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Heavy-quark axial charges to nonleading order

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We combine Witten's renormalization group with the matching conditions of Bernreuther and Wetzel to calculate at next-to-leading order the complete heavy-quark contribution to the neutral-current axial-charge measurable in neutrino-proton elastic scattering. Our results are manifestly renormalization group invariant.

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This paper announces results for the next-to-leading-order (NLO) heavy-quark corrections to the axial charge $g_A^{(Z)}$ for protons to couple to the weak neutral current

$$J^Z_{\mu 5}=\frac{1}{2}\left\{\sum_{q=u,c,t}-\sum_{q=d,s,b}\right\}\bar{q}\gamma_\mu\gamma_5 q. \tag{1}$$

The calculation is performed by decoupling heavy quarks $h=t,b,c$ sequentially, i.e. one at a time. An extension to simultaneous decoupling of t,b,c quarks is foreshadowed in our concluding remarks.

The charge $g_A^{(Z)}$ receives contributions from both light u,d,s and heavy c,b,t quarks,

$$2g_A^{(Z)}=(\Delta u-\Delta d-\Delta s)+(\Delta c-\Delta b+\Delta t) \tag{2}$$

where Δq refers to expectation value $\langle p,s|\bar{q}\gamma_\mu\gamma_5 q|p,s\rangle=2m_p s_\mu \Delta q$ for a proton of spin s_μ and mass m_p. It governs parity-violating effects due to Z^0 exchange at low energies in elastic νp and $\bar{\nu}p$ scattering [1,2] or in light atoms [3,4]. A definitive measurement of νp elastic scattering may be possible using the MiniBooNE setup at Fermilab [5].

Once heavy-quark corrections [2,6,7] have been taken into account, $g_A^{(Z)}$ is related (modulo the issue of δ-function terms at $x=0$ [8]) to the flavor-singlet axial charge, defined scale invariantly and extracted from polarized deep inelastic scattering:

$$g_A^{(0)}|_{\rm inv}=0.2-0.35. \tag{3}$$

The small value of this quantity has inspired vast experimental and theoretical activity to understand the spin structure of the proton [9]. As a result, new experiments are being planned to map out the spin-flavor structure of the proton. These include polarized proton-proton collisions at the BNL Relativistic Heavy Ion Collider (RHIC) [10], semi-inclusive polarized deep inelastic scattering, and polarized ep collider studies [11]. Full NLO analyses are essential for a consistent interpretation of these experiments.

Many techniques for decoupling a single heavy quark are available. We rely on Witten's method [12], where the renormalization scheme is mass independent and improved Callan-Symanzik equations [13] can be exploited. In such schemes, the decoupling of heavy particles required by the Appelquist-Carrazone theorem [14] is not manifest. However, correct decoupling is ensured by applying the matching conditions of Bernreuther and Wetzel [15]; these relate coupling constant, mass and operator normalizations before and after the decoupling of a heavy quark. The advantages of this approach are its rigor and the fact that the final results are expressed in terms of renormalization group (RG) invariants. These invariants are Witten-style running couplings $\tilde{\alpha}_h$, one for each heavy quark $h=t,b,c$, and axial charges for nucleons in the residual theory with three light flavors.

We find that, when first t, then b, and finally c are decoupled from Eq. (2), the full NLO result is

$$2g_A^{(Z)}=(\Delta u-\Delta d-\Delta s)_{\rm inv}+\mathcal{P}(\Delta u+\Delta d+\Delta s)_{\rm inv}+O(m_{t,b,c}^{-1}) \tag{4}$$

where $\mathcal{P}$ is a polynomial in the running couplings $\tilde{\alpha}_h$,

$$\mathcal{P}=\frac{6}{23\pi}(\tilde{\alpha}_b-\tilde{\alpha}_t)\left\{1+\frac{125663}{82800\pi}\tilde{\alpha}_b+\frac{6167}{3312\pi}\tilde{\alpha}_t-\frac{22}{75\pi}\tilde{\alpha}_c\right\}-\frac{6}{27\pi}\tilde{\alpha}_c-\frac{181}{648\pi^2}\tilde{\alpha}_c^2+O(\tilde{\alpha}_{t,b,c}^3) \tag{5}$$

and $(\Delta q)_{\rm inv}$ denotes the scale-invariant version of Δq defined in the following way.

