Noncommutative oscillators and the commutative limit

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It is shown in first order perturbation theory that anharmonic oscillators in noncommutative space behave smoothly in the commutative limit just as harmonic oscillators do. The noncommutativity provides a method for converting a problem in degenerate perturbation theory to a nondegenerate problem.

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In the last few years theories in noncommutative space [1–5] have been studied extensively. While the motivation for this kind of space with noncommuting coordinates is mainly theoretical, it is possible to look experimentally for departures from the usually assumed commutativity among the space coordinates [6–8]. So far no clear departure has been found, but it is clear that any experiment can only provide a bound on the amount of noncommutativity, and that more precise experiments in the future can reveal a small amount. Meanwhile, there are some theoretical issues which have arisen in the course of these investigations. If one calls the noncommutativity parameter θ, so that two noncommuting spatial coordinates $\hat{x},\hat{y}$ satisfy the relation

$$[\hat{x},\hat{y}]=i\theta,$$

one would expect ordinary commutative space to emerge in the limit $\theta\to 0$. In many field theoretical and quantum mechanical problems, however, the passage from the noncommutative space to its commutative limit has *not* appeared to be smooth [9–12]. The literature is replete with expressions where θ appears in the denominator. The simplest system is the two-dimensional harmonic oscillator. As in commutative space, this quantum mechanical problem is again exactly solvable [13–16], and the spectrum is in fact smooth in the commutative limit, but the literature is not very clear about the situation: there seems to be a lack of smoothness in the generic case [17,18]. For a clarification of this limit, we will first review the two-dimensional harmonic oscillator and then go over to a perturbation $(\hat{x}^2+\hat{y}^2)^2$ to see if the smoothness survives. The anharmonic problem cannot be solved exactly even in commutative space. But a perturbative treatment shows that the quartic terms do not destroy the smoothness of the $\theta\to 0$ limit. It is interesting to note that the commutative oscillator here involves a degenerate perturbation problem, while the noncommutative one is nondegenerate.

Let us write the two-dimensional anharmonic oscillator potential in the form

$$\frac{1}{2}m\omega^2(\hat{x}^2+\hat{y}^2)+\alpha(\hat{x}^2+\hat{y}^2)^2. \tag{1}$$

The noncommuting coordinates can be expressed in terms of commuting coordinates and their momenta in the form

$$\hat{x}=x-\frac{1}{2\hbar}\theta p_y,$$

$$\hat{y}=y+\frac{1}{2\hbar}\theta p_x. \tag{2}$$

The Hamiltonian for the unperturbed system is

$$\begin{aligned}H_{HO}&=\frac{1}{2m}(p_x^2+p_y^2)+\frac{1}{2}m\omega^2(\hat{x}^2+\hat{y}^2)\\&=\frac{1}{2m}(p_x^2+p_y^2)+\frac{1}{2}m\omega^2\left[\left(x-\frac{1}{2\hbar}\theta p_y\right)^2+\left(y+\frac{1}{2\hbar}\theta p_x^2\right)^2\right]\\&=\left(\frac{1}{2m}+\frac{m\theta^2\omega^2}{8\hbar^2}\right)(p_x^2+p_y^2)+\frac{1}{2}m\omega^2(x^2+y^2)\\&\quad-\frac{m\omega^2\theta}{2\hbar}(xp_y-yp_x).\end{aligned} \tag{3}$$

It is convenient to set $(1/2m+m\theta^2\omega^2/8\hbar^2)\equiv 1/2M$ and $m\omega^2\equiv M\Omega^2$. One can introduce the ladder operators through the equations

$$\begin{aligned}a_x&=\sqrt{\frac{M\Omega}{2\hbar}}\left(x+\frac{ip_x}{M\Omega}\right),\quad a_x^\dagger=\sqrt{\frac{M\Omega}{2\hbar}}\left(x-\frac{ip_x}{M\Omega}\right),\\a_y&=\sqrt{\frac{M\Omega}{2\hbar}}\left(y+\frac{ip_y}{M\Omega}\right),\quad a_y^\dagger=\sqrt{\frac{M\Omega}{2\hbar}}\left(y-\frac{ip_y}{M\Omega}\right),\\x&=\sqrt{\frac{\hbar}{2M\Omega}}(a_x+a_x^\dagger),\quad p_x=\frac{1}{i}\sqrt{\frac{M\Omega\hbar}{2}}(a_x-a_x^\dagger),\\y&=\sqrt{\frac{\hbar}{2M\Omega}}(a_y+a_y^\dagger),\quad p_y=\frac{1}{i}\sqrt{\frac{M\Omega\hbar}{2}}(a_y-a_y^\dagger).\end{aligned} \tag{4}$$

