (8) for the cases $k=0,1$ which results in a change of sign in the expressions of the Killing vectors as shown in Kramer *et al.* [28].

TABLE I. Bianchi types of the possible subgroups G_3 on S_3 according to the values $\{\epsilon,k,n\}$ of metric (11).

Bianchi types	ϵ	n	k
I, III	0	0	-1
VII_0	0	0	0
VIII, IX	0	1	-1
II	0	1	0
III	0	1	1
V, VII_h	1	0	0

$$\frac{\nabla_\rho(\vec{\eta}_1^{\,2})\nabla^\rho(\vec{\eta}_1^{\,2})}{4\,\vec{\eta}_1^{\,2}} \tag{16}$$

tends to 1 as $y\to 0$. The axial Killing vector $\vec{\eta}_1$ together with $\vec{\eta}_2=\partial_x-\epsilon y\partial_y$ generates an Abelian subgroup G_2 on S_2. For the purposes of this paper it is desirable to express the metric in a form adapted to both $\vec{\eta}_1$ and $\vec{\eta}_2$. To do this, we perform the following coordinate transformations to cylindrical coordinates:

$$\begin{aligned}x&\to z+nk\varphi,\\ y&\to re^{-\epsilon z},\\ w&\to\varphi,\\ t&\to t,\end{aligned} \tag{17}$$

which brings the compact metric (11) into the form

$$ds^2=-dt^2+a^2(t)\boldsymbol{\theta}_{nk}^2+b^2(t)[(dr-\epsilon r dz)^2+\Sigma(r,k)^2d\varphi^2], \tag{18}$$

where

$$\boldsymbol{\theta}_{nk}=dz+n(F(r,k)+k)d\varphi. \tag{19}$$

To our knowledge, this is a new form of presenting all the G_4 on S_3 LRS spacetimes in compact form. The axial Killing vector is then given by

$$\vec{\eta}_1=\partial_\varphi, \tag{20}$$

while the other three Killing vectors $\vec{\eta}_i$, $i=2$ to 4, are taken to be

$$\begin{aligned}\vec{\eta}_2&=\partial_z,\\ \vec{\eta}_3&=e^{\epsilon z}[\sin\varphi\,\partial_r+\cos\varphi(f(r)\partial_\varphi+g(r)\partial_z)],\\ \vec{\eta}_4&=e^{\epsilon z}[\cos\varphi\,\partial_r-\sin\varphi(f(r)\partial_\varphi+g(r)\partial_z)],\end{aligned} \tag{21}$$

where we have defined $f(r)=\Sigma_{,r}/\Sigma$ and $g(r)=n(\Sigma-f(F+k))$.

IV. THE MATCHING HYPERSURFACE

Our aim is to match a spacetime corresponding to the metric (18) and a static OT cylindrically symmetric spacetime. The metric (18) can be cast in the more general form, say g^+, given by

$$ds^{2+} = -\hat{A}^2 dt^2 + \hat{B}^2 dr^2 - 2\epsilon r \hat{B}^2 dr dz + \hat{C}^2 d\varphi^2 + 2\hat{E} dz d\varphi + \hat{D}^2 dz^2, \tag{22}$$

where $\hat{A}$, $\hat{B}$, $\hat{C}$, $\hat{D}$ and $\hat{E}$ are functions of t and r. The line element (18) is recovered by making the identifications

$$\begin{aligned}
\hat{A}^2(t,r) &= 1,\\
\hat{B}^2(t,r) &= b^2(t),\\
\hat{C}^2(t,r) &= b^2(t)\Sigma^2(r,k) + na^2(t)(F(r,k)+k)^2,\\
\hat{D}^2(t,r) &= a^2(t) + \epsilon r^2 b^2(t),\\
\hat{E}(t,r) &= na^2(t)(F(r,k)+k).
\end{aligned} \tag{23}$$

The usefulness of writing Eq. (18) as Eq. (22) using Eq. (23) will become clear in the next section. In the following we shall take the functions in Eq. (22) to be arbitrary functions, with

$$\epsilon \hat{E} = 0,$$

which follows from Eq. (14).

The metric g^- is assumed to be static and cylindrically symmetric [14], admitting, in principle, a maximal group[7] G_3 on T_3 containing an Abelian subgroup G_2 on S_2 which includes an axial symmetry [26,27]. The orbits of this G_2 subgroup are also assumed to generate orthogonal surfaces, i.e., the group G_2 is assumed to act orthogonally transitively (OT). This is the analogue of the "circularity condition," usually used within the context of stationary axisymmetric interior problems, where it implies nonconvectivity in fluids [30]. This assumption is in fact a consequence of the existence of an axis of symmetry in spacetimes with certain types of matter content, including vacuum [14,30].

Now one can always find a coordinate system $\{T,\rho,\tilde{\varphi}',\tilde{z}'\}$ adapted to the Killing vectors $\{\partial/\partial T, \partial/\partial\tilde{\varphi}', \partial/\partial\tilde{z}'\}$, where $\partial/\partial\tilde{\varphi}'$ is the axial Killing vector, such that the metric g^- is given by

$$ds^{2-} = -\check{A}^2 dT'^2 + \check{B}^2 d\rho^2 + \check{C}^2 d\tilde{\varphi}'^2 + \check{D}^2 d\tilde{z}'^2 + 2\check{E} d\tilde{\varphi}' d\tilde{z}', \tag{24}$$

where $\check{A}$, $\check{B}$, $\check{C}$, $\check{D}$ and $\check{E}$ are functions of ρ.

Following the matching procedure specified in Sec. II we proceed by specifying the two embeddings $\sigma^\pm$. The embedding of σ^+ can be defined by choosing the appropriate coordinates on σ denoted by

$$\{\xi^a\} = \{\lambda, \phi, \zeta\}. \tag{25}$$

The coordinate ϕ is chosen such that the vector field $\partial/\partial\phi$ is mapped, by the push-forward $d\Phi^+$, at every point in σ, into the restriction of the axial Killing vector $\vec{\eta}_1 = \partial/\partial\varphi$ on σ^+, that is,

$$d\Phi^+\left(\frac{\partial}{\partial\phi}\right) = \left.\frac{\partial}{\partial\varphi}\right|_{\sigma^+} \equiv \vec{e}_2^{\,+},$$

and thus

$$\frac{\partial\Phi^{0+}}{\partial\phi} = \frac{\partial\Phi^{1+}}{\partial\phi} = \frac{\partial\Phi^{3+}}{\partial\phi} = 0, \quad \frac{\partial\Phi^{2+}}{\partial\phi} = 1. \tag{26}$$

Since we want the matching to preserve the cylindrical symmetry (i.e., a G_2 on S_2 containing the axial symmetry generated by $\vec{\eta}_1$), there must exist a vector field $\vec{\gamma}$ in σ which is mapped to the restriction on σ^+ of a Killing vector that, together with $\vec{\eta}_1$, generates a G_2 on S_2. The only possibility is for this Killing vector to be a linear combination of $\vec{\eta}_1$ and $\vec{\eta}_2$. In other words, we have $d\Phi^+(\vec{\gamma}) = a\,\vec{\eta}_1|_{\sigma^+} + b\,\vec{\eta}_2|_{\sigma^+}$ where a and b are arbitrary constants with $b \neq 0$. We can now use the fact that the group which is preserved is automatically inherited by the hypersurface σ [25,31] in which it has the same algebraic type [31]. Since the G_2 generated by $\vec{\eta}_1$ and $\vec{\eta}_2$ is Abelian then the vectors $\partial/\partial\phi$ and $\vec{\gamma}$, which are Killing vectors in σ, commute.

