

Debye screening at finite temperature reexamined

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We present an alternative way to calculate the screening of the static potential between two charges in (non-)Abelian gauge theories at high temperatures. Instead of a loop expansion of a gauge boson self-energy, we evaluate the energy shift of the vacuum to order e^2 after applying an external static magnetic field and extract a temperature- and momentum-dependent dielectric permittivity. The hard thermal loop gluon and photon Debye masses are recovered from the lowest lying Landau levels of the perturbed vacuum. In QED, the complete calculation exhibits an interesting cancellation of terms, resulting in a logarithmic running $\alpha(T)$. In QCD, a Landau pole in α_s arises in the infrared from the sign of the gluon contribution, as in more sophisticated thermal renormalization group calculations.

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I. INTRODUCTION

In quantum field theory, fluctuations of the vacuum give rise to the production of pair quanta which tend to screen (or antiscreen) the charge of a heavy test particle. If one perturbatively calculates the nonrelativistic potential $V(r)$ between two unlike static charges, say, in QED, the usual Coulomb-like behavior is modified by the photon self-energy $\Pi(K^2)$ such that

$$V(r) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{r}} \frac{-e^2}{k^2 + \Pi(K^2 = -k^2)}, \quad (1)$$

where $k = |\vec{k}|$ and $K = (k^0, \vec{k})$. Inserting the textbook result for $\Pi(K^2)$ and expanding for small distances $k^2 \gg m_e^2$, the quantum fluctuations lead to an effective coupling constant

$$\alpha_{\text{eff}}(k) = \frac{\alpha}{1 - \frac{\alpha}{3\pi} \log\left(\frac{k^2}{\Lambda^2}\right)}, \quad (2)$$

where $\Lambda = \exp(5/3)m_e$ is a scale related to the electron mass m_e . This is, of course, the familiar result of the running coupling in QED which is commonly obtained using renormalization group methods. In Refs. [1,2], it has been shown that the running of a coupling constant at $T=0$ can be understood in physical terms by the polarizability of the vacuum. The effects of fluctuations can be incorporated to a certain extent in a scale-dependent dielectric permittivity ϵ that defines an effective charge $\alpha_{\text{eff}} = \alpha/\epsilon$. In vacuum, Lorentz invariance dictates that

$$\mu\epsilon = 1, \quad (3)$$

where μ is the magnetic permeability. Calculating $\mu(k)$ at the momentum scale k and extracting the leading log contribution, one finally recovers the familiar expressions for the running couplings in QED and QCD. Then, asymptotic freedom can be interpreted in terms of a paramagnetic ground state.

In this work, we extend the approach of [1,2] to finite temperature and calculate an effective coupling constant $\alpha_{\text{eff}}(k, T)$. Instead of loop expansion, we evaluate the energy shift of the vacuum to order e^2 after applying an external (chromo)magnetic field H . The connection of magnetic permeability and dielectric permittivity at finite temperature is made by invoking a renormalization group argument. QCD with a magnetic background field at finite temperature has been studied in a number of works [3]. In contrast with previous approaches, we lay out a nontechnical calculation of charge screening without reference to propagators or self-energies, resorting to entities that have an immediate physical interpretation (energy densities and susceptibilities). Our work then allows an alternative, although slightly more phenomenological, view of screening at high temperature.

A possible dissolution of bound quarkonia states, e.g., J/ψ , was proposed long ago as an experimental signature of the quark-gluon plasma in heavy-ion collisions. The T dependence of the interquark potential in QCD is therefore of great interest, and simulations of the potential in lattice gauge theory do indeed show a strong screening [4]. In perturbation theory, the quantity that enters the Fourier transform of the potential at finite temperature is the static limit of the longitudinal gauge boson self-energy $\Pi_L(k^0, k; T)$ [5]:

$$V(r, T) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{r}} \frac{-e^2}{k^2 + \Pi_L(0, k; T)}. \quad (4)$$

Equivalently, one can define a dielectric permittivity by [6]

$$\epsilon(k, T) = 1 + \frac{\Pi_L(0, k, T)}{k^2}. \quad (5)$$

The perturbative one-loop thermal contribution to Π_L was calculated long ago as [7]

$$\Pi_L(0, k, T) = \frac{e^2 T^2}{3} \equiv (m_D^e)^2 \quad \text{for QED} \quad (6)$$

and

$$\Pi_L(0,k,T) = \left(N_c + \frac{N_f}{2} \right) \frac{g^2 T^2}{3} \equiv (m_D^e)^2 \quad \text{for QCD} \quad (7)$$

which define screening masses m_D . Here, e and g are the electromagnetic and strong coupling, respectively, N_c is the number of colors, and N_f is the number of thermally active flavors. Since the static limits of the self-energies are momentum independent, the poles of the expression in Eq. (4) are simply the gauge invariant Debye masses m_D defined in Eqs. (6) and (7) and lead to an exponential damping of the potential $V(r) \sim \exp(-m_D r)/r$. In particular, this form of Π_L has the consequence that gluons *screen* the strong interaction, in contrast to the zero temperature case, over long distances. However, the formula for the running QCD coupling constant commonly used in finite temperature calculations assumes that typical momentum transfers are of the order of the temperature; hence,

