PHYSICAL REVIEW D **66**, 041701(R) (2002)

Stabilizing textures with magnetic fields

R. S. Ward*
Department of Mathematical Sciences, University of Durham, Durham DH1 3LE, United Kingdom
(Received 27 May 2002; published 9 August 2002)

The best-known way of stabilizing textures is by Skyrme-like terms, but another possibility is to use gauge fields. The semilocal vortex may be viewed as an example of this, in two spatial dimensions. In three dimensions, however, the idea (in its simplest form) does not work—the link between the gauge field and the scalar field is not strong enough to prevent the texture from collapsing. Modifying the $|D\Phi|^2$ term in the Lagrangian (essentially by changing the metric on the Φ space) can strengthen this link, and lead to stability. Furthermore, there is a limit in which the gauge field is entirely determined in terms of the scalar field, and the system reduces to a pure Skyrme-like one. This is described for the gauge group U(1), in dimensions two and three. The non-Abelian version is discussed briefly, but as yet no examples of texture stabilization are known in this case.

DOI: 10.1103/PhysRevD.66.041701 PACS number(s): 11.27.+d, 05.45.Yv, 11.10.Lm

I. INTRODUCTION

Textures are classical solutions which are characterized by a nonzero homotopy group $\pi_d(T)$, d being the number of space dimensions. The relevant systems typically involve a scalar field Φ taking values in the target space T. With a Lagrangian such as $|\partial_\mu\Phi|^2$, and for $d\geqslant 2$, configurations are prone to implode (by the usual Derrick scaling argument). In an expanding universe, textures might be stabilized by the cosmological expansion; but we are interested here in cases where gravitational effects are negligible, and we take spacetime to be flat. In flat space, the best-known way of stabilizing textures is to add a Skyrme term involving higher powers of $\partial_\mu\Phi$.

By contrast, vortices or monopoles correspond to a nontrivial $\pi_{d-1}(T)$, and (in their "local" versions) are stabilized by gauge fields. Many similarities between textures and vortices or monopoles have been noted. For example, multi-Skyrmions and Bogomol'nyi-Prasad-Sommerfield (BPS) multimonopoles (located at a single point in space) each have a polyhedral structure corresponding to an appropriate subgroup of O(d), and this has been partly understood in terms of rational maps from the Riemann sphere to itself [1]. The purpose of this paper is to investigate the stabilization of textures by gauge fields, and so in particular it explores a different sort of relation between the two classes of topological solitons, generalizing the example provided by the semilocal vortex [2,3].

The idea of stabilizing textures with gauge fields has been investigated before. One motivation has been the fact that Skyrme terms are non-renormalizable, whereas gauge theories may have better quantum behavior; but in this paper the considerations are entirely classical. For the extended Abelian-Higgs model (with the Higgs field being a complex doublet), it was pointed out in [4] that an expansion in field gradients produced a Skyrme-like term, which suggested stability; at the time, this was not investigated in detail. More recent numerical simulations [5] seemed to show that stability was indeed present (although, as reported below, we have not been able to confirm this result). In a different Abelian system (involving a triplet of real scalar fields and a massive Abelian gauge field) no stable textures could be found [6]. For the non-Abelian case, scaling arguments again suggest stability (cf. [7]); but detailed investigation such as [8] have produced negative results. The conclusion seems to be that the scalar field and the gauge field have to be linked to each other sufficiently strongly in order to prevent each from collapsing independently; and in "standard" systems, this link is not strong enough.

