Constructing bidimensional scalar field theory models from zero mode fluctuations

G. Flores-Hidalgo*
Centro Brasileiro de Pesquisas Fisicas-CBPF, Rua Dr. Xavier Sigaud 150, 22290-180 Rio de Janeiro, RJ, Brazil

N. F. Svaiter†
Center of Theoretic Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

(Received 6 December 2001; revised manuscript received 30 April 2002; published 31 July 2002)

In this paper we review how to construct scalar field theories in two-dimensional space-time that support kink or bouncelike solutions starting from solvable Schrödinger equations. Three different Schrödinger potentials are analyzed. We obtain two new models starting from the Morse and Scarf II hyperbolic potentials, i.e. the $U(\phi)=\phi^2\ln^2(\phi^2)$ and $U(\phi)=\phi^2\cos^2\ln(\phi^2)$ models, respectively. Also we give a closed expression for the (renormalized) kink quantum mass corrections in the case of the second model and evaluate it numerically.

DOI: 10.1103/PhysRevD.66.025031 PACS number(s): 11.30.Pb, 03.65.Fd, 11.10.Ef

I. INTRODUCTION

Solitons are solutions of nonlinear equations that have the following fundamental properties: their profile is stable, the energy associated with them is finite and also they behave as particles in the sense that multisolitonic solutions behave as independent one-soliton solutions as time goes to infinity [1]. Also, there is a less restricted class of solutions for nonlinear equations that has the same properties of solitonic solutions except the property of retaining their shape after collision. In this case such solutions are called solitary waves. In general solitons can exist in any $(d+1)$-dimensional space-time. In the $(1+1)$-dimensional case, the static solutions are called kinks. These solutions link two degenerate trivial vacua of the theory. An important property of these solutions is that they remain stable when quantum corrections are taken into account. On the other hand, there are solutions that become unstable when quantum corrections are taken into account. These solutions are called lumps or bounces.

In $1+1$ dimensions any scalar field-theoretical model is renormalizable. In order to render the theory finite it suffices to use a normal ordering prescription. But, as is well known, one of the main interests in $(1+1)$-dimensional quantum field theory is the possibility of understanding their nonperturbative aspects. And this is the reason why we should not consider arbitrary models, since we will not be always able to perform a nonperturbative analysis. One of the nonperturbative aspects is the solitonic sector of a model. In this sector, there are some models (the so-called integrable models) that can be solved exactly at the classical and quantum level. The best known example is the sine-Gordon model [2]. On the other hand, the $(1+1)$ ϕ^4 kink model admits exact kink-like solutions at the classical level and admits an analytical treatment at the semiclassical level; i.e. we can compute quantum corrections at order $\hbar$. In this paper we are interested in the construction of models of this type. In order to do this, we make use of the fact that the first quantum corrections around static soliton-like solutions are given by a one-dimensional Schrödinger equation (SE) and that the ground state eigenfunction of this SE is equal to the spatial derivative of the kink-like solution. Also we will be interested in the construction of models that permit lump-like solutions. In this case we use the fact that the first excited eigenfunction of the SE is equal to the spatial derivative of the lump-like solution. In both cases the eigenvalue of these eigenfunctions is zero and this is the reason why they are called zero mode eigenfunctions. We call stable and unstable models the models that permit, respectively, kink and lump-like (or bounce-like) solutions.

In the case of stable models the SE describes the excited states of the soliton quantum state as well as the scattering of particles by the soliton quantum state. Also the quantum mass correction (at order $\hbar$) for the soliton quantum state is given in terms of the eigenvalues of the SE [3]. In the case of unstable models, the decay rate of the false vacuum (in the so-called thin wall approximation) is given in terms of the eigenvalues of the SE [4]. Anyway, in both cases, in order to obtain analytical information it will be necessary that the SE be exactly solvable. Then it will be interesting to obtain field-theoretical models starting from exactly solvable SE's since in this case we have a chance to perform analytical calculations. We believe this fact was first stressed in Ref. [5]. More recently, using supersymmetric quantum mechanics this research program was continued in Refs. [6] and [7]. For other interesting references see, for instance, [8,9] and [10]. We would like to stress that previously to these works, in Ref. [11], the author suggested the construction of solitonic profiles using isospectral Hamiltonians.