$$\alpha_s(T) \sim \frac{1}{(11N_c - 2N_f) \log(T)}. \quad (8)$$

In this expression, gluons therefore retain their antiscreening property, reflecting the ultraviolet sector of the theory. The transition to Debye screening is not obvious. Another troublesome feature of QCD screening at finite temperature is the behavior of the Debye mass at next-to-leading order, which reads [8]

$$m_D^{(2)}(T) = m_D^{(1)} + \frac{N_c g^2 T}{4\pi} \log\left(\frac{C}{g}\right), \quad (9)$$

with $m_D^{(1)}$ given by Eq. (7). Here C is a constant arising from the *ad hoc* removal of infrared singularities involving chromomagnetic static modes. The appearance of the nonperturbative logarithmic term questions the applicability of loop calculations somewhat. Furthermore, whereas in QED the self-energy tensor $\Pi_{\mu\nu}$ is gauge independent, this is not the case in QCD, which makes the very definition of a Debye mass conceptually difficult. Finally, due to the nonlinear coupling of the gluons, relation (5) remains valid only within certain gauges (like the temporal axial gauge) [9]. An evaluation of the effective charge and its possible screening not relying on a Feynman graph expansion is therefore desirable.

II. THE ZERO TEMPERATURE CASE

In this section, we define our notation and briefly review the calculation of Refs. [1,2]. To obtain a scale-dependent μ , let us look at the change in the energy E of the vacuum when an external magnetic field H is applied:

$$\Delta E = -\frac{1}{2} [4\pi\chi(H)] V H^2 - E_{\text{vac}}, \quad (10)$$

where $\mu(H) = 1 + 4\pi\chi(H)$, and $\chi(H)$ is the field-dependent magnetic susceptibility. As soon as the energies at $H=0$ and finite H are known to some approximation, a field- or equiva-

lently scale-dependent $\mu(H)$ can be extracted. Later, the external field $2eH$ is identified with the scale K^2 at which the physical process is probed.

For charged scalar fields, the general expression for the energy spectrum of a single Fourier mode reads

$$E_{n,k}^\pm = \omega_k \left(n_k^\pm + \frac{1}{2} \right), \quad (11)$$

distinguishing between particles (+) and antiparticles (-). The dispersion relation $\omega_k = k$ follows from the positive energy solution of the Klein-Gordon equation for massless, noninteracting particles. At $T=0$, the occupation number n_k^\pm for the ground state is zero. Summing over particle and antiparticle states, we recover the familiar divergent zero-point vacuum energy $E_0 = \sum_k \omega_k$. For massless spin- $\frac{1}{2}$ fermions, the energy without an external field becomes

$$E_0^f = -2 \sum_k \omega_k. \quad (12)$$

The factor 2 arises from the spin summation; the factor -1 stems from the anticommutation relation obeyed by fermionic annihilation and creation operators. In the presence of the magnetic field H , we substitute $\partial_\mu \rightarrow D_\mu = \partial_\mu - igqA_\mu$, where q is the charge of the (anti)particle in units of the coupling g . Choosing the orientation of the H field along the z axis, we construct a vector potential as $A_\mu = (0, 0, x_1 H, 0)$. This choice for A_μ obeys $\partial_\mu A^\mu = 0$. In the following, we treat QED and QCD in parallel and define $e = qg$. We have to solve for the energy spectrum of $i\not{D}\psi(x) = 0$, which is basically a relativistic version of the Landau theory for the diamagnetic properties of an electron gas. The solution for the energy of a single Fourier mode becomes

$$E_{n,k_3,s_3} = \sqrt{k_3^2 + 2eH(n + 1/2 + s_3)}. \quad (13)$$

In addition, the x_1 space variable is shifted by $-k_2/(eH)$. Note that the energy depends only on two quantum numbers. The third is ‘‘hidden’’ in the mentioned x_1 shift. Here $s_3 = \pm \frac{1}{2}$, the z component of the spin. The Hs_3 term clearly shows the coupling of the spin to the external field, and hence, if the spin of the fermion is antiparallel to the H field, the energy is lowered. For QCD, there is also an implicit sum over the color charges q hidden in $e = qg$. Finally, for a vector gauge boson the H -independent energy is the same as for a scalar field, except that there is an additional factor of 2 counting the transverse spin degrees of freedom:

$$E_0^g = 2 \sum_k \omega_k. \quad (14)$$

The sum over color degrees of freedom yields an additional multiplicative factor of $N_c^2 - 1$. In the presence of the magnetic field, we separate the field A_μ into the classical background part A_μ^b and the fluctuating quantum part A_μ^q . The equations of motion become $D_\mu G^{\mu\nu} = 0$, where $G^{\mu\nu}$ is the gluon field strength tensor. With a suitable choice of back-

ground gauge, the energy for the two physical degrees of freedom of A_μ can be written as

$$E_{n,k_3,s_3} = \sqrt{k_3^2 + 2eH(n+1/2+s_3)}, \quad (15)$$

the same as in the fermionic case, but now with $s_3 = \pm 1$. Again, summation over the color charges is implicitly assumed.