The general framework is as follows. Suppose we have a system involving a gauge field (with gauge group G), and a multiplet Φ of scalar fields coupled to it. The "basic" Lagrangian of the system has the form

$$\mathcal{L}=\tfrac{1}{2}|D_\mu\Phi|^2-\tfrac{1}{4}(F_{\mu\nu})^2-V(\Phi). \tag{1}$$

For space dimension $d=2$, the system defined by Eq. (1) may admit stable static solutions (for example, semilocal vortices); but for $d=3$ it seems not to—some modification is needed. The idea pursued here is that the term $|D_\mu\Phi|^2$ in the Lagrangian involves a choice of metric on the space T in which Φ takes its values, and we can change this metric. For example, if Φ is a complex vector, then the standard Euclidean metric is $|D\Phi|^2=(D\Phi^\dagger)(D\Phi)$, where $D\Phi^\dagger$ denotes the complex-conjugate transpose of $D\Phi$. A natural modification of this (see the following section) is to add a term $\kappa^2|\Phi^\dagger D_\mu\Phi|^2$, where κ is a constant. So we now have a family of systems, parametrized by κ. Taking the limit $\kappa\to\infty$ enforces the constraint

$$\Phi^\dagger D_\mu\Phi=0, \tag{2}$$

which (under favorable circumstances) determines the gauge potential in terms of Φ. So we have a family of systems where, in an appropriate limit, the gauge degrees of freedom disappear, and the Maxwell or Yang-Mills term $(F_{\mu\nu})^2$ becomes a Skyrme term. This enables us to track a soliton solution as it changes from a gauge-stabilized texture into a Skyrme-stabilized texture. In an appropriate limit of param-

*Email address: richard.ward@durham.ac.uk

eters $\kappa, \ldots \to \infty$, one gets a Skyrme system which certainly admits stable solitons; one question is for which *finite* values of these parameters there are stable solitons.

Non-trivial examples of this idea have only been found in the Abelian case $G=\mathrm{U}(1)$, and these are described in Secs. II and III (for $d=2$ and $d=3$, respectively). A discussion of the non-Abelian case $[G=\mathrm{SU}(2)$ in $d=3]$ is given in the Sec. IV. The conclusion, therefore, is that textures can be stabilized by (Abelian) magnetic fields, but no non-Abelian version of this appears to be known.

It might be noted that the idea of adding a term $\kappa^2|\Phi^\dagger D_\mu\Phi|^2$, and investigating how solitons depend on the parameter κ, has been investigated before; the simplest example (in a somewhat different context) is that of the CP^1 model with no gauge field [9].

II. SEMILOCAL VORTICES AND PLANAR SKYRMIONS

In this section we take $d=2$ (so space is the plane $\mathbf{R}^2$), and gauge group U(1). Let the Higgs field Φ be a complex doublet $\Phi=[\Phi^1\ \Phi^2]^t$. The resulting extended Abelian-Higgs system admits semilocal vortex solutions [2,3]; and in the limit $\kappa\to\infty$ it becomes, as we shall see, a Skyrme version of the CP^1 model. The generalization with Φ being an M-tuplet, leading in the limit to a Skyrme version of the CP^{M-1} model, is straightforward; but for simplicity we shall restrict ourselves here to the CP^1 case.

The standard Lagrangian is

$$\mathcal{L}=\tfrac{1}{2}(D_\mu\Phi)^\dagger(D^\mu\Phi)-\tfrac{1}{4}(F_{\mu\nu})^2-\tfrac{1}{8}\lambda(1-\Phi^\dagger\Phi)^2, \quad (3)$$

where $D_\mu\Phi=\partial_\mu\Phi-\mathrm{i}A_\mu\Phi$. For the semilocal vortex solution, the gauge field provides a "hard core" which prevents the soliton from shrinking. If $0<\lambda<1$, the single soliton is stable; but for $\lambda>1$ it is unstable (it expands without limit) [10,11]. For $\lambda=1$ there is a one-parameter family of static solutions saturating a Bogomolny bound, but these solitons are marginally unstable [12]. One member of this family is (an embedding of) the standard Nielsen-Olesen vortex.

Various relations between this system and the CP^1 model have been noted before (cf. [4,13]). For example, imposing the constraint $\Phi^\dagger\Phi=1$ (this corresponds to letting the parameter λ tend to infinity), and scaling away the $(F_{\mu\nu})^2$ term leaves the CP^1 model [13]. But in order to have stable semilocal vortices which become Skyrmions as a limiting case, one needs to make some modifications.