We can now choose a coordinate ζ such that $\vec{\gamma} = \partial/\partial\zeta$ and use a coordinate transformation $\zeta' = b\zeta$, $\phi' = \phi + a\zeta$ in σ in order to obtain (after dropping the primes)

$$d\Phi^+\left(\frac{\partial}{\partial\zeta}\right) = \left.\frac{\partial}{\partial z}\right|_{\sigma^+} \equiv \vec{e}_3^{\,+},$$

and thus

$$\frac{\partial\Phi^{0+}}{\partial\zeta} = \frac{\partial\Phi^{2+}}{\partial\zeta} = \frac{\partial\Phi^{1+}}{\partial\zeta} = 0, \quad \frac{\partial\Phi^{3+}}{\partial\zeta} = 1, \tag{27}$$

leaving Eq. (26) unchanged. Finally, the remaining coordinate λ can always be chosen such that the image of $\partial/\partial\lambda$ through $d\Phi^+$ is orthogonal to $\vec{e}_2^{\,+}$ and $\vec{e}_3^{\,+}$. This implies

$$\frac{\partial\Phi^{2+}}{\partial\lambda} \overset{\sigma}{=} -\frac{\hat{E}}{\hat{C}^2}\frac{\partial\Phi^{3+}}{\partial\lambda}$$

and

$$\frac{\partial\Phi^{3+}}{\partial\lambda} \overset{\sigma}{=} \epsilon\Phi^{1+}\frac{\hat{B}^2\hat{C}^2}{\hat{\Delta}}\frac{\partial\Phi^{1+}}{\partial\lambda},$$

where $\hat{\Delta} \equiv \hat{C}^2\hat{D}^2 - \hat{E}^2$, and thus

$$\frac{\partial\Phi^{2+}}{\partial\lambda} = 0.$$

[7]The G_3 group is then taken to be Abelian. Other algebraic types require the existence of more symmetries [14], and therefore need to be studied separately.

Hence we obtain the following expression for $\vec{e}_1^{\,+}$:

$$d\Phi^{+}\left(\frac{\partial}{\partial\lambda}\right)\overset{\sigma}{=}\frac{\partial\Phi^{0+}}{\partial\lambda}\left.\frac{\partial}{\partial t}\right|_{\sigma^+}+\frac{\partial\Phi^{1+}}{\partial\lambda}\left.\frac{\partial}{\partial r}\right|_{\sigma^+}+\epsilon\Phi^{1+}\frac{\hat{B}^2\hat{C}^2}{\hat{\Delta}}\frac{\partial\Phi^{1+}}{\partial\lambda}\left.\frac{\partial}{\partial z}\right|_{\sigma^+}\equiv\vec{e}_1^{\,+}. \quad (28)$$

By denoting the embedding $\{\Phi^{0+},\Phi^{1+},\Phi^{2+},\Phi^{3+}\}$ in Eq. (1) as $\{t,r,\varphi,z\}$, the matching surface σ^+ is parametrized as

$$\sigma^+=\{(t,r,\varphi,z):t=t(\lambda),\quad r=r(\lambda),\quad \varphi=\phi,\quad z=\zeta+f_z(\lambda)\}, \quad (29)$$

where $t(\lambda)$ and $r(\lambda)$ are functions of λ restricted by the fact that $d\Phi^+$ has to be of rank 3, that is $\dot{t}^2+\dot{r}^2\neq 0$, and

$$\dot{f}_z(\lambda)\overset{\sigma}{=}\epsilon r(\lambda)\dot{r}(\lambda)\frac{\hat{B}^2\hat{C}^2}{\hat{\Delta}}, \quad (30)$$

where the dot denotes differentiation with respect to λ. For $\epsilon=0$ we can choose $f_z=0$ without loss of generality.

We now consider the embedding Φ^-. The preservation of the cylindrical symmetry implies that the axial Killing vectors from both sides have to coincide at the matching hypersurface. Since the tangent spaces of both σ^+ and σ^- are going to be identified (see Sec. II), we only have to impose $d\Phi^-(\partial/\partial\phi)=\partial/\partial\tilde{\varphi}'|_{\sigma^-}\equiv\vec{e}_2^{\,-}$. The image of the vector $\partial/\partial\zeta$ must complete the basis of some G_2 on S_2 subgroup of the G_3 on T_3 admitted by $(\mathcal{V}^-,g^-)$. Therefore the image of $\partial/\partial\zeta$ takes the form

$$d\Phi^{-}\left(\frac{\partial}{\partial\zeta}\right)=a\left.\frac{\partial}{\partial\tilde{\varphi}'}\right|_{\sigma^-}+b\left.\frac{\partial}{\partial\tilde{z}'}\right|_{\sigma^-}+c\left.\frac{\partial}{\partial T'}\right|_{\sigma^-}\equiv\vec{e}_3^{\,+}, \quad (31)$$

where a, b and c are arbitrary constants, such that the inequality

$$[-c^2\check{A}^2\check{C}^2+b^2(\check{C}^2\check{D}^2-\check{E}^2)]|_{\sigma^-}>0 \quad (32)$$

gives the necessary and sufficient condition for having spacelike orbits, implying $b\neq 0$. Now condition (32) will be automatically satisfied once the preliminary junction conditions are satisfied since $\vec{e}_3^{\,+}$ is always spacelike, which makes $\vec{e}_3^{\,-}$ spacelike.

In order to simplify Eq. (31), we perform a change in $(\mathcal{V}^-,g^-)$ to a new coordinate system $\{T,\rho,\tilde{\varphi},\tilde{z}\}$ defined by

$$T=T'-\frac{c}{b}\tilde{z}',\quad \tilde{\varphi}=\tilde{\varphi}'-\frac{a}{b}\tilde{z}',\quad \tilde{z}=\frac{1}{b}\tilde{z}',$$

which is still adapted to the axial Killing vector, that is, $\partial/\partial\tilde{\varphi}=\partial/\partial\tilde{\varphi}'$. In these new coordinates the line element for g^- reads

$$ds^{2-}=-A^2dT^2-2cA^2dTd\tilde{z}+B^2d\rho^2+C^2d\tilde{\varphi}^2+2Ed\tilde{\varphi}d\tilde{z}+D^2d\tilde{z}^2, \quad (33)$$

where all the "nonhatted" functions depend only on ρ and we have set $A\equiv\check{A}$, $B\equiv\check{B}$, $C\equiv\check{C}$ and

$$D^2\equiv-c^2\check{A}^2+a^2\check{C}^2+b^2\check{D}^2+2ab\check{E},$$

$$E\equiv a\check{C}^2+b\check{E}. \quad (34)$$

Note that this change of coordinates is well defined whenever Eq. (32) holds, since this implies $C^2D^2-E^2>0$. Now, taking the embedding $\Phi^{\alpha-}$ in the new coordinates $\{T,\rho,\tilde{\varphi},\tilde{z}\}$, the images of $\partial/\partial\phi$ and $\partial/\partial\zeta$ simplify to

$$d\Phi^{-}\left(\frac{\partial}{\partial\phi}\right)=\left.\frac{\partial}{\partial\tilde{\varphi}}\right|_{\sigma^-}\equiv\vec{e}_2^{\,-},$$

with

$$\frac{\partial\Phi^{0-}}{\partial\phi}=\frac{\partial\Phi^{1-}}{\partial\phi}=\frac{\partial\Phi^{3-}}{\partial\phi}=0,\quad \frac{\partial\Phi^{2-}}{\partial\phi}=1, \quad (35)$$

$$d\Phi^{-}\left(\frac{\partial}{\partial\zeta}\right)=\left.\frac{\partial}{\partial\tilde{z}}\right|_{\sigma^-}\equiv\vec{e}_3^{\,-},$$

with

$$\frac{\partial\Phi^{0-}}{\partial\zeta}=\frac{\partial\Phi^{1-}}{\partial\zeta}=\frac{\partial\Phi^{2-}}{\partial\zeta}=0,\quad \frac{\partial\Phi^{3-}}{\partial\zeta}=1. \quad (36)$$