We want to extract the leading $\log(H)$ contribution to the energy shift induced by the external field. With the total spin s of the particle considered and $i=f,g$:

$$\Delta E^i = (-1)^{2s} \left(\sum_{n,k_2,k_3,s_3} E_{n,k_3,s_3}^i - \sum_{k_1,k_2,k_3,s_3} \omega_k \right), \quad (16)$$

where

$$E_{n,k_3,s_3}^i = \sqrt{k_3^2 + 2eH(n+1/2+s_3)}. \quad (17)$$

Introducing a quantization volume $V=L^3$, we replace the sum over k_2 and k_3 by an integral weighted with the density of states. Taking into account that the x_1 variable was shifted, k_2 is restricted to $0 \leq k_2 \leq LeH$. Then,

$$\sum_{k_2,k_3} \rightarrow \frac{L}{2\pi} \int dk_3 \frac{L}{2\pi} (eH \cdot L) = \frac{V}{4\pi^2} (eH) \int dk_3. \quad (18)$$

To regularize the divergence, we will use a uv cutoff Λ such that $0 \leq n \leq \Lambda^2/2eH = n_\Lambda$ and $k_3^2 \leq \Lambda^2$. The first idea would be to replace the sum over n by an integral. However, if we perform the shift $n' = 2eHn$, we find that the integral would be independent of H to leading order. That is, we would have recovered the vacuum result, in the absence of the field H . So what we need is the *correction* to the replacement of a sum with an integral. Such a correction term suitable for our case here is provided by a specific Euler sum rule

$$\sum_{n=n_1}^{n_2} f(n+1/2) = \int_{n_1}^{n_2} f(x) dx - \frac{1}{24} f'(x) \Big|_{n_1}^{n_2}.$$

We may now redefine the energy shift as

$$\Delta E^i = (-1)^{2s} \sum_{s_3} \left\{ \sum_{n=0}^{n_\Lambda} f(n+1/2+s_3) - \int_0^{n_\Lambda} dn f(n+1/2+s_3) \right\},$$

where

$$f(x) = \frac{V}{2\pi^2} (eH) \int_0^\Lambda dk_3 \sqrt{k_3^2 + 2eHx}.$$

Since we are not interested in the soft modes of the order of eH (the leading log behavior is dominated by the uv behavior of the theory), we split the sum into two pieces ($N \ll n_\Lambda$)

$$\sum_{n=0}^{n_\Lambda} = \sum_{n=0}^N + \sum_{n=N}^{n_\Lambda}.$$

Let us treat s_3 formally as a continuous variable. Taylor expanding in s_3 (since $n \geq N \geq s_3$), we are left with

$$\Delta E^i = (-1)^{2s} \sum_{s_3} \sum_{n=N}^{n_\Lambda} \left(f(n+1/2) - s_3 f'(n+1/2) + \frac{s_3^2}{2} f''(n+1/2) + \dots \right) + \Phi(eH, N). \quad (19)$$

Now $\Phi(eH, N)$, which represents the contributions from soft modes only, does not depend on Λ . It is thus proportional to $(eH)^2$ for dimensional reasons, a small nonleading log contribution, and may be safely neglected. The linear term in s_3 vanishes upon summation, and resubstituting $e = gq$, we find

$$\Delta E^i = -\frac{1}{2} V (gH)^2 \left[\frac{q^2 (-1)^{2s}}{2\pi^2} \sum_{s_3} \left(\frac{s_3^2}{2} - \frac{1}{24} \right) \log \left(\frac{\Lambda^2}{2eH} \right) \right]. \quad (20)$$

The sum over a $SU(N_c)$ multiplet of the squared charges q^2 is $N_f/2$ for the fundamental representation (N_f quark flavors) and $N_c/2$ for the adjoint representation (the gluons). For QCD, the susceptibility becomes

$$4\pi\chi \rightarrow -g^2 \frac{11N_c - 2N_f}{48\pi^2} \log \left(\frac{2eH}{\Lambda^2} \right),$$

which reproduces the expression obtained by renormalization group calculations if we identify $2eH = K^2$. For QED, the sum over the charge(s) is simply 1, so we obtain

$$4\pi\chi \rightarrow + \frac{e^2}{12\pi^2} \log \left(\frac{2eH}{\Lambda^2} \right),$$

again in accordance with Eq. (2). Having outlined the calculation of [1,2], we now proceed to the main part of the paper and switch on temperature.

