Investigations of the πN total cross sections at high energies using a new finite-energy sum rule: $\log \nu$ or $(\log \nu)^2$

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We propose to use rich information on πp total cross sections below $N(\sim 10\ \text{GeV})$ in addition to high-energy data in order to discern whether these cross sections increase like $\log \nu$ or $(\log \nu)^2$ at high energies, since it is difficult to discriminate between asymptotic $\log \nu$ and $(\log \nu)^2$ fits from high-energy data alone. A finite-energy sum rule (FESR) which is derived in the spirit of the P' sum rule as well as the $n=1$ moment FESR is required to constrain the high-energy parameters. We then search for the best fit of $\sigma_{\text{tot}}^{(+)}$ above 70 GeV in terms of high-energy parameters constrained by these two FESR's. We can show from this analysis that the $(\log \nu)^2$ behavior is preferred to the $\log \nu$ behavior.

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The high energy behavior of πN total cross sections has been one of the long standing problems in particle physics. The sum of $\pi^+ p$ and $\pi^- p$ total cross sections has a tendency to increase above 70 GeV experimentally [1]. It is well known as the Froissart unitarity bound [2] that the increase of total cross sections is at most $\log^2 \nu$. It has not been possible [3], however, to discriminate between asymptotic $\log \nu$ and $\log^2 \nu$ fits if one uses high-energy data alone above ~ 70 GeV.

The purpose of this paper is to propose to use the rich information of πp total cross sections at low and intermediate energy regions *through new finite-energy sum rules (FESR) as constraints* in addition to high-energy data, in order to discriminate the high energy behavior of πp total cross sections above 70 GeV.

Such a kind of attempt has been initiated in Ref. [4]. The s-wave πN scattering length $a^{(+)}$ of the crossing-even amplitude has been expressed as

$$\left(1+\frac{\mu}{M}\right)a^{(+)}=-\frac{g_r^2}{4\pi}\left(\frac{\mu}{2M}\right)^2\frac{1}{M}\,\frac{1}{1-\left(\frac{\mu}{2M}\right)^2}+\frac{1}{2\pi^2}\int_0^\infty dk[\sigma_{\text{tot}}^{(+)}(k)-\sigma_{\text{tot}}^{(+)}(\infty)] \tag{1}$$

with pion mass μ under the assumption that there are no singularities with the vacuum quantum numbers in the J plane except for the Pomeron (P). The evidence that this sum rule had not been satisfied led us to the prediction of the P' trajectory with $\alpha_{P'}(0)\approx 0.5$, and soon the f meson $[f_2(1275)]$ has been uncovered on this P' trajectory.

[FESR(1)]: Taking into account the present situation of increasing total cross section data, we derive FESR in the spirit of the P' sum rule [4]. We consider the crossing-even (spin-averaged) forward scattering amplitude for πp scattering [5]

$$f^{(+)}(\nu)=\frac{1}{4\pi}[A^{(+)}(\nu)+\nu B^{(+)}(\nu)]. \tag{2}$$

We assume

$$\text{Im}\ f^{(+)}(\nu)\simeq \text{Im}\,R(\nu)+\text{Im}\ f_{P'}(\nu)$$
$$=\frac{\nu}{\mu^2}\left(c_0+c_1\log\frac{\nu}{\mu}+c_2\log^2\frac{\nu}{\mu}\right)+\frac{\beta_{P'}}{\mu}\left(\frac{\nu}{\mu}\right)^{\alpha_{P'}(0)} \tag{3}$$

at high energies $(\nu \geqslant N)$. Since this amplitude is crossing even, we have

$$R(\nu)=\frac{i\nu}{2\mu^2}\left\{2c_0+c_2\pi^2+c_1\left(\log\frac{e^{-i\pi}\nu}{\mu}+\log\frac{\nu}{\mu}\right)+c_2\left(\log^2\frac{e^{-i\pi}\nu}{\mu}+\log^2\frac{\nu}{\mu}\right)\right\}, \tag{4}$$

$$f_{P'}(\nu)=-\frac{\beta_{P'}}{\mu}\left(\frac{(e^{-i\pi}\nu/\mu)^{\alpha_{P'}(0)}+(\nu/\mu)^{\alpha_{P'}(0)}}{\sin\pi\alpha_{P'}(0)}\right), \tag{5}$$

and subsequently we obtain

$$\text{Re}\,R(\nu)=\frac{\pi\nu}{2\mu^2}\left(c_1+2c_2\log\frac{\nu}{\mu}\right), \tag{6}$$

$$\text{Re}\,f_{P'}(\nu)=-\frac{\beta_{P'}}{\mu}\left(\frac{\nu}{\mu}\right)^{\alpha_{P'}(0)}\cot\frac{\pi\alpha_{P'}(0)}{2}$$

