Nonperturbative effects from the resummation of perturbation theory

Taekoon Lee
Department of Physics, Korea Advanced Institute of Science and Technology, Daejon 305-701, Korea
(Received 1 April 2002; published 26 August 2002)

Using the general argument in Borel resummation of perturbation theory that links the divergent perturbation theory to the nonperturbative effect I argue that the nonperturbative effect associated with the perturbation theory should have a branch cut only along the positive real axis in the complex coupling plane. The component in the weak coupling expansion of the nonperturbative amplitude that gives rise to the branch cut can be calculated in principle from the perturbation theory combined with the exactly calculable properties of the nonperturbative effect. The realization of this mechanism is demonstrated in the double well potential and the two-dimensional $O(N)$ nonlinear sigma model. In these models the leading term in the weak coupling expansion of the nonperturbative effect can be obtained with a good accuracy from the first terms of perturbation theory. Applying this mechanism to the infrared renormalon induced nonperturbative effect in QCD, I suggest some of the QCD condensate effects can be calculated in principle from the perturbation theory.

DOI: 10.1103/PhysRevD.66.034027 PACS number(s): 12.38.Cy, 11.15.Bt, 11.15.Tk, 12.38.Aw

I. INTRODUCTION

The usual perturbation in the weak coupling constant in field theory is an asymptotic expansion. When the perturbation series is sign alternating it may be resummed, for example, in the manner of Borel resummation. However, when the series is not sign alternating, it usually implies the presence of a genuine nonperturbative effect, and the Borel resummation of the perturbation series alone is in principle not sufficient for an adequate description of the true amplitude.

The nonperturbative effect and the perturbation expansion are not totally independent; the former controls the large order behavior of the latter. Even with this relation, however, explicit calculations show that the nonperturbative effect cannot be calculated from the perturbation series even when the latter is known to all orders.

The purpose of this paper is to argue, by taking a closer look at the general but heuristic argument that relates the nonperturbative effect with the perturbation theory, that some parts of the nonperturbative effect, which usually include the leading piece in weak coupling expansion, can be calculated in principle from the Borel resummation of perturbation theory along with the exactly calculable properties of the nonperturbative effect.

II. CALCULABLE COMPONENT IN NONPERTURBATIVE EFFECT

A general but heuristic argument that relates the nonperturbative effect with perturbation theory goes as follows.[1] Let $A(\alpha)$ be an amplitude with perturbation expansion in the coupling constant α:

$$A(\alpha)=\sum_0^\infty a_n\alpha^{n+1} \tag{1}$$

and assume that $A(\alpha)$ be real for $\alpha>0$. I shall further assume that a_n at large orders is nonalternating in sign. In general, a_n diverges factorially due to renormalons or instanton–anti-instanton pairs, and in principle Eq. (1) is meaningless unless some kind of resummation is performed on the divergent series. To do a resummation, consider a new series

$$B(\alpha)=\sum_0^\infty a_n(-1)^{n+1}\alpha^{n+1} \tag{2}$$

which is obtained from Eq. (1) by flipping the sign of the coupling constant. Since for $\alpha>0$ this series is alternating in sign, it can now be Borel resummed, yielding a resummed amplitude $B_{\rm PT}(\alpha)$,

$$B_{\rm PT}(\alpha)\equiv\int_0^\infty db\, e^{-(b/\alpha)}\tilde{B}(b) \tag{3}$$

where

$$\tilde{B}(b)=\sum_0^\infty \frac{(-1)^{n+1}a_n}{n!}b^n. \tag{4}$$

One would expect that $A(\alpha)$ could be obtained from the Borel resummed $B_{\rm PT}(\alpha)$ by analytic continuation from the positive real axis to the negative real axis in the complex α plane. The problem is, however, $B_{\rm PT}(\alpha)$ is expected to have a branch cut along the negative axis, and consequently $B_{\rm PT}(-\alpha\pm i\epsilon)$ will have an imaginary part. Therefore analytic continuation alone of $B_{\rm PT}(\alpha)$ cannot reproduce $A(\alpha)$ which is by definition real for $\alpha>0$. The expected resolution of this problem is that the true amplitude $A(\alpha)$ has a nonperturbative amplitude $A_{\rm NP}(\alpha)$ in addition to the $B_{\rm PT}(-\alpha)$, so when they are added together the imaginary parts from each amplitude cancel each other, rendering the total amplitude to be real. That is,

$$A(\alpha)=A_{\rm PT}(\alpha)+A_{\rm NP}(\alpha) \tag{5}$$

[1] A good introduction can be found in [1,2].

with

$$\mathrm{Im}[A_{\mathrm{PT}}(\alpha\pm \mathrm{i}\epsilon)]+\mathrm{Im}[A_{\mathrm{NP}}(\alpha\pm \mathrm{i}\epsilon)]=0, \tag{6}$$

where $A_{\mathrm{PT}}(\alpha)\equiv B_{\mathrm{PT}}(-\alpha)$, which may be called the perturbative amplitude, has now a branch cut along the positive real axis in the α plane. Performing the analytic continuation in Eq. (3) I obtain

$$A_{\mathrm{PT}}(\alpha\pm \mathrm{i}\epsilon)=\int_{0\pm \mathrm{i}\epsilon}^{\infty\pm \mathrm{i}\epsilon} db e^{-(b/\alpha)}\tilde{A}(b) \tag{7}$$

where[2]

$$\begin{aligned}\tilde{A}(b)&=-\tilde{B}(-b)\\ &=\sum_{0}^{\infty}\frac{a_n}{n!}b^n.\end{aligned} \tag{8}$$

Note that, as I make analytic continuation of $B_{\mathrm{PT}}(\alpha)$ from the positive real axis to the negative real axis in the α plane counterclockwise (clockwise) in the upper (lower) half plane, the integration contour also should rotate counterclockwise (clockwise), hence the $\pm$ sign in Eq. (7).

Using the dispersion relation on the resummed $A_{\mathrm{PT}}(\alpha)$ and Eq. (6), we have

$$\begin{aligned}A_{\mathrm{PT}}(\alpha)&=\frac{1}{\pi}\int_0^\infty\frac{\mathrm{Im}[A_{\mathrm{PT}}(\alpha'+\mathrm{i}\epsilon)]}{\alpha'-\alpha}d\alpha'\\ &=-\frac{1}{\pi}\sum_{n=0}^{\infty}\left[\int_0^\infty\frac{\mathrm{Im}[A_{\mathrm{NT}}(\alpha'+\mathrm{i}\epsilon)]}{\alpha'^{n+2}}\right]\alpha^{n+1}.\end{aligned} \tag{9}$$

Thus the perturbative coefficient in Eq. (1) can be written as

$$a_n=-\frac{1}{\pi}\int_0^\infty\frac{\mathrm{Im}[A_{\mathrm{NP}}(\alpha+\mathrm{i}\epsilon)]}{\alpha^{n+2}}d\alpha, \tag{10}$$

which makes the relation between the nonperturbative effect and the perturbation theory explicit. At large values of n the dominant contribution in Eq. (10) comes from the small α region, and so for large order behavior only the weak coupling limit of the nonperturbative effect is required.

Now with Eqs. (5) and (6) the amplitude $A(\alpha)$ for $\alpha>0$ can be written as the sum of the real parts of the perturbative and the nonperturbative terms:

$$A(\alpha)=\mathrm{Re}[A_{\mathrm{PT}}(\alpha\pm \mathrm{i}\epsilon)]+\mathrm{Re}[A_{\mathrm{NP}}(\alpha\pm \mathrm{i}\epsilon)]. \tag{11}$$

Equation (6) also shows that the imaginary part of the nonperturbative amplitude $A_{\mathrm{NP}}(\alpha)$ can be calculated in principle from the Borel resummation of the perturbation theory. The real part, however, is in general not calculable from the perturbation theory.