III. THE TEMPERATURE-DEPENDENT PART

At finite temperature T , the occupation number n_k^\pm appearing in Eq. (11) does not vanish any more for the thermal ground state; instead $n_k = [\exp(\beta\omega_k) - 1]^{-1} = n_{BE}$, the usual Bose-Einstein distribution function ($\beta = 1/T$). For fermions, $n_k = [\exp(\beta\omega_k) + 1]^{-1} = n_{FD}$, the Fermi-Dirac distribution function. Thus, when summing over the infinitely many degrees of freedom, we find for the total vacuum energy of a charged scalar field

$$E_0^s = \sum_k \omega_k (1 + 2n_{BE}(\omega_k)). \quad (21)$$

The result clearly separates into the divergent vacuum part already treated and a finite, T -dependent part. In the case of a finite magnetic field H , the higher energy modes (17) are occupied with their respective thermal probabilities, and we can write ($i=f,g$)

$$\Delta E^i = E^i - E_0^i, \quad (22)$$

$$E^i = \sum_{n,k_2,k_3,s_3} E_{n,k_3,s_3}^i \times \left[(-1)^{2s} + \frac{2}{\exp(\beta E_{n,k_2,s_3}^i) - (-1)^{2s}} \right], \quad (23)$$

$$E_0^i = \sum_{k_1,k_2,k_3,s_3} \omega_k \left[(-1)^{2s} + \frac{2}{\exp(\beta \omega_k) - (-1)^{2s}} \right]. \quad (24)$$

Again, we need to extract the leading thermal contribution to $\mu(k)$. However, at finite temperature, relation (3) does not hold any more. One could imagine starting with

$$\mu(k)\epsilon(k) = n(k)^2, \quad (25)$$

where n is the index of refraction. This quantity is related to the photon or gluon phase velocity by $v_p = 1/n$, and v_p could be extracted from the (full) dispersion relation of the corresponding gauge boson since $v_p = \omega_k/k$. However, Eq. (25) holds only for ‘‘on-shell’’ propagating gauge bosons. Since Lorentz invariance is formally broken by the presence of the heat bath, μ and ϵ become functions of k^0 and k , and Eq. (25) reads, more explicitly, $\mu(\omega_k, k)\epsilon(\omega_k, k) = n(k)^2$. Here, we consider an off-shell external field, so a relation between $\mu(0, k)$ and $\epsilon(0, k)$ is required.

The total energy density of the system can be written as the sum of the field and the induced medium energy density:

$$\mathcal{E}_{\text{tot}} = \frac{1}{2} H^2 + \frac{1}{V} \sum_i \Delta E^i. \quad (26)$$

Introducing an effective field H_{eff} , we rewrite \mathcal{E} as

$$\mathcal{E}_{\text{tot}} = \frac{1}{2} H_{\text{eff}}^2 = \frac{1}{2} \frac{(eH)^2}{e_{\text{eff}}^2}.$$

In the last step we made use of the fact that eH has to be renormalization group invariant, so $eH = e_{\text{eff}} H_{\text{eff}}$. The effective coupling constant is now defined by [2,3]

$$\frac{1}{e_{\text{eff}}^2} \equiv 2 \frac{\partial \mathcal{E}_{\text{tot}}}{\partial (eH)^2} = \frac{1}{e^2} [1 - 4\pi\chi(2eH, T)],$$

using Eqs. (10) and (26). Replacing $2eH$ by k^2 , as at $T=0$, our master formula hence reads

$$\alpha_{\text{eff}}(k, T) = \frac{\alpha}{\epsilon(k, T)} = \frac{\alpha}{1 - 4\pi\chi(k, T)}. \quad (27)$$

The thermal piece of Eq. (23) can be compactly rewritten as

$$E^{th}(T, b, s, s_3) = VT^4 \left(\frac{b}{2\pi^2} \sum_{s_3} \sum_{n=0}^{\infty} \times \int_0^{\infty} dx \frac{\sqrt{x^2 + b(n+1/2+s_3)}}{\exp(\sqrt{x^2 + b(n+1/2+s_3)} - (-1)^{2s})} \right), \quad (28)$$

where x is dimensionless and $b = 2eH/T^2$ is a measure for the ratio of quantum and thermal effects. We consider the high temperature limit $b \ll 1$ in the rest of the paper.

A. A first (incomplete) approximation

The sum appearing in expression (28) obviously cannot be evaluated exactly. It is instructive to work out the first intuitive approximation to the sum although we will show in the next section that it is too crude.

Consider the fermionic part. Note that the factor $b(n+1/2+s_3)$ plays the role of a mass term in the integral in Eq. (28), so the contribution of the terms in the sum becomes exponentially suppressed as n increases. In contrast to the $T=0$ case we are therefore interested in the behavior of the sum at *small* n where the s_3 spin component is not negligible. Thus we cannot apply a Taylor expansion in s_3 , as done in Eq. (19), and need an exact summation over s_3 . Isolating then the lowest lying Landau mode ($n=0, s_3 = -1/2$) and combining the remaining expressions into a single sum, we find

$$\frac{E^f}{VT^4} \equiv \tilde{E}^f = \frac{b}{24} + \frac{b}{\pi^2} \sum_{n=0}^{\infty} \int_0^{\infty} dx \frac{\sqrt{x^2 + b(n+1)}}{\exp(\sqrt{x^2 + b(n+1)} + 1)}. \quad (29)$$