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In terms of these operators the unperturbed Hamiltonian takes the form

$$H_{HO}=\hbar\Omega(a_x^\dagger a_x+a_y^\dagger a_y+1)-\frac{M\Omega^2\theta}{2i}(a_x^\dagger a_y-a_y^\dagger a_x). \tag{5}$$

In view of the Schwinger representation for the angular momentum,

$$J_1=\frac{1}{2}(a_x^\dagger a_y+a_y^\dagger a_x),$$

$$J_2=\frac{1}{2i}(a_x^\dagger a_y-a_y^\dagger a_x),$$

$$J_3=\frac{1}{2}(a_x^\dagger a_x-a_y^\dagger a_y), \tag{6}$$

the part $a_x^\dagger a_y-a_y^\dagger a_x$ in H_{HO} is seen to be $2iJ_2$. Under the unitary transformation

$$\begin{pmatrix} a_x \\ a_y \end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ i & -1 \end{pmatrix}\begin{pmatrix} a_x' \\ a_y' \end{pmatrix}, \tag{7}$$

in which this piece takes the diagonal form $2iJ_3'$, the Hamiltonian becomes

$$H_{HO}=\hbar\Omega(a_x'^\dagger a_x'+a_y'^\dagger a_y'+1)-\frac{M\Omega^2\theta}{2}(a_x'^\dagger a_x'-a_y'^\dagger a_y'). \tag{8}$$

If $\hat{N}_x=a_x'^\dagger a_x'$ and $\hat{N}_y=a_y'^\dagger a_y'$ are the transformed number operators of the harmonic oscillators in the x and y directions, respectively, one can write

$$H_{HO}=\hbar\Omega(\hat{N}_x+\hat{N}_y+1)-\frac{M\Omega^2\theta}{2}(\hat{N}_x-\hat{N}_y). \tag{9}$$

The eigenvalues of the unperturbed Hamiltonian are therefore

$$E^0_{n_x,n_y}=\hbar\Omega(n_x+n_y+1)-\frac{M\Omega^2\theta}{2}(n_x-n_y), \tag{10}$$

where n_x,n_y are non-negative integers. In terms of m and ω, the eigenvalues can be written as

$$E^0_{n_x,n_y}=m\omega^2\hbar^2\left(\frac{1}{m^2\omega^2\hbar^2}+\frac{\theta^2}{4\hbar^4}\right)^{1/2}(n_x+n_y+1)-\frac{m\omega^2\theta}{2}(n_x-n_y). \tag{11}$$

These eigenvalues are generically nondegenerate. In the limit $\theta\to 0$ they smoothly reduce to the standard degenerate expression

$$E^0_{n_x,n_y}\to\hbar\omega(n_x+n_y+1). \tag{12}$$

Let us now introduce a perturbation of the form $\alpha(\hat{x}^2+\hat{y}^2)^2$. In terms of the ladder operators,

$$\hat{x}=\sqrt{\frac{\hbar}{2M\Omega}}(a_x+a_x^\dagger)-\frac{1}{i}\sqrt{\frac{M\Omega\theta^2}{8\hbar}}(a_y-a_y^\dagger),$$

$$\hat{y}=\sqrt{\frac{\hbar}{2M\Omega}}(a_y+a_y^\dagger)+\frac{1}{i}\sqrt{\frac{M\Omega\theta^2}{8\hbar}}(a_x-a_x^\dagger). \tag{13}$$

But under the unitary transformation (7),

$$\hat{x}=\frac{1}{2}\left(\sqrt{\frac{\hbar}{M\Omega}}-\sqrt{\frac{M\Omega\theta^2}{4\hbar}}\right)a_x'^\dagger+\frac{i}{2}\left(\sqrt{\frac{\hbar}{M\Omega}}+\sqrt{\frac{M\Omega\theta^2}{4\hbar}}\right)a_y'^\dagger,$$
$$=\beta(a_x'+a_x'^\dagger)-i\gamma(a_y'-a_y'^\dagger), \tag{14}$$

where $\beta\equiv\frac{1}{2}(\sqrt{\hbar/M\Omega}-\sqrt{M\Omega\theta^2/4\hbar})$ and $\gamma\equiv\frac{1}{2}(\sqrt{\hbar/M\Omega}+\sqrt{M\Omega\theta^2 4\hbar})$. One also has

$$\hat{y}=i\beta(a_x'-a_x'^\dagger)-\gamma(a_y'+a_y'^\dagger). \tag{15}$$

Hence,

$$(\hat{x}^2+\hat{y}^2)=4\beta^2a_x'^\dagger a_x'+4\gamma^2a_y'^\dagger a_y'+2(\beta^2+\gamma^2)-4i\beta\gamma a_x'a_y'+4i\beta\gamma a_x'^\dagger a_y'^\dagger. \tag{16}$$