The argument hitherto is well known, perhaps except for Eq. (7) which allows me to relate the imaginary part from the analytic continuation in the α plane to that arising from the Borel integral. Now, my observation, which will play a crucial role throughout the paper, is that Eq. (6) suggests $A_{\mathrm{NP}}(\alpha)$ have a branch cut along the positive real axis in the complex coupling plane, in order to cancel the imaginary part coming from the branch cut in the Borel resummed perturbative amplitude. This rather straightforward observation can have an important consequence: it renders some part of the nonperturbative amplitude to be calculable from the perturbation theory.[3] In the weak coupling limit ($\alpha\to 0$) the approximate functional form of $A_{\mathrm{NP}}(\alpha)$ can be rather easily determined from other nonperturbative techniques such as renormalization group argument or instanton calculations, and the component, typically the leading one in weak coupling expansion, that could give rise to a branch cut along the positive real axis can be easily identified. I then further fix this component to a more specific form by *demanding that it give a branch cut only along the positive real axis*; I expect the nonperturbative effect should not have other branch cut, for instance, such as one along the negative real axis, since it would imply that the perturbation series (1) is not Borel-summable even for $\alpha<0$. The constraint on the functional form from this step turns out to be sufficient enough for me to relate the real part of the above-mentioned component to its imaginary part, consequently rendering the former to be calculable from the Borel resummation of the perturbation theory through Eq. (6).

In the next two sections I consider a few definite examples and show how this procedure can be realized in model calculations. In these examples I shall focus on the nonperturbative effect associated with the first singularity on the positive real axis in the Borel plane.

III. THE DOUBLE WELL POTENTIAL

The quantum mechanical double well potential with the action

$$S=\int\left[\frac{1}{2}\dot{q}^2-\frac{1}{2}q^2(1-\lambda q)^2\right]dt \tag{12}$$

has instanton solutions, and the nonperturbative effects due to the instantons give rise to singularities on the positive real axis on the Borel plane, causing same sign perturbation series.

Consider, for example, $E(\alpha)$ defined by

$$E(\alpha)=\frac{1}{2}[E_+(\alpha)+E_-(\alpha)-1], \tag{13}$$

[2]Throughout the paper whenever a Borel transform is defined through a perturbation series like Eq. (8) it is assumed that the value of the Borel transform at a point beyond the convergence disk is obtained by analytic continuation.

[3]The idea of rebuilding nonperturbative effects from perturbation theory is an old story. To the author's knowledge it was first speculated in [3], and an approach similar to mine was observed in [4].

where $\alpha \equiv \lambda^2$ and $E_-(\alpha)$, $E_+(\alpha)$ are, respectively, the energies of the ground and the first excited states. The Borel transform

$$\tilde{E}(b)=\sum_{n=0}^{\infty}\frac{a_n}{n!}b^n \tag{14}$$

of the perturbation series for $E(\alpha)$,

$$E(\alpha)=\sum_{n=0}^{\infty}a_n\alpha^{n+1}, \tag{15}$$

is expected to have multi-instanton caused singularities at $b=2nS_0$, where $n=1,2,3,\ldots,$ and $S_0=1/6$ is the one-instanton action.

The large order behavior of the perturbation (15) is controlled by the first singularity at $b=1/3$. The nonperturbative effect that causes this singularity is due to the contributions of instanton–anti-instanton pairs, and can be calculated from the potential of an instanton–anti-instanton pair, and reads [5]

$$E_{\mathrm{NP}}(\alpha)=\frac{1}{\pi\alpha}e^{-1/3\alpha}\left[\left(\ln\left(\frac{-2}{\alpha}\right)+\gamma_{\mathrm{E}}\right)\left(1-\frac{53}{6}\alpha\right)-\frac{23}{2}\alpha+O(\alpha^2\ln\alpha)\right], \tag{16}$$

where γ_{E} is the Euler constant.

Notice that $E_{\mathrm{NP}}(\alpha)$ has a branch cut along the positive real axis in the α plane, in agreement with our argument in the previous section. The minus sign in the argument of the logarithmic term, which causes the branch cut, arises from the required sign flip in the coupling constant to pick up the nonperturbative effect from the attractive potential of the instanton–anti-instanton pair [6].

The imaginary and the real parts of $E_{\mathrm{NP}}(\alpha)$ now read

$$\mathrm{Im}[E_{\mathrm{NP}}(\alpha\pm i\epsilon)]=\pm\frac{1}{\alpha}e^{-1/3\alpha}\left[1+\frac{53}{6}\alpha+O(\alpha^2)\right], \tag{17}$$

$$\mathrm{Re}[E_{\mathrm{NP}}(\alpha\pm i\epsilon)]=\frac{1}{\pi\alpha}e^{-1/3\alpha}[-\ln(\alpha)+\ln(2)+\gamma_{\mathrm{E}}+O(\alpha)]. \tag{18}$$

Note that the real part has terms, for example, the constant terms within the bracket in Eq. (18), that have nothing to do with the imaginary part. These terms represent genuine nonperturbative effect and cannot be calculated from Borel resummation of perturbation theory.

Since in this example the nonperturbative effect can be calculated in weak coupling expansion there is no real need to attempt to extract the nonperturbative effect from the perturbation theory. However, for the sake of argument, let us suppose that we knew only that the nonperturbative effect in weak coupling expansion was given in the form:

$$E_{\mathrm{NP}}(\alpha)\propto\frac{1}{\alpha}e^{-1/3\alpha}[\ln(\alpha)+\text{subleading terms}]. \tag{19}$$

In any event, inferring this form may not be so difficult since the preexponential factor $1/\alpha$ can be obtained by counting the number of (quasi)zero modes, in this case two, and the logarithmic term arises from the instanton–anti-instanton potential at large distance.

We can now improve the form of the nonperturbative effect (19) by demanding that it have a branch cut only along the positive real axis in the α plane. Because the branch cut can arise only from the logarithmic term we can immediately see that $E_{\mathrm{NP}}(\alpha)$ must assume the following form:

$$E_{\mathrm{NP}}(\alpha)=-\frac{C}{\pi\alpha}e^{-1/3\alpha}[\ln(-\alpha)+\text{subleading terms}] \tag{20}$$

with C an unknown real constant. Of course, a comparison with Eq. (16) shows that the true value of the constant must be $C=1$. I now show that the leading term, the logarithmic term, in the real part (18) can be calculated from the perturbation theory starting from the ansatz (20). The imaginary and the real parts from this expression (20) are then

$$\mathrm{Im}[E_{\mathrm{NP}}(\alpha\pm i\epsilon)]=\pm\frac{C}{\alpha}e^{-1/3\alpha}[1+\text{subleading terms}], \tag{21}$$

$$\mathrm{Re}[E_{\mathrm{NP}}(\alpha\pm i\epsilon)]=-\frac{C}{\pi\alpha}e^{-1/3\alpha}[\ln(\alpha)+\text{subleading terms}]. \tag{22}$$

To determine the leading term in the real part we now only need to fix the constant C. This constant becomes the residue, up to a calculable normalization, of the first singularity in the Borel plane, and can be calculated in perturbation theory [7,8]. In fact, for the Borel resummation

$$E_{\mathrm{PT}}(\alpha\pm i\epsilon)=\int_{0\pm i\epsilon}^{\infty\pm i\epsilon}e^{-(b/\alpha)}\tilde{E}(b)db \tag{23}$$

to have imaginary parts that can cancel the imaginary terms in Eq. (21) the Borel transform $\tilde{E}(b)$ must have a singularity at $b=1/3$ of the form

$$\tilde{E}(b)=-\frac{9C}{\pi(1-3b)^2}[1+O(1-3b)]. \tag{24}$$

To determine the residue we now consider a function $R(b)$ defined by

$$R(b)=\tilde{E}(b)(1-3b)^2. \tag{25}$$

The difference between $R(b)$ and $\tilde{E}(b)$ is that the former has a much softer singularity. Although it may appear that $R(b)$ is regular at $b=1/3$, it is easy to see that this is not the case. In fact, for the imaginary parts from the Borel integral Eq. (23) to cancel the imaginary parts in (17), $\tilde{E}(b)$ should have an expansion around the singularity

TABLE I. Sum of the first $N+1$ terms of the perturbation series for the normalized residue ($C_\infty=\tilde{C}_\infty=1$).