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$$=-\frac{\beta_{P'}}{\mu}\left(\frac{\nu}{\mu}\right)^{0.5}, \tag{7}$$

substituting $\alpha_{P'}(0)=\frac{1}{2}$ in Eq. (5). Let us define

$$\tilde{f}^{(+)}(\nu)=f^{(+)}(\nu)-R(\nu)-f_{P'}(\nu)\sim\nu^{\alpha(0)} \quad [\alpha(0)<0], \tag{8}$$

and write a dispersion relation for $\tilde{f}^{(+)}(\nu)/\nu-\mu$. Since this amplitude is superconvergent, we obtain

$$\begin{aligned}\mathrm{Re}\,\tilde{f}^{(+)}(\mu)&=\frac{P}{\pi}\int_{-\infty}^{\infty}d\nu'\frac{\mathrm{Im}\,\tilde{f}^{(+)}(\nu')}{\nu'-\mu}\\&=\frac{2P}{\pi}\int_0^{\infty}\frac{\nu'\ \mathrm{Im}\,\tilde{f}^{(+)}(\nu')}{k'^2}d\nu'.\end{aligned} \tag{9}$$

Using Eqs. (8) and (9), we have

$$\begin{aligned}\mathrm{Re}\,f^{(+)}(\mu)=&\,\mathrm{Re}\,R(\mu)+\mathrm{Re}\,f_{P'}(\mu)-\frac{g_r^2}{4\pi}\left(\frac{\mu}{2M}\right)^2\\&\times\frac{1}{M}\frac{1}{1-\left(\frac{\mu}{2M}\right)^2}+\frac{1}{2\pi^2}\int_0^{\bar{N}}\sigma_{\mathrm{tot}}^{(+)}(k)dk\\&-\frac{2P}{\pi}\int_0^N\frac{\nu}{k^2}\left\{\mathrm{Im}\,R(\nu)+\frac{\beta_{P'}}{\mu}\left(\frac{\nu}{\mu}\right)^{0.5}\right\}d\nu,\end{aligned} \tag{10}$$

where $\bar{N}\equiv\sqrt{N^2-\mu^2}\simeq N$. Let us call Eq. (10) as the FESR(1) which we use as the first constraint. It is important to notice that Eq. (10) reduces to the P' sum rule in Ref. [4] if $c_1,c_2\to 0$.

The FESR ([6–8])

$$\int_0^N d\nu\,\nu^n\ \mathrm{Im}\ f(\nu)=\sum_i\ \beta_i\frac{N^{\alpha_i(0)+n+1}}{\alpha_i(0)+n+1} \tag{11}$$

holds for even positive integer n when $f(\nu)$ is crossing odd, and holds for odd positive integer n when $f(\nu)$ is crossing even. We can also derive the negative-integer moment FESR. The only significant FESR is one for $f^{(+)}(\nu)/\nu$ corresponding to $n=-1$. FESR(1) belongs to this case.

It is important to emphasize that the FESR should not depend so much on the value of N.

[FESR(2)]: The second FESR corresponding to $n=1$ is

$$\begin{aligned}\pi\mu\left(\frac{g_r^2}{4\pi}\right)\left(\frac{\mu}{2M}\right)^3+\frac{1}{4\pi}\int_0^{\bar{N}}dk\,k^2\sigma_{\mathrm{tot}}^{(+)}(k)\\=\int_0^N\nu\ \mathrm{Im}\,R(\nu)d\nu+\int_0^N\nu\ \mathrm{Im}\ f_{P'}(\nu)d\nu.\end{aligned} \tag{12}$$

We call Eq. (12) as the FESR(2). It is to be noticed that the contribution from higher energy regions is enhanced.

(Data): The numerical values,

$$\begin{aligned}-\frac{g_r^2}{4\pi}\left(\frac{\mu}{2M}\right)^2\frac{1}{M}\frac{1}{1-\left(\frac{\mu}{2M}\right)^2}&=-0.0854\ \ \mathrm{GeV}^{-1},\\ \pi\mu\frac{g_r^2}{4\pi}\left(\frac{\mu}{2M}\right)^3&=0.0026\ \ \mathrm{GeV},\end{aligned} \tag{13}$$

have been evaluated using $g_r^2/4\pi=14.4$.

$$\begin{aligned}\mathrm{Re}\,f^{(+)}(\mu)&=\left(1+\frac{\mu}{M}\right)a^{(+)}\\&=\left(1+\frac{\mu}{M}\right)\frac{1}{3}(a_{1/2}+2a_{3/2})\\&=-(0.014\pm0.026)\ \mathrm{GeV}^{-1}\end{aligned} \tag{14}$$

was obtained from [9] $a_{1/2}=(0.171\pm0.005)\mu^{-1}$ and $a_{3/2}=-(0.088\pm0.004)\mu^{-1}$.

