Null string evolution in black hole and cosmological spacetimes

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We discuss the problem of the motion of classical strings in some black hole and cosmological spacetimes. In particular, the null string limit (zero tension) of tensile strings is considered. We present some new exact string solutions in a Reissner-Nordström black hole background as well as in the Einstein static universe and in the Einstein-Schwarzschild (a black hole in the Einstein static universe) spacetime. These solutions can give some insight into the general nature of the propagation of strings (cosmic and fundamental) in curved backgrounds.

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I. INTRODUCTION

Fundamental string theory is undoubtedly the most serious candidate for unification of gauge interactions with gravity [1]. Its effects should clearly be visible in extremely high gravitational fields of black holes and in the early universe. It is not an easy task to study quantum string propagation in these background fields and this gives us motivation to study the motion of classical strings in these fields first in order to determine some really ‘‘stringy’’ properties of a quantum theory. On the other hand, the classical motion of strings gives an appropriate formalism to study the dynamics of cosmic strings which appear naturally in grand unified theory (GUT) models [2]. This is why we will study the classical motion of strings in some black hole and cosmological spacetimes.

The classical motion of strings which evolve in curved spacetimes can be described by a system of the second-order nonlinear coupled partial differential equations [3,4]. The nonlinearity of these equations causes a complication which leads to their nonintegrability and possibly chaos [5]. It is well known that various types of nonlinearities appear in Newtonian as well as relativistic systems and so they can deliver chaos. On the other hand, some types of nonlinear equations can be integrable and their solutions are not chaotic. It seems that the theory of relativity is ideal for producing chaotic behavior since its basic equations are highly nonlinear. However, the problem is not as easy as one would think, because most of the systems under study possess some symmetries which simplify the problem. This also refers to a single particle obeying either Newtonian or relativistic equations. Simply, a single particle which moves in the gravitational field of a source of gravity cannot move chaotically. However, two particles which form a three-body system including the source can move in a chaotic way, though still not for all possible configurations.

The admission of extended objects such as strings causes another complication which, roughly, can be compared to the fact that now we have a many-body system which can obviously be chaotic on the classical level. An extended character of a string is reflected by the equations of motion which become a very complicated nonlinear system from the very beginning. Thus, no wonder chaos can appear for classical evolution of strings around the simplest sources of gravity, such as Schwarzschild black holes. This, in fact, was explicitly proven [6,7]. However, in a similar way as for other types of nonlinear sets of equations, there exist integrable configurations. The investigation of such explicit configurations can give an interesting insight into the problem of the general evolution of extended objects in various sources of gravity. Of course, it is justified, provided we do not consider back reaction of these extended objects onto the source field, i.e., if we consider test strings in analogy to test particles which do not ‘‘disturb’’ sources’ gravitational fields.

Studies of exact configurations can give much insight into the problem. One useful example is when unstable periodic orbits (UPO) appear. Their emergence becomes a signal for a possible chaotic behavior of the general system [8].

The task of this paper is to study some exact configurations for strings moving in simple spacetimes of general relativity. Unfortunately, for strings, the main complication refers to their self-interaction reflected in the equations of motion by a nonzero value of tension (tensile strings). However, one is able to study simpler extended configurations for which tension vanishes called null (tensionless) strings [9–11]. Their equations of motion are null geodesic equations of general relativity appended by an additional ‘‘stringy’’ constraint. Many exact null string configurations in various curved spacetimes have already been studied [11–18]. One of the advantages of the null string approach is the fact that one may consider null strings as null approximation in various perturbative schemes for tensile strings [10,19–21].

In Sec. II we present tensile and tensionless string equations of motion. In Sec. III we obtain exact null string configurations both in Reissner-Nordström and Schwarzschild spacetime while in Sec. IV we derive string configurations in a static Einstein universe. In Sec. V we discuss the evolution

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of strings in the Einstein-Schwarzschild (Vadiya) universe. In Sec. VI we discuss our solutions.

II. TENSILE AND NULL STRINGS IN CURVED SPACETIMES

A free string which propagates in a flat Minkowski spacetime sweeps out a world-sheet (two-dimensional surface) in contrast to a point particle, whose history is a world line. The world-sheet action for a free, closed string is given by the formula [22]

$$S=\frac{T}{2}\int d\tau d\sigma\sqrt{-h}h^{ab}\eta_{\mu\nu}\partial_a X^\mu\partial_b X^\nu, \tag{1}$$

where $T=1/2\pi\alpha'$ is the string tension, α' the Regge slope, τ and σ are the (spacelike and timelike, respectively) string coordinates, h^{ab} is a two-dimensional world-sheet metric $(a,b=0,1)$, $h=\det(h_{ab})$, $X^\mu(\tau,\sigma)$ $(\mu,\nu=0,1,\dots,D-1)$ are the coordinates of the string world sheet in D-dimensional Minkowski spacetime with metric $\eta_{\mu\nu}$.