Since the image of $\partial/\partial\lambda$ by $d\Phi^+$ is orthogonal to $\vec{e}_2^{\,+}$ and $\vec{e}_3^{\,+}$, then its image by $d\Phi^-$, i.e., $\vec{e}_1^{\,-}$, must be orthogonal to $\vec{e}_2^{\,+}$ and $\vec{e}_3^{\,+}$, essentially because of the preliminary junction conditions, thus resulting in

$$\frac{\partial\Phi^{3-}}{\partial\lambda}\overset{\sigma}{=}\frac{cA^2C^2}{\Delta}\frac{\partial\Phi^{0-}}{\partial\lambda},\quad \frac{\partial\Phi^{2-}}{\partial\lambda}\overset{\sigma}{=}-\frac{E}{C^2}\frac{\partial\Phi^{3-}}{\partial\lambda}, \quad (37)$$

where we have defined $\Delta\equiv C^2D^2-E^2$, so that we have

$$d\Phi^{-}\left(\frac{\partial}{\partial\lambda}\right)=\frac{\partial\Phi^{0-}}{\partial\lambda}\left.\frac{\partial}{\partial T}\right|_{\sigma^-}+\frac{\partial\Phi^{1-}}{\partial\lambda}\left.\frac{\partial}{\partial\rho}\right|_{\sigma^-}-\frac{cA^2}{\Delta}\frac{\partial\Phi^{0-}}{\partial\lambda}\left.\left(E\frac{\partial}{\partial\tilde{\varphi}}-C^2\frac{\partial}{\partial\tilde{z}}\right)\right|_{\sigma^-}\equiv\vec{e}_1^{\,-}.$$

As a result the matching surface σ^- is parametrized by

$$\sigma^-=\{(T,\rho,\tilde{\varphi},\tilde{z}):T=T(\lambda),\rho=\rho(\lambda),\ \tilde{\varphi}=\phi+f_{\tilde{\varphi}}(\lambda),\tilde{z}=\zeta+f_{\tilde{z}}(\lambda)\}, \quad (38)$$

where $T(\lambda)$ and $\rho(\lambda)$ are arbitrary functions of λ, and $f_{\tilde{\varphi}}(\lambda)$ and $f_{\tilde{z}}(\lambda)$ are related to $T(\lambda)$ by Eq. (37). In this way, we have obtained the most general parametrizations for the interior and exterior matching surfaces σ^+ and σ^-.

The matching procedure described above involves some free parameters in the metric g^- which account for the possible inequivalent ways the two spacetimes $(\mathcal{V}^{\pm},g^{\pm})$ can be joined [25,32]. Some of these parameters may encode the freedom in choice of coordinates, while others may imply physical differences [32].

Thus, in practical situations where the metric (24) is given, we must use the new "nonhatted" functions obtained from the relations (34), which then depend on the parameters *a, b* and *c*, in the equations arising from the junction conditions. On the other hand, if the spacetime metric g^- is unknown, then according to the way we have set up the problem, the matching problem must be treated using Eq. (33). In that case, to recover the form (24), we need to invert Eqs. (34) in order to obtain the hatted functions appearing in Eq. (33). Only after this inversion is performed is it possible to determine whether different values of the parameter *c* corresponding to different ways of joining the spacetimes give rise to equivalent matchings.

V. JUNCTION CONDITIONS

We recall that in order to derive the junction conditions we have to calculate the first and second fundamental forms for both σ^+ and σ^-. For the g^+ metric (22), the parametric form of σ^+ (29) gives $dt|_{\sigma^+}=\dot{t}d\lambda$, $dr|_{\sigma^+}=\dot{r}d\lambda$, $d\varphi|_{\sigma^+}=d\phi$ and $dz|_{\sigma^+}=d\zeta+\dot{f}_z d\lambda$. Using Eq. (30), the first fundamental form on σ^+ can be written as

$$ds^{2+}|_{\sigma^+}\overset{\sigma}{=}(-\hat{A}^2\dot{t}^2+\mathcal{B}^2\dot{r}^2)d\lambda^2+\hat{C}^2d\phi^2+2\hat{E}d\phi d\zeta+\hat{D}^2d\zeta^2, \tag{39}$$

where

$$\mathcal{B}^2\equiv\hat{B}^2\left(1-\epsilon r^2\frac{\hat{B}^2\hat{C}^2}{\hat{\Delta}}\right),$$

which can be reduced to $\hat{B}^2(1-\epsilon r^2\hat{B}^2/\hat{D}^2)$ making use of $\epsilon\hat{E}=0$. Similarly for the g^- metric (33), the first fundamental form on σ^- (38) is given by

$$ds^{2-}|_{\sigma^-}\overset{\sigma}{=}(-\mathcal{A}^2\dot{T}^2+B^2\dot{\rho}^2)d\lambda^2+C^2d\phi^2+2Ed\zeta d\phi+D^2d\zeta^2, \tag{40}$$

where

$$\mathcal{A}^2=A^2\left(1+c^2\frac{A^2C^2}{\Delta}\right).$$

The equality of the first fundamental forms (39) and (40) gives

$$-\hat{A}^2\dot{t}^2+\mathcal{B}^2\dot{r}^2\overset{\sigma}{=}-\mathcal{A}^2\dot{T}^2+B^2\dot{\rho}^2, \tag{41}$$

$$\hat{D}\overset{\sigma}{=}D, \tag{42}$$

$$\hat{C}\overset{\sigma}{=}C, \tag{43}$$

$$\hat{E}\overset{\sigma}{=}E. \tag{44}$$

In order to derive the remaining junction conditions we need the normal forms to $\sigma^{\pm}$, which can be written as

$$\mathbf{n}^+=\hat{A}\mathcal{B}(-\dot{r}dt+\dot{t}dr)|_{\sigma^+},$$

$$\mathbf{n}^-=\gamma\mathcal{A}B(-\dot{\rho}dT+\dot{T}d\rho)|_{\sigma^-} \tag{45}$$

so that they have the same norm on σ and where $\gamma=\pm1$ defines the two possible relative orientations. The rigging vectors can be obtained from Eqs. (5) and (6), and a suitable choice is

$$\vec{l}^+=-\frac{\dot{r}}{\hat{A}^2}\frac{\partial}{\partial t}\bigg|_{\sigma^+}+\frac{\dot{t}}{\mathcal{B}^2}\frac{\partial}{\partial r}\bigg|_{\sigma^+}, \tag{46}$$