N	0	1	2	3	4	5
$C_{\rm N}$	0.349	0.175	0.339	0.487	0.631	0.759
$\tilde{C}_{\rm N}$	0.349	0.109	0.502	0.650	0.862	0.994

$$\tilde{E}(b)=-\frac{9}{\pi(1-3b)^2}\left[1-\frac{53}{18}(1-3b)+O[(1-3b)^2\ln(1-3b)]\right]. \tag{26}$$

$R(b)$ is, therefore, logarithmically [multiplied by $(1-3b)^2$] singular at $b=1/3$, but bounded.

In terms of $R(b)$ the constant is given by

$$C=-\frac{\pi}{9}R\left(\frac{1}{3}\right). \tag{27}$$

The essential point for the perturbative calculation of the residue is that the right-hand side of Eq. (27) can be written as a convergent series. The perturbation series for $R(b)$,

$$R(b)=\sum_{n=0}^{\infty} r_n b^n, \tag{28}$$

is convergent on the disk $|b|\leqslant 1/3$ (note the boundary is included). Being bounded, though singular, at $b=1/3$, $R(b)$ can be evaluated in series at $b=1/3$.

We can now do some numerical checks to see how rapidly the series (28),when evaluated at $b=1/3$, converges to the known exact value. From the perturbative coefficients for $E(\alpha)$ given in [9] the coefficients r_n in Eq. (28) can be obtained. In Table I I give the first terms of $C_{\rm N}$ defined by

$$C_{\rm N}=-\frac{\pi}{9}\sum_{n=0}^{\rm N} r_n\left(\frac{1}{3}\right)^n. \tag{29}$$

Note that $C_\infty=1$. The numbers show that from C_1 to C_5 the series approaches the true value in a steady pattern.

Since the positions of the singularities for $\tilde{E}(b)$ are known I can improve the convergence using an "optimal" conformal mapping [10]

$$w=w(b). \tag{30}$$

An optimal mapping for our case is that $R[b(w)]$ become as smooth as possible within the convergence disk of the perturbation expansion in the w plane

$$R[b(w)]=\sum_{n=0}^{\infty}\tilde{r}_n w^n. \tag{31}$$

An obvious strategy for an optimal mapping is to push all other singularities in the Borel plane except for the first one far away from the origin. Here we consider a mapping

$$w=\frac{1-\sqrt{1-3b/2}}{1+\sqrt{1-3b/2}}. \tag{32}$$

This maps the first singularity to $w=w_0$, where

$$w_0=\frac{\sqrt{2}-1}{\sqrt{2}+1}\approx 0.171, \tag{33}$$

and all other singularities to the unit circle. Because the singularities (other than the first one) in the w plane are relatively farther away from the origin than in the b plane, we expect the series in the w plane to give better convergence. In fact, the first terms shown in Table I of $\tilde{C}_{\rm N}$, which is defined by

$$\tilde{C}_{\rm N}=-\frac{\pi}{9}\sum_{n=0}^{\rm N}\tilde{r}_n w_0^n, \tag{34}$$

show a definite improvement in convergence (note again $\tilde{C}_\infty=1$).

The result of this exercise shows that the leading term in the nonperturbative effect caused by the instanton–anti-instanton pairs on the ground state energy can be calculated accurately (99% accuracy) with only the first six terms of the perturbation series for the ground state energy.

IV. THE TWO-DIMENSIONAL $O(N)$ NONLINEAR SIGMA MODEL

This model is exactly solvable in $1/N$ expansion and mimics many interesting features of quantum chromodynamics (QCD). It has an asymptotic freedom, dimensional transmutation, and most interestingly for us the infrared (IR) and ultraviolet (UV) renormalons at the next leading order in $1/N$ expansion. Moreover, it was through the studies of this model [3,11–14] that the link between operator product expansion (OPE) and IR renormalons suggested by Parisi [15] has become more transparent. This model can also provide a nontrivial test of our proposed mechanism.

The two-dimensional $O(N)$ nonlinear sigma model is defined by the (Euclidean) action

$$S=\frac{1}{2}\int d^2x\sum_{a=1}^{N}\left[\partial_\mu\sigma^a(x)\partial_\mu\sigma^a(x)+\frac{\alpha(x)}{\sqrt{N}}\left(\sigma^a(x)\sigma^a(x)-\frac{N}{4\pi f}\right)\right], \tag{35}$$

where $\alpha(x)$ is an auxiliary field, and f is the coupling constant. At the leading order in $1/N$ the σ fields get dynamical mass

$$m^2=\mu^2 e^{-1/f(\mu)}, \tag{36}$$

where μ is the renormalization scale, through the vacuum condensate of the auxiliary field

$$\langle 0|\alpha(0)|0\rangle=-\sqrt{N}m^2. \tag{37}$$

Equation (36) also defines the renormalization group (RG) running of the coupling constant. The β function in the leading order in $1/N$ is therefore given by

$$\beta(f)=\mu^2\frac{df}{d\mu^2}=-f^2. \tag{38}$$

I shall now test my proposed mechanism with the truncated two-point function of the σ fields

$$\Gamma(p^2)=p^2+\Sigma(p^2), \tag{39}$$

which is known to all orders in OPE at order $1/N$ via the exact calculation of the self-energy $\Sigma(p^2)$ in [16]. $\Gamma(p^2)$ can be expanded in powers of m^2/p^2, which corresponds to an OPE, as

$$\Gamma(p^2)=p^2\left[C_0[f(p)]+C_1[f(p)]\frac{m^2}{p^2}+O\left(\frac{m^4}{p^4}\right)\right]. \tag{40}$$

I keep here only the first two terms because I am focused on the nonperturbative effect associated with the first IR renormalon. The first term contains the usual perturbation expansion, and the second term, which comes from the vacuum condensate of $\alpha(x)$, is the nonperturbative amplitude that gives rise to the first IR renormalon. The terms of higher powers in m^2/p^2 are associated with the higher renormalon singularities, and shall be ignored.

At the leading order in $1/N$, $C_0=C_1=1$. At order $1/N$ they have a rich structure and read [16]

$$C_0[f(p)\pm i\epsilon]=\frac{1}{N}\int_{0\pm i\epsilon}^{\infty\pm i\epsilon}db\left[e^{-b/f(p)}\left(\frac{1}{f(p)}F^{(0)}(b)+G^{(0)}(b)\right)-H^{(0)}(b)\right],$$

$$C_1[f(p)\pm i\epsilon]=-\frac{1}{N}\int_{0\pm i\epsilon}^{\infty\pm i\epsilon}db\left[e^{-b/f(p)}\left(\frac{1}{f(p)}F^{(1)}(b)+G^{(1)}(b)\right)-H^{(1)}(b)\right], \tag{41}$$

where

$$F^{(0)}(b)=1,$$

$$G^{(0)}(b)=\frac{1}{b}+\frac{1}{b-1}-\psi(1+b)-\psi(2-b)-2\gamma_E,$$

$$H^{(0)}(b)=\frac{1}{b}+B_1(b), \tag{42}$$

and

$$F^{(1)}(b)=b^2-1,$$

$$G^{(1)}(b)=-\frac{1}{b}+2b^2-4b+1+(1-b^2)\times[\psi(1+b)+\psi(2-b)+2\gamma_E],$$

$$H^{(1)}(b)=-\frac{1}{b}-\frac{1}{b-1}-\frac{1}{1+b}+B_0(b). \tag{43}$$

$B_0(b),B_1(b)$, whose exact forms are not important for us, are analytic functions.