If instead of the flat Minkowski background one takes any *curved* spacetime with metric $g_{\mu\nu}$, then the action (1) changes into

$$S=-\frac{T}{2}\int d\tau d\sigma\sqrt{-h}h^{ab}g_{\mu\nu}\partial_a X^\mu\partial_b X^\nu. \tag{2}$$

The action (2) is usually called the Polyakov action [22]. It is fully equivalent to the so-called Nambu-Goto action which contains a square root and is simply the surface area of the string world-sheet

$$S=T\int d\tau d\sigma\sqrt{-h}. \tag{3}$$

It is useful to present the relation between the background (target space) metric $g_{\mu\nu}$ and the induced world-sheet metric h_{ab} embedded in $g_{\mu\nu}$:

$$h_{ab}=g_{\mu\nu}\partial_a X^\mu\partial_b X^\nu. \tag{4}$$

In Eq. (2) one can then apply the conformal gauge

$$\sqrt{-h}h^{ab}=\eta^{ab}, \tag{5}$$

which allows the two-dimensional world-sheet metric h^{ab} to be taken as flat metric η^{ab}. This is because the action is invariant under Weyl (conformal) transformations $h'^{ab}=f(\sigma)h^{ab}$ and the h^{ab} dependence can be gauged away. However, Weyl transformations rescale invariant intervals, hence there is no invariant notion of distance between two points. In conformal gauge the action (2) takes the form

$$S=\frac{T}{2}\int d\tau d\sigma\,\eta^{ab}g_{\mu\nu}\partial_a X^\mu\partial_b X^\nu. \tag{6}$$

In fact, the action (6) describes a nontrivial quantum field theory (QFT), known as the nonlinear σ model [22,23].

The variation of the action (6) gives equations of motion of a tensile string $(T\neq 0)$ and the conformal gauge condition (5) gives the constraints equations.

However, the action (6) has a disadvantage. Alike the point particle case with its zero mass limit, one cannot take the limit of zero tension $T\to 0$ here. In order to avoid this one has to apply a different action which contains a Lagrange multiplier $E(\tau,\sigma)$ [19,20]:

$$S=\frac{1}{2}\int d\tau d\sigma\left[\frac{g_{\mu\nu}h^{ab}\partial_a X^\mu\partial_b X^\nu}{E^2(\tau,\sigma)}-\frac{E(\tau,\sigma)}{\alpha'^2}\right]. \tag{7}$$

Varying this action (7) with respect to E gives the condition

$$E=\alpha'\sqrt{-h}. \tag{8}$$

Substiting Eq. (8) back into Eq. (7) gives simply the Nambu-Goto action (3).

By the introduction of a new constant γ with the dimension of (length)^2 we define a parameter

$$\varepsilon=\frac{\gamma}{\alpha'}. \tag{9}$$

Finally, after imposing the gauge

$$E=-\gamma(g_{\mu\nu}X'^\mu X'^\nu), \tag{10}$$

together with the orthogonality condition

$$g_{\mu\nu}\dot{X}^\mu X'^\nu=0, \tag{11}$$

we get the equations of motion and the constraint for the action (7) [13,15,19,20]

$$\ddot{X}^\mu+\Gamma^\mu_{\nu\rho}\dot{X}^\nu\dot{X}^\rho=\varepsilon^2(X''^\mu+\Gamma^\mu_{\nu\rho}X'^\nu X'^\rho), \tag{12}$$

$$g_{\mu\nu}\dot{X}^\mu\dot{X}^\nu=-\varepsilon^2 g_{\mu\nu}X'^\mu X'^\nu, \tag{13}$$

where: $(\cdots)^\cdot\equiv\partial/\partial\tau$, $(\cdots)'\equiv\partial/\partial\sigma$, and $\mu,\nu,\rho=0,1,2,3$ from now on.

Now it makes sense to take the limits:

(i) $\varepsilon^2\to 0$ $(T\to 0)$ for tensionless (null) strings whose world-sheet is placed on the light cone.

(ii) $\varepsilon^2\to 1$ for tensile strings whose world-sheet is placed inside the light cone.

(iii) $\varepsilon=\gamma/\alpha'\ll 1$ for the perturbative scheme for the tensile strings expanded out of the null strings [19–21].

These equations can also be obtained using the gauge as proposed by Bozhilov [24]. Another approach to the null string expansion has been performed in [10,11].

An important characteristic for both null and tensile strings is their invariant size defined by (for closed strings) [22]

$$S(\tau)=\int_0^{2\pi}S(\tau,\sigma)d\sigma, \tag{14}$$

where

$$S(\tau,\sigma)=\sqrt{-g_{\mu\nu}X'^{\mu}X'^{\nu}}. \tag{15}$$

III. THE EVOLUTION OF STRINGS IN BLACK HOLE SPACETIMES

We start with the study of the evolution of strings in a charged black hole spacetime or Reissner-Nordström spacetime which generalizes Schwarzschild spacetime [26]. Reissner-Nordström spacetime is a spherically symmetric charged black hole with metric (t,r,θ,ϕ spacetime coordinates):

$$ds^2=\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)dt^2-\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)^{-1}dr^2 -r^2(d\theta^2+\sin^2\theta d\varphi^2), \tag{16}$$

where M is the mass, Q is the charge. In order to get the Schwarzschild black hole one has to put $Q=0$. For $Q^2<M^2$ there exists an event horizon at $r=r_+=M+\sqrt{M^2-Q^2}$ and a Cauchy horizon at $r=r_-=M-\sqrt{M^2-Q^2}$. For $Q^2=M^2, r_+=r_-=M$, and for $Q^2>M^2$ there are no horizons [25].