The organization of this paper is the following: In Sec. II we show how to construct scalar field theory models from zero mode solutions. In Sec. III we analyze the models that arise from the Rosen-Morse II hyperbolic potential. In Secs. IV and V we perform the same analysis for the Morse and the Scarf II hyperbolic potentials, respectively. In Sec. VI we give a renormalized expression for the quantum mass corrections for the model constructed in Sec. V and evaluate it numerically. Conclusions are given in Sec. VII. The reader that is more interested in the computation of quantum mass

*Email address: gflores@cbpf.br

†Permanent address: Centro Brasileiro de Pesquisas Fisicas, Rio de Janeiro, RJ 22290-180, Brazil. Email address: svaiter@lns.mit.edu, nfuxsvai@cbpf.br

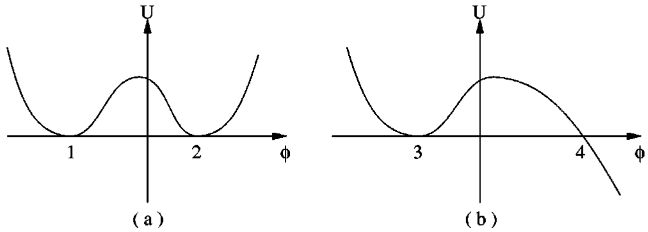

FIG. 1. (a) $U(\phi)$ with two degenerate vacua and (b) with a false vacuum.

corrections can go directly to Sec. VI. Throughout this paper we use the words kink and soliton as synonymous. Also we use natural units $\hbar = c = 1$.

II. CONSTRUCTING THE FIELD THEORY MODELS

In this section we briefly review how to construct scalar field theory models starting from solvable Schrödinger equations. We start from a Lagrangian

$$L = \int dx \left(\frac{1}{2} \partial_\mu \phi \ \ \partial^\mu \phi - U(\phi) \right), \tag{2.1}$$

where $\mu = 0{,}1$ and $U(\phi)$ is a density potential having at least two degenerate absolute minima as showed in Fig. 1(a) or a local minima (a false vacuum) as showed in Fig. 1(b). The classical equation of motion for static configurations are given from Eq. (2.1)

$$\frac{d^2\phi}{dx^2} = U'(\phi). \tag{2.2}$$

Equation (2.2) can be analyzed making use of a particle mechanical analogy. Suppose that ϕ describes the position of a particle and x is the time. Consequently, Eq. (2.2) is the equation of motion of a particle in a conservative potential $-U(\phi)$. In order to analyze Eq. (2.2) we have to take into account only the possible trajectories of the ''particle'' in the inverted potential. We are interested only in solutions with a finite energy, in other words, solutions that have a finite interval of motion in ϕ but that are not oscillatory. From the inverted potential $-U(\phi)$ depicted in Fig. 1(a) it is easy to see that such a requirement is satisfied only for the motion that takes place between the absolute minima given by points 1 and 2. Using the same argument for the case described by Fig. 1(b) we see that the only allowed motion is that which starts in point 3, bounces in 4 and returns to point 3. In the first case the static solution is know as a kink, while in the second case such solution is called a lump or a bounce. From Figs. 1(a) and 1(b) we see that these solutions are integrals of motion with zero energy, using the particle mechanical analogy. Then, from Eq. (2.2) we obtain

$$\frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 = U(\phi), \tag{2.3}$$

an equation that is known as the Bogomol'nyi condition [12]. From Eq. (2.3) it is straightforward to obtain the kinks or lumps (we denote them as ϕ_c) by solving the integral

$$x - x_0 = \int^{\phi_c(x)} \frac{d\phi}{\sqrt{2U(\phi)}} \tag{2.4}$$

and inverting it.

Next, we can consider the first quantum corrections around the classical static field configuration. For such purpose, we expand the time-dependent field $\phi(x,t)$ around the static field configuration, i.e. $\phi(x,t) = \phi_c(x) + \eta(x,t)$, replace it in Eq. (2.1) and retaining only quadratic terms (this approximation is valid at order $\hbar$) in η we obtain the following Lagrangian:

$$L = L[\phi_c] + \int dx \left[\frac{1}{2} \frac{d^2\eta}{dt^2} - \frac{1}{2}\eta\left(-\frac{d^2}{dx^2} + U''[\phi_c(x)] \right)\eta \right]. \tag{2.5}$$