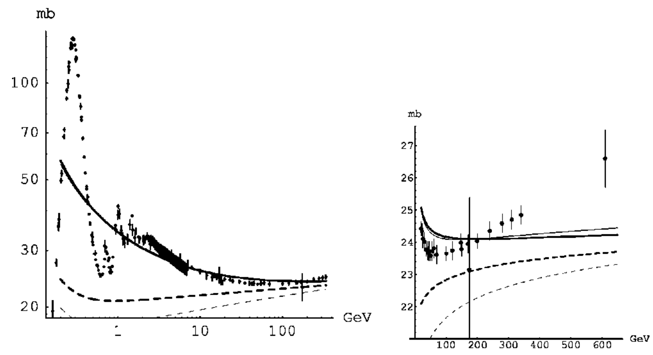
We have used rich data [1] of σ^{π^+p} and σ^{π^-p} to evaluate the relevant integrals of cross sections appearing in FESR(1) and (2). We connect each data point [10] of $\sigma^{\pi^\pm p}(k)$ or $k^2\sigma^{\pi^\pm p}(k)$ with the next point by a straight line in order, from $k=0$ to $k=\bar{N}$, and regard the area of this polygonal line graph as the relevant integral in the region $0\leqslant k\leqslant\bar{N}$. The integrals of $\sigma_{\mathrm{tot}}^{(+)}(k)[k^2\sigma_{\mathrm{tot}}^{(+)}(k)]$ are given by averaging those of $\sigma^{\pi^+p}(k)$ and $\sigma^{\pi^-p}(k)$ $[k^2\sigma^{\pi^+p}(k)$ and $k^2\sigma^{\pi^-p}(k)]$. We have obtained

$$\begin{aligned}\frac{1}{2\pi^2}\int_0^{\bar{N}}dk\ \sigma_{\mathrm{tot}}^{(+)}(k)&=38.75\pm0.25\ \ \mathrm{GeV}^{-1},\\ \frac{1}{4\pi}\int_0^{\bar{N}}dk\,k^2\sigma_{\mathrm{tot}}^{(+)}(k)&=1817\pm31\ \ \mathrm{GeV}\end{aligned} \tag{15}$$

for $\bar{N}=10$ GeV. The errors of relevant integrals, which are from the error of each data point, are very small $(\sim1\%)$, and thus we regard the central values as exact ones in the following analysis.

When σ^{π^+p} and σ^{π^-p} data points are listed at the same value of k, we make the $\sigma_{\mathrm{tot}}^{(+)}(k)$ data point by averaging these values. Totally, 183 points are obtained in the region $0.16\leqslant k\leqslant340$ GeV as $\sigma_{\mathrm{tot}}^{(+)}(k)$ data. There are 12 points in the $k\geqslant70$ GeV region, which will be used in the following analysis.

(Analysis): The FESR(1) and (2) are our starting points. Armed with these two, we expressed high-energy parameters c_0, c_1, c_2, and $\beta_{P'}$ in terms of the Born term and the πN scattering length $a^{(+)}$ as well as the total cross sections up to N. We then attempt to fit the $\sigma_{\mathrm{tot}}^{(+)}$ above 70 GeV. We set $N=10$ GeV (corresponding to $\sqrt{s_{p\pi}}=4.43$ GeV) since there are no resonances above this energy. The FESR(2) also has contributions from the lower trajectory P'' which may pass



(a) whole-energy regions (b) high-energy regions

FIG. 1. Fit to the $\sigma_{\rm tot}^{(+)}$ data above 70 GeV by the log ν model. A thick (thin) solid line shows the result in the case of $\alpha_{P'}=0.5$ (0.586). Correspondingly, the contribution from Im $R(\nu)$ (with $c_2=0$) is shown by a thick (thin) dashed line. Recently a datum [12] for the $\pi^- N$ total cross section at very high energy ($k=610$ GeV) was reported by the SELEX collaboration. This point is included in (b). The log ν model with $\alpha_{P'}(0)=0.5$ (0.586) predicts 24.2 (24.4) mb for $\sigma_{\rm tot}^{(+)}$ at 610 GeV which is inconsistent with their value on $\pi^- N$, (26.6 ± 0.9) mb.

through $f_2(1810)$. Since $\alpha_{P''}(0)$ is expected to be around -1, we can assume P'' contribution to be suppressed compared with that from P'.