Using the notation $X^0=t(\tau,\sigma)$, $X^1=r(\tau,\sigma)$, $X^2=\theta(\tau,\sigma)$, $X^3=\varphi(\tau,\sigma)$, the equations of motion for a string in Reissner-Nordström spacetime are

$$\ddot{t}-\varepsilon^2t''+\frac{2M}{r^3}\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)^{-1}\left(r-\frac{Q^2}{M}\right)\times(\dot{r}\dot{t}-\varepsilon^2r't')=0, \tag{17}$$

$$\ddot{r}-\varepsilon^2r''+\frac{M}{r^3}\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)\left(r-\frac{Q^2}{M}\right)(\dot{t}^2-\varepsilon^2t'^2) -\frac{M}{r^3}\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)^{-1}\left(r-\frac{Q^2}{M}\right)(\dot{r}^2-\varepsilon^2r'^2) -r\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)(\dot{\theta}^2-\varepsilon^2\theta'^2) -r\sin^2\theta\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)(\dot{\varphi}^2-\varepsilon^2\varphi'^2)=0, \tag{18}$$

$$\ddot{\theta}-\varepsilon^2\theta''+\frac{2}{r}(\dot{r}\dot{\theta}-\varepsilon^2r'\theta')-\sin\theta\cos\theta(\dot{\varphi}^2-\varepsilon^2\varphi'^2)=0, \tag{19}$$

$$\ddot{\varphi}-\varepsilon^2\varphi''+\frac{2}{r}(\dot{r}\dot{\varphi}-\varepsilon^2r'\varphi')+2\cot\theta(\dot{\theta}\dot{\varphi}-\varepsilon^2\theta'\varphi')=0, \tag{20}$$

whereas the constraints are given by

$$\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)\dot{t}^2-\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)^{-1}\dot{r}^2-r^2(\dot{\theta}^2+\sin^2\theta\dot{\varphi}^2) =-\varepsilon^2\left[\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)t'^2-\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)^{-1}r'^2-r^2(\theta'^2+\sin^2\theta\varphi'^2)\right], \tag{21}$$

$$\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)\dot{t}t'-\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)^{-1}\dot{r}r'-r^2(\dot{\theta}\theta'+\sin^2\theta\dot{\varphi}\varphi')=0. \tag{22}$$

If one takes $Q=0$ one gets the equations for the neutral Schwarzschild spacetime [13].

For a null circular string ($\varepsilon^2\to0$) with circular ansatz:

$$t=t(\tau),\quad r=r(\tau),\quad \theta=\theta(\tau),\quad \varphi=\sigma,$$

one gets, from Eqs. (17)–(20),

$$\dot{t}=\frac{E(\sigma)}{1-\frac{2M}{r}+\frac{Q^2}{r^2}}, \tag{23}$$

$$\dot{r}^2-E^2(\sigma)+\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)\frac{1}{r^2}[K(\sigma)+L^2(\sigma)]=0, \tag{24}$$

$$\dot{\theta}^2=r^{-4}\sin^{-2}\theta[K(\sigma)\sin^2\theta-L^2(\sigma)\cos^2\theta], \tag{25}$$

$$\dot{\varphi}=\frac{L(\sigma)}{r^2\sin^2\theta}, \tag{26}$$

where $K(\sigma)$ is Carter's constant of motion (a constant which refers to coordinate θ [13]). It is easy to notice that an energy

E, an angular momentum L, and a constant K for a null string do not depend on coordinate σ.

A. A circular null string with $K=L=0$ in Reissner-Nordström spacetime

First, we study the evolution of a null circular string for $K=L=0$. From Eqs. (23)–(26) we obtain

$$\dot{t}=\frac{E}{1-\frac{2M}{r}+\frac{Q^2}{r^2}}, \tag{27}$$

$$\dot{r}^2=E^2, \tag{28}$$

$$\dot{\theta}=0, \tag{29}$$

$$\dot{\varphi}=0, \tag{30}$$

and the constraints are automatically satisfied.

In analogy with a null circular string that moves in Schwarzschild spacetime [13], we notice that Eqs. (27)–(30) describe a "cone" string and its trajectory is, for $Q^2>M^2$ given by

$$\theta=\text{const}, \tag{31}$$

$$r-r_0+M\ln\left|\frac{r^2-2Mr+Q^2}{r_0-2Mr+Q^2}\right|+\frac{2M^2-Q^2}{\sqrt{Q^2-M^2}}\times\left(\arctan\frac{r-M}{\sqrt{Q^2-M^2}}-\arctan\frac{r_0-M}{\sqrt{Q^2-M^2}}\right)=\pm(t-t_0), \tag{32}$$

for $Q^2<M^2$ given by

$$\theta=\text{const}, \tag{33}$$

$$r-r_0+M\ln\left|\frac{r^2-2Mr+Q^2}{r_0-2Mr+Q^2}\right|+\frac{2M^2-Q^2}{2\sqrt{M^2-Q^2}}\ln\left|\frac{(r-M-\sqrt{M^2-Q^2})(r_0-M+\sqrt{M^2-Q^2})}{(r-M+\sqrt{M^2-Q^2})(r_0-M-\sqrt{M^2-Q^2})}\right|=\pm(t-t_0), \tag{34}$$

and for $Q^2=M^2$ given by

$$\theta=\text{const}, \tag{35}$$

$$r-r_0-M+2M\ln\left|\frac{r-M}{r_0-M}\right|-\frac{M^2}{r-M}+\frac{M^2}{r_0-M}=\pm(t-t_0). \tag{36}$$

Cone strings start with a finite size and sweep out a cone of a constant angle θ (Fig. 1). An observer traveling together with a "cone" string would approach the event horizon at $r=r_+$ after a finite time and then he would fall onto the singularity (which, in fact, can be escaped from since it is timelike in Reissner-Nordström spacetime). On the other hand, an observer at spatial infinity is not able to notice the moment of passing the event horizon by the string. The observer sees that the string moves more and more slowly, in fact, an infinite time to pass the event horizon, or eventually, fall.