$$\vec{l}^-=G\left[-\alpha^2\frac{\dot{\rho}}{\mathcal{A}^2}\frac{\partial}{\partial t}+\frac{\dot{T}}{B^2}\frac{\partial}{\partial\rho}+\frac{1}{\Delta}\left(\alpha^2\dot{\rho}\frac{cA^2}{\mathcal{A}^2}+\dot{t}\frac{\epsilon r\hat{B}^2}{\mathcal{B}^2G}\right)\left(E\frac{\partial}{\partial\tilde{\varphi}}-C^2\frac{\partial}{\partial\tilde{z}}\right)\right]\bigg|_{\sigma^-}, \tag{47}$$

where $G\neq0$ and α are functions that satisfy

$$\frac{1}{\hat{A}\mathcal{B}}(\mathcal{B}^2\dot{r}^2+\hat{A}^2\dot{t}^2)\overset{\sigma}{=}\frac{\gamma G}{\mathcal{A}B}(\alpha^2B^2\dot{\rho}^2+\mathcal{A}^2\dot{T}^2), \tag{48}$$

$$2\dot{r}\dot{t}\overset{\sigma}{=}G(\alpha^2+1)\dot{T}\dot{\rho}. \tag{49}$$

The explicit expressions for the junction conditions $H^+_{ab}\overset{\sigma}{=}H^-_{ab}$ can be written, using Eqs. (42)–(44), in the forms

$$H_{\lambda\lambda}:\dot{r}\ddot{t}+\dot{t}\ddot{r}+\dot{r}\left(\frac{\hat{A}_{,t}}{\hat{A}}\dot{t}^2+2\frac{\hat{A}_{,r}}{\hat{A}}\dot{r}\dot{t}+\frac{\mathcal{B}\mathcal{B}_{,t}}{\hat{A}^2}\dot{r}^2\right)+\dot{t}\left(\frac{\mathcal{B}_{,r}}{\mathcal{B}}\dot{r}^2+2\frac{\mathcal{B}_{,t}}{\mathcal{B}}\dot{r}\dot{t}+\frac{\hat{A}\hat{A}_{,r}}{\mathcal{B}^2}\dot{t}^2\right)$$

$$\overset{\sigma}{=}G\left\{\alpha^2\dot{\rho}\ddot{T}+\dot{T}\ddot{\rho}+2\alpha^2\dot{\rho}^2\dot{T}\frac{\mathcal{A}_{,\rho}}{\mathcal{A}}+\dot{T}\left[\dot{\rho}^2\frac{B_{,\rho}}{B}+\dot{T}^2\frac{\mathcal{A}\mathcal{A}_{,\rho}}{B^2}\right]\right\}, \tag{50}$$

$$H_{\lambda\phi}:\quad 0\overset{\sigma}{=}c(2EC_{,\rho}-E_{,\rho}C), \tag{51}$$

$$H_{\lambda\zeta}:\quad -\frac{\gamma\epsilon}{\hat{A}\mathcal{B}}\left(\frac{\mathcal{B}^2}{\hat{B}^2}\right)_{,t}\overset{\sigma}{=}\frac{r}{\mathcal{A}Bc}\left(\frac{\mathcal{A}^2}{A^2}\right)_{,\rho}, \tag{52}$$

$$H_{\phi\phi}:\quad \dot{r}\frac{\hat{C}_{,t}}{\hat{A}^2}-\dot{t}\frac{\hat{C}_{,r}}{\mathcal{B}^2}\overset{\sigma}{=}-G\dot{T}\frac{C_{,\rho}}{B^2}, \tag{53}$$

$$H_{\zeta\zeta}:\quad \dot{r}\frac{\hat{D}_{,t}}{\hat{A}^2}-\dot{t}\frac{\hat{D}_{,r}}{\mathcal{B}^2}\overset{\sigma}{=}-G\dot{T}\frac{D_{,\rho}}{B^2}, \tag{54}$$

$$H_{\phi\zeta}:\quad \dot{r}\frac{\hat{E}_{,t}}{\hat{A}^2}-\dot{t}\frac{\hat{E}_{,r}}{\mathcal{B}^2}\overset{\sigma}{=}-G\dot{T}\frac{E_{,\rho}}{B^2}, \tag{55}$$

where in Eqs. (51), (52) we have used the fact that, since $\epsilon\hat{E}=0$, then $\epsilon E_{,\rho}\dot{\rho}\overset{\sigma}{=}0$. Note that although there is a factor c^{-1} in Eq. (52), the right-hand side of this equation vanishes identically for $c=0$. The usefulness of using Eq. (22) for Eq. (18) becomes clear from the symmetry of the above equations.

With the exception of Eq. (52), the set of junction conditions (41)–(44) and (48)–(54), is formally the same as those given in [13,25], where the conditions for a matching preserving the G_2 symmetry of two OT cylindrically symmetric spacetimes, one assumed to be static, were studied. Following [25], the combination of Eqs. (42)–(44), their derivatives along σ Eq. (49) and Eqs. (53)–(55) lead to

(i) $\hat{C}_{,t}\overset{\sigma}{=}\hat{C}_{,r}\overset{\sigma}{=}0\Leftrightarrow C_{,\rho}\overset{\sigma}{=}0$.

(ii) $\hat{D}_{,t}\overset{\sigma}{=}\hat{D}_{,r}\overset{\sigma}{=}0\Leftrightarrow D_{,\rho}\overset{\sigma}{=}0$.

(iii) $\hat{E}_{,t}\overset{\sigma}{=}\hat{E}_{,r}\overset{\sigma}{=}0\Leftrightarrow E_{,\rho}\overset{\sigma}{=}0$.

(iv) $\dot{r}=0\Leftrightarrow\dot{\rho}=0$, and then necessarily $\hat{C}_{,t}=\hat{D}_{,t}=\hat{E}_{,t}=0$.

(v) $\dot{t}=0\Leftrightarrow\dot{T}=0$, and then necessarily $\hat{C}_{,t}\overset{\sigma}{=}\hat{D}_{,t}\overset{\sigma}{=}\hat{E}_{,t}\overset{\sigma}{=}=0$.

It must be stressed that due to (iv) we cannot have a matching across σ defined by $\dot{\rho}=0$, and equivalently by $\dot{r}=0$, since from Eq. (23) this would imply a static LRS region. We summarize this latter result in the following lemma:

Lemma 5.1. A nonstatic G_4 on S_3 LRS spacetime (18) cannot be matched to a static OT cylindrically symmetric spacetime (24) across a hypersurface with $\dot{\rho}=0$ (or equivalently $\dot{r}=0$), preserving the cylindrical symmetry. ■

Furthermore, the combination of equations that led to the previous statements, also imply the following important conditions on σ:

$$\text{(I)}\quad \hat{D}_{,t}\hat{C}_{,r}-\hat{D}_{,r}\hat{C}_{,t}\overset{\sigma}{=}0,$$

$$\text{(II)}\quad \hat{E}_{,t}\hat{D}_{,r}-\hat{E}_{,r}\hat{D}_{,t}\overset{\sigma}{=}0,$$

$$\text{(III)}\quad \hat{E}_{,t}\hat{C}_{,r}-\hat{E}_{,r}\hat{C}_{,t}\overset{\sigma}{=}0. \tag{56}$$

These are the so-called *exterior* conditions, which led to the impossibility of the cylindrically symmetric analogues to the Einstein-Straus model in [13,25]. We emphasize that these conditions involve only the coefficients of the g^+ metric. Three possibilities may arise: (a) they cannot be satisfied, and thus the matching is impossible, (b) they impose constraints on the matching, and in fact determine σ^+ if the functions $\hat{C}$, $\hat{D}$, $\hat{E}$ are given, and (c) they are satisfied automatically and therefore give no information.