Recall, first, the symmetry of this system [3]. The ungauged system has an SO(4) global symmetry. On gauging a U(1) subgroup, this SO(4) is reduced to the product of the local U(1) and a global SU(2); the field Φ belongs to the fundamental representation of this SU(2). The most general SU(2)-invariant metric on $T=\mathbf{C}^2$ is $h_{PQ}D\Phi^P D\bar{\Phi}^Q$, where

$$h_{PQ}(\Phi,\bar{\Phi})=g(\xi)\,\delta_{PQ}+\tilde{g}(\xi)\bar{\Phi}_P\Phi_Q \quad (4)$$

and $\xi=\bar{\Phi}^P\Phi^P=\Phi^\dagger\Phi$. The two functions g and $\tilde{g}$ are arbitrary. But in the limit $\lambda\to\infty$, which is of particular interest here, we have $\xi\equiv1$; so let us take g and $\tilde{g}$ to be constants, scaling g to unity and writing $\tilde{g}=\kappa^2$. Using this modified metric instead of the standard Euclidean one amounts to replacing Eq. (3) by

$$\mathcal{L}=\tfrac{1}{2}(D_\mu\Phi)^\dagger(D^\mu\Phi)+\tfrac{1}{2}\kappa^2|\Phi^\dagger D_\mu\Phi|^2-\tfrac{1}{4}(F_{\mu\nu})^2 -\tfrac{1}{8}\lambda(1-\Phi^\dagger\Phi)^2. \quad (5)$$

The second modification is as follows. In order to have stability for $\lambda>1$, we need an extra potential term, which necessarily breaks the SU(2) global symmetry (see for example [14,15]). We shall add to Eq. (5) the term $\alpha|\Phi^2|^2$, where α is a positive constant. In the Bogomolny case ($\kappa=0$ and $\lambda=1$), there is now a unique minimium: it has $\Phi^2=0$, and is the Nielsen-Olesen vortex with energy $E=\pi$.

With these two modifications, the static energy density $\mathcal{E}$ of the system is given by

$$2\mathcal{E}=(D_j\Phi)^\dagger(D_j\Phi)+\kappa^2|\Phi^\dagger D_j\Phi|^2+(B_j)^2+V(\Phi), \quad (6)$$

where $V(\Phi)=\frac{1}{4}\lambda(1-\Phi^\dagger\Phi)^2+2\alpha|\Phi^2|^2$, and where $B_j=\epsilon_{jkl}\partial_k A_l$ is the magnetic field strength.

The boundary conditions are chosen to ensure finite energy. At spatial infinity, one must have (a) $A_j=f^{-1}\partial_j f$, where $|f|=1$; (b) $D_j\Phi=0\Rightarrow\Phi=f^{-1}K$, where K is a constant 2-vector; and (c) $V(\Phi)=0\Rightarrow K=[k\ 0]^t$ with $|k|=1$. Because of (b) and (c), Φ cannot be zero at spatial infinity; and in order for Φ to be single valued, f has to be single valued. Hence f is a map from the circle at spatial infinity to the gauge group U(1), and the degree of f is the soliton number N. The total magnetic flux is proportional to N, in the usual way. The fact that there is nontrivial topology does not necessarily mean that there are stable solitons; but the numerical work described below indicates that there are, at least for certain ranges of the parameters α, λ and κ.

Taking the limit $\lambda\to\infty$ enforces the constraint $\Phi^\dagger\Phi=1$ (so Φ takes values in S^3). If in addition $\kappa\to\infty$, then the minimum-energy configuration approaches one for which $\Phi^\dagger D_j\Phi=0$, and hence

$$A_j=-\mathrm{i}\Phi^\dagger\partial_j\Phi. \quad (7)$$

With A_j given in terms of Φ by this expression, $(D_j\Phi)^\dagger(D_j\Phi)$ becomes the standard CP^1 energy, and $(B_j)^2$ becomes a Skyrme term. We can reexpress this as an O(3) sigma model in the usual way: define a unit 3-vector field $\vec{\phi}$ by $\vec{\phi}=\Phi^\dagger\vec{\sigma}\Phi$, where σ^a are the Pauli matrices. This corresponds to the standard Hopf map from S^3 (the space $\Phi^\dagger\Phi=1$) to S^2. Strictly speaking, the Φ field is a vortex (winding at spatial infinity); but the $\vec{\phi}$ field, obtained from it by projection, is a texture (constant at spatial infinity). Then in the $\lambda,\kappa\to\infty$ limit, the energy density $\mathcal{E}$ is given by