The "cone" string is an analogue of a point particle moving on a radial geodesic; however, it does not move in a plane through the origin of coordinates $r=0$ but it moves perpendicularly to the equatorial plane, except for the moment when it is captured. Moreover, one can find that rotation of such a string is forbidden by the constraints.

Taking the limit $Q=0$, Eq. (34) gives exactly the same result for a cone string as in Schwarzschild spacetime [13]. The equations of motion for Kerr spacetime have been studied in [16].

B. A circular null string with $K\neq0$, $L=0$ in Reissner-Nordström spacetime

Another interesting example of an exact solution is a circular null string with $K\neq0$, $L=0$ and the impact parameter $D=3\sqrt{3}M$ (the impact parameter for strings is defined as $D\equiv\sqrt{L^2+K}/E$) [13]. For $D=3\sqrt{3}M$ there exists a photon sphere with radius r_{ph} (an unstable photon orbit) in a Reissner-Nordström spacetime. In fact, when

$$r=r_{ph}=1{,}5M\left[1+\left(1-\frac{8Q^2}{9M^2}\right)^{1/2}\right], \tag{37}$$

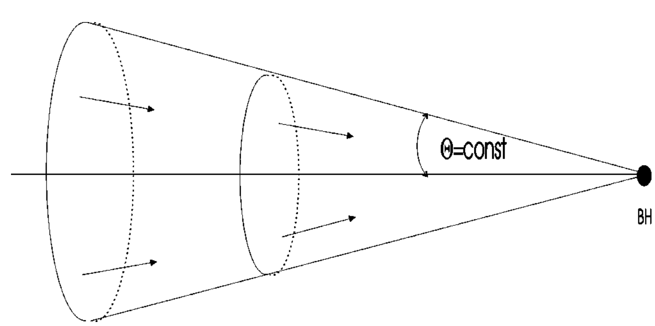

FIG. 1. The evolution of a cone string in a black hole (BH) spacetime.

$$Q^2<\frac{9}{8}M^2, \tag{38}$$

one obtains that equations of motion of a string (17)–(20) are solved by

$$t=3E\tau+\frac{2Q^2E\tau}{3M^2+3M^2\left(1-\frac{8Q^2}{9M^2}\right)^{1/2}-2Q^2}, \tag{39}$$

$$\theta=\pm\frac{E\tau}{\left[1{,}5M^2+1{,}5M^2\left(1-\frac{8Q^2}{9M^2}\right)^{1/2}-Q^2\right]^{1/2}}+\theta_0, \tag{40}$$

$$\varphi=\sigma. \tag{41}$$

The string oscillates an infinite number of times between the poles of the photon sphere. Its coordinate radius is given by Eq. (37) with the restriction (38) and its invariant size (14) is given by

$$S(\tau)=2\pi r_{ph}\sin\left\{\pm\frac{E\tau}{\left[1.5M^2+1.5M^2\left(1-\frac{8Q^2}{9M^2}\right)^{1/2}-Q^2\right]^{1/2}}+\theta_0\right\}. \tag{42}$$

In the limit $Q\rightarrow 0$ one gets the solution for a string moving on the photon sphere in Schwarzschild spacetime [13]. In analogy to a point particle case one is able to say that these solutions are unstable with respect to small perturbations.

In the special case $Q^2=M^2$, $r_{ph}=2M$, $r_+=M$, the Eqs. (39), (40) vastly simplify to give (Fig. 2)

$$t=4E\tau, \tag{43}$$

$$\theta=\pm\frac{E\tau}{M}+\theta_0, \tag{44}$$

and the invariant string size is

$$S(\tau)=4\pi M\sin\left(\pm\frac{E\tau}{M}+\theta_0\right), \tag{45}$$

so that it reduces to zero at poles and to a maximum value at the equatorial plane. Note that we have to consider the angle θ as a multiply covering angle for the Reissner-Nordström coordinate θ in metric (16) because the string timelike coordinate extends from $-\infty<\tau<\infty$.

Let us stress that the solution for a string moving on the photon sphere is not the only one with a constant r. We can find another solution given as

$$t=\tau, \tag{46}$$

$$r=r_+=M+\sqrt{M^2-Q^2}, \tag{47}$$

$$\theta=\text{const}=\theta_0, \tag{48}$$

$$\varphi=\sigma, \tag{49}$$

which is analogous to the solution for a null string on the event horizon [12] in Schwarzschild spacetime which is a stable solution. Such a string is placed exactly on the event horizon $r=r_+$. Contrary to a string moving dynamically on the photon sphere, the string described by the Eqs. (46)–(49) is stationary. A similar solution exists for a string placed on the Cauchy horizon