Let $\alpha_f=g_f^2/4\pi$ and $\beta_f(\alpha_f)$ be the gluon coupling and beta function for $\overline{\rm MS}$ renormalized quantum chromodynamics (QCD) with f flavors and $N_c=3$ colors, and let $\gamma_f(\alpha_f)$ be the gamma function for the singlet current

$$(\bar{u}\gamma_\mu\gamma_5 u+\bar{d}\gamma_\mu\gamma_5 d+\cdots)_f=\sum_{k=1}^{f}(\bar{q}_k\gamma_\mu\gamma_5 q_k)_f. \tag{6}$$

A scale-invariant current $(S_{\mu 5})_f$ is obtained when Eq. (6) is multiplied by

$$E_f(\alpha_f)=\exp\int_0^{\alpha_f}dx\,\frac{\gamma_f(x)}{\beta_f(x)}. \tag{7}$$

Up to $O(m_h^{-1})$ corrections, the invariant singlet charge (3) is given by

$$g_A^{(0)}|_{\rm inv}=E_3(\alpha_3)(\Delta u+\Delta d+\Delta s)_3 \\ =(\Delta u+\Delta d+\Delta s)_{\rm inv}. \quad (8)$$

Flavor-dependent, scale-invariant axial charges $\Delta q|_{\rm inv}$ such as

$$\Delta s|_{\rm inv}=\tfrac{1}{3}\,(g_A^{(0)}|_{\rm inv}-g_A^{(8)}) \quad (9)$$

can then be obtained from linear combinations of Eq. (8) and

$$g_A^{(3)}=\Delta u-\Delta d=(\Delta u-\Delta d)_{\rm inv}$$

$$g_A^{(8)}=\Delta u+\Delta d-2\Delta s=(\Delta u+\Delta d-2\Delta s)_{\rm inv}. \quad (10)$$

Here $g_A^{(3)}=1.267\pm 0.004$ is the isotriplet axial charge measured in neutron beta-decay, and $g_A^{(8)}=0.58\pm 0.03$ is the octet charge measured independently in hyperon beta decay. Taking $\tilde{\alpha}_t=0.1$, $\tilde{\alpha}_b=0.2$ and $\tilde{\alpha}_c=0.35$ in Eq. (5), we find a small heavy-quark correction factor $\mathcal{P}=-0.02$, with LO terms dominant.

Our results extend and make more precise the well known work of Collins, Wilczek and Zee [6] and Kaplan and Manohar [2], where heavy-quark effective theory was used to estimate $g_A^{(Z)}$ in leading order (LO) for sequential decoupling of t,b and t,b,c respectively. Our analysis is also influenced by a discussion of [6] by Chetyrkin and Kühn [16], who considered some aspects of NLO decoupling of the t quark from the neutral current and in particular, the requirement that the result be scale invariant. Related work has been done on heavy-quark production in polarized deep inelastic scattering using the QCD parton model [17] and in high-energy polarized γp and pp at NLO [18].

The plan of this paper is as follows. First is a brief review of Witten's application of improved Callan-Symanzik equations [13] to the decoupling of a heavy quark in mass-independent renormalization schemes. Next, we combine it with matching conditions [15] to deal with next-to-leading-order (NLO) calculations involving axial-vector currents. Following is then a direct derivation of Eq. (5) from Eq. (1) for the neutral current. Our concluding remarks indicate the result of extending Eq. (5) to simultaneous decoupling of t,b,c—done not only for numerical reasons, but also to check that the t,b contributions cancel for $m_t=m_b$.

We begin by considering mass-independent schemes, such as the modified minimal subtraction scheme $(\overline{\rm MS})$, where renormalized masses behave like coupling constants. This key property is exploited in Witten's method.

Let μ be the scale used to define dimensional regularization and renormalization. Then the $\overline{\rm MS}$ scale is

$$\bar{\mu}=\mu\sqrt{4\pi}e^{-\gamma/2}, \quad \gamma=0.5772\ldots. \quad (11)$$

We choose the same scale $\bar{\mu}$ irrespective of the number of flavors f being considered, and so hold $\bar{\mu}$ fixed as the heavy quarks (masses m_h) decouple:

$$F\to f \text{ flavors}, \ m_h\to\infty.$$

Also held fixed in this limit are the coupling α_f and light-quark masses m_{lf} of the *residual* f-flavor theory, and all momenta $\mathbf{p}$. Feynman diagrams for amplitudes