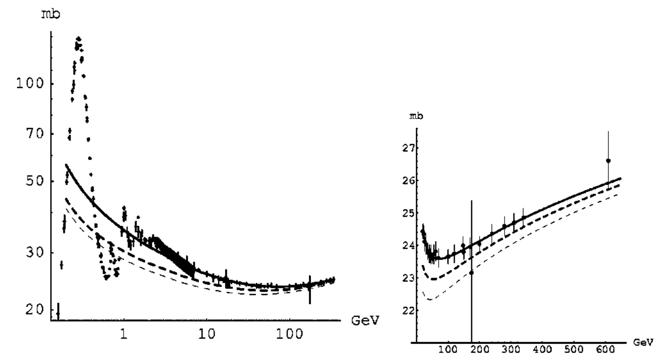
Let us first define the $\log^2 \nu$ model and the log ν model. The $\log^2 \nu$ model is a model for which the imaginary part of $f^{(+)}(\nu)$ behaves as $a+b\log\nu + c(\log\nu)^2$ as ν becomes large [11]. The log ν model is a model for which the imaginary part of $f^{(+)}(\nu)$ behaves as $a'+b'\log\nu$ for large ν. So we generally assume that the Im $f^{(+)}(\nu)$ behaves as Eq. (3) at high energies ($\nu \geqslant N$).

(1) log ν model: This model has three parameters c_0, c_1, and $\beta_{P'}$ with two constraints FESR (1), (2). (Note that the number of independent parameters is one.) We set $N=10$ GeV and expressed both $c_0,\beta_{P'}$ as a function of c_1 using the FESR(1) and (2). We obtained

$$c_0(c_1)=0.0879-4.94c_1,$$
$$\beta_{P'}(c_1)=0.1290+7.06c_1. \tag{16}$$

We then tried to fit 12 data points of $\sigma_{\rm tot}^{(+)}(k)$ between 70 and 340 GeV. The result is shown by a thick solid line in Fig. 1. The best fit we obtained is $c_1=0.001\,85$ which gives $c_0=0.0787$ and $\beta_{P'}=0.142$ with the bad "reduced χ^2," $\chi^2/(N_{\rm data}-N_{\rm param})=29.04/(12-1)\simeq 2.6$. Therefore it turned out that this model had difficulties in reproducing the experimental increase of πp total cross sections above 70 GeV [see the thick solid line in Fig. 1(b)]. In this log ν fit, the results also depend on the value of N, which is not so good.

(2) $\log^2\nu$ model: This model has four parameters c_0, c_1, c_2, and $\beta_{P'}$ with two constraints FESR(1),(2). (So the number of independent parameters is two.) We again set $N=10$ GeV and required both FESR(1) and (2) as constraints. Then $c_0,\beta_{P'}$ are expressed as functions of c_1 and c_2 as



(a) whole-energy regions (b) high-energy regions

FIG. 2. Fit to the $\sigma_{\rm tot}^{(+)}$ data above 70 GeV by the $\log^2\nu$ model. The result in the case of $\alpha_{P'}=0.5$ is shown by a thick solid line, which overlaps in all energy regions with the result of $\alpha_{P'}=0.586$ shown by a thin solid line, and both results cannot be distinguished from each other. The contribution from Im $R(\nu)$ with $c_2>0$ is shown by a thick (thin) dashed line. A datum at 610 GeV obtained by SELEX collaboration is included in (b). Our $\log^2\nu$ model predicts 25.9 mb for $\sigma_{\rm tot}^{(+)}$ at 610 GeV which is consistent with their value on $\pi^- N$, (26.6 ± 0.9) mb.

$$c_0(c_1,c_2)=0.0879-4.94c_1-21.50c_2,$$
$$\beta_{P'}(c_1,c_2)=0.1290+7.06c_1+41.46c_2. \tag{17}$$

We then searched for the fit to 12 data points of $\sigma_{\rm tot}^{(+)}(k)$ above 70 GeV. The result is shown by a thick solid line in Fig. 2. The best fit in terms of two parameters c_1 and c_2 led us to a greatly improved value of "reduced χ^2," $\chi^2/(N_{\rm data}-N_{\rm param})=0.746/(12-2)\simeq 0.075$ for $c_1=-0.0215<0$ and $c_2=0.00182>0$ which give $c_0=0.155$ and $\beta_{P'}=0.0524$. This is an excellent fit to the data [see the thick solid line in Fig. 2(b)].