$$t=\tau, \tag{50}$$

$$r=r_-=M-\sqrt{M^2-Q^2}, \tag{51}$$

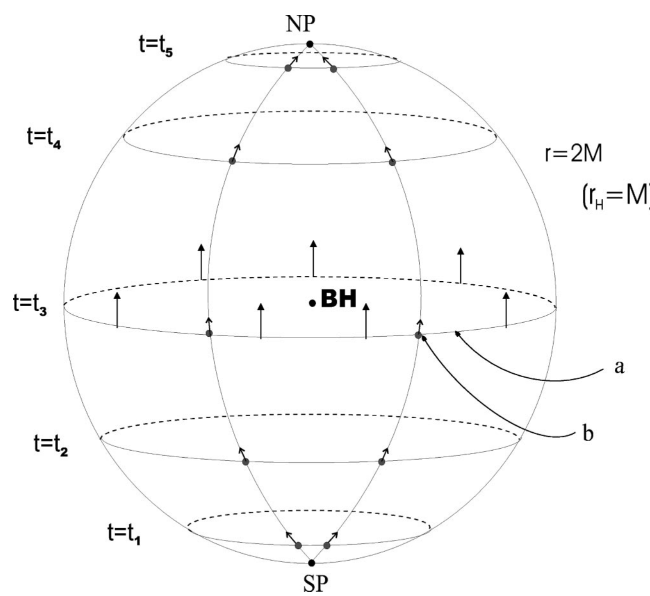

FIG. 2. The evolution of a string on the photon sphere in an extreme Reissner-Nordström spacetime with $Q^2=M^2$: (a) a string in a moment of passing the equatorial plane, (b) a point of the string moving all the time in the plane through the origin of coordinate $r=0$. BH is the black hole singularity (here timelike), $r=2$M is a radius of the photon sphere, $r_H=M$ is the event horizon.

$$\theta = \text{const} = \theta_0, \tag{52}$$

$$\varphi = \sigma, \tag{53}$$

which is unstable. This is possible since both surfaces of the event horizon and the Cauchy horizon are null (isotropic). The problem of the evolution of strings in Reissner-Nordström spacetime has been studied in both tensile and null context in Refs. [11,27,28]. It has been shown that inside the horizon instabilities appear due to the repulsive effect of a charge. However, for an extreme black hole ($Q^2 = M^2$) instabilities do not appear.

IV. THE EVOLUTION OF STRINGS IN THE STATIC EINSTEIN UNIVERSE

The metric of the static Einstein universe is [25]

$$ds^2 = dt^2 - R^2\left[\frac{dr^2}{1-r^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right] \tag{54}$$

$$= dt^2 - R^2[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\varphi^2)], \tag{55}$$

where $R = \text{const}$ is a radius of the universe, $r = \sin\chi$ and the proper distance in the universe is $l = R\chi$, where $0 \leqslant \chi \leqslant \pi$ which corresponds to $0 \leqslant r \leqslant 1$. The easiest way to study model (54) is when one introduces the spherical coordinates

$$x = R\sin\chi\sin\theta\cos\phi, \tag{56}$$

$$y = R\sin\chi\sin\theta\sin\phi, \tag{57}$$

$$z = R\sin\chi\cos\theta, \tag{58}$$

$$w = R\cos\chi, \tag{59}$$

where $0 \leqslant \theta \leqslant \pi, 0 \leqslant \phi \leqslant 2\pi$. In these coordinates one is able to embed the 4-sphere $x^2 + y^2 + z^2 + w^2 = R^2$ in a four-dimensional Euclidean space with metric $dS^2 = dx^2 + dy^2 + dz^2 + dw^2$, or, if one includes a time coordinate, in a five-dimensional space with metric $d\tilde{S}^2 = -dt^2 + dS^2$. In such a background the equations of motion for a propagating string are, in general, given as

$$\ddot{t} = \varepsilon^2 t'', \tag{60}$$

$$\ddot{r} + \frac{r}{1-r^2}\dot{r}^2 - r(1-r^2)\dot{\theta}^2 - r(1-r^2)\sin^2\theta\dot{\varphi}^2$$
$$= \varepsilon^2\left[r'' + \frac{r}{1-r^2}r'^2 - r(1-r^2)\theta'^2 - r(1-r^2)\sin^2\theta\varphi'^2\right], \tag{61}$$

$$\ddot{\theta} - \varepsilon^2\theta'' + \frac{2}{r}(\dot{r}\dot{\theta} - \varepsilon^2 r'\theta') - \sin\theta\cos\theta(\dot{\varphi}^2 - \varepsilon^2\varphi'^2)$$
$$= 0 \tag{62}$$

$$\ddot{\varphi} - \varepsilon^2\varphi'' + \frac{2}{r}(\dot{r}\dot{\varphi} - \varepsilon^2 r'\varphi') + 2\frac{\cos\theta}{\sin\theta}(\dot{\varphi}\dot{\theta} - \varepsilon^2\varphi'\theta')$$
$$= 0. \tag{63}$$

The parameter ε^2, as before, distinguishes between null and tensile strings. The constraints are

$$\dot{t}^2 - \frac{R^2}{1-r^2}\dot{r}^2 - R^2r^2\dot{\theta}^2 - R^2r^2\sin^2\theta\dot{\varphi}^2$$
$$= -\varepsilon^2\left[t'^2 - \frac{R^2}{1-r^2}r'^2 - R^2r^2\theta'^2 - R^2r^2\sin^2\theta\varphi'^2\right], \tag{64}$$

$$\dot{t}t' - \frac{R^2}{1-r^2}\dot{r}r' - R^2r^2\dot{\theta}\theta' - R^2r^2\sin^2\theta\dot{\varphi}\varphi' = 0. \tag{65}$$

The invariant size (15) of a string in the Einstein static universe is given by

$$S(\tau) = \int_0^{2\pi}\left(-t'^2 + \frac{R^2}{1-r^2}r'^2 + R^2r^2\theta'^2 + R^2r^2\sin^2\theta\varphi'^2\right)^{1/2} d\sigma. \tag{66}$$