$$\mathcal{A}_F=\mathcal{A}_F(\mathbf{p},\bar{\mu},\alpha_F,m_{lF},m_h) \quad (12)$$

give rise to power series in m_h^{-1} modified by polynomials in $\ln(m_h/\bar{\mu})$. We consider just the leading power $\tilde{\mathcal{A}}_F$:

$$\mathcal{A}_F=\tilde{\mathcal{A}}_F\{1+O(1/m_h)\}. \quad (13)$$

As m_h tends to infinity, logarithms in $\tilde{\mathcal{A}}_F$ can be produced by any 1PI (one-particle irreducible) subgraph which contains at least one heavy-quark propagator and whose divergence by power counting is at least logarithmic. The effect is equivalent to shrinking all contributing 1PI parts of each diagram to a point. This means [14] that the F-flavor amplitudes $\tilde{\mathcal{A}}_F$ are the same as amplitudes $\mathcal{A}_f$ in the residual f-flavor theory, apart from m_h-dependent renormalizations of the coupling constant, light masses, and amplitudes:

$$\tilde{\mathcal{A}}_F(\mathbf{p},\bar{\mu},\alpha_F,m_{lF},m_h)$$

$$=\sum_{\mathcal{A}'}\mathcal{Z}_{\mathcal{A}\mathcal{A}'}(\alpha_F,m_h/\bar{\mu})\mathcal{A}'_f(\mathbf{p},\bar{\mu},\alpha_f,m_{lf}) \quad (14)$$

$$\alpha_f=\alpha_f(\alpha_F,m_h/\bar{\mu}), \quad m_{lf}=m_{lF}D(\alpha_F,m_h/\bar{\mu}). \quad (15)$$

Eventually, we will have to invert Eq. (15), i.e. use α_f and m_{lf} as dependent variables instead of α_F and m_{lF}, because we hold α_f and m_{lf} fixed as $m_h\to\infty$.

For any number of flavors f (including F), let

$$\mathcal{D}_f=\bar{\mu}\frac{\partial}{\partial\bar{\mu}}+\beta_f(\alpha_f)\frac{\partial}{\partial\alpha_f}+\delta_f(\alpha_f)\sum_{k=1}^{f}m_{kf}\frac{\partial}{\partial m_{kf}} \quad (16)$$

be the corresponding Callan-Symanzik operator. Then the amplitude $\mathcal{A}_F$ and hence its leading power $\tilde{\mathcal{A}}_F$ both satisfy an F-flavor improved Callan-Symanzik equation:

$$\{\mathcal{D}_F+\gamma_F(\alpha_F)\}\tilde{\mathcal{A}}_F=0. \quad (17)$$

In general, both γ_F and $\mathcal{Z}=(\mathcal{Z}_{\mathcal{A}\mathcal{A}'})$ are matrices.

If we substitute Eq. (14) in Eq. (17) and change variables,

$$\mathcal{D}_F=\bar{\mu}\frac{\partial}{\partial\bar{\mu}}+(\mathcal{D}_F\alpha_f)\frac{\partial}{\partial\alpha_f}+\sum_{k=1}^{f}(\mathcal{D}_F m_{kf})\frac{\partial}{\partial m_{kf}} \quad (18)$$

the result is an improved Callan-Symanzik equation for each residual amplitude,

$$\{\mathcal{D}_f+\gamma_f(\alpha_f)\}\mathcal{A}_f=0 \quad (19)$$

where the functions [12,15]

$$\beta_f(\alpha_f)=\mathcal{D}_F\alpha_f \quad (20)$$

$$\delta_f(\alpha_f)=\mathcal{D}_F \ln m_l \tag{21}$$

$$\gamma_f(\alpha_f)=\mathcal{Z}^{-1}[\gamma_F(\alpha_F)+\mathcal{D}_F]\mathcal{Z} \tag{22}$$

depend on α_f *alone*. The lack of m_l dependence of the renormalization factors in Eqs. (14) and (15) ensures mass-independent renormalization for the residual theory.