Since $b \ll 1$, the terms in the sum vary slowly with n , so we can again try to trade the sum for an integral over n :

$$\tilde{E}^f = \frac{b}{24} + \frac{2}{\pi^2} \int_0^{\infty} dr \frac{r^2 \sqrt{r^2 + b}}{\exp(\sqrt{r^2 + b}) + 1}. \quad (30)$$

Neglecting terms of order b in the integral, we obtain as a first approximation

$$E_{(0)}^f = -\frac{1}{2} VH^2 \left[-\frac{(m_D^e)^2}{2eH} \right] + \frac{7}{4} \frac{\pi^2}{15} VT^4 \quad (31)$$

for QED with the hard thermal loop (HTL) Debye mass defined in Eq. (6). The second term is simply the energy $E_0^f(T)$

of a thermally excited, noninteracting massless fermion-antifermion pair, i.e., the thermal energy of the *unperturbed* vacuum that has to be subtracted [cf. Eq. (22)]. This means that we have recovered within our simple framework the perturbative one-loop HTL result from the lowest Landau level contribution to the energy of the magnetically perturbed thermal vacuum. The energy difference that enters in Eq. (10) already yields $4\pi\chi(H,T)$ as the expression in square brackets, and the effective coupling constant reads, following Eq. (27),

$$\alpha_{\text{eff}}(k,T) = \frac{\alpha}{\epsilon(k,T)} = \frac{\alpha}{1 + \frac{(m_D^e)^2}{k^2}},$$

as within HTL perturbation theory, Eq. (5).

For QCD with N_f flavors, we obtain

$$E_{(0)}^f = -\frac{1}{2}VH^2 \left[-\frac{m_{D,f}^2}{2eH} \right] + N_f \frac{7\pi^2}{60} VT^4 \quad (32)$$

with the fermionic part of the squared QCD Debye mass (7), $m_{D,f}^2 = N_f/6g^2T^2$. For the total evaluation of the QCD susceptibility, we need to add the contribution from the gauge bosons. At zero temperature, contributions from ‘‘unphysical’’ gluon states in the calculation of the energy spectrum, Eq. (15), are exactly cancelled by Fadeev-Popov ghost contributions within the background gauge condition used here. Since we only consider excitations of energy levels that were evaluated at $T=0$, no ambiguity arises and we still work only with physical gluon degrees of freedom with two polarization states. We proceed in close analogy to the fermionic case: first, we sum over $s_3 = \pm 1$. A subtlety arises since the combination $n=0$ and $s_3 = -1$ in Eq. (28) gives a negative value under the square root for small x . This ‘‘tachyonic’’ mode is related to a possible instability of the vacuum [10]. Its effect on the magnetic field over large distances, however, is negligible; the sum over n for $s_3 = -1$ therefore starts only at $n=1$. Isolating again the lowest lying physical Landau level ($n=1, s_3 = -1$) contribution to the sum, we are left with

$$\begin{aligned} \frac{E^g}{VT^4} \equiv \tilde{E}^g &= \frac{b}{2\pi^2} \int_0^\infty dx \frac{\sqrt{x^2 + b/2}}{\exp(\sqrt{x^2 + b/2}) - 1} \\ &+ \frac{b}{\pi^2} \sum_{n=0}^\infty \int_0^\infty dx \frac{\sqrt{x^2 + b(n+3/2)}}{\exp(\sqrt{x^2 + b(n+3/2)}) - 1}. \end{aligned} \quad (33)$$

Replacing the sum by an integration, setting $b=0$ in the integrals, and summing over color, the result becomes

$$E_{(0)}^g = -\frac{1}{2}VH^2 \left[-\frac{m_{D,g}^2}{2eH} \right] + 2(N_c^2 - 1) \frac{\pi^2}{15} VT^4. \quad (34)$$

Again, the last term is the thermal energy $E_0^g(T)$ of the unperturbed $SU(N_c)$ gluon vacuum. The expression in square brackets exactly corresponds to the gluonic part of the squared QCD Debye mass, $m_{D,g}^2 = N_c/3g^2T^2$. Putting all the pieces together, the effective coupling becomes

$$\alpha_{s,\text{eff}}(k,T) = \frac{\alpha_s}{1 + \frac{(m_D^c)^2}{k^2}},$$

very similar to the QED case. In our model, the hard thermal loop (chromo)electric Debye masses therefore appear as the lowest Landau level contribution to the energy difference that arises when one probes the thermal vacuum by a (chromo)magnetic field. It is worth noting that, in this approximation, the alignment of an external field always *increases* the thermal energy of the vacuum, regardless of the non-Abelian structure of the theory. Therefore $\chi(k,T)$ is always negative and we conclude, using Eq. (27), that the static potential would become screened by both fermions and gauge bosons.