($\alpha_{P'}$ dependence): So far, we have assumed the intercept of the P' trajectory $\alpha_{P'}(0)$ to be 0.5. The value $\alpha_{P'}(0)$ is estimated to be 0.586 according to the Chew-Frautschi plot, using the universal slope $\alpha'=1/1.15$ GeV^{-2} and the mass of f_2 to be 1275 MeV. Let us check if the results change for this value of $\alpha_{P'}(0)$.

Suppose we take $\alpha_{P'}(0)=0.586$ and discuss the two cases, log ν and $\log^2\nu$.

(1) log ν model: We again set $N=10$ GeV and expressed both $c_0,\beta_{P'}$ as functions of c_1 using FESR(1),(2). We then obtained

$$c_0(c_1)=0.0817-5.28c_1,$$
$$\beta_{P'}(c_1)=0.1238+6.77c_1. \tag{18}$$

We then searched for the fit to 12 data points of $\sigma_{\rm tot}^{(+)}(k)$ above 70 GeV. The result is shown by a thin solid line in Fig. 1. The best fit we obtained is $c_1=0.003\,53$ which gives $c_0=0.0630$ and $\beta_{P'}=0.148$ with "reduced χ^2" $\chi^2/(N_{\rm data}-N_{\rm param})=22.30/(12-1)=2.03$. So, this model has difficulties again reproducing the experimental increase of πp total cross sections above 70 GeV [see the thin solid line in Fig. 1(b)].

(2) $\log^2 \nu$ model: We also set $N=10$ GeV and required both FESR(1) and (2) as constraints. Then we obtained

$$c_0(c_1,c_2)=0.0817-5.28c_1-23.50c_2,$$
$$\beta_{P'}(c_1,c_2)=0.1238+6.77c_1+39.80c_2. \tag{19}$$

We again searched for the fit to 12 points of $\sigma_{\rm tot}^{(+)}(k)$ above 70 GeV. The best fit in terms of two parameters c_1 and c_2 again led us to a greatly improved value of "reduced χ^2," $\chi^2/(N_{\rm data}-N_{\rm param})=0.750/(12-2)=0.075$ for $c_1=-0.0197<0$ and $c_2=0.001\,73>0$ which give $c_0=0.145$ and $\beta_{P'}=0.0593$. This is again an excellent fit to the data (see the caption of Fig. 2).

We have also searched for $\alpha_{P'}(0)=0.543$ (average of 0.5 and 0.586) and for $\alpha_{P'}(0)=0.642$ due to the Particle Data Group [1]. We found that the results do not change so much.

It is remarkable to notice that the wide range of data ($k \geqslant 5$ GeV) have been reproduced within the error even in the region where the fit has not been made [see Figs. 2(a) and 2(b)]. The results do not change so much for the value of N. The increase of $\sigma_{\rm tot}^{(+)}$ above 50 GeV is explained via $\log^2 \nu/\mu$ ($c_2>0$) and the decrease between 5 and 50 GeV is explained by $\log \nu/\mu$ ($c_1<0$). It should also be emphasized that the comparison of Fig. 1 ($\log \nu$ model) and Fig. 2 ($\log^2 \nu$ model) clearly indicates the latter model to be preferred. A similar conclusion is also obtained by a completely different approach [13].

Recently a datum [12] for a $\pi^- N$ total cross section at very high energy ($k=610$ GeV) [14] was reported by the SELEX Collaboration. Our $\log^2 \nu$ model ($\log \nu$ model) with $\alpha_{P'}(0)=0.5$ predicts 25.9 mb (24.2 mb) for $\sigma_{\rm tot}^{(+)}$ at 610 GeV which is consistent (inconsistent) with their value on $\pi^- N$, (26.6 ± 0.9) mb. This fact also suggests the validity of the $\log^2 \nu$ model.

Therefore we can conclude that our analysis in terms of high-energy parameters constrained by the FESR(1),(2) prefers the $\log^2 \nu/\mu$ behaviors satisfying the Froissart unitarity bound. Finally we should add a note that the origin of the $\log^2 \nu$ behavior of the amplitude at high energy is argued to be explained from the effect of gluon saturation [15].

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[10] We take the error Δy for each data point y as $\Delta y=\sqrt{(\Delta y)^2_{\rm stat}+(\Delta y)^2_{\rm syst}}$. When several data points, denoted as y_i with error Δy_i ($i=1,\ldots,n$), are listed at the same value of k, these points are replaced by $\overline{y}$ with $\Delta\overline{y}$, given by $\overline{y}=[\Sigma_i y_i/(\Delta y_i)^2]/[\Sigma_i 1/(\Delta y_i)^2]$ and $\Delta\overline{y}=\sqrt{1/[\Sigma_i 1/(\Delta y_i)^2]}$.

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