For the null circular string $t = t(\tau), r = r(\tau), \theta = \theta(\tau), \varphi = \sigma$ one gets

$$S(\tau) = 2\pi Rr\sin\theta. \tag{67}$$

First, we consider the following ansatz:

$$t = t(\tau), \quad r = r(\tau) = \sin\chi(\tau), \quad \theta = \text{const}, \quad \varphi = \sigma \tag{68}$$

(a null circular string with a variable r). The solution of the field equations (60)–(63) is

$$t = E\tau, \tag{69}$$

$$\varphi = \sigma, \tag{70}$$

$$\theta = \text{const} = \theta_0, \tag{71}$$

$$\chi = \pm\frac{E\tau}{R} + \chi_0, \tag{72}$$

where we have explicitly used the metric (55) and the constraints (64), (65) which are, in fact, automatically fulfilled. The invariant string size is

$$S(\tau) = 2\pi R\left[\sin\left(\pm\frac{E\tau}{R} + \chi_0\right)\right]\sin\theta_0. \tag{73}$$

The solution (69)–(72) is a cosmological analogue of the solution (37), (39)–(41) [or in simpler form (43)] which represented a null string on the photon sphere in Schwarzschild spacetime. It has gotten the following physical interpretation: suppose we send a bunch of photons in all spatial directions from the point $\chi=0(r=0)$ (assuming that $\chi_0=0$) at the moment $t=0$. These photons form a spherical wave-front of which we consider only a circular bunch of constant θ_0—*a null string*. The string (the bunch) then starts from zero size at $\chi=0(r=0)$, expands to a maximum size $S=2\pi R\sin\theta_0$ that happens for

$$\tau=\mp\frac{R}{E}\left(\frac{\pi}{2}-\chi_0\right), \tag{74}$$

and finally contracts to zero size again when it reaches the antipodal point at $\chi=\pi$. Then the string starts from the antipodal point, reaches maximum size, and eventually comes back to the initial point $\chi=0$ it started with. This means it returned to the place it was sent after it has traveled throughout the whole universe. This cycle can then be repeated infinitely many times. Since the Einstein static universe can be represented as a cylinder in flat space it can be reminded that each individual point of string will move on a spiral which winds around this cylinder [25]. Using the embedding equations (56) one can show, for instance, that the point $\varphi=\sigma=0$ is rotating in the hypersurface (x,z,w) while the point $\varphi=\sigma=\pi/2$ is rotating in the hypersurface (y,z,w), etc.

Now, starting with the equations of motion (60)–(63) we consider the possibility of having tensile strings $(\varepsilon^2=1)$ with a constant radial coordinate $r=\sin\chi=$const (circular ansatz). Imposing this condition the equations (60)–(63) simplify to

$$\ddot{t}=\varepsilon^2 t'', \tag{75}$$

$$\dot{\theta}^2+\sin^2\theta\dot{\varphi}^2=\varepsilon^2[\theta'^2+\sin^2\theta\varphi'^2], \tag{76}$$

$$\ddot{\theta}-\sin\theta\cos\theta\dot{\varphi}^2=\varepsilon^2[\theta''-\sin\theta\cos\theta\varphi'^2], \tag{77}$$

$$\ddot{\varphi}+2\frac{\cos\theta}{\sin\theta}(\dot{\varphi}\dot{\theta}-\varepsilon^2\varphi'\theta')=\varepsilon^2\varphi'', \tag{78}$$

although the constraints (64), (65) do not reduce so vastly.

The analysis of the equations (75)–(78) shows that tensile strings with a constant radial coordinate cannot exist. This is due to self-interaction of strings (cf. [13]).

V. STRINGS IN EINSTEIN-SCHWARZSCHILD SPACETIME

In this section we consider the evolution of strings in the Einstein-Schwarzschild (Vadiya) spacetime [29,30]. It describes a point mass m which is placed in the static Einstein universe of Sec. IV. The metric reads as

$$ds^2=\left[1-\frac{2m}{R}\cot\left(\frac{r}{R}\right)\right]dt^2-\frac{dr^2}{1-\frac{2m}{R}\cot\left(\frac{r}{R}\right)}$$
$$-R^2\sin^2\left(\frac{r}{R}\right)(d\theta^2-\sin^2\theta d\varphi^2). \tag{79}$$

It is easy to notice that the coordinate χ in Eq. (54) now reads as $\chi=r/R$ and the role of the radial coordinate similar as in Reissner-Nordström or Einstein solution is now played by

$$\bar{r}=\sin\frac{r}{R}. \tag{80}$$

Another point is that the Einstein metric (55) is obtained in the limit $m\to 0$ while the Schwarzschild metric is allowed in the limit $R\to\infty$. The properties of spacetime (79) have been discussed carefully in [30]. It is interesting to learn that there exist two curvature singularities: one at $r=0$ and another at $r=\pi R$. The former is spacelike in full analogy to Schwarzschild singularity while the latter is timelike (naked) in analogy to Reissner-Nordström singularity of Sec. III. Therefore, the metric (79) describes the Einstein static universe with two antipodal black hole singularities: a spacelike and a timelike singularity.