Although these equations hold for any $f<F$, their practical application is straightforward only when heavy quarks are decoupled one at a time. So we set $F=f+1$, where just one quark h is heavy. Then it is convenient to introduce a running coupling [12]

$$\tilde{\alpha}_h=\tilde{\alpha}_h(\alpha_F,\ln(m_h/\bar{\mu})) \tag{23}$$

associated with the $\overline{\mathrm{MS}}_F$ renormalized mass m_h:

$$\ln(m_h/\bar{\mu})=\int_{\alpha_F}^{\tilde{\alpha}_h} dx[1-\delta_F(x)]/\beta_F(x). \tag{24}$$

It satisfies the constraints

$$\tilde{\alpha}_h(\alpha_F,0)=\alpha_F,\tilde{\alpha}_h(\alpha_F,\infty)=0 \tag{25}$$

the latter being a consequence of the asymptotic freedom of the F-flavor theory ($F\leqslant 16$). Also, Eqs. (16), (20) and (24) imply that $\tilde{\alpha}_h$ is renormalization group (RG) invariant:

$$\mathcal{D}_F\tilde{\alpha}_h=0. \tag{26}$$

Witten's solution of Eq. (22) for the matrix $\mathcal{Z}$ is

$$\mathcal{Z}(\alpha_F,m_h/\mu)=\exp\left\{\int_{\alpha_F}^{\tilde{\alpha}_h}dx\frac{\gamma_F(x)}{\beta_F(x)}\right\}_{\mathrm{ord}}\mathcal{Z}(\tilde{\alpha}_h,1)$$
$$\times\exp-\left\{\int_{\alpha_f}^{\alpha_f(\tilde{\alpha}_h,1)}dx\frac{\gamma_f(x)}{\beta_f(x)}\right\}_{\mathrm{ord}} \tag{27}$$

where "ord" indicates x-ordering of matrix integrands in the exponentials. Note that it is the *relative* scaling between the initial and residual theories which matters.

For our NLO calculation, we need the formulas

$$\beta_f(x)=-\frac{x^2}{3\pi}\left(\frac{33}{2}-f\right)-\frac{x^3}{12\pi^2}(153-19f)+O(x^4)$$

$$\gamma_f(x)=\frac{x^2}{\pi^2}f+\frac{x^3}{36\pi^3}(177-2f)f+O(x^4)$$

$$\delta_f(x)=-\frac{2x}{\pi}+O(x^2) \tag{28}$$

where γ_f refers to the f-flavor singlet current (6) and includes the three-loop term found by Larin [19] and Chetyrkin and Kühn [16].

Our matching procedure amounts to evaluating to NLO accuracy the quantities $\tilde{\alpha}_h$, $\alpha_f(\tilde{\alpha}_h,1)$ and $\mathcal{Z}(\tilde{\alpha}_h,1)$ in Eq. (27), such that the answers depend on α_f and not α_F.

Bernreuther and Wetzel [15] applied the Appelquist-Carrazone decoupling theorem [14] to the gluon coupling constant α_Q^{MO} renormalized at space-like momentum Q,

$$\alpha_Q^{\mathrm{MO}}\big|_{\mathrm{with}\ h}=\alpha_Q^{\mathrm{MO}}\big|_{\mathrm{no}\ h}+O(m_h^{-1}) \tag{29}$$

and compared calculations of α_Q^{MO} in the $F=f+1$ and f-flavor $\overline{\mathrm{MS}}$ theories. This reduces to a determination of the leading power of the one-h-loop $\overline{\mathrm{MS}}_F$ gluon self-energy. The result is a matching condition

$$\alpha_F^{-1}-\alpha_f^{-1}=C_{\mathrm{LO}}\ln(m_h/\bar{\mu})+C_{\mathrm{NLO}}+O(\alpha_f,m_h^{-1}) \tag{30}$$

with α_f-independent LO and NLO coefficients given by

$$C_{\mathrm{LO}}=\frac{1}{3\pi},\quad C_{\mathrm{NLO}}=0. \tag{31}$$

As a result, we find

$$\alpha_f(\tilde{\alpha}_h,0)=\tilde{\alpha}_h+O(\tilde{\alpha}_h^3)\underset{\mathrm{NLO}}{=}\tilde{\alpha}_h. \tag{32}$$