Finally, following [13,25] we find that, after the substitution of G and α, the complete set of matching conditions can be written as

$$C_{,\rho}\dot{T}\frac{\mathcal{A}}{B}\overset{\sigma}{=}\gamma\left(\hat{A}\frac{\hat{C}_{,r}}{\mathcal{B}}\dot{t}+\mathcal{B}\frac{\hat{C}_{,t}}{\hat{A}}\dot{r}\right), \tag{57}$$

$$C_{,\rho}^2B^{-2}\overset{\sigma}{=}\frac{\hat{C}_{,r}^2}{\mathcal{B}^2}-\frac{\hat{C}_{,t}^2}{\hat{A}^2}, \tag{58}$$

$$\begin{aligned}\dot{T}C_{,\rho}^2\mathcal{A}_{,\rho}B^{-3}\overset{\sigma}{=}&\left(\frac{\hat{C}_{,r}^2}{\mathcal{B}^2}-\frac{\hat{C}_{,t}^2}{\hat{A}^2}\right)\left(\frac{\hat{A}_{,r}}{\mathcal{B}}\dot{t}+\frac{\mathcal{B}_{,t}}{\hat{A}}\dot{r}\right)\\ &-\frac{\hat{C}_{,t}}{\hat{A}}\frac{\hat{C}_{,r}}{\mathcal{B}}\left(\frac{\mathcal{B}_{,t}}{\mathcal{B}}\dot{t}+\frac{\mathcal{B}_{,r}}{\mathcal{B}}\dot{r}\right)\\ &+\frac{\hat{C}_{,t}}{\hat{A}}\frac{\hat{C}_{,r}}{\mathcal{B}}\left(\frac{\hat{A}_{,t}}{\hat{A}}\dot{t}+\frac{\hat{A}_{,r}}{\hat{A}}\dot{r}\right)\\ &-\frac{\hat{C}_{,r}}{\mathcal{B}}\left(\frac{\hat{C}_{,tt}}{\hat{A}}\dot{t}+\frac{\hat{C}_{,tr}}{\hat{A}}\dot{r}\right)\\ &+\frac{\hat{C}_{,t}}{\hat{A}}\left(\frac{\hat{C}_{,tr}}{\mathcal{B}}\dot{t}+\frac{\hat{C}_{,rr}}{\mathcal{B}}\dot{r}\right),\end{aligned} \tag{59}$$

plus the analogous forms for D and E [i.e., changing C by D and E respectively in the expressions (57)–(59)], together with Eqs. (42)–(44), (51), (52) and the exterior conditions (56). Note that not all these equations are independent since the set of equations for $C(\rho)$, Eqs. (57)–(59), is related to their analogues for D and E by

$$\hat{D}_{,t}C_{,\rho}\overset{\sigma}{=}\hat{C}_{,t}D_{,\rho},\quad \hat{E}_{,t}C_{,\rho}\overset{\sigma}{=}\hat{C}_{,t}E_{,\rho},\quad \hat{E}_{,t}D_{,\rho}\overset{\sigma}{=}\hat{D}_{,t}E_{,\rho}.$$

A. The explicit conditions

We shall now study the explicit matching conditions across nonspacelike hypersurfaces (so that neither $\dot{t}$ nor $\dot{T}$ can vanish on σ) for the LRS homogeneous spacetimes (18) by substituting the metric functions (23) in the above junction conditions. We start by considering the exterior conditions (56). Condition (I) implies that the only possible matchings are those satisfying

$$(a_{,t}\overset{\sigma}{=}0)\vee(b^2\Sigma_{,r}+a^2n(F+k)\overset{\sigma}{=}0). \tag{60}$$

Similarly, condition (II) gives

$$(a_{,t}\overset{\sigma}{=}0)\vee(n=0). \tag{61}$$

Combining conditions (60) and (61) and excluding matchings that hold only across a single value for r corresponding to $\Sigma_{,r}\overset{\sigma}{=}0$ (see lemma 5.1), we find that

$$a_{,t}\overset{\sigma}{=}0 \tag{62}$$

is a *necessary condition* for the required matching which, since $\dot{t}\neq0$ on σ, implies $a_{,t}=0$ and thus

$$a(t)=\text{const}(\equiv\beta). \tag{63}$$

Considering the remaining exterior condition (III) and assuming Eq. (62), we find

$$(b_{,t}\overset{\sigma}{=}0)\vee(n=0). \tag{64}$$

As a result, for $n\neq 0$, a nonstatic metric (18) cannot be matched to the static metrics (22) across a nonspacelike hypersurface.

Since we are interested in a nonstatic LRS region, we shall concentrate on the $n=0$ case. In this case the matching across a nonspacelike σ is in principle possible for $a=\text{const}$ and the LRS metric coefficients take the form

$$\hat{A}=1,\ \ \hat{B}=b(t),\ \ \hat{C}=b(t)\Sigma(r,k),$$
$$\hat{D}^2=\beta^2+\epsilon r^2 b^2(t)(=\beta^2+\epsilon\hat{C}^2),\ \ \hat{E}=0. \tag{65}$$

Regarding the static region we find that the preliminary junction conditions (42)–(44) for $\dot{\rho}\neq 0$, together with Eq. (65), imply that *in a neighborhood of* σ, the coefficient $D(\rho)$ is uniquely determined in terms of $C(\rho)$ by

$$D^2(\rho)=\beta^2+\epsilon C^2(\rho), \tag{66}$$

and that

$$E(\rho)=0. \tag{67}$$

Furthermore, from statement (i) above we cannot have $C_{,\rho}\overset{\sigma}{=}0$, since otherwise the only possible LRS regions would have to be static. Since we can set $B^2=1$ using the freedom to choose ρ, then only A and C remain free in Eq. (33). Using Eqs. (65)–(67), the complete set of junction conditions translates into

$$C\overset{\sigma}{=}b\Sigma,$$
$$C_{,\rho}\dot{T}\mathcal{A}\overset{\sigma}{=}\gamma\frac{\beta}{\sqrt{\beta^2+\epsilon r^2 b^2}}\left[\Sigma_{,r}\left(1+\epsilon r^2\frac{b^2}{\beta^2}\right)+\dot{t}+bb_{,t}\Sigma\dot{r}\right],$$
$$C^2_{,\rho}\overset{\sigma}{=}\left(1+\epsilon r^2\frac{b^2}{\beta^2}\right)\Sigma^2_{,r}-\Sigma^2 b^2_{,t}, \tag{68}$$
$$C^2_{,\rho}\dot{T}\mathcal{A}_{,\rho}\overset{\sigma}{=}-\beta\Sigma\Bigg\{[\sqrt{\beta^2+\epsilon r^2b^2}\Sigma_{,r}b_{,tt}+\epsilon r^2 bb^2_{,t}]\dot{t}+\frac{\beta^2 b_{,t}\Sigma}{(\beta^2+\epsilon r^2b^2)^{3/2}}(b^2_{,t}+k)^2\dot{r}\Bigg\},$$

plus Eq. (52), which now explicitly reads

$$\frac{2\epsilon\gamma\beta r}{(\beta^2+\epsilon r^2b^2)^{3/2}}b_{,t}\overset{\sigma}{=}\frac{c}{\mathcal{A}}\left(\frac{A^2}{\beta^2+\epsilon C^2}\right)_{,\rho}. \tag{69}$$

This equation shows that when $\epsilon=1$ we also need $c\neq 0$ in order to have a nonstatic LRS part. Note that the degree of freedom (i.e., c) introduced by the matching in Eq. (33) allows the LRS metrics (18) with $\epsilon=1$ to be matched to static OT cylindrically symmetric spacetimes. On the other hand, if $\epsilon=0$ we either have $c=0$ or $A=\text{const}$. But if $A=\text{const}$, the static region (33) with Eqs. (66) and (67) admits at least one more isometry (see [25]), and the matching procedure would then require a different treatment from the outset.