The equations of motion (12), (13) for a string moving in the field of metric (79) are given by

$$\ddot{t}-\varepsilon^2 t''+2\frac{m}{R^2\sin^2\left(\frac{r}{R}\right)\left[1-\frac{2m}{R}\cot\left(\frac{r}{R}\right)\right]}(\dot{t}\dot{r}-\varepsilon^2 t'r')=0, \tag{81}$$

$$\ddot{r}-\varepsilon^2 r''+\frac{m}{R^2\sin^2\left(\frac{r}{R}\right)\left[1-\frac{2m}{R}\cot\left(\frac{r}{R}\right)\right]}(\dot{r}^2-\varepsilon^2 r'^2)$$
$$+\frac{m}{R^2}\frac{\left[1-\frac{2m}{R}\cot\left(\frac{r}{R}\right)\right]}{\sin^2\left(\frac{r}{R}\right)}(\dot{t}^2-\varepsilon^2 t'^2)$$
$$-R\sin\frac{r}{R}\cos\frac{r}{R}\left[1-\frac{2m}{R}\cot\left(\frac{r}{R}\right)\right](\dot{\theta}^2-\varepsilon^2\theta'^2)$$
$$-R\sin\left(\frac{r}{R}\right)\cos\left(\frac{r}{R}\right)\left[1-\frac{2m}{R}\cot\left(\frac{r}{R}\right)\right]$$
$$\times\sin^2\theta(\dot{\varphi}^2-\varepsilon^2\varphi'^2)=0, \tag{82}$$

$$\ddot{\theta}-\varepsilon^2\theta''-\sin\theta\cos\theta(\dot{\varphi}^2-\varepsilon^2\varphi'^2)$$
$$+\frac{2}{R}\cot\left(\frac{r}{R}\right)(\dot{r}\dot{\theta}-\varepsilon^2 r'\theta')=0, \tag{83}$$

$$\ddot{\varphi}-\varepsilon^2\varphi''+2\cot\theta(\dot{\varphi}\dot{\theta}-\varepsilon^2\varphi'\theta') + \frac{2}{R}\cot\left(\frac{r}{R}\right)(\dot{r}\dot{\varphi}-\varepsilon^2 r'\varphi')=0. \tag{84}$$

The constraints read as

$$-\left[1-\frac{2m}{R}\cot\left(\frac{r}{R}\right)\right](\dot{t}^2+\varepsilon^2 t'^2) + \left[1-\frac{2m}{R}\cot\left(\frac{r}{R}\right)\right]^{-1}(\dot{r}^2+\varepsilon^2 r'^2)+R^2\sin^2\frac{r}{R}[\dot{\theta}^2 + \varepsilon^2\theta'^2+\sin^2\theta(\dot{\varphi}^2+\varepsilon^2\varphi'^2)]=0, \tag{85}$$

$$-\left[1-\frac{2m}{R}\cot\left(\frac{r}{R}\right)\right]\dot{t}t'+\left[1-\frac{2m}{R}\cot\left(\frac{r}{R}\right)\right]^{-1}\dot{r}r' + R^2\sin^2\frac{r}{R}(\dot{\theta}\theta'+\sin^2\theta\dot{\varphi}\varphi')=0. \tag{86}$$

For the null strings $(\varepsilon^2=0)$, one has

$$\ddot{t}+2\frac{m}{R^2\sin^2\left(\frac{r}{R}\right)\left[1-\frac{2m}{R}\cot\left(\frac{r}{R}\right)\right]}\dot{t}\dot{r}=0, \tag{87}$$

$$\ddot{r}+\frac{m}{R^2\sin^2\left(\frac{r}{R}\right)\left[1-\frac{2m}{R}\cot\left(\frac{r}{R}\right)\right]}\dot{r}^2 + \frac{m}{R^2}\frac{\left[1-\frac{2m}{R}\cot\left(\frac{r}{R}\right)\right]}{\sin^2\left(\frac{r}{R}\right)}\dot{t}^2 - R\sin\left(\frac{r}{R}\right)\cos\left(\frac{r}{R}\right)\left[1-\frac{2m}{R}\cot\left(\frac{r}{R}\right)\right] \times(\dot{\theta}^2+\sin^2\theta\dot{\varphi}^2)=0, \tag{88}$$

$$\ddot{\theta}-\sin\theta\cos\theta\dot{\varphi}^2+\frac{2}{R}\cot\left(\frac{r}{R}\right)\dot{r}\dot{\theta}=0, \tag{89}$$

$$\ddot{\varphi}+2\cot\theta\dot{\varphi}\dot{\theta}+\frac{2}{R}\cot\left(\frac{r}{R}\right)\dot{r}\dot{\varphi}=0, \tag{90}$$

and

$$-\left[1-\frac{2m}{R}\cot\left(\frac{r}{R}\right)\right]\dot{t}^2+\left[1-\frac{2m}{R}\cot\left(\frac{r}{R}\right)\right]^{-1}\dot{r}^2 + R^2\sin^2\frac{r}{R}(\dot{\theta}^2+\sin^2\theta\dot{\varphi}^2)=0, \tag{91}$$

$$-\left[1-\frac{2m}{R}\cot\left(\frac{r}{R}\right)\right]\dot{t}t'+\left[1-\frac{2m}{R}\cot\left(\frac{r}{R}\right)\right]^{-1}\dot{r}r' + R^2\sin^2\frac{r}{R}(\dot{\theta}\theta'+\sin^2\theta\dot{\varphi}\varphi')=0. \tag{92}$$