As a next step, we use the expansion $\eta(x,t) = \Sigma_n q_n(t)\psi_n(x)$, and choosing the complete basis $\{\psi_n\}$ as solutions of the Schrödinger equation

$$\left[-\frac{d^2}{dx^2} + U''[\phi_c(x)] \right]\psi_n(x) = \omega_n^2 \psi_n(x), \tag{2.6}$$

we reduce the Lagrangian given by Eq. (2.5) to

$$L = L[\phi_c] + \frac{1}{2}\sum_n (\dot{q}_n^2 - \omega_n^2 q_n^2). \tag{2.7}$$

From Eq. (2.7) we see that the problem is reduced to a system of uncoupled harmonic oscillators. Now, the quantization program can be implemented (at $\hbar$ order) in a standard way. In particular, the (bare) zero point energy is given by [1]

$$H = H[\phi_c] + \frac{1}{2}\sum_n \omega_n. \tag{2.8}$$

Taking the derivative of Eq. (2.2) it is easy to show that Eq. (2.6) admits a zero mode ($\omega^2 = 0$) solution with eigenfunction given by

$$\psi(x) = \frac{d}{dx}\phi_c(x). \tag{2.9}$$

From Fig. 1(a) one can see that $(d/dx)\phi_c$ (the velocity in the particle mechanical analogy) is zero only in the limits $x \to \pm\infty$; i.e. the zero mode eigenfunction has no nodes and then the $\omega^2 = 0$ is the lowest eigenvalue. All the ω's are real and consequently the kink remain stable when quantum corrections are taken into account. On the other hand, from Fig. 1(b) one can see that $(d/dx)\phi_c$ is zero for some finite $x = x_0$ in the returning point 4 (we can always choose this point by translational invariance as corresponding to $x_0 = 0$). In this case the the zero mode eigenfunction has a node, and then $\omega^2 = 0$ is not the lowest eigenvalue. There exist one negative eigenvalue $\omega^2 < 0$ and then one imaginary ω. In this situation, the lump becomes unstable by quantum corrections. In this case Eq. (2.8) has no direct physical interpretation, but its imaginary part signalizes the decay of the false vacuum [13,14].

In both cases, to go further we have to solve a one-dimensional SE that in general cannot be solved analytically. Instead of trying to solve general SE's, we can adopt a different approach. We can start from an exactly solvable SE to obtain the field theory model associated with it. The steps in this program are the following: first, solve Eq. (2.9) for $\phi_c(x)$,

$$\phi_c(x) = \int^x \psi(y)\, dy. \tag{2.10}$$

The second one is to invert Eq. (2.10) with respect to x obtaining $x = x(\phi_c)$. Third, we replace Eq. (2.9) in Eq. (2.3) to obtain $U(\phi_c)$:

$$U(\phi_c) = \frac{1}{2}\left\{\frac{d\phi_c}{dx}\right\}^2 = \frac{1}{2}\{\psi_0[x(\phi_c)]\}^2. \tag{2.11}$$

Finally we can remove the subscript "c," obtaining in this way the scalar field-theoretic model. Since the zero mode eigenfunction $\psi(x)$, as given by Eq. (2.9), is not normalized we shall obtain the field-theoretic models modulo the coupling constants. With the functional form of $U(\phi)$ in hand we can choose the coupling constants adequately. This is the reason why in many places we will drop out deliberately some numerical factors.

In the following sections we will construct field-theoretic models starting from integrable SE's with three different potentials [15]: the Rosen-Morse II hyperbolic potential

$$V(x) = A^2 + \frac{B^2}{A^2} - \frac{A(A+1)}{\cosh^2(x)} + 2B\tanh(x),$$
$$|B| < A^2, \tag{2.12}$$

the Morse potential

$$V(x) = A^2 + B^2\exp(2x) - B(2A+1)\exp(x) \tag{2.13}$$

and finally the Scarf II hyperbolic potential

$$V(x) = A^2 + \frac{(B^2 - A^2 - A)}{\cosh^2(x)} + B(2A+1)\frac{\tanh(x)}{\cosh(x)}. \tag{2.14}$$

Before constructing the field theory models starting from these potentials, we would like to clarify about the exact relation between $V(x)$ and $U''[\phi_c(x)]$ that appears in Eq. (2.6). For all the above potentials the ground state eigenfunction has zero eigenvalue [15]. Then, in the case of stable models we have $U''[\phi_c(x)] = m^2 V(mx)$ and in the case of unstable models we have $U''[\phi_c(x)] = m^2[V(mx) - \lambda^2]$, where λ^2 is the first excited eigenvalue associated with $V(x)$ and m is a mass scale factor. The above relation between $V(x)$ and $U''[\phi_c(x)]$ in the case of unstable models is necessary in order to satisfy the requirement that the first excited eigenfunction associated with $U''[\phi_c(x)]$ has zero eigenvalue.