We also note that given the LRS region, i.e., given $b(t)$, the static region is not uniquely specified by this matching. This is due to the fact that the exterior conditions are in this case identically satisfied and hence, as mentioned above, they do not prescribe the matching hypersurface σ^+ [i.e., $t(\lambda)$ and $r(\lambda)$].

VI. CONSEQUENCES OF THE MATCHING CONDITIONS

We briefly summarize the results obtained in Sec. V A. The condition $n=0$ rules out the metric forms (8) which include the Bianchi types II, VIII, and IX, where the last is an anisotropic generalization of FLRW $k=+1$ metrics. From Eq. (64) one therefore has:

Proposition 6.1. A nonstatic G_4 on S_3 LRS spacetime admitting a simply transitive subgroup G_3 of Bianchi types II, VIII or IX cannot be matched to an OT cylindrically symmetric static spacetime across a nonspacelike hypersurface preserving the cylindrical symmetry. ■

Therefore we are left with the metric forms (7) and (9). Now Eq. (7) with $k=+1$ is the Kantowski-Sachs metric. The cases corresponding to $k=0$ and $k=-1$ for $\epsilon=0$ include Bianchi types III, I and VII_0, while the case $\epsilon=1$ includes Bianchi types V and VII_h. These models could be of cosmological interest since the former generalize $k=0$ FLRW and the latter the $k=-1$ FLRW metrics. As shown in the previous section, the LRS metric coefficients were severely restricted to Eq. (65) and the possible resulting metrics can be summarized as follows:

Theorem 6.1. The only possible nonstatic G_4 on S_3 LRS spacetimes that can be matched to an OT cylindrically symmetric static spacetime across a nonspacelike hypersurface preserving the cylindrical symmetry are given by

$$ds^2=-dt^2+\beta^2dz^2+b^2(t)[(dr-\epsilon r dz)^2+\Sigma^2(r,k)d\varphi^2], \tag{70}$$

where β is a constant, Σ and k are given by Eq. (10), and $\epsilon=0,1$ is such that $\epsilon k=0$. ■

Remark. The line element for the static region then becomes

$$ds^2=-A^2dT^2-2cA^2dTd\tilde{z}+d\rho^2+C^2d\tilde{\varphi}^2+(\beta^2+\epsilon C^2)d\tilde{z}^2,$$

where A and C are functions of ρ and c is constant. The matching hypersurface is given by Eq. (29) with $\dot{f}_z(\lambda)\overset{\sigma}{=}\epsilon r\dot{r}b^2/(\beta^2+\epsilon r^2b^2)$ and by Eq. (38) with $\dot{f}_{\tilde{\varphi}}(\lambda)=0$ and $\dot{f}_{\tilde{z}}(\lambda)\overset{\sigma}{=}cA^2/(\beta^2+\epsilon r^2b^2)$, but it is not fully determined in general. The set of equations to be satisfied are those given in Eqs. (68) and (69).

This demonstrates that the possible LRS models (70) are extremely special. In fact, condition $a(t)=\beta$ on Eq. (18) with $n=0$ poses a strong constraint, whereby the timelike surfaces Ω in Eq. (70) parametrized by $\{\lambda_1,\lambda_2\}$ and defined by $\{t=\lambda_1, z=\lambda_2, \varphi=\varphi_0, r=r_0 e^{\epsilon\lambda_2}\}$, where φ_0 and r_0 are constants, have no dependence on time. As mentioned in Sec. III, in the $n=0$ cases of Eq. (18), and thus in Eq. (70), the surfaces Ω_S spanned by r and φ (at constant t and z) are surfaces of constant curvature, since there is a G_3 acting multiply transitively, which is generated by $\vec{\eta}_1$, $\vec{\eta}_3$ and $\vec{\eta}_4$ (20), (21). When $\epsilon=0$, the family of surfaces Ω are just the family of orthogonal surfaces to the Ω_S orbits.

Another way of looking at this is that one of the components of the expansion tensor $\theta_{\alpha\beta}(=\nabla_{(\alpha}u_{\beta)}$ of the flow given by $\vec{u}=\partial_t$ vanishes. To be more precise, the only nonvanishing components of $\theta_{\alpha\beta}$ in the natural orthonormal tetrad

$$\boldsymbol{\theta}_0=dt,\quad \boldsymbol{\theta}_1=b(dr-\epsilon r dz),\quad \boldsymbol{\theta}_2=b\Sigma d\varphi,\quad \boldsymbol{\theta}_3=\beta dz, \tag{71}$$

are

$$\theta_{11}=\theta_{22}=b_{,t}/b. \tag{72}$$

This is a strong constraint as far as cosmologically interesting models are concerned, since there is no expansion along the spacelike direction spanned by $\partial_z+\epsilon r\partial_r$ (which is orthogonal to Ω_S iff $\epsilon=0$).

This result can be seen in two ways: either as a consequence of the assumption that the metric g^- is static and cylindrically symmetric, or as a consequence of the homogeneity in the evolving spacetime, which prohibits the norm of the Killing vector $\partial/\partial z$ to be space dependent. The condition $a(t)=$const may not be necessary if either the assumption of the cylindrical symmetry on g^- or the homogeneity of the metric g^+ are relaxed, although one might still expect strong constraints on g^+ leading to restrictions on the possible matter content there. We shall return to these questions in a future publication.

So far we have not restricted the source fields in the matching spacetimes. We shall now consider the particular case of a perfect fluid LRS metric.

A. Perfect-fluid LRS region

The Einstein tensor for Eq. (70) in the natural orthonormal tetrad (71) has the form

$$G_{00}=\frac{k+b_{,t}^2}{b^2}-3\frac{\epsilon}{\beta^2},$$

$$G_{03}=-2b_{,t}\frac{\epsilon}{\beta b},$$

$$G_{11}=G_{22}=-\frac{b_{,tt}}{b}+\frac{\epsilon}{\beta^2},$$

$$G_{33}=-2\frac{b_{,tt}}{b}-\frac{k+b_{,t}^2}{b}+\frac{\epsilon}{\beta^2}. \tag{73}$$

The allowed Segre types are {1, 1(11)}, {2(11)} together with their degeneracies. We are interested in the perfect-fluid type, i.e., {1,(111)}. The first condition for this type of source is $(G_{03})^2=(G_{00}+G_{22})(G_{33}-G_{22})$ (see [25,33]), which can explicitly be cast in the form

$$(bb_{,tt}-b_{,t}^2-k)[\beta^2(bb_{,tt}+b_{,t}^2+k)+2\epsilon b^2]=0, \tag{74}$$

so that $\rho=G_{22}+G_{00}-G_{33}$ and $p=G_{22}$. The vanishing of the first term in Eq. (74) results in $G_{00}+G_{22}=-2\epsilon/\beta^2$. In order to have a perfect fluid we also need $G_{00}+G_{22}\neq 0$ to have the same sign as $\rho+p$. As a result, we are left with the case $\epsilon=1$, which implies $\rho+p<0$. In this case both ρ and p are constants such that $\rho+3p=0$.