$$8\mathcal{E}_{\lambda,\kappa\to\infty}=(\partial_j\vec{\phi})\cdot(\partial_j\vec{\phi})+(\vec{\phi}\cdot\partial_1\vec{\phi}\times\partial_2\vec{\phi})^2+4\alpha(1-\vec{n}\cdot\vec{\phi}), \quad (8)$$

where $\vec{n}$ is a constant unit vector. This is a planar Skyrme system [16,17]. The energy of the Skyrmion solutions de-

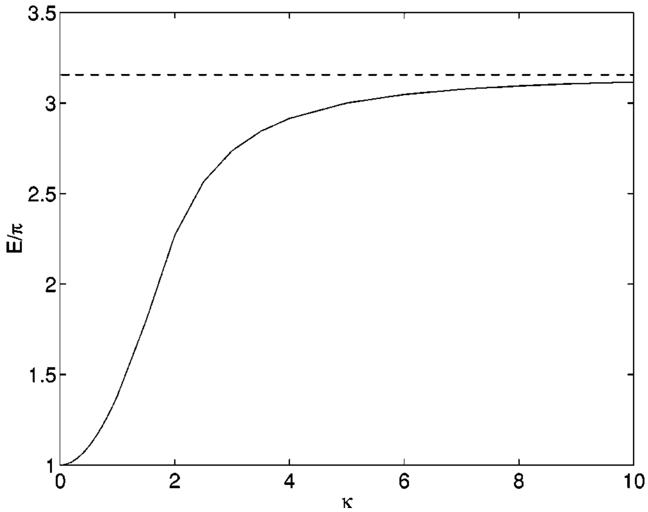

FIG. 1. The energy E of the 1-soliton on $\mathbf{R}^2$, as a function of κ, with $\alpha=1$ and $\lambda=1+\kappa^2$. The dashed line is the energy of the planar Skyrmion obtained in the limit $\kappa\to\infty$.

pends on α, and can be found by numerical minimization; for the 1-soliton with $\alpha=1$ it is $E=3.1557\pi$.

The energy (and the stability) of the solitons in the system (6) may be investigated numerically, as a function of the three parameters α, λ and κ, and of the soliton number N. This has been done for the $N=1$ case, with $\alpha=1$ and $\lambda=1+\kappa^2$. The result is summarized in Fig. 1, which shows the energy E as a function of $\kappa\geqslant 0$. It was obtained by assuming the standard form for O(2)-symmetric fields, namely $\Phi^1=f(r)\exp(\mathrm{i}N\theta)$, $\Phi^2=g(r)$, $A_r=0$ and $A_\theta=a(r)$, where f, g, and a are real valued. The discrete version of the energy functional $E[f,g,a]$ was then minimized numerically, using a conjugate-gradient method. For each value of κ, a stable solution was found. Note that, as expected, E goes from $E=\pi$ (the Nielsen-Olesen vortex) at $\kappa=0$ and $\lambda=1$, to $E=3.1557\pi$ (the planar Skyrmion) as $\kappa\to\infty$ and $\lambda\to\infty$.

III. VORTEX RINGS AND HOPF TEXTURES

In this section we investigate the same system (5) as before, but in spatial dimension $d=3$. The extra potential term is omitted (in other words, $\alpha=0$), so the global SU(2) symmetry is unbroken. One may form a texture configuration by taking a finite length of semilocal vortex with its ends joined together to form a loop in 3-space, and it has previously been speculated that such a texture might be stable [4,5].