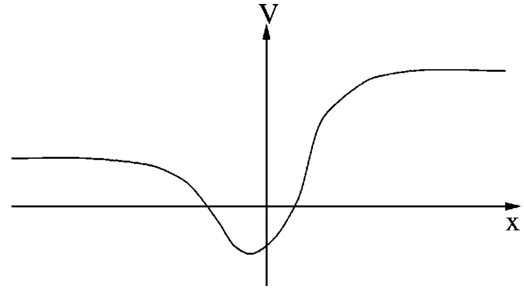

FIG. 2. The Rosen-Morse II hyperbolic potential.

III. THE ROSEN-MORSE II HYPERBOLIC POTENTIAL

As we have discussed in the preceding section, the Rosen-Morse II hyperbolic potential is given by

$$V(x) = A^2 + \frac{B^2}{A^2} - \frac{A(A+1)}{\cosh^2(x)} + 2B\tanh(x),$$
$$|B| < A^2. \tag{3.1}$$

This potential is showed in Fig. 2 assuming $B > 0$. This potential admits discrete and continuous eigenfunctions. The discrete eigenfunctions and eigenvalues are given, respectively, by [15]

$$\psi_n(x) = (1-y)^{a/2}(1+y)^{b/2}\ P_n^{(a,b)}(y) \tag{3.2}$$

and

$$\lambda_n^2 = A^2 - (A-n)^2 + \frac{B^2}{A^2} - \frac{B^2}{(A-n)^2},$$
$$n = 0,1,2, \ldots < A. \tag{3.3}$$

In Eq. (3.2) we have $a = A - n + B/(A-n)$, $b = A - n - B/(A-n)$, $y = \tanh(x)$ and $P_n^{(a,b)}(y)$ are the Jacobi polynomials [16]. The continuous (scattering) eigenfunctions behave asymptotically as (assuming $B > 0$)

$$\psi_k(x) = \begin{cases} e^{ik_- x} + a_k e^{-ik_- x}, & x \to -\infty \\ b_k e^{-\kappa x}, & x \to \infty \end{cases}$$
$$\text{for } A^2 - 2B < \lambda_k^2 < A^2 + 2B \tag{3.4a}$$

$$\psi_k(x) = \begin{cases} e^{ik_- x} + c_k e^{-ik_- x}, & x \to -\infty \\ d_k e^{ik_+ x}, & x \to \infty \end{cases}$$
$$\text{for } A^2 + 2B < \lambda_k^2 < \infty \tag{3.4b}$$

and where the continuous eigenvalues λ_k^2 are expressed in terms of k_-, k_+ and κ, respectively, by

$$\lambda_k^2 = k_-^2 + (A^2 - 2B), \tag{3.5}$$

$$\lambda_k^2 = k_+^2 + (A^2 + 2B), \tag{3.6}$$

and

$$\lambda_k^2 = -\kappa^2 + (A^2 + 2B). \tag{3.7}$$

The coefficients a_k, b_k, c_k and d_k that appear in Eqs. (3.4b) can be computed exactly [17].

To construct field-theoretic models that support kink-like solutions (stable models) from the above potential, we have to work with the ground state, the zero node eigenfunction $\psi_0(x)$. On the other hand, to construct field theory models that support lump-like solutions we have to consider the eigenfunction with one node $\psi_1(x)$.

A. Stable models

To obtain the kinks from the potential given by Eq. (3.1) we have to integrate the ground state eigenfunction $\psi_0(x)$ that can be obtained from Eq. (3.2) taking $n=0$:

$$\psi_0(x) = (1-y)^{a/2}(1+y)^{b/2}. \tag{3.8}$$

Using the above expression in Eq. (2.10) we obtain

$$\phi_c(x) = \int^{\tanh(x)} \frac{(1-y)^{a/2}(1+y)^{b/2}}{1-y^2} dy, \tag{3.9}$$

and from Eq. (2.11) we obtain, for $U(\phi_c)$,

$$U(\phi_c) = (1-y)^a (1+y)^b. \tag{3.10}$$

In general, the integral in Eq. (3.9) cannot be performed analytically. Consequently, we restrict ourselves to the following cases.