Therefore, in order to have a perfect fluid satisfying the dominant energy condition (i.e., $\rho+p>0$) we can only consider the vanishing of the second term in Eq. (74). We shall consider the cases $\epsilon=1,0$ in turn. For the case $\epsilon=1$, the equation $\beta^2(bb_{,tt}+b_{,t}^2)+2b^2=0$ gives $b(t)=c_1\sqrt{\sin(2t/\beta)}$ (after rescaling t), and hence $\rho=p-8/\beta^2=1/\beta^2(1/\sin^2(2t/\beta)-6)$. The energy density changes sign at $\sin(2t/\beta)=1/\sqrt{6}$, and thus the weak energy condition cannot be satisfied over the whole spacetime.

Therefore, we are left with the case $\epsilon=0$, in which case the equation for $b(t)$ becomes $b_{,t}^2+b_{tt}b+k=0$, giving

$$b(t)=\sqrt{\alpha t-kt^2}, \tag{75}$$

where α is an arbitrary constant that can be taken to be positive without loss of generality. This corresponds to a stiff perfect fluid given by

$$\rho=p=\frac{\alpha^2}{4t^2(\alpha-kt)^2}, \tag{76}$$

which ensures that the energy conditions are satisfied. We recall that this solution can also be interpreted as applying to the case with a minimally coupled scalar field as the source. These results are summarized in the following theorem:

Theorem 6.2. The only possible nonstatic G_4 on S_3 LRS perfect-fluid spacetimes satisfying the dominant energy condition that can be matched to an OT cylindrically symmetric static metric across a nonspacelike hypersurface preserving the symmetry are given by

$$ds^2=-dt^2+dz^2+(\alpha t-kt^2)[dr^2+\Sigma(r,k)^2 d\varphi^2], \tag{77}$$

where α is a constant and Σ is defined as in Eq. (10). The equation of state is that of a stiff fluid and is given by Eq. (76).

This amounts to a no-go result, namely that there are no evolving G_4 on S_3 LRS perfect-fluid spacetimes with $\rho\neq p$, satisfying the dominant energy condition, that match a locally OT cylindrically symmetric static region across a nonspacelike matching hypersurface preserving the symmetry.

So far we have studied the matching between a G_4 on S_3 LRS region and an OT cylindrically symmetric static region across a nonspacelike matching hypersurface preserving the symmetry. This treatment has been local and has not dealt with matching in the specific context of a particular configuration. Consequently, our results can be used in a number of different settings. For example, they can be employed to study the generalization of the Einstein-Straus result concerning the embedding of a static region in a LRS cosmological model, by taking the static part as describing *locally* an interior region and the LRS part as its exterior. But given the interior-exterior duality in the matching procedure they can also be used to study the question of the existence of an astrophysical evolving object described *locally* by a LRS metric which is surrounded by an OT cylindrically symmetric static background. In this way our results can be used to consider generalizations of the Oppenheimer-Snyder [7] collapsing model.

The no-go result above then tells us that *a G_4 on an S_3 LRS evolving perfect-fluid model cannot contain a locally OT cylindrically symmetric static cavity* except for the very particular stiff-fluid case mentioned above. Theorem 6.2 rules out not only static cosmological strings in LRS cosmological backgrounds, but also static cavities which are locally cylindrically symmetric, as for instance, bottle or coin-shaped objects. Furthermore, it implies that *no astrophysical object described by a nonstiff perfect fluid type G_4 on S_3 LRS metric can be embedded into a locally OT cylindrically symmetric static background*. Importantly, these results hold irrespective of the matter content in the static part.

Concerning global configurations, as discussed above, we can go further and apply our results to the case of spatially homogeneous nonstatic exterior spacetimes. This follows from the fact that the model for a spatially bounded interior region whose bounding surface is topologically S^2 preserves the existence of an axis of symmetry across this border. This implies that the exterior homogeneous part has to be locally rotational symmetric, and thus admit a further isometry becoming a G_4 on an S_3 LRS region. We shall demonstrate this result in the following section.

VII. BIANCHI SPACETIMES: AXIALLY SYMMETRIC GLOBAL MODELS

We recall that a spacetime admits a cyclical symmetry if its metric is invariant under an effective realization of the one-dimensional torus on the manifold [34]. Axial symmetry arises when the set of fixed points is nonempty (i.e., the generator of the isometry, say $\vec{\xi}$, vanishes). In fact, it has been shown [35] that any nonempty set W_2 of fixed points in a four-dimensional spacetime is a timelike two-dimensional surface. Furthermore, the axial Killing vector field $\vec{\xi}$ is spacelike in a neighborhood of the axis and satisfies the regularity condition, i.e., the expression (16) for $\vec{\xi}$ tends to 1 on W_2. Here we shall concentrate on the preservation of an axial symmetry across a matching surface. The models for the common compact and simply connected astrophysical objects usually consist of an interior region whose spatial boundary is topologically a two-sphere S^2. The boundary σ (and thus σ^+ and σ^-) is then taken to be nonspacelike and homeomorphic to $S^2\times I$, where I is an open interval of the real line. In other words, it is assumed that σ can be foliated by a set of spacelike two-surfaces S_τ homeomorphic to S^2. We refer to [16,17] for a detailed general construction. The surfaces S_τ are embedded into $\mathcal{V}^+$ (respectively $\mathcal{V}^-$) by the maps $\Phi^+_\tau \equiv \Phi^+ \circ i_\tau$ (respectively $\Phi^-_\tau \equiv \Phi^- \circ i_\tau$) where $i_\tau : S_\tau \to \sigma$ are the natural inclusion for each surface into σ.

Let us denote by $\vec{\xi}^+$ the generator of a spacelike (cyclic) isometry on $(\mathcal{V}^+, g^+)$. Since we wish to preserve this symmetry across σ, there exists a vector field $\vec{\gamma}$ defined in σ such that $d\Phi^+(\vec{\gamma}) = \vec{\xi}^+|_{\sigma^+}$. That is, the restriction of $\vec{\xi}^+$ on σ^+ is tangent to σ^+ everywhere. Let us now take $(\mathcal{V}^+, g^+)$ to be a spatially homogeneous spacetime. We can then construct a natural foliation of the manifold $\mathcal{V}^+$ by taking the homogeneous spacelike hypersurfaces, say $\{t=\text{const}\}$ spanned by the orbits of the simply transitive G_3 on S_3 group of isometries. By construction, the restrictions of our Killing vector field to the orbits $\vec{\xi}^+|_{\{t\}}$ are tangent to these hypersurfaces. Since σ^+ is nonspacelike everywhere,[8] we can now define the following foliation $\{S^+_t\}$ of $\sigma^+ : S^+_t \equiv \sigma^+ \cap \{t=\text{const}\}$, where S^+_t is taken to be the image of S_t through Φ^+_t. The restriction of $\vec{\xi}^+$ on S^+_t, $\vec{\xi}^+|_{S^+_t}$, is clearly tangent to the surfaces S^+_t, and therefore there is a vector field $\vec{\gamma}_t$ defined in S_t such that $d\Phi^+_t(\vec{\gamma}_t) = \vec{\xi}^+|_{S^+_t}$.