The energy density is given by Eq. (6), with $\alpha=0$; so the system depends on the two parameters λ and κ. In the limit $\lambda,\kappa\to\infty$, we again get an S^3-valued scalar field Φ, with the gauge potential being given by Eq. (7); it has previously been pointed out (cf. [18,19]) that this limit is equivalent to the Faddeev-Hopf system [20–26]. So there are stable ring like solitons in the limit; the question here is whether they are stable for finite values of κ and λ.

The boundary conditions imply, as before, that $\Phi=[\Phi^1\ \Phi^2]^t=f^{-1}K$ at $r=\infty$, where K is a constant 2-vector; so $W=\Phi^1/\Phi^2$ is constant at spatial infinity. Thinking of W as a stereographic coordinate for CP^1 therefore shows that

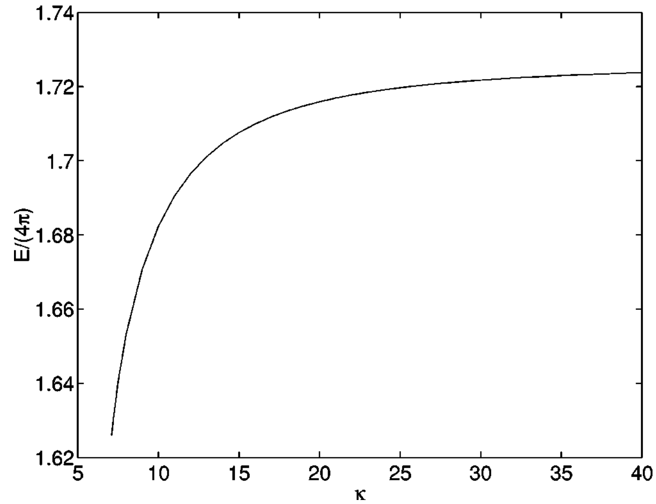

FIG. 2. The energy E of the 1-soliton on $\mathbf{R}^3$, as a function of κ, with $\lambda=1+\kappa^2$.

Φ (provided it is nowhere zero) defines a map from $\mathbf{R}^3$ to S^2 which is constant at infinity, and hence is classified topologically by the Hopf number $N\in\pi_3(S^2)$. For $N=1$, the field resembles a single vortex ring. The stability of such $N=1$ configurations has been investigated numerically, again by minimization of the energy functional. The solitons cannot be spherically symmetric, but one expects that for small values of N they will be axially symmetric [21,22]. So one can reduce the problem to a two-dimensional one which is not too difficult computationally. More precisely, one can use cylindrical coordinates, and impose an $SO(2)\times SO(2)$-invariant ansatz, as for example in [5].

Minima were sought for the one-parameter family of systems obtained by setting $\lambda=\kappa^2+1$, and stable solitons were found for $\kappa\geqslant 7.1$. Their energy is plotted in Fig. 2. For $\kappa\leqslant 7$, however, the radius of the vortex ring shrinks to zero, and the field unwinds: there is no stable minimum. When κ (and therefore λ) tend to infinity, the normalized energy $E'=E/4\pi$ approaches the value $E'_\infty=1.73$ (obtained by extrapolation of the data in Fig. 2). This is exactly the energy of the single Hopf soliton: from [26], and allowing for different coupling constants, we get the value $E'=1.22\sqrt{2}=1.73$.

The analogous computation previously reported in [5] for the $\kappa=0$ case suggested that one might have stability for fairly small values of λ (of order unity). The results described above do not confirm this, and in fact no stable solution could be found for $\kappa=0$, even with λ quite large. But (as emphasized in [5]), there might be local minima in the configuration space which are difficult to detect, and which require an initial condition which is very close to the actual solution. So it remains an open question as to whether stable vortex rings exist for small values of κ and λ. It is, however, the case that the configuration which is stable for $\kappa=7.1$, $\lambda=51.4$ collapses if κ and λ are reduced to $\kappa=7$, $\lambda=50$.