B. A better approximation

However, additional contributions to Eqs. (31), (32), and (34) of the *same order in e^2* arise from two sources. First, the expansion of the integrals (30) and (33) in b is similar to the high temperature expansion of loop integrals with massive particles in the small quantity m_0/T . The Appendix contains the relevant formula. Second, the correction to the replacement of the sum by an integral yields terms to order b and b^2 that are provided by the Euler-MacLaurin formula

$$\begin{aligned} \sum_{n=0}^N f(n) &= \int_0^N f(x) dx + \frac{1}{2}[f(N) + f(0)] \\ &+ \frac{1}{12}[f'(N) - f'(0)] + \dots, \end{aligned} \quad (35)$$

where the dots denote terms with higher derivatives in $f(n)$. For our purposes, Eq. (35), taking $N \rightarrow \infty$, is sufficient, as long as $f(x) \in \mathcal{C}^2$ for $x \in [0, N]$. When calculating the thermal contribution to the vacuum energy, we include these correction terms to the integral in the following and expand all integrals in the small parameter b , using the relations presented in the Appendix. The summation of all terms to order e^2 then alters the results in Eqs. (32) and (34) *qualitatively*.

C. Results for QED

For fermions, we start with Eq. (29). Defining $\delta^2 = b$, we obtain to order δ^4

$$\begin{aligned} \tilde{E}_{(1)}^f = & \frac{\delta^2}{24} + \frac{2}{\pi^2} \int_0^\infty dr \frac{r^2 \sqrt{r^2 + \delta^2}}{\exp(\sqrt{r^2 + \delta^2}) + 1} + \frac{1}{2} \left[\frac{\delta^2}{\pi^2} \int_0^\infty dr \frac{\sqrt{r^2 + \delta^2}}{\exp(\sqrt{r^2 + \delta^2}) + 1} \right] \\ & + \frac{1}{12} \left[-\frac{\delta^4}{2\pi^2} \int_0^\infty dx \left\{ \frac{1}{\sqrt{x^2 + \delta^2}} - \frac{1}{1 + \exp(-\sqrt{x^2 + \delta^2})} \right\} \frac{1}{\exp(\sqrt{x^2 + \delta^2}) + 1} \right]. \end{aligned} \quad (36)$$

Using the functions $f_i(y)$ and $g^+(y)$ defined in the Appendix, we rewrite

$$\begin{aligned} \tilde{E}_{(1)}^f = & \frac{\delta^2}{24} + \frac{2}{\pi^2} f_5(\delta) + \frac{5\delta^2}{2\pi^2} f_3(\delta) + \frac{\delta^4}{2\pi^2} f_1(\delta) \\ & - \frac{\delta^4}{24\pi^2} g^+(\delta). \end{aligned} \quad (37)$$

Expanding in δ and keeping all terms up to $\mathcal{O}(\delta^4)$, surprisingly all terms of order δ^2 cancel, and we are left with

$$\tilde{E}_{(1)}^f = \frac{7\pi^2}{60} + \frac{\delta^4}{96\pi^2} \log(\mathcal{A}_f \delta^2) \quad (38)$$

with $\gamma = 0.5772\dots$ and the constant $\mathcal{A}_f = \exp(2\gamma - 1)/\pi^2 \approx 0.12$. The first term is the well-known thermal part of the vacuum energy in the absence of the field H . Since $\delta^2 \ll 1$, the alignment of a magnetic field hence *decreases* the energy of the vacuum at finite temperature, in contrast to the result of the previous section. The susceptibility in QED therefore becomes

$$4\pi\chi(H, T) = -\frac{\alpha}{3\pi} \log\left(\mathcal{A}_f \frac{2eH}{T^2}\right). \quad (39)$$

Note that the prefactor of the logarithm is the same as in the zero-temperature case. Including this pure quantum correction for the QED running coupling, Eq. (2), all momentum dependence drops out, and we finally obtain with Eq. (27)

$$\alpha_{\text{eff}}(k, T) = \frac{\alpha}{1 - \frac{\alpha}{3\pi} \log\left(\frac{T^2}{\Lambda_T^2}\right)}$$

with

$$\Lambda_T = m_e \frac{e^{\gamma + 7/6}}{\pi} \approx \frac{\Lambda}{3}, \quad (40)$$

which is valid to order e^2 and for momenta

$$m_e \ll k \ll T. \quad (41)$$

Using the zero-temperature coupling $\alpha_{\text{eff}}(k)$ from Eq. (2), the effective coupling can, to this order, be rewritten as

$$\alpha_{\text{eff}}(k, T) = \alpha_{\text{eff}}(T) = \alpha_{\text{eff}}(\langle k \rangle \approx 3T). \quad (42)$$

So the common practice used in perturbation theory of simply taking the running of the coupling at zero temperature and setting as the scale the thermally averaged momentum scale $\langle k \rangle \approx 3T$ does indeed find support from our calculation for QED.