Bernreuther and Wetzel showed that it is possible to deduce *all* LO and NLO terms in Eq. (30) from Eq. (31) and β_f and δ_f in Eq. (28). We have done the calculation explicitly:

$$\alpha_{f+1}^{-1}\underset{\mathrm{NLO}}{=}\alpha_f^{-1}+\frac{1}{3\pi}\ln\frac{m_h}{\bar{\mu}}+c_f\ln\left[1+\frac{\alpha_f}{3\pi}\ln\frac{m_h}{\bar{\mu}}\right]$$
$$+d_f\ln\left[1+\frac{\alpha_f}{3\pi}\left(\frac{33}{2}-f\right)\ln\frac{m_h}{\bar{\mu}}\right]$$
$$c_f=\frac{142-19f}{2\pi(31-2f)},\quad d_f=\frac{57+16f}{2\pi(33-2f)(31-2f)}. \tag{33}$$

From Eq. (24), we have also found $\tilde{\alpha}_h$ in NLO,

$$\tilde{\alpha}_h^{-1}\underset{\mathrm{NLO}}{=}\alpha_f^{-1}+\frac{1}{3\pi}\left(\frac{33}{2}-f\right)\ln\frac{\bar{m}_h}{\bar{\mu}}+\frac{153-19f}{2\pi(33-2f)}$$
$$\times\ln\left[1+\frac{\alpha_f}{3\pi}\left(\frac{33}{2}-f\right)\ln\frac{m_h}{\bar{\mu}}\right] \tag{34}$$

where $\bar{m}_h$ is Witten's RG invariant mass:

$$\bar{m}_h=m_h\exp\int_{\alpha_F}^{\tilde{\alpha}_h}dx\,\delta_F(x)/\beta_F(x). \tag{35}$$

If desired, $\ln(\bar{m}_h/\bar{\mu})$ can be eliminated by substituting

$$\ln\frac{\bar{m}_h}{\bar{\mu}}\underset{\mathrm{LO}}{=}\ln\frac{m_h}{\bar{\mu}}-\frac{12}{31-2f}\ln\left[1+\frac{\alpha_f}{3\pi}\left(\frac{31}{2}-f\right)\ln\frac{m_h}{\bar{\mu}}\right]. \tag{36}$$

Therefore the asymptotic formula for $\tilde{\alpha}_h$ as $m_h\to\infty$ is

$$\tilde{\alpha}_h \sim 3\pi \Big/ \left\{ \left(\frac{33}{2}-f\right)\ln\frac{m_h}{\bar{\mu}} + k_f \ln\ln\frac{m_h}{\bar{\mu}} + O(1) \right\}$$

$$k_f = \frac{3(153-19f)}{2(33-2f)} - \frac{6(33-2f)}{31-2f}. \tag{37}$$

To find the matrix $\mathcal{Z}(\tilde{\alpha}_h,1)$ in NLO, we need a matching condition for the $\overline{\text{MS}}$ amplitude $\Gamma_{\mu 5}$ for $\bar{h}\gamma_\mu\gamma_5 h$ to couple to a light quark l. We have calculated the leading power due to the two-loop diagram  :

$$\Gamma_{\mu 5} = \left(\frac{\alpha_F}{\pi}\right)^2 \gamma_\mu\gamma_5 \left(\ln\frac{m_h}{\bar{\mu}} + \frac{1}{8}\right) + O(\alpha_F^3, m_h^{-1}). \tag{38}$$

Consequently, there is a NLO term $\tilde{\alpha}_h^2/8\pi^2$ in $\mathcal{Z}(\tilde{\alpha}_h,1)$ for $\bar{h}\gamma_\mu\gamma_5 h$ to produce $\bar{l}\gamma_\mu\gamma_5 l$ as $m_h \to \infty$.

Now we consider the special case where heavy quarks are decoupled from the weak neutral axial current. Let us adopt the shorthand notation q_f for $\overline{\text{MS}}$ currents $(\bar{q}\gamma_\mu\gamma_5 q)_f$ in the f-flavor theory, e.g. the neutral current $J_{\mu 5}^{(\mathrm{Z})}$ and the scale-invariant singlet current $(S_{\mu 5})_f$:

$$J^{\mathrm{Z}} = \tfrac{1}{2}(t-b+c-s+u-d)_6 \tag{39}$$

$$S_f = E_f(\alpha_f)(u+d+s+\cdots)_f. \tag{40}$$

We begin by decoupling the t quark. Because of

$$(c-s+u-d)_6 = (c-s+u-d)_5 + O(1/m_t) \tag{41}$$

we see that Eq. (27) is nontrivial only for

$$(t-b)_6 = \mathcal{Z}_{6\to 5}(u+d+s+c+b)_5 + \tfrac{1}{5}(u+d+s+c-4b)_5 + O(1/m_t). \tag{42}$$