IV. NON-ABELIAN GAUGE FIELD

As mentioned in the Introduction, the question of whether textures can be stabilized by a non-Abelian gauge field has previously been investigated; there are suggestions based on simple scaling arguments (cf. [7]), but more detailed studies have yielded negative results (cf. [8]). Let us look at the three-dimensional case ($d=3$), with gauge group SO(3). The field Φ belongs to some representation Γ of SO(3); so we have to choose Γ, as well as an appropriate potential function $V(\Phi)$. For example, for the 't Hooft–Polyakov monopole one uses the fundamental representation $\Gamma=\mathbf{3}$. The simplest extension of this is the four-dimensional representation $\Gamma=\mathbf{1}\oplus\mathbf{3}$. The corresponding system admits monopole-like soliton solutions which have been referred to as semilocal monopoles [3]. (Another simple extension is $\Gamma=\mathbf{3}\oplus\mathbf{3}$, the corresponding solitons being referred to as colored monopoles [11].)

Let us look at the $\mathbf{1}\oplus\mathbf{3}$ case: so $\Phi=(\phi_0,\vec{\phi})$ is a four-vector. Take the potential function to be $\lambda(1-|\Phi|^2)^2$; so for large λ, we get the constraint $|\Phi|^2\approx 1$. One may then impose texture boundary conditions (rather than monopole boundary conditions): namely, Φ tends to a constant as $r\to\infty$ in $\mathbf{R}^3$. So Φ is effectively a map from S^3 to S^3, and it has a winding number N. The stability of spherically symmetric $N=1$ configurations has been studied numerically—the details are as follows.

For simplicity, we shall take the $\lambda\to\infty$ limit, so $|\Phi|^2\equiv 1$; and the metric on Φ space to be flat (no extra term analogous to $\kappa^2|\Phi^\dagger D_j\Phi|^2$). The energy density is

$$\mathcal{E}=\tfrac{1}{2}|D_j\Phi|^2+\tfrac{1}{4}(F_{jk})^2, \tag{9}$$

where $D_j\Phi=(\partial_j\phi_0,\partial_j\phi^a-2\epsilon^{abc}A_j^b\phi^c)$. To implement spherical symmetry, we take Φ and the gauge potential A to have the standard "hedgehog" form

$$A_j^a=\epsilon_{jak}x^k f(r)/r^2, \quad \phi_0=\cos g(r), \quad \phi^a=x^a\sin g(r)/r, \tag{10}$$

with the boundary conditions $f(0)=0$, $f(\infty)=\frac{1}{2}$, $g(0)=\pi$, $g(\infty)=0$. The energy density then becomes

$$\mathcal{E}=\frac{f_r^2}{r^2}+\frac{2f^2(f-1)^2}{r^4}+\frac{g_r^2}{2}+\frac{\sin^2 g}{r^2}[1+4f(f-1)]. \tag{11}$$

One can then minimize the energy numerically; this was done using a conjugate-gradient method, with various initial conditions. But no smooth minimum could be found—in every case, both f and g collapse towards being zero almost everywhere.

One can see this collapse analytically, in the following highly simplified version (involving just two degrees of freedom α and β). Let α and β be the values of r such that $f(\beta)=1/4$ and $g(\alpha)=\pi/4$. In other words, α and β are the "radii" of the scalar field and the gauge field, respectively. More explicitly, take f and g to have the form

$$f(r)=\begin{cases}\dfrac{r^2}{4\beta^2} & \text{for } 0\leqslant r\leqslant\beta\\[2ex] \dfrac{1}{2}-\dfrac{\beta}{4r} & \text{for } r\geqslant\beta\end{cases},$$

$$\cos g(r)=\begin{cases}-1+\dfrac{r^2}{\alpha^2} & \text{for } 0\leqslant r\leqslant\alpha\\[2ex] 1-\dfrac{\alpha^2}{r^2} & \text{for } r\geqslant\alpha\end{cases}.$$