D. Results for QCD

For QCD, the fermionic contribution takes a form similar to the QED result,

$$4\pi\chi(H, T) = -\frac{g^2 N_f}{24\pi^2} \log\left(\mathcal{A}_f \frac{2eH}{T^2}\right). \quad (43)$$

Note that the prefactor of the logarithm is again the same as at zero temperature. The calculation of the gluonic part of χ runs along the same lines. Starting with Eq. (33) and setting $\delta^2 = b/2$, we obtain by use of the functions $h_i(y)$ and $g^-(y)$

$$\begin{aligned} \tilde{E}^g = & \frac{2}{\pi^2} h_5(\sqrt{3}\delta) + \frac{\delta^2}{\pi^2} [7h_3(\sqrt{3}\delta) + h_3(\delta)] \\ & + \frac{\delta^4}{\pi^2} \left[h_1(\delta) + 3h_1(\sqrt{3}\delta) - \frac{1}{6} g^-(\sqrt{3}\delta) \right]. \end{aligned} \quad (44)$$

Expanding, we find that all terms of order δ^3 cancel and the result becomes

$$\tilde{E}^g = \frac{2\pi^2}{15} - \frac{b}{12} - \frac{5b^2}{128\pi^2} \log(\mathcal{A}_g b), \quad (45)$$

with the constant $\mathcal{A}_g = \exp[2\gamma - 13/10 + (11/5)\log 3]/(32\pi^2) \approx 0.03$. As for the fermionic part, the alignment of a chromomagnetic field hence always *lowers* the energy of the gluonic vacuum; however, this time the difference goes linearly in b and not only logarithmically. Finally summing over color, the gluonic susceptibility reads

$$4\pi\chi(H, T) = \frac{N_c}{3} \frac{g^2 T^2}{2eH} + 5N_c \frac{g^2}{48\pi^2} \log\left(\mathcal{A}_g \frac{2eH}{T^2}\right). \quad (46)$$

In this expression the sign is reversed as compared to Eq. (43), and $4\pi\chi$ is always positive. The total thermal result for QCD, excluding the $T=0$ contribution, becomes

$$\alpha_{s,\text{eff}}(k,T) = \frac{\alpha_s}{1 - \alpha_s \frac{4\pi N_c}{3} \frac{T^2}{k^2} - \frac{\alpha_s}{12\pi} \left[5N_c \log\left(\frac{\mathcal{A}_g k^2}{T^2}\right) - 2N_f \log\left(\frac{\mathcal{A}_f k^2}{T^2}\right) \right]}. \quad (47)$$

Since the gluon contribution dominates by far over the logarithmic fermionic, there is *antiscreening* at high temperature and long distances. This result is in contrast to expectation and lattice results on the interquark potential [4]. Extrapolating Eq. (47) beyond the kinematical region $g \ll k/T \ll 1$ where our approximations are valid, a Landau pole appears in the infrared region $k/T \approx g$. Figure 1 shows the ratio $\alpha_{s,\text{eff}}/\alpha_s$ as a function of k/T for a weak coupling $g=0.1$, compared to the usual HTL result. A similar behavior is also found in more sophisticated renormalization group analyses of the running coupling at finite temperature (see, e.g., [3,11–13]). We note that our results compare quite well with the numerical solutions obtained in Ref. [12]. Since Eq. (47) depends only on the dimensionless quantity k/T , taking the limit $T \rightarrow \infty$ at large k is in a sense equivalent to probing the infrared region $k \rightarrow 0$ at smaller T , indicating that nonperturbative physics plays an important role even at high T . The necessity for a soft magnetic mass $\sim g^2 T$ as an infrared regulator in loop calculations [see Eq. (9)] supports this line of reasoning.

IV. CONCLUSIONS

We presented an alternative way to calculate the screening of the static potential between two charges in (non-)Abelian gauge theories at finite temperature by looking at the magnetic properties of the vacuum. Instead of a loop expansion, we calculated the energy shift of the vacuum at finite temperature to order e^2 after applying an external (chromo)magnetic field H as a probe. Magnetic permeability and dielectric permittivity were connected by a renormalization group argument.

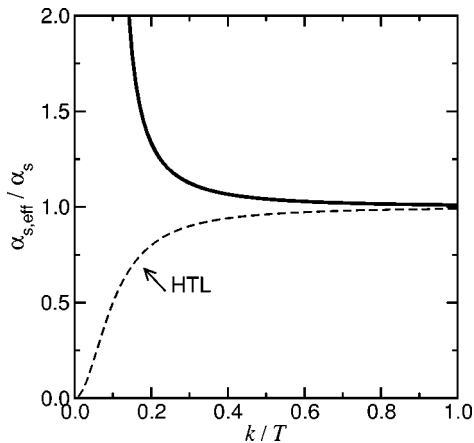


FIG. 1. The ratio $\alpha_{s,\text{eff}}/\alpha_s$ of Eq. (47) as a function of k/T for a coupling $g=0.1$ and $N_f=0$ (solid line). For comparison, the corresponding ratio in the HTL calculation, using the Debye mass of Eq. (7), is also plotted (dashed line).

Using a high temperature expansion $H/T^2 \ll 1$, the gluon and photon Debye masses appearing in the HTL calculation were recovered in a first, though incomplete approximation, originating from the lowest lying Landau level contribution to the thermal energy. Taking into account all contributions to order e^2 in QED, the final expression in the kinematic region (41) shows a logarithmic, momentum-independent running of α with temperature, as expected from simply inserting the average thermal momentum $\langle k \rangle \approx 3T$ in the zero-temperature running coupling.