Note the renormalon poles at $b=n$, where n is a nonzero integer, in $G^{(0)},H^{(1)}$. To avoid these poles the integration contour can be either on the upper or the lower half plane; for consistency, however, an identical contour should be taken for C_0 and C_1.

I shall identify the first term in Eq. (40) as the perturbative amplitude $\Gamma_{\rm PT}$ and the second term as the nonperturbative amplitude $\Gamma_{\rm NP}$. Performing the integration over b in Eq. (41) it is then easy to see that $\Gamma_{\rm NP}$ at order $1/N$ is given by

$$N\Gamma_{\rm NP}[f(p)\pm i\epsilon]=-\frac{m^2}{p^2}[\ln f(p)\mp i\pi+\text{real const}+O(f)] \tag{44}$$

$$=-e^{-1/f(p)}\{\ln[-f(p)\mp i\epsilon]+\text{real const}+O(f)\}, \tag{45}$$

where in the last step we have used Eq. (36). Thus to the leading order in weak coupling

$$N\Gamma_{\rm NP}(f)=-e^{-1/f}\ln(-f), \tag{46}$$

which has a branch cut along the positive real axis in the coupling plane, again in agreement with my proposed mechanism. The origin of the imaginary part in Eq. (44) lies with the ambiguity in obtaining the renormalized condensate, in this case the condensate of the auxiliary field $\alpha(x)$, from the dimensionally regularized condensate in $2+\epsilon$ dimension [3,11,12]. When the perturbative and the nonperturbative amplitudes are added together this imaginary part is canceled by the imaginary part in $\Gamma_{\rm PT}[f(p)\pm i\epsilon]$ coming from the pole at $b=1$ in $G^{(0)}(b)$.

From Eq. (41) the perturbation expansion for $\Gamma_{\rm PT}[f(p)]$ can be easily obtained,

$$\Gamma_{\rm PT}[f(p)]=\ln f(p)+\text{const}+\sum_{n=0}^{\infty}a_nf(p)^{n+1}, \tag{47}$$

with

$$a_0=-2,$$

$$a_n=n!\{[1+(-1)^n]\zeta(n+1)-2\}\quad(\text{for } n\geqslant 1), \tag{48}$$

where ζ denotes the Riemann ζ function. The logarithmic term in Eq. (47) arises from the anomalous dimension of the σ fields.

It is instructive to see now that the logarithmic dependence in Eq. (44) can be obtained without knowing the exact solution for the σ self-energy. To see this let us make the renormalization scale dependence of Γ explicit, which was hitherto implicitly suppressed.[4] $\Gamma[p^2,\mu^2,f(\mu)]$ can be expanded in OPE as

$$\Gamma[p^2,\mu^2,f(\mu)]=p^2\Bigg\{\bar{C}_0[\mu^2/p^2,f(\mu)] + \frac{1}{p^2}\bar{C}_1[\mu^2/p^2,f(\mu)]\langle 0|\alpha|0\rangle_{|\mu} + \text{higher dimension terms}\Bigg\}, \qquad (49)$$

where $\bar{C}_i$ are the Wilson coefficients, and the ignored terms involve operators of dimension four or higher. From the RG equations for the coefficient $C_1[\mu^2/p^2,f(\mu)]$ and the condensate $\langle 0|\alpha|0\rangle$ the second term in Eq. (49) can be easily written up to a momentum independent factor as

$$\bar{C}_1[1,f(p)]e^{-1/f(p)} \times \exp\left[-\int^{f(p)}[2\gamma_\sigma(f')+\gamma_\alpha(f')]/\beta(f')df'\right], \qquad (50)$$

where $\gamma_\sigma(f)$ and $\gamma_\alpha(f)$ denote the anomalous dimensions for the σ and α fields, respectively. Comparing this term with the second one in Eq. (40) we can identify

$$C_1[f(p)]=\bar{C}_1[1,f(p)]\exp\left[-\int^{f(p)}[2\gamma_\sigma(f') + \gamma_\alpha(f')]/\beta(f')df'\right]. \qquad (51)$$

Therefore the logarithmic term in Eq. (44) can arise from a logarithmic term at order $1/N$ in the Wilson coefficient $\bar{C}_1[1,f(p)]$, and the anomalous dimensions $\gamma_{\{\sigma,\alpha\}}$, which are nonvanishing at order $1/N$:

$$\gamma_{\{\sigma,\alpha\}}(f)\propto\frac{1}{N}[f+O(f^2)]. \qquad (52)$$

What is noteworthy here is that the contributions from these sources when added together conspire to absorb the imaginary parts in Eq. (44) into a cut-function. If, for example, the logarithmic term had a different coefficient than that given in Eq. (44), then after the absorption of the imaginary parts into a cut-function $[\ln(-f)]$ there would have remained a logarithmic piece $[\ln(f)]$ which has a cut along the negative real axis in the coupling plane. This would have failed my argument that the nonperturbative amplitude has a cut only along the positive real axis.

[4]This was allowable because the renormalization scale dependence in $\Gamma(p^2)$ can be factored out.

Now, once I know the form of the anomalous dimensions for the σ and α fields, and the logarithmic dependence in the Wilson coefficient at order $1/N$, the nonperturbative amplitude can be written with no help from the exact solution for the σ self-energy as

$$N\Gamma_{\rm NP}[f(p)]\propto e^{-1/f(p)}[\ln f(p)+\text{subleading terms}]. \qquad (53)$$

Then demanding $\Gamma_{\rm NP}(f)$ have a branch cut along the positive real axis in the coupling plane I can refine Eq. (53) to

$$N\Gamma_{\rm NP}[f(p)]=Ce^{-1/f(p)}\{\ln[-f(p)]+\text{subleading terms}\}, \qquad (54)$$

with C an undetermined real constant, from which I can obtain a relation valid in leading order in weak coupling:

$$\begin{aligned}\mathrm{Re}[\Gamma_{\rm NP}(f)] &= \mp\frac{1}{\pi}\mathrm{Im}[\Gamma_{\rm NP}(f\pm i\epsilon)]\ln f(p)\\ &= \pm\frac{1}{\pi}\mathrm{Im}[\Gamma_{\rm PT}(f\pm i\epsilon)]\ln f(p).\end{aligned} \qquad (55)$$

Because the imaginary part of the perturbative amplitude can be calculated from Borel resummation this relation renders the real part of the nonperturbative amplitude to be calculable from the perturbation theory.

The leading term in $\Gamma_{\rm NP}(f)$ depends only on the constant C. I can calculate this constant from perturbation theory using the method used in the previous section for a similar purpose. First, I note that for the $\Gamma_{\rm PT}(f)$ to cancel the imaginary part coming from Eq. (54) at positive $f(p)$ the Borel transform of $\Gamma_{\rm PT}(f)$ should have the following singularity at $b=1$ (the first IR renormalon):

$$\tilde{\Gamma}_{\rm PT}(b)=\frac{C}{1-b}[1+O(1-b)]. \qquad (56)$$

This shows that the constant C becomes the residue of the renormalon singularity and can be written as [7,8]

$$C=R(1) \qquad (57)$$

with

$$R(b)\equiv(1-b)\tilde{\Gamma}_{\rm PT}(b). \qquad (58)$$

Because of the UV renormalon at $b=-1$ the residue C cannot be directly evaluated by the perturbation expansion of $R(b)$ around the origin. To map away other renormalons than the first IR renormalon I introduce a conformal mapping

$$w=\frac{\sqrt{1+b}-\sqrt{1-b/2}}{\sqrt{1+b}+\sqrt{1-b/2}} \qquad (59)$$

which sends the first IR renormalon to $w=1/3$ and all other renormalons to the unit circle. Because in the w plane the first IR renormalon is the closest singularity to the origin, C

TABLE II. Sum of the first $N+1$ terms of the perturbation series for the renormalon residue ($C_\infty=-1$).