The first integrals of Eqs. (87)–(90) are (compare [13])

$$\dot{t}=\frac{E(\sigma)}{1-\frac{2m}{R}\cot\left(\frac{r}{R}\right)}, \tag{93}$$

$$\dot{\varphi}=\frac{L(\sigma)}{\sin^2\theta\sin^2\left(\frac{r}{R}\right)}, \tag{94}$$

$$\sin^4\left(\frac{r}{R}\right)\sin^2\theta\dot{\theta}^2=-L^2(\sigma)\cos^2\theta+K(\sigma)\sin^2\theta, \tag{95}$$

and

$$\dot{r}^2+V(r)=0, \tag{96}$$

where

$$V(r)=-E^2(\sigma)+\frac{R^2}{\sin^2\left(\frac{r}{R}\right)}\left[1-\frac{2m}{R}\cot\left(\frac{r}{R}\right)\right] \times[L^2(\sigma)+K(\sigma)]. \tag{97}$$

There exists a solution with a constant r given by

$$t=\tau, \tag{98}$$

$$r=r_H=R\ \arctan\left(\frac{2m}{R}\right), \tag{99}$$

$$\theta=\mathrm{const}=\theta_0, \tag{100}$$

$$\varphi=\sigma, \tag{101}$$

which is analogous to the solution for a null string on the event horizon (so that it should be stable) in the Reissner-Nordström spacetime given by Eqs. (46)–(49). Here the event horizon is at r_H.

It is interesting to notice that apparently there should exist another solution with a constant r which would be for the photon sphere r_{ph} in Einstein-Schwarzschild spacetime given by

$$t=3E\tau, \tag{102}$$

$$r=r_{ph}=R\ \arctan\left(\frac{3m}{R}\right), \tag{103}$$

$$\theta = \frac{E\tau}{\sqrt{3}m}\sqrt{1+\frac{9m^2}{R^2}}:, \tag{104}$$

$$\varphi = \sigma, \tag{105}$$

which would be analogous to the solution for a null string on the photon sphere in Schwarzschild spacetime [13] which can be obtained in the limit $R\to\infty$ [or by taking $Q\to 0$ in the solution (46)–(49) for a null string on the photon sphere in Reissner-Nordström spacetime]. However, it is a simple exercise to show that *this solution is a contradiction*, namely there is a conflict between the field equation (87) and the constraint (91). The physical reason for this is similar to those which produce stationary strings in the de Sitter spacetime (cf. [31])—since there is no string tension which can balance local gravity a stationary or better static string cannot exist.

On the other hand, there exists a solution for a ‘‘cone’’ string given by

$$t = E\int \frac{d\tau}{1-\frac{2m}{R}\cot\left(\pm\frac{E}{R}\tau\right)}, \tag{106}$$

$$r = \pm E\tau, \tag{107}$$

$$\theta = \text{const} = \theta_0, \tag{108}$$

$$\phi = \sigma. \tag{109}$$

This is analogous to the solution (69)–(72) in the Einstein static universe. It is also easy to prove in the same way as in Schwarzschild and Reissner-Nordström and Einstein spacetimes that there exist no tensile circular strings of constant radius.

VI. CONCLUSION

In this paper we have found some exact string configurations in black hole and cosmological spacetimes which apply both for fundamental and for cosmic strings. We generalized previously found solutions of Ref. [13] for a ‘‘cone’’ string and for a string moving on the photon sphere into a Reissner-Nordström spacetime which is also related to the discussion of the behavior of strings in this spacetime given in Refs. [11,27,28]. We also generalized an event horizon solution and presented a Cauchy horizon solution for the Reissner-Nordström spacetime. We found a solution for a null string moving around the Einstein static universe and two completely new solutions for strings evolving in the Einstein-Schwarzschild spacetime (a black hole in the Einstein static universe).

First, we briefly presented a formalism which allowed us to take the limit of null strings in an appropriate action. Then, we studied the evolution of strings in Reissner-Nordström, Einstein static, and Einstein-Schwarzschild spacetimes. The exact configurations we found can be grouped geometrically into a couple of classes. There is a class of solutions which describe null strings residing on the null surfaces of these spacetimes, i.e., event and Cauchy horizons. There is also a class of solutions which describe strings sweeping out the light cones of a particular spacetime. Another class is for strings which reside on the surface of the photon sphere (an unstable periodic orbit for zero point particles). This class exists both in Schwarzschild and Reissner-Nordström spacetimes and, in an adapted form, in the Einstein static universe, but not in the Einstein-Schwarzschild spacetime.

As far as the physical properties are concerned we found that some of our solutions are unstable (for instance, a string on the photon sphere in Reissner-Nordström spacetime) and some are stable (e.g., a string on the event horizon). According to Ref. [31], multistring solutions appear whenever the world-sheet time τ is a multivalued function of the physical time and they are possible, for instance, in the positive cosmological constant models such as the de Sitter space. In our paper only the Einstein static universe admits a positive cosmological constant and because of that one should perhaps expect some multistring solutions admissible. However, our solutions of Secs. IV and V do not possess this property. On the other hand, some of our solutions [a string on the photon sphere [Eqs. (39)–(41)] and a string in the Einstein static universe [Eqs. (69)–(72)]] have an invariant string size described by multiply covering an azimuthal angle because of an infinite domain of the timelike string coordinate τ.

The existence of the photon sphere, i.e., an unstable periodic orbit (UPO), together with other special solutions suggests that a general evolution of a tensile (or perhaps even a null) string in these simple curved backgrounds is chaotic. This statement is obviously true for Schwarzschild spacetime [7], and the solutions we have found are straightforward generalizations of exact configurations in Schwarzschild spacetime.

The results we gained can give some insight into the nature of motion of strings in extremely high gravitational fields of black holes and in the early universe in fully quantum string theory.

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