In QCD, we found indications for a Landau pole at small k/T that arises, as in more sophisticated thermal renormalization group calculations, from the sign of the gluon contribution, despite well controlled approximations and a completely different approach as compared to conventional perturbation theory. Our calculation may serve as yet another indication that an expansion in a presumably small coupling g at high temperatures ceases to yield sensible results for some quantities, and that this failure is not specific to a Feynman graph expansion. Truly nonperturbative input that is probably linked to the understanding of confinement is then called for.

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APPENDIX

Here we present the formulas used to evaluate the δ -dependent integrals appearing in Eqs. (44) and (37). Fermionic integrals of the form

$$f_n(y) = \int_0^\infty dx \frac{x^{n-1}}{\sqrt{x^2+y^2}} \frac{1}{\exp(\sqrt{x^2+y^2})+1}$$

can be expanded for small y (note our slightly different convention compared to [9]). In particular,

$$f_1(y) = -\frac{1}{2} \left[\log\left(\frac{y}{\pi}\right) + \gamma \right] + \dots, \quad (A1)$$

$$f_3(y) = \frac{\pi^2}{12} + \frac{y^2}{4} \left[\log\left(\frac{y}{\pi}\right) + \gamma - \frac{1}{2} \right] + \dots, \quad (A2)$$

$$f_5(y) = \frac{7\pi^4}{120} - \frac{\pi^2}{8} y^2 - \frac{3}{16} y^4 \left[\log\left(\frac{y}{\pi}\right) + \gamma - \frac{3}{4} \right] + \dots \quad (A3)$$

where $\gamma=0.5772\dots$ is the Euler-Mascheroni constant. For bosons,

$$h_n(y) = \int_0^\infty dx \frac{x^{n-1}}{\sqrt{x^2+y^2}} \frac{1}{\exp(\sqrt{x^2+y^2}) - 1}.$$

The corresponding expansions read

$$h_1(y) = \frac{\pi}{2y} + \frac{1}{2} \left[\log\left(\frac{y}{4\pi}\right) + \gamma \right] + \dots, \quad (\text{A4})$$

$$h_3(y) = \frac{\pi^2}{6} - \frac{\pi}{2}y - \frac{y^2}{4} \left[\log\left(\frac{y}{4\pi}\right) + \gamma - \frac{1}{2} \right] + \dots, \quad (\text{A5})$$

$$h_5(y) = \frac{\pi^4}{15} - \frac{\pi^2}{4}y^2 + \frac{\pi}{2}y^3 + \frac{3}{16}y^4 \left[\log\left(\frac{y}{4\pi}\right) + \gamma - \frac{3}{4} \right] + \dots \quad (\text{A6})$$

For the evaluation of derivative terms, we need the leading $\log(y)$ behavior of integrals such as

$$\begin{aligned} g^\pm(y) &= g_1^\pm(y) + g_2^\pm(y) \\ &= \int_0^\infty dx \left(\frac{1}{\sqrt{x^2+y^2}} - \frac{1}{1 \pm \exp(-\sqrt{x^2+y^2})} \right) \\ &\quad \times \frac{1}{\exp(\sqrt{x^2+y^2}) \pm 1}. \end{aligned} \quad (\text{A7})$$

The expansion of the first term in parentheses, g_1^\pm , is known since $g_1^+ = f_1$ and $g_1^- = h_1$. For the evaluation of the second a trick is convenient. Introduce a parameter α to write

$$g_2^\pm(y; \alpha) = - \int_0^\infty dx \frac{\exp(\alpha\sqrt{x^2+y^2})}{[\exp(\alpha\sqrt{x^2+y^2}) \pm 1]^2}. \quad (\text{A8})$$

Obviously, $g_2^\pm(y; 1)$ is the sought quantity. Now $g_2^\pm(y; \alpha)$ can also be written as

$$\frac{\partial}{\partial \alpha} \left[\int_0^\infty dx \frac{1}{\sqrt{x^2+y^2}} \frac{1}{\exp(\alpha\sqrt{x^2+y^2}) \pm 1} \right] = \frac{d}{d\alpha} g_1^\pm(\alpha y). \quad (\text{A9})$$

Expanding Eq. (A8) for small y hence yields

$$g_2^-(y; \alpha) = - \frac{1}{\alpha^2} \frac{\pi}{2y} + \frac{1}{2\alpha} + \dots \quad \text{for bosons} \quad (\text{A10})$$

and

$$g_2^+(y; \alpha) = - \frac{1}{2\alpha} + \dots \quad \text{for fermions.} \quad (\text{A11})$$

Setting $\alpha = 1$ and putting the pieces together, the leading-log behavior of Eq. (A7) is

$$g^+(y) = - \frac{1}{2} \left[\log\left(\frac{y}{\pi}\right) + \gamma + 1 \right] + \dots \quad \text{for fermions} \quad (\text{A12})$$

and

$$g^-(y) = + \frac{1}{2} \left[\log\left(\frac{y}{4\pi}\right) + \gamma + 1 \right] + \dots \quad \text{for bosons.} \quad (\text{A13})$$

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