N	0	1	2	3	4	5
$-C_N$	2.000	2.000	0.100	0.945	0.916	1.052

can now be evaluated by plugging $w=1/3$ into the following series expansion of $R[b(w)]$:[5]

$$R[b(w)]=\sum_{n=0}^{\infty} r_n w^n. \tag{60}$$

The first terms of C_N, which are defined by

$$C_N=\sum_{n=0}^{N} r_n\left(\frac{1}{3}\right)^n, \tag{61}$$

were calculated using the perturbative coefficients (48). The numbers in Table II show that the residue can be determined with good accuracy from the first terms of the perturbation theory.

V. THE QCD CONDENSATE EFFECTS

I now come to the potentially most interesting application of my proposed mechanism. Because of the nonperturbative nature of the QCD vacuum, operator condensates appear ubiquitously in low energy QCD phenomenology, especially in the Shifman-Vainshtein-Zakharov (SVZ) sum rule formalism [17,18]. The effects of these condensates become stronger at lower energies and become phenomenologically more important. They are in general not calculable, and treated as free parameters to be fitted by experimental data. The condensates are generally introduced through OPE. In Borel resummation of the QCD perturbation theory they appear as the nonperturbative amplitudes which are required to cancel the imaginary parts arising from the IR renormalon singularities. The purpose of this section is to see the implication of my proposed mechanism on the nonperturbative effects caused by these condensates.

In general the form of a nonperturbative effect due to the condensates can be determined by OPE and the RG equations for the associated Wilson coefficients and the condensates. Once the form is determined then I can further refine it by demanding the nonperturbative amplitude have a branch cut only along the positive real axis in the coupling plane. This will then allow me to write the real part of the amplitude in terms of the imaginary part that can be calculated from Borel resummation.

Of course, one should remember that not all nonperturbative effects in QCD could be related to perturbation theory. It is clear, for example, that perturbation theory cannot have any bearing on the nonperturbative effects arising in chirality violating processes.

For definiteness, I shall consider the Adler D function in massless QCD defined by

$$D(Q^2)=-4\pi^2 Q^2\frac{d\Pi(-Q^2)}{dQ^2}-1, \tag{62}$$

where $\Pi(q^2)$ ($q^2\equiv -Q^2$) is the vacuum polarization function in the Euclidean region ($q^2<0$) of the current $j^\mu(x)=\overline{u}(x)\gamma^\mu d(x)$, with u,d denoting the *up* and *down* quarks. $D(Q^2)$ can be expanded in OPE as

$$D(Q^2)=C_0(Q^2)+C_4(Q^2)\frac{\langle 0|O_4|0\rangle}{Q^4} + \text{higher dimension terms}. \tag{63}$$

As before I focus on the nonperturbative effect associated with the closest singularity to the origin on the positive real axis in the Borel plane, in this case the first IR renormalon, and ignore terms of dimension six or higher since they are associated with the higher IR renormalons. $\langle 0|O_4|0\rangle$ is the renormalization scale invariant gluon condensate of the dimension-four operator:

$$O_4\equiv -\frac{1}{\pi\beta_0}\left[\frac{\beta(\alpha_s)}{\alpha_s}G^a_{\mu\nu}G^{a\mu\nu}\right], \tag{64}$$

where $G^a_{\mu\nu}$ is the gluon field strength tensor, and α_s is the strong coupling constant. $\beta(\alpha_s)$ is the QCD β function:

$$\beta(\alpha_s)=\mu^2\frac{d\alpha_s(\mu)}{d\mu^2}=-\alpha_s^2[\beta_0+\beta_1\alpha_s+O(\alpha_s^2)], \tag{65}$$

where for N_c colors and N_f quark flavors

$$\beta_0=\frac{1}{4\pi}\left(\frac{11}{3}N_c-\frac{2}{3}N_f\right),$$

$$\beta_1=\frac{1}{(4\pi)^2}\left(\frac{34}{3}N_c^2-\frac{N_c^2-1}{N_c}N_f-\frac{10}{3}N_cN_f\right). \tag{66}$$

The Wilson coefficients C_0,C_4 can be expanded in power series in the strong coupling $\alpha_s(Q)$:

$$C_0(Q^2)=\sum_{n=0}^{\infty} d_n^{(0)}\alpha_s(Q)^{n+1}, \tag{67}$$

$$C_4(Q^2)=\frac{2\pi^2}{3}\left(1+\sum_{n=1}^{\infty} w_n^{(0)}\alpha_s(Q)^n\right), \tag{68}$$

where the coefficients $d_n^{(0)},w_n^{(0)}$ are real numbers.

I shall now identify the first term in Eq. (63) as the perturbative amplitude $D_{\rm PT}[\alpha_s(Q)]$ and the second term of gluon condensate as the nonperturbative amplitude $D_{\rm NP}[\alpha_s(Q)]$. $D_{\rm PT}$ can be expressed as a Borel resummation of the perturbation series (67)

[5]More precisely, it is the expansion of $R(b)$ of the Borel transform for $[\Gamma_{\rm PT}(f)-\ln f-\text{const}]$ in Eq. (47) rather than that of the Borel transform for $\Gamma_{\rm PT}(f)$. Either of the Borel transforms can be used because both have an identical renormalon singularity at $b=1$.

$$D_{\rm PT}[\alpha_s(Q)\pm {\rm i}\epsilon]=\frac{1}{\beta_0}\int_{0\pm {\rm i}\epsilon}^{\infty\pm {\rm i}\epsilon}db e^{-b/\beta_0\alpha_s(Q)}\tilde{D}_{\rm PT}(b), \tag{69}$$

where $\tilde{D}_{\rm PT}(b)$ is defined by

$$\tilde{D}_{\rm PT}(b)=\sum_{n=0}^{\infty}\frac{d_n^{(0)}}{n!}\left(\frac{b}{\beta_0}\right)^n. \tag{70}$$

This series is expected to have a finite radius of convergence ($|b|=1$) set by the UV renormalon at $b=-1$. Beyond the radius of convergence $\tilde{D}(b)$ is assumed to be obtained by analytic continuation. $D_{\rm PT}$ is now expected to have an imaginary part with sign ambiguity due to the first IR renormalon at $b=2$ in the Borel integral. This imaginary part is to be canceled by the imaginary part from $D_{\rm NP}$. Now $D_{\rm NP}$, the second term in Eq. (63), can be written as

$$D_{\rm NP}[\alpha_s(Q)]\propto\alpha_s(Q)^{-\nu}e^{-2/\beta_0\alpha_s(Q)}\tilde{C}_4[\alpha_s(Q),\beta_i,w_i^{(0)}] \tag{71}$$

via the weak coupling expansion of

$$\frac{\langle 0|O_4|0\rangle}{Q^4}\propto\exp\left[-2\int^{\alpha_s(Q)}\frac{d\alpha'}{\beta(\alpha')}\right], \tag{72}$$

which comes from the RG invariance of the gluon condensate. Here,

$$\nu=2\beta_1/\beta_0^2, \tag{73}$$

which is noninteger for most of the combinations of N_c, N_f. $\tilde{C}_4$, which comes from C_4 and the weak coupling expansion of Eq. (72), is real and calculable in perturbation, and can be expanded in power series:[6]

$$\tilde{C}_4[\alpha_s(Q),\beta_i,w_i^{(0)}]=1+\tilde{w}_1^{(0)}\alpha_s(Q)+\tilde{w}_2^{(0)}\alpha_s(Q)^2+\cdots, \tag{74}$$

where $\tilde{w}_n^{(0)}$ depends on β_{i+1} and $w_i^{(0)}$, $i\leqslant n$, respectively, in Eqs. (65) and (68).