One can compute the energy $E(\alpha,\beta)$ of this configuration exactly: it is a rational function of α and β. In particular, for $\beta=1/\alpha$ the energy has the form $E(\alpha,1/\alpha)=\alpha\times$(polynomial in α). The salient point about this form is that its minimum occurs when $\alpha=0$; this corresponds to the scalar field shrinking to zero width, while the gauge field spreads out. As one sees from the usual Derrick scaling argument used in [7], the contribution to the energy from the $|D_j\Phi|^2$ term can be reduced by scaling one way, while the contribution from $(F_{jk})^2$ can be reduced by scaling the other way. But the system as a whole can never reach a balance: the gauge field and the scalar field are just not sufficiently strongly coupled to each other to prevent each one from collapsing separately. As was remarked before, these results do not actually prove the absence of a stable solution: there might still be a local minimum somewhere in the configuration space. But it seems rather unlikely that this system does admit a stable texture.

As in the preceding sections, one can modify the metric on Φ space, and this may improve the stability properties. That possibility has not yet been fully investigated; but certainly there is no gauge-invariant extra term, the vanishing of which determines the gauge potential as in the Abelian case (7). So the idea of obtaining the usual Skyrme model as a limit does not quite work in this non-Abelian case. Using other representations Γ, and for that matter other gauge groups, opens up many more possibilities, which are still to be explored. But for the time being, it remains the case that there are no known examples of three-dimensional systems in which a texture is stabilized by a non-Abelian gauge field.

[1] C.J. Houghton, N.S. Manton, and P.M. Sutcliffe, Nucl. Phys. **B510**, 507 (1998).
[2] T. Vachaspati and A. Achúcarro, Phys. Rev. D **44**, 3067 (1991).
[3] J. Preskill, Phys. Rev. D **46**, 4218 (1992).
[4] M. Hindmarsh, Nucl. Phys. **B392**, 461 (1993).
[5] A.J. Niemi, K. Palo, and S. Virtanen, Phys. Rev. D **61**, 085020 (2000).
[6] L. Perivolaropoulos and T.N. Tomaras, Phys. Rev. D **62**, 025012 (2000).
[7] L. Perivolaropoulos, Phys. Rev. D **62**, 047301 (2000).
[8] C. Bachas, P. Tinyakov, and T.N. Tomaras, Phys. Lett. B **385**, 237 (1996).

[9] W.J. Zakrzewski, Lett. Math. Phys. **12**, 283 (1986).
[10] M. Hindmarsh, Phys. Rev. Lett. **68**, 1263 (1992).
[11] A. Achúcarro, K. Kuijken, L. Perivolaropoulos, and T. Vachaspati, Nucl. Phys. **B388**, 435 (1992).
[12] R.A. Leese, Phys. Rev. D **46**, 4677 (1992).
[13] B.J. Schroers, Nucl. Phys. **B475**, 440 (1996).
[14] M. Axenides and L. Perivolaropoulos, Phys. Rev. D **56**, 1973 (1997).
[15] M. Axenides, L. Perivolaropoulos, and T.N. Tomaras, Phys. Rev. D **58**, 103512 (1998).
[16] B.M.A.G. Piette, B.J. Schroers, and W.J. Zakrzewski, Z. Phys. C **65**, 165 (1995).
[17] B.M.A.G. Piette, B.J. Schroers, and W.J. Zakrzewski, Nucl. Phys. **B439**, 205 (1995).
[18] P. van Baal and A. Wipf, Phys. Lett. B **515**, 181 (2001).
[19] E. Babaev, L.D. Faddeev, and A.J. Niemi, Phys. Rev. B **65**, 100512 (2002).
[20] L. Faddeev and A.J. Niemi, Nature (London) **387**, 58 (1997).
[21] J. Gladikowski and M. Hellmund, Phys. Rev. D **56**, 5194 (1997).
[22] R.A. Battye and P.M. Sutcliffe, Proc. R. Soc. London **455**, 4305 (1999).
[23] J. Hietarinta and P. Salo, Phys. Lett. B **451**, 60 (1999).
[24] J. Hietarinta and P. Salo, Phys. Rev. D **62**, 081701 (2000).
[25] R.S. Ward, Nonlinearity **12**, 241 (1999).
[26] R.S. Ward, Phys. Lett. B **473**, 291 (2000).