Since $(t-b)_6$ is scale invariant, we have $\gamma_F = 0$ in Eq. (27):

$$\mathcal{Z}_{6\to 5}(\alpha_6, m_t/\bar{\mu}) \underset{\text{NLO}}{=} \mathcal{Z}_{6\to 5}(\tilde{\alpha}_t,1)\exp - \int_{\alpha_5}^{\tilde{\alpha}_t} dx \frac{\gamma_5(x)}{\beta_5(x)}. \tag{43}$$

The operator matching condition (38) corresponds to

$$t_6 = \frac{\alpha_6^2}{\pi^2}\left(\ln\frac{m_t}{\bar{\mu}} + \frac{1}{8}\right)(u+d+s+c+b)_5 + O(\alpha_6^3, m_t^{-1}) \tag{44}$$

and so we conclude:

$$\mathcal{Z}_{6\to 5}(\tilde{\alpha}_t,1) = -\tfrac{1}{5} + (8\pi^2)^{-1}\tilde{\alpha}_t^2 + O(\tilde{\alpha}_t^3). \tag{45}$$

Equation (43) is to be expanded about $\tilde{\alpha}_t \sim 0$ with α_5 held fixed. In that limit, the exponential tends to the constant factor $E_5(\alpha_5)$ of Eq. (7). This factor combines with the singlet current in Eq. (42) to form the scale-invariant operator S_5, as required by $\mathrm{RG}_{f=5}$ invariance. The full NLO result is then obtained by writing

$$(t-b)_6 \underset{\text{NLO}}{=} \mathcal{Z}_{6\to 5}(\tilde{\alpha}_t,1)\exp\left\{-\int_0^{\tilde{\alpha}_t} dx \frac{\gamma_5(x)}{\beta_5(x)}\right\} S_5 + \tfrac{1}{5}(u+d+s+c-4b)_5 \tag{46}$$

and expanding in $\tilde{\alpha}_t$, keeping all quadratic terms:

$$(t-b)_6 = \left\{-\frac{1}{5} - \frac{6}{23}\frac{\tilde{\alpha}_t}{\pi}\left(1 + \frac{6167}{3312}\frac{\tilde{\alpha}_t}{\pi}\right) + O(\tilde{\alpha}_t^3)\right\} S_5 + \tfrac{1}{5}(u+d+s+c-4b)_5 + O(1/m_t). \tag{47}$$

Next we decouple the b quark. Here, it is natural to define five-flavor quantities $\tilde{\alpha}_{b_5}$ and $\bar{m}_{b_5}$ analogous to the six-flavor running coupling $\tilde{\alpha}_t$ and mass $\bar{m}_t$ for the top quark:

$$\ln\frac{m_{b5}}{\bar{\mu}} = \int_{\alpha_5}^{\tilde{\alpha}_{b_5}} dx \frac{1-\delta_5(x)}{\beta_5(x)},$$

$$\ln\frac{\bar{m}_{b_5}}{m_{b_5}} = \int_{\alpha_5}^{\tilde{\alpha}_{b_5}} dx \frac{\delta_5(x)}{\beta_5(x)}. \tag{48}$$

Equations (20) and (21) imply that $\tilde{\alpha}_{b_5}$ and $\bar{m}_{b_5}$ are both $\mathrm{RG}_{f=5}$ and $\mathrm{RG}_{f=6}$ invariant

$$\mathcal{D}_5\tilde{\alpha}_{b_5} = 0 = \mathcal{D}_6\tilde{\alpha}_{b_5}, \quad \mathcal{D}_5\bar{m}_{b_5} = 0 = \mathcal{D}_6\bar{m}_{b_5} \tag{49}$$

and hence physically significant in the original six-flavor theory. So we write $\tilde{\alpha}_b$ and $\bar{m}_b$ for $\tilde{\alpha}_{b_5}$ and $\bar{m}_{b_5}$.