Following the argument in Sec. II I demand the imaginary part in $D_{\rm NP}$ come from a branch cut along the positive real axis in the coupling plane. Presence of UV renormalons, which gives rise to a sign alternating large order behavior, does not affect this requirement, since UV renormalons can be mapped away using a conformal mapping from the Borel integration contour, thus causing no essential problem for Borel resummation [15,19]. Since in the nonperturbative amplitude (71) a branch cut can arise only from the factor $\alpha_s(Q)^{-\nu}$, with ν a noninteger number, I conjecture that the nonperturbative amplitude is given in the form:

$$D_{\rm NP}[\alpha_s(Q)]=C[-\alpha_s(Q)]^{-\nu}e^{-2/\beta_0\alpha_s(Q)}\times\tilde{C}_4[\alpha_s(Q),\beta_i,w_i^{(0)}], \tag{75}$$

where C is an undetermined, dimensionless, real constant. In this specific case the power of the coupling constant in the preexponential factor depends on ν only. But, in general, the power depends not only on ν but also on the one-loop anomalous dimensions of the associated operators. In such a case, I propose that the branch cut arises likewise from the preexponential power term in the coupling, and the correct form for the nonperturbative amplitude can be obtained by flipping the sign of the coupling constant in the preexponential factor.

The argument leading to Eq. (75) shows that when ν takes an integer value for a particular combination of N_c and N_f, the nonperturbative amplitude $D_{\rm NP}$ cannot have an imaginary part. This implies disappearance of the first IR renormalon singularity in the Borel plane for an integer ν. A further comment on this point will follow shortly.

Now with Eq. (75) we have the real and the imaginary parts,

$${\rm Re}\{D_{\rm NP}[\alpha(Q)\pm{\rm i}\epsilon]\}=C\cos(\nu\pi)\alpha_s(Q)^{-\nu}e^{-2/\beta_0\alpha_s(Q)}\tilde{C}_4, \tag{76}$$

$${\rm Im}\{D_{\rm NP}[\alpha(Q)\pm{\rm i}\epsilon]\}=\pm C\sin(\nu\pi)\alpha_s(Q)^{-\nu}\times e^{-2/\beta_0\alpha_s(Q)}\tilde{C}_4, \tag{77}$$

from which a relation between the real and the imaginary parts is obtained:

$${\rm Re}\{D_{\rm NP}[\alpha_s(Q)\pm{\rm i}\epsilon]\}=\pm\cot(\nu\pi){\rm Im}\{D_{\rm NP}[\alpha_s(Q)\pm{\rm i}\epsilon]\}. \tag{78}$$

This has an important implication. It relates the usually incalculable real part of the nonperturbative amplitude to its imaginary part that is calculable from Borel resummation. Moreover, this relation is not for some part only of the amplitude as in the previous two examples, but holds to all orders in perturbative expansion of $\tilde{C}_4$. This implies that as far as the gluon condensate effect is concerned the nonperturbative effect can be calculated completely from the Borel resummation of the perturbation theory.

From Eq. (78) and the fact that the imaginary parts in $D_{\rm PT}[\alpha_s(Q)\pm{\rm i}\epsilon]$ and $D_{\rm NP}[\alpha_s(Q)\pm{\rm i}\epsilon]$ cancel each other, I can write the Adler function in terms of $D_{\rm PT}$ only:

$$D(Q^2)=[{\rm Re}\mp\cot(\nu\pi){\rm Im}]D_{\rm PT}[\alpha_s(Q)\pm{\rm i}\epsilon]. \tag{79}$$

Thus both the real and the imaginary parts of the Borel resummation are required to rebuild the true amplitude from the perturbation theory. The imaginary part comes from the region beyond the first IR renormalon ($b\geqslant 2$), and for its calculation analytic continuation to the region beyond the radius of convergence of the perturbative Borel transform (70) is required. A more convenient method, though equiva-

[6]When the higher renormalons are taken into account the series expansion for $\tilde{C}_4$ is an asymptotic expansion, and should be Borel resummed. However, since I ignore all higher renormalons $\tilde{C}_4$ is assumed to be well defined by the series expansion. The error on $\tilde{C}_4$ by this assumption is $O(e^{-1/\beta_0\alpha(Q)})$ which is due to the dimension six condensates.

lent to the analytic continuation, is to use a conformal mapping to map, for example, all the renormalon singularities, or all the renormalon singularities except for the first IR renormalon, to the unit circle. A conformal mapping of the first kind was used in rebuilding the imaginary part of a metastable-vacuum energy in a quantum mechanical model [20], and the second kind was recently used in Borel resummation of the real part of the Adler function [21], which may also be used for the calculation of the imaginary part. Here, instead, I shall try to evaluate the constant C which would give a rough estimate of the nonperturbative amplitude $D_{\rm NP}[\alpha_s(Q)]$.

As in the previous examples, this constant becomes the residue, up to a calculable constant, of the Borel transform at the renormalon singularity at $b=2$. In fact, for the Borel resummation (69) to have an imaginary part that can cancel the imaginary part (77) the Borel transform $\tilde{D}_{\rm PT}(b)$ should have the following singularity at $b=2$:

$$\tilde{D}_{\rm PT}(b)=\frac{C}{\Gamma(-\nu)}(\beta_0/2)^{1+\nu}(1-b/2)^{-1-\nu}\times[1+O(1-b/2)]+\text{analytic part}, \tag{80}$$

where "analytic part" denotes terms that are analytic at $b=2$. In general the analytic part cannot be calculated. Note that, as previously mentioned, the renormalon singularity disappears when ν takes an integer value. This is obvious for a negative integer ν, which happens, for example, at $N_c=2, N_f=8$ with $\nu=-10$, or at $N_c=3, N_f=15$ with $\nu=-176$. The disappearance of the renormalon singularity for the latter case was noticed before [22]. What seems to have been unexpected is that the singularity also disappears for a non-negative integer ν, for example, at $N_c=6, N_f=12$ with $\nu=1$. In this case the singularity disappears because of the vanishing residue. The residue vanishes since the constant C should always be bounded, for the nonperturbative amplitude (75) cannot be divergent.

With Eq. (80), C can now be obtained by [7,8]

$$C=\Gamma(-\nu)(2/\beta_0)^{1+\nu}R(2) \tag{81}$$

with

$$R(b)=(1-b/2)^{1+\nu}\tilde{D}_{\rm PT}(b). \tag{82}$$

As in the previous examples $R(2)$ may be evaluated as a perturbation series. However, it is unlikely to obtain a good estimate of the constant C by directly following the procedures in the previous examples, since too few perturbative coefficients are known. Only the first three terms of the perturbation series for the Adler function are presently known. To improve the situation we will exploit the renormalization scale independence of the Adler function. To do this, we replace in Eqs. (67)–(69), and (74) the coupling $\alpha(Q)$ with the running coupling $\alpha(\xi Q)$, with ξ defined by the renormalization scale $\mu^2=\xi^2Q^2$. Then the perturbative coefficients in these equations should change accordingly as $d_n^{(0)}\to d_n(\xi)$, $w_n^{(0)}\to w_n(\xi)$, and $\tilde{w}_n^{(0)}\to\tilde{w}_n(\xi)$, with $d_n(\xi=1)=d_n^{(0)}$, $w_n(\xi=1)=w_n^{(0)}$, and $\tilde{w}_n(\xi=1)=\tilde{w}_n^{(0)}$. Also the Borel transform (70) should be redefined as

$$\tilde{D}_{\rm PT}(b,\xi)=\sum_{n=0}^{\infty}\frac{d_n(\xi)}{n!}\left(\frac{b}{\beta_0}\right)^n. \tag{83}$$

Now the renormalization scale invariance of the gluon condensate allows me to rewrite Eq. (75) as

$$D_{\rm NP}[\alpha_s(\xi Q)]=C\xi^4[-\alpha_s(\xi Q)]^{-\nu}e^{-2/\beta_0\alpha_s(\xi Q)}\times\tilde{C}_4[\alpha_s(\xi Q),\beta_i,w_i(\xi)], \tag{84}$$

and consequently the renormalon singularity corresponding to Eq. (80) is given by

$$\tilde{D}_{\rm PT}(b,\xi)=\frac{C\xi^4}{\Gamma(-\nu)}(\beta_0/2)^{1+\nu}(1-b/2)^{-1-\nu}\times[1+O(1-b/2)]+\text{analytic part}. \tag{85}$$

Note that C in Eqs. (84) and (85) is the same one defined in Eq. (75), and is independent of the scale ξ but dependent on the renormalization scheme.