1. B=0

In this case $a=b=A$, and Eqs. (3.9), (3.10) become, respectively,

$$\phi_c(x) = \int^{\tanh(x)} (1-y^2)^{(A-2)/2} dy, \tag{3.11}$$

$$U(\phi_c) = (1-y^2)^A. \tag{3.12}$$

The integral in Eq. (3.11) can be performed in terms of elementary functions only for the A integer. For $A=1$, we obtain

$$\phi_c(x) = \sin^{-1}(y). \tag{3.13}$$

Solving the above equation and substituting in Eq. (3.12) we obtain (after deleting the subscript c)

$$U(\phi) = \frac{1}{2}[1-\cos(2\phi)], \tag{3.14}$$

i.e. we recover the sine-Gordon model. For $A=2$, we obtain from Eq. (3.11) $y=\phi_c(x)$ and using this result in Eq. (3.12) we get

$$U(\phi) = (\phi^2-1)^2, \tag{3.15}$$

i.e. we recover the ϕ^4 kink model. As we have mentioned the integral given by Eq. (3.11) can be computed for any integer value of A. But for $A>2$ the resulting expression cannot be inverted, and then it will not be possible to recover the field-theoretical model for this case. We would like to stress the following point. We are considering the case in which $V(x) = -A(A+1)/\cosh^2(x)$ for A integer. Such potentials have both discrete and continuous modes with the advantageous property of being reflectionless. The first quantum corrections for the (bare) mass of the kinks are given by Eq. (2.8) after subtracting the zero point energy of the trivial vacuum. To sum the continuous modes we have to know the density of states that can be given in terms of the phase shift of the one-dimensional scattering problem. In general this sum is logarithmically divergent and we need to renormalize the theory. But in two-dimensional scalar field theories such divergences can be eliminated using a normal ordering prescription only. This property was used in Ref. [18] to find a finite result for the quantum corrections to the mass of the static solitons. Moreover, in the case of reflectionless potentials, the authors obtained the quantum mass corrections only in terms of the discrete eigenvalues of the associated SE. Then in the present case it will be possible to obtain the kink quantum mass corrections without explicit knowledge of the theoretical models that permit such kink solutions. This task has been done in Ref. [19].

2. B≠0

In this case $a \neq b$. We can rewrite Eq. (3.9) as

$$\phi_c(x) = \int^{\tanh(x)} (1-y)^{(a-2)/2}(1+y)^{(b-2)/2} dy. \tag{3.16}$$

Choosing $(a-2)/2=r$ and $(b-2)/2=s$, we have

$$\phi_c(x) = \int^{\tanh(x)} (1-y)^r (1+y)^s dy. \tag{3.17}$$

Let us consider the case in which $r=0$. Then, from Eq. (3.17) we obtain

$$\phi_c(x) = [1+\tanh(x)]^{s+1}, \tag{3.18}$$

which can be solved for $y=\tanh(x)$. Replacing this solution in Eq. (3.10) we obtain

$$U(\phi) = \phi^2(2-\phi^{1/(s+1)})^2. \tag{3.19}$$

We see that for the values of s such that $1/(s+1)$ is fractionary, we will have in some cases [for example, when $1/(s+1)=1/2$] complex values for $U(\phi)$. For such values of s, we can redefine $\phi^{1/(s+1)}$ as $(\phi^2)^{1/2(s+1)}$ to make $U(\phi)$ a real-valued function, but in this way we will generate discontinuities in the derivatives of $U(\phi)$. If we take s such that $1/(s+1)=2l$, with l integer, we get

$$U(\phi)=\phi^2(2-\phi^{2l})^2, \tag{3.20}$$

i.e. we obtain a polynomial field-theoretical model with three degenerate vacua. The case $l=1$ is the ϕ^6 model with three degenerate vacua. This model was considered in Ref. [20], where the author obtained an expression for the renormalized mass of the soliton. It is interesting to point out that in Ref. [4], the authors obtained the same kink-like solutions associated with this model by studying the vacuum decay rate in the massive ϕ^6_{3D} model in the thin wall approximation. If we set $1/(s+1)=(2l+1)$ with an l integer we get

$$U(\phi)=\phi^2(2-\phi^{2l+1})^2, \tag{3.21}$$

i.e. we obtain polynomial field-theoretic models with two degenerate vacua. The case $l=1$ is the ϕ^8 model with two degenerate vacua that was obtained also in Ref. [7]. If we consider the case in which $s=0$, we obtain the same configurations as that in the case $r=0$. If we consider the case in which both r and s are integers we can still integrate Eq. (3.17), but in this case it will not be possible to invert the resulting expression.