This is a nondiagonal operator in the basis in which the unperturbed eigenstates of the Hamiltonian are diagonal, but its effect on the eigenvalues can be studied in first order perturbation theory. In view of the nondegeneracy of the unperturbed eigenvalues, it is sufficient to calculate the expectation values of $(\hat{x}^2+\hat{y}^2)^2$ in the states $|n_x,n_y\rangle$. Thus,

$$\alpha\langle n_x,n_y|(\hat{x}^2+\hat{y}^2)^2|n_x,n_y\rangle=\alpha[16\beta^4n_x(n_x+1)+16\gamma^4n_y(n_y+1)+32\beta^2\gamma^2(n_x+n_y)+64\beta^2\gamma^2n_xn_y+4(\beta^4+\gamma^4)+24\beta^2\gamma^2]. \tag{17}$$

The expression on the right-hand side gives the first order correction to the eigenvalues caused by the anharmonicity. In the $\theta\to 0$ limit, β,γ behave smoothly and this correction goes over smoothly to a finite value:

$$E^1_{n_x,n_y}\to\alpha\left(\frac{\hbar}{m\omega}\right)^2(n_x^2+n_y^2+4n_xn_y+3n_x+3n_y+2). \tag{18}$$

The smooth transition should make it obvious that the same correction must be obtained in the case of the commutative anharmonic oscillator. However, the eigenvalues of the unperturbed commutative oscillator are degenerate, so it may be more convincing if the agreement is shown explicitly after doing a degenerate perturbation theory calculation.

In the commutative case $\theta=0$, both β,γ reduce to $\frac{1}{2}\sqrt{\hbar/m\omega}\equiv\beta_0$. The operator of interest is

$$(x^2+y^2) = 4\beta_0^2(a_x'^\dagger a_x' + a_y'^\dagger a_y' + 1 - i a_x' a_y' + i a_x'^\dagger a_y'^\dagger), \tag{19}$$

where the unitarily transformed ladder operators are used for ease of comparison with the previous calculation. The matrix element of the square of this operator has to be calculated between degenerate eigenstates of the unperturbed Hamiltonian. Any choice of basis for the degenerate states is permissible: it is convenient to use the eigenstates of $a_x'^\dagger a_x', a_y'^\dagger a_y'$ instead of the untransformed number operators. Thus, states with different values of n_x, n_y but *the same values of* n_x+n_y are to be considered. Now $(x^2+y^2)^2|n_x,n_y\rangle$ contains the states $|n_x,n_y\rangle, |n_x-2,n_y-2\rangle, |n_x+2,n_y+2\rangle, |n_x-1,n_y-1\rangle, |n_x+1,n_y+1\rangle$. Out of these, only $|n_x,n_y\rangle$ has the original value of n_x+n_y, while all the other states have different values. Thus, although the operator of interest is not completely diagonal in the energy basis, it is diagonal in the subspace of states with fixed n_x+n_y. The diagonal value is

$$16\beta_0^4[(n_x+n_y+1)^2+(n_x+1)(n_y+1)+n_x n_y] = \left(\frac{\hbar}{m\omega}\right)^2(n_x^2+n_y^2+4n_x n_y+3n_x+3n_y+2). \tag{20}$$

These diagonal values are also the eigenvalues of the matrix, so that the correction to the degenerate unperturbed energy eigenvalue is $\alpha(\hbar/m\omega)^2(n_x^2+n_y^2+4n_x n_y+3n_x+3n_y+2)$, which agrees with the $\theta\to 0$ limit (18) of the noncommutative calculation. One could also carry out the calculation in the untransformed occupation number basis, where the matrix in the space of the degenerate eigenvectors is not diagonal to begin with, but on diagonalization, the same eigenvalues are obtained.

Thus, not only the exactly solvable harmonic oscillator but even the first order perturbation theory result for the eigenvalues of the two-dimensional noncommutative anharmonic oscillator behave smoothly in the commutative limit. It is conceivable that, as is widely believed, all problems may not show this smoothness. But there clearly is a class of Hamiltonians, not just an isolated Hamiltonian, whose eigenvalues have a smooth θ dependence.

A by-product of this demonstration is the emergence of a method of handling the degenerate perturbation theory through conversion to a nondegenerate problem. This happens through the introduction of the parameter θ, which can be regarded as a mathematical trick from the point of view of commutative theory. The calculation may be done for nonzero θ and then the limit $\theta\to 0$ taken.

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