With Eq. (85), C can be written as

$$C=\frac{1}{\xi^4}\Gamma(-\nu)(2/\beta_0)^{1+\nu}R(b=2,\xi) \tag{86}$$

with

$$R(b,\xi)=(1-b/2)^{1+\nu}\tilde{D}_{\rm PT}(b,\xi). \tag{87}$$

We now proceed to evaluate $R(b=2,\xi)$ as a perturbation series. Because of the UV renormalon at $b=-1$ the evaluation point $b=2$ is beyond the convergence radius of the series for $R(b,\xi)$ around the origin, hence we have to map away all other renormalons except for the first IR renormalon, using a conformal mapping like

$$w=\frac{\sqrt{1+b}-\sqrt{1-b/3}}{\sqrt{1+b}+\sqrt{1-b/3}}, \tag{88}$$

which maps the first IR renormalon to $w=1/2$ and all other renormalons to the unit circle. Now in the w plane C can be expressed as a convergent power series, that is, $C=C_\infty$ where C_N is defined by

$$C_N(\xi^2)=\frac{1}{\xi^4}\Gamma(-\nu)(2/\beta_0)^{1+\nu}\sum_{n=0}^{N}r_n(\xi)\left(\frac{1}{2}\right)^n \tag{89}$$

with $r_n(\xi)$ coming from the expansion

$$R[b(w),\xi]=\sum_{n=0}^{\infty}r_n(\xi)w^n. \tag{90}$$

Although C is ξ independent, in general C_N will have a ξ dependence because of its finite order summation. This property is generic for any finite order QCD perturbation series

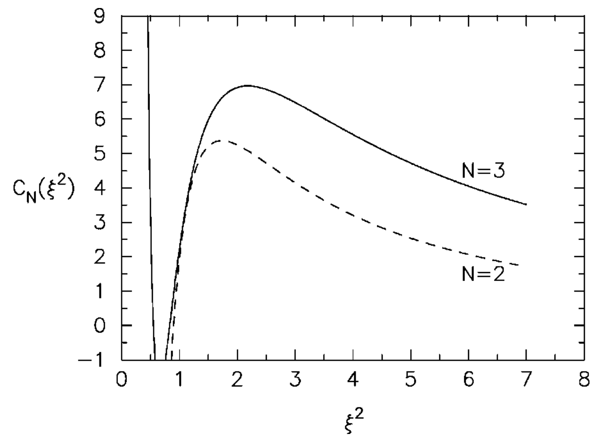

FIG. 1. Renormalon residue vs renormalization scale ξ^2.

and can be used to improve the convergence of the series by demanding that at an optimal ξ the scale dependence of the series be minimal [23]. Applying this idea to our problem we can hope that a better estimate of the constant C can be achieved by taking $C_N(\xi^2)$ at an optimal ξ_0 at which the unphysical ξ dependence disappears locally,

$$\left.\frac{d\,C_N(\xi^2)}{d\,\xi^2}\right|_{\xi=\xi_0}=0. \tag{91}$$

Using the calculated next-next-leading order Adler function [24,25] and the estimated $O(\alpha_s^4)$ coefficient [21,26], in the $\overline{\mathrm{MS}}$ scheme at $N_f=3$ quark flavors ($\xi=1$)[7]

$$D(Q^2)=a(Q)+1.6398\,a(Q)^2+6.3710\,a(Q)^3 +d_3^{(0)}a(Q)^4+O(a^5), \tag{92}$$

where $a(Q)\equiv\alpha_s(Q)/\pi$, I give the last two calculable terms for C_N in the $\overline{\mathrm{MS}}$ scheme:

$$C_2(1.7)=5.37, \quad C_3(2.2)=6.96. \tag{93}$$

In this calculation I took the estimated value $d_3^{(0)}=25$, which is from the recent estimate using a technique called "bilocal expansion of Borel amplitude" [21]. This value is also in consistency with the well-known estimate in [26]. Note that the optimal ξ for C_N is at $\xi_0^2\approx1.7$ for C_2 and $\xi_0^2\approx2.2$ for C_3 (see Fig. 1).

Because C is evaluated with the perturbation series (90) at $w=1/2$ which is on the boundary of the convergence disk, C_N in Eq. (93) should be regarded only as a rough estimate. The speed of convergence of the sequence C_N is expected to be sensitive on the size of the analytic part in Eq. (85), since the size of the singular term of $R(b,\xi)$ at $b=2$ is determined by this analytic part.

Using this estimate of C we can now evaluate the real part of the nonperturbative amplitude. From Eqs. (76)

$$\begin{aligned}\mathrm{Re}\{D_{\mathrm{NP}}[\alpha_s(Q)\pm i\epsilon]\} &\approx C_3(2.2)\cos(\nu\pi)\,\alpha_s(Q)^{-\nu}e^{-2/\beta_0\alpha_s(Q)}\tilde{C}_4\\ &\approx C_3(2.2)\cos(\nu\pi)\frac{\Lambda_{\overline{\mathrm{MS}}}^4}{Q^4}[1+w_1^{(0)}\alpha_s(Q)+O(\alpha_s^2)],\end{aligned} \tag{94}$$

where $w_1^{(0)}$ is defined in Eq. (68) and we have substituted $C_3(2.2)$ for C, $\Lambda_{\overline{\mathrm{MS}}}$ denotes the $\overline{\mathrm{MS}}$ renormalization scale.

Now this nonperturbative amplitude can be translated to a gluon condensate in the OPE (63). I define the gluon condensate by this nonperturbative amplitude by

$$\begin{aligned}\left\langle\frac{\alpha_s}{\pi}G^a_{\mu\nu}G^{a\mu\nu}\right\rangle_{\mathrm{NP}} &\equiv\frac{3Q^4}{2\pi^2}\mathrm{Re}\{D_{\mathrm{NP}}[\alpha_s(Q)\pm i\epsilon]\}\\ &\approx\frac{3}{2\pi^2}C_3(2.2)\cos(\nu\pi)\Lambda_{\overline{\mathrm{MS}}}^4\\ &\approx0.005\ \mathrm{GeV}^4,\end{aligned} \tag{95}$$

where I have used $\Lambda_{\overline{\mathrm{MS}}}\approx370$ MeV for $N_f=3$ quark flavors [27].

One should not compare the estimated value (95) directly with the phenomenologically fitted gluon condensate, for example, from the QCD sum rule. In the QCD sum rule, the difference between the Borel resummed perturbative amplitude and the sum of the first terms in the perturbation series is approximated by power corrections, and therefore the phenomenologically fitted condensate includes contributions not only from the nonperturbative amplitude but also from the perturbative amplitude. One may try to extract the gluon condensate effect in the resummed perturbative amplitude through Eq. (69), for example, by computing the minimal term of the perturbation series with the large order behavior given by the renormalon singularity (80). I believe, however, that this is not necessary, and also not a good way to handle the renormalon effect. A better approach to incorporate the renormalon effect, with only the first few terms of the perturbation series available, is to write using Eq. (82) the Borel transform $\tilde{D}_{\mathrm{PT}}(b)$ in the Borel integral (69) as

$$\tilde{D}_{\mathrm{PT}}(b)=\frac{R(b)}{(1-b/2)^{1+\nu}} \tag{96}$$

and do perturbation on $R(b)$ instead of doing perturbation directly on $\tilde{D}_{\mathrm{PT}}(b)$. This way the Borel transform can be better described in the most important region in the Borel integral, i.e., between the origin and the first IR renormalon singularity and the region just beyond the singularity [21].