Consider decoupling the b quark from Eq. (47). The NLO matching condition (38) becomes

$$b_5 = \frac{\alpha_5^2}{\pi^2}\left(\ln\frac{\bar{m}_{b_5}}{\bar{\mu}} + \frac{1}{8}\right)(u+d+s+c)_4 + O(\alpha_5^3, m_{b5}^{-1}) \tag{50}$$

so the nonsinglet current in Eq. (47) can be written

$$(u+d+s+c-4b)_5 = \{1-(\tilde{\alpha}_b^2/2\pi^2)\}E_4^{-1}(\tilde{\alpha}_b)S_4 + O(\tilde{\alpha}_b^3, m_{b5}^{-1}). \tag{51}$$

For the singlet current S_5 in Eq. (47), we find

$$S_5 = E_5(\tilde{\alpha}_b)\left\{1 + \frac{\tilde{\alpha}_b^2}{8\pi^2}\right\}E_4^{-1}(\tilde{\alpha}_b)S_4 + O(\tilde{\alpha}_b^3, m_{b5}^{-1}) \tag{52}$$

taking into account the definitions (7) and (40). Then we expand Eqs. (51) and (52) in $\tilde{\alpha}_b$, keeping quadratic terms:

$$(t-b)_6 = \frac{6}{23\pi}(\tilde{\alpha}_b - \tilde{\alpha}_t)\left\{1 + \frac{125663}{82800\pi}\tilde{\alpha}_b + \frac{6167}{3312\pi}\tilde{\alpha}_t\right\} S_4 + O(\tilde{\alpha}_{t,b}^3, m_{t,b}^{-1}). \tag{53}$$

The same technique can be applied to decouple the c quark from S_4 in Eq. (53) and $(c-s+u-d)_4$ [the result of decoupling b from Eq. (41)]. That yields the final results (4) and (5) given in the introduction.

Notice that our results depend on two key features:

(i) Like previous workers in this area, we decouple heavy quarks sequentially, i.e. one at a time.

(ii) Our running couplings $\tilde{\alpha}_t$, $\tilde{\alpha}_b$ and $\tilde{\alpha}_c$, which correspond to Witten's prescription [12], are all renormalization group invariant.

The restriction to sequential decoupling is numerically reasonable for the t quark, but dubious for the b and c quarks, because it amounts to an assumption that $\ln(m_c/\bar{\mu})$ is negligible compared with $\ln(m_b/\bar{\mu})$. This inhibits detailed comparison of NLO results with data, which ought to be carried out with NLO accuracy [20].

There is also a theoretical issue here: one would like to check that, in the limit $m_t = m_b$, the t and b contributions cancel. However, that is outside the region of validity $\ln(m_t/\bar{\mu}) \gg \ln(m_b/\bar{\mu})$ for sequential decoupling.

For these reasons, we have extended our analysis to the case of simultaneous decoupling, where the mass logarithms are allowed to grow large together: $\ln(m_c/\bar{\mu}) \sim \ln(m_b/\bar{\mu}) \sim \ln(m_t/\bar{\mu}) \to$large. This requires a considerable theoretical development of matching conditions and the renormalization group, which we will present separately. It involves the construction of running couplings α_t, α_b, α_c with the following properties: (i) They are renormalization group invariant; (ii) they are defined for $m_t \geqslant m_b \geqslant m_c$, and can have a nontrivial dependence on more than one heavy-quark mass; (iii) in the special case of sequential decoupling, they agree with $\tilde{\alpha}_t$, $\tilde{\alpha}_b$ and $\tilde{\alpha}_c$ to NLO; and (iv) for the case of equal masses, they coincide, e.g.

$$\alpha_t = \alpha_b \quad \text{for} \quad m_t = m_b . \tag{54}$$

Then we find that the result for the simultaneous decoupling of the t,b,c quarks from the neutral current is of the same form (4) as the sequential answer, but with the sequential running couplings in Eq. (5) replaced by our simultaneous couplings α_t, α_b, and α_c:

$$\mathcal{P} = \frac{6}{23\pi}(\alpha_b - \alpha_t)\left\{1 + \frac{125663}{82800\pi}\alpha_b + \frac{6167}{3312\pi}\alpha_t - \frac{22}{75\pi}\alpha_c\right\} - \frac{6}{27\pi}\alpha_c - \frac{181}{648\pi^2}\alpha_c^2 + O(\alpha_{t,b,c}^3). \tag{55}$$

Notice the factorization of the terms depending on α_t and α_b. Given Eq. (54), the factor $\alpha_b - \alpha_t$ ensures that all contributions from b and t quarks cancel (as they should) for $m_t = m_b$.

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[20] This includes matching conditions for the b and c masses, to be discussed elsewhere.