We now demand that this foliation is such that S_t is homeomorphic to S^2, that is, we take S_t to be the S_τ above.[9] In the following we shall refer to ''spatially compact'' as ''spatially compact according to the homogeneous slicing.'' There must then exist a point where $\vec{\gamma}_t$ vanishes [36]. So, for every t there exists a point $w_t \in S_t$ where $\vec{\gamma}_t = \vec{0}$, and hence

$$\vec{\xi}^+|_{w^+_t} = \vec{0}, \tag{78}$$

where $w^+_t = \Phi^+_t(w_t)$. The existence of a fixed point for the cyclic symmetry (generated by $\vec{\xi}^+$) ensures the existence of a timelike surface of fixed points W^+_2. Furthermore, there is a neighborhood around any point in W^+_2 where $\vec{\xi}^+$ is spacelike and vanishes only at W^+_2 [35]. This can be used to show that the points w^+_t, for all t, are in W^+_2. Actually, because $\vec{\gamma}$ generates a cyclic symmetry in σ, which is inherited from the embedding (see, for instance, [31]), the set(s) of fixed points of $\vec{\gamma}$ must be timelike curves in σ,[10] defined as $W \equiv \{w_t ; \forall t \in I\}$. Also, since $d\Phi^+$ is a rank-three map, $\vec{\xi}^+|_{\sigma^+}$ can only vanish where $\vec{\gamma} = 0$, i.e., on the curve W, and thus W^+_2 cannot be contained in σ^+. It follows that $\mathcal{V}^+ \backslash \sigma^+$ contains points on the axis of symmetry W^+_2 of the cyclic (in this case axial) symmetry generated by $\vec{\xi}^+$. As a consequence $(\mathcal{V}^+, g^+)$ cannot be completely anisotropic and spatially homogeneous, as

[8]This assumption can in fact be replaced by a less restrictive one; see [17].

[9]This might not be necessary, as in most cases one may be able to find a diffeomorphism between a previously constructed foliation S_τ and the surfaces given by S_t.

[10]This comes from the fact that $\nabla_a \gamma_b$ is of rank 2. See [37].

otherwise there would not be a Killing vector field with zero points; $(\mathcal{V}^+,g^+)$ must at least admit one isotropy, which is generated by the Killing vector field $\vec{\xi}^+$.

Furthermore, since $d\Phi_t^-(\vec{\gamma}_t)=0$ at $w_t^-=\Phi_t^-(w_t)$, we will also have a nonempty set of points where one generator, say $\vec{\xi}^-$, of the cyclic symmetry we are preserving on $(\mathcal{V}^-,g^-)$ will vanish. These points w_t^- are precisely those that are to be identified with w_t^+. The same argument as above can then be used for the existence of an axis W_2^- in $\mathcal{V}^-\backslash\sigma^-$. We have then shown the following:

Lemma 7.1. Let $(\mathcal{V},g)$ be a spacetime resulting from the matching of two cyclically symmetric spacetimes $(\mathcal{V}^+,g^+)$ and $(\mathcal{V}^-,g^-)$ preserving the symmetry. If one part, say $(\mathcal{V}^+,g^+)$, is spatially homogeneous and either part represents a spatially compact and simply connected region, then both $(\mathcal{V}^+,g^+)$ and $(\mathcal{V}^-,g^-)$ must be axially symmetric. In particular, $(\mathcal{V}^+,g^+)$ is locally rotationally symmetric admitting a G_4 on an S_3 group of isometries. ∎

Note that since this lemma relies on the topology of the matching boundary, the spatially homogeneous region $(+)$ does not necessarily correspond to an exterior region, and therefore $(-)$ and $(+)$ can be interpreted as either interior or exterior.

Using this lemma we can apply the results given in the previous sections for G_4 on S_3 LRS spacetimes to Bianchi spacetimes, once one of the regions of the matching represents a bounded object without holes. Since in this paper we have mainly focused on the generalization of the Einstein-Straus model, we shall, in the following statements, consider the static region to be the spatially bounded cavity surrounded by a homogeneous background. From proposition 6.1 we obtain:

Corollary 7.1. A nonstatic homogeneous Bianchi II, VIII or IX spacetime cannot be matched to a spatially compact and simply connected locally cylindrically symmetric static region across a nonspacelike hypersurface preserving the symmetry.

And similarly, from theorem 6.1 we have:

Corollary 7.2. The only possible nonstatic spatially homogeneous spacetimes that can be matched to a spatially compact and simply connected locally cylindrically symmetric static region across a nonspacelike hypersurface preserving the symmetry are given by metric (70) with $k=0,-1$.

The same remarks made about theorem 6.1 regarding the interior apply here. Finally, from theorem 6.2 we obtain:

Corollary 7.3. The only possible nonstatic spatially homogeneous perfect-fluid spacetimes, satisfying the dominant energy condition, that can be matched to a spatially compact and simply connected locally cylindrically symmetric static region, across a nonspacelike hypersurface preserving the symmetry, is given by metric (77). The possible Bianchi types of the G_3 on S_3 are I, III or VII_0 and the equation of state is that of a stiff fluid.

The last corollary amounts to a no-go result, namely that there is no possible evolving perfect-fluid Bianchi spacetimes with $\rho\neq p$ satisfying the dominant energy condition and containing a locally OT cylindrically symmetric static cavity preserving the symmetry.

The above corollaries have focused on the existence of static cavities in homogeneous backgrounds. Given the interior-exterior duality and taking into account the remark after lemma 7.1, the same results also apply when the spatially homogeneous region is taken to be the bounded region which is embedded in a locally OT cylindrically symmetric static background. They therefore provide a no-go result concerning the anisotropic generalization of the Oppenheimer-Snyder model.

VIII. CONCLUSION

We have studied a generalization of the Einstein-Straus model, by considering a locally cylindrically symmetric static cavity embedded in an expanding LRS region. We have derived the matching conditions for such space-times and have found that they impose strong constraints on the LRS metrics, by implying that $a_{,t}=0$ and $n=0$. The former implies that no dynamical evolution is allowed along a spacelike direction as seen by the observer ∂_t. This direction is orthogonal to the orbits of the subgroup G_3 on S_2 of the LRS when $\epsilon=0$. Condition $n=0$ implies that it is impossible to have an exterior metric of Bianchi types II, VII or IX. Our main result in this connection, expressed in theorem 6.1 and corollary 7.2, is that the exteriors can only take very particular forms within the Bianchi types I, III, V, VII_0, VII_h or Kantowski-Sachs metrics. These results make no reference to the matter content and are therefore, in this sense, completely general.

To study the effects of including matter contents, we also considered perfect fluid sources for the metrics allowed in theorem 6.1, and found that such embeddings are only possible when the matter content is a stiff fluid. This is our second main result, which is summarized in theorem 6.2 (and corollary 7.3).

We have also proved that if the nonstatic spacetime is assumed to be spatially homogeneous (not necessarily LRS) and if the static spacetime represents a spatially compact and simply-connected region, then in order to perform the matching preserving the cyclic symmetry the nonstatic part must be LRS. As a consequence, we were able to reformulate our results with the weaker assumption of homogeneity, instead of local rotational symmetry.

Given that deviations from isotropy and sphericity are expected to be present in the universe, these results are of potential interest, since they make it impossible to embed locally OT cylindrical static objects (which are compact and simply connected) in homogeneous universes and at the same time have a reasonable exterior cosmological evolution.

Because of the interior-exterior duality, our results also apply to the cases of bounded objects described by spatially homogeneous metrics embedded in locally cylindrically symmetric static backgrounds. In particular, this would have the interesting consequence that the Oppenheimer-Snyder model for collapse cannot be generalized in this way.

Finally it would be of interest to study the inhomogeneous generalizations of our results. We hope to return to this question in a future work.

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