Nonetheless, it is interesting to observe that the estimated value (95) is remarkably close to the recent estimate of gluon condensate 0.006 ± 0.012 GeV^4 [28] which was obtained by fitting the spectral function of hadronic τ decay using the QCD sum rule.

[7] $d_n(\xi)$ in terms of $d_n^{(0)}$ can be found in [21].

Finally, some comments are in order. A full amplitude in QCD in general has infinitely many cut singularities in the complex coupling plane as shown by 't Hooft [29]. One may wonder how this can be compatible with our proposed mechanism that is based on the proposition that the nonperturbative amplitude as well as the Borel resummed amplitude of perturbation theory have cuts only along the positive real axes. The resolution of this question lies probably with the nonconvergence of the OPE. Since the OPE is expected to be an asymptotic expansion [30], each term in the OPE, to the Wilson coefficient of which the Borel resummation is applied, does not have to have the same singularities of the true amplitude. Also, throughout this paper, the nonperturbative effects due to the higher dimensional operator condensates were consistently ignored. I expect, however, there should be no fundamental difficulty in incorporating them along the lines described in this section. I conjecture that the nonperturbative effect by the condensate of a dimension $2n$ operator in the Adler function can be written as

$$D_{\rm NP}(\alpha_s)=C_n[-\alpha_s]^{-\nu_n}\exp(-n/\beta_0\alpha_s)\tilde{C}_{2n}(\alpha_s), \quad (97)$$

where ν_n is a constant calculable from the RG equations on the Wilson coefficient and the condensate, and $\tilde{C}_{2n}$ is a modified Wilson coefficient defined in a similar fashion as the $\tilde{C}_4$ in Eq. (74). Here $\tilde{C}_{2n}$ should be Borel resummed in the manner described in this section. The unknown constant C_n can then be determined by demanding that the imaginary part of order $\alpha_s^{-\nu_n}\exp(-n/\beta_0\alpha_s)$ be canceled by the imaginary parts coming from the amplitudes associated with the operators of lower dimensions.

VI. SUMMARY

Based on the general argument on Borel resummation of a same sign perturbation series I have argued that the nonperturbative effect associated with the divergence of the perturbation series should have a branch cut only along the positive real axis in the coupling plane. Demanding that the nonperturbative amplitude have such a branch cut constrains the form of a part of the nonperturbative amplitude, the part from which the branch cut arises, sufficiently that a relation can be established between the usually incalculable real part of the nonperturbative amplitude and the imaginary part that is calculable from the Borel resummation. This way part of the real part of the nonperturbative amplitude, which usually includes the leading term in weak coupling expansion, can be calculated from Borel resummation of the perturbation theory.

As a nontrivial test, this mechanism was applied to the ground state energy of the double well potential and the two-point function in the two-dimensional $O(N)$ nonlinear sigma model at order $1/N$. In agreement with my proposed mechanism the nonperturbative amplitudes in these models have branch cuts only along the positive real axis in the coupling plane. With this mechanism the leading terms of the nonperturbative amplitudes in these models could be calculated with good accuracy from the first terms of the corresponding perturbation series.

I then applied this mechanism to the QCD condensate effects, particularly the gluon condensate effect, and suggested that some of the condensate effects can be calculated from perturbation theory, and gave an estimate of the nonperturbative amplitude induced by the gluon condensate using the known perturbative calculations of the Adler function. I observed that this mechanism could be applied to QCD, despite the fact that a true QCD amplitude has an infinite number of cut singularities in the coupling plane, since the OPE to which the Borel resummation is applied is not a convergent expansion.

ACKNOWLEDGMENTS

The author is thankful to Gorazd Cvetic for a careful reading of the manuscript and useful comments. Part of the work was completed while the author was visiting Universidad Tecnica Federico Santa Maria, Valparaiso, Chile. The warm hospitality by the members of the physics department is gratefully acknowledged. This work was supported by BK21 Core Project.

[1] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena* (Clarendon, Oxford, 1989).
[2] J. C. Le Guillou and J. Zinn-Justin, *Large Order Behavior of Perturbation Theory* (North-Holland, Amsterdam, 1990).
[3] F. David, Nucl. Phys. **B209**, 433 (1982).
[4] G. Grunberg, Phys. Lett. B **325**, 441 (1994).
[5] J. Zinn-Justin, Nucl. Phys. **B218**, 333 (1983).
[6] E.B. Bogomolny, Phys. Lett. **91B**, 431 (1980).
[7] T. Lee, Phys. Rev. D **56**, 1091 (1997).
[8] T. Lee, Phys. Lett. B **462**, 1 (1999).
[9] E. Brezin, G. Parisi, and J. Zinn-Justin, Phys. Rev. D **16**, 408 (1977).
[10] A. H. Mueller, *The QCD Perturbation Series*. Talk given at Workshop on QCD: 20 Years Later, Aachen, Germany, 1992 (World Scientific, Singapore, 1992), pp. 162–171 (QCD161;W586;1992).
[11] F. David, Nucl. Phys. **B234**, 237 (1984).
[12] F. David, Nucl. Phys. **B263**, 637 (1986).
[13] V.A. Novikov, M.A. Shifman, A.I. Vainshtein, and V.I. Zakharov, Nucl. Phys. **B249**, 445 (1985).
[14] V.A. Novikov, M.A. Shifman, A.I. Vainshtein, and V.I. Zakharov, Phys. Rep. **116**, 103 (1984).
[15] G. Parisi, Nucl. Phys. **B150**, 163 (1979).
[16] M. Beneke, V.M. Braun, and N. Kivel, Phys. Lett. B **443**, 308 (1998).
[17] M.A. Shifman, A.I. Vainshtein, and V.I. Zakharov, Nucl. Phys. **B147**, 385 (1979).
[18] M.A. Shifman, A.I. Vainshtein, and V.I. Zakharov, Nucl. Phys. **B147**, 448 (1979).
[19] A.H. Mueller, Phys. Lett. B **308**, 355 (1993).

[20] H. Suzuki and H. Yasuta, Phys. Lett. B **400**, 341 (1997).
[21] G. Cvetic and T. Lee, Phys. Rev. D **64**, 014030 (2001).
[22] C.N. Lovett-Turner and C.J. Maxwell, Nucl. Phys. **B432**, 147 (1994).
[23] P.M. Stevenson, Phys. Rev. D **23**, 2916 (1981).
[24] S.G. Gorishnii, A.L. Kataev, and S.A. Larin, Phys. Lett. B **259**, 144 (1991).
[25] L.R. Surguladze and M.A. Samuel, Phys. Rev. Lett. **66**, 560 (1991).
[26] A.L. Kataev and V.V. Starshenko, Mod. Phys. Lett. A **10**, 235 (1995).
[27] R. Barate *et al.*, Eur. Phys. J. C **4**, 409 (1998).
[28] B.V. Geshkenbein, B.L. Ioffe, and K.N. Zyablyuk, Phys. Rev. D **64**, 093009 (2001).
[29] G. 't Hooft, in *The Whys of Subnuclear Physics.* Proceedings of the 1977 International School of Subnuclear Physics, Erice, Trapani, Sicily, 1977, edited by A. Zichichi (Plenum, New York, 1979).
[30] B.L. Ioffe, JETP Lett. **58**, 876 (1993); M.A. Shifman, ''Theory of pre-asymptotic effects in weak inclusive decays,'' hep-ph/9405246; N. Isgur, S. Jeschonnek, W. Melnitchouk, and J.W. Van Orden, Phys. Rev. D **64**, 054005 (2001).