Before constructing the unstable models we would like to comment about the stable models constructed in this section. As we have commented in the Introduction, the continuous solutions of the SE describe the scattering of particles (associated with the trivial vacua) by the soliton quantum state [3]. The (squared) mass of the particles are given by $U''[\phi_c(\pm\infty)]$ since the asymptotic values of the kink solution are equal to the trivial vacua of the theory. Setting the mass scale factor $m=1$ we can identify the mass of the particles with $V(\pm\infty)$ (in this case we have $\omega_k^2=\lambda_k^2$). In the case of the models associated with the Rosen-Morse II hyperbolic potential, the scattering data can be obtained from Eqs. (3.4b). For $B=0$ this potential is symmetric ($k_-=k_+$ and the particles have the same mass, $m^2=A^2$), and only the solutions for $A^2<\lambda_k^2<\infty$ make sense. In this case, for the A integer the potential is reflectionless ($c_k=0$). Then, in the case of the sine-Gordon ($A=1$) and ϕ^4 kink ($A=2$) models, incoming particles from $x\rightarrow-\infty$ are completely transmitted to $x\rightarrow\infty$. The only effect of the scattering is the appearance of a phase shift in their wave functions ($d_k=e^{i\delta(k)}$). For $B\neq 0$ (the case of the ϕ^6 and ϕ^8 kink models) the (squared) mass of the particles is different and given by $m_-^2=(A^2-2B)$, $m_+^2=(A^2+2B)$. The scattering picture in this case is as follows: incoming particles from $x\rightarrow-\infty$ with (squared) energies ω_k^2, such that $m_-^2<\omega_k^2<m_+^2$, are completely reflected by the soliton quantum state into particles of the same mass. On the other hand, incoming particles from $x\rightarrow-\infty$ with energies such that $\omega_k^2>m_+^2$, are scattered by the soliton quantum state into transmitted and reflected particles of different masses. The transmitted particles have mass m_+ while the reflected particles have mass m_- equal to the mass of the incoming particles. Note that Eq. (3.4b) only describes the scattering of particles incoming from $x\rightarrow-\infty$. For the scattering of particles incoming from $x\rightarrow\infty$ the solution is qualitatively similar to the case in which $\omega_k^2>m_+^2$ but is quantitatively different [30].

In the next section we will analyze the unstable models (bounces).

B. Unstable models

In this case the bounce-like solutions are obtained integrating the $n=1$ case of Eq. (3.2):

$$\psi_1(x)=(1-y)^{a/2}(1+y)^{b/2}[(a+b+2)y+a-b]; \tag{3.22}$$

then integrating the above equation we obtain

$$\phi_c(x)=\int^{\tanh(x)}\frac{(1-y)^{a/2}(1+y)^{b/2}[(a+b+2)y+a-b]}{1-y^2}dy, \tag{3.23}$$

and the field-theoretical unstable models are given by

$$U(\phi)=(1-y)^a(1+y)^b[(a+b+2)y+a-b]^2. \tag{3.24}$$

We can perform the integration in Eq. (3.23) in the following cases.

1. B=0

In this case $a=b$. Consequently we have, for $\psi_1(x)$,

$$\psi_1(x)=2A\frac{\sinh(x)}{\cosh^A(x)}, \tag{3.25}$$

and the integral in Eq. (3.23) can be easily performed. We obtain

$$\phi_c(x)=\frac{1}{\cosh^{A-1}(x)}. \tag{3.26}$$

Solving in the above equation for $y=\tanh(x)$ and substituting in Eq. (3.24) we obtain the field-theoretic models described by the class of density potentials

$$U(\phi)=(A-1)\phi^2(1-\phi^{2/(A-1)}). \tag{3.27}$$

For $A=2$ we have the unstable ϕ^4 model. This model has been used in Ref. [21] as a field-theoretic model for the study of the kinematics of first order phase transitions. For $A=3$ we obtain the unstable ϕ^3 model. This last model has been used as a laboratory for computing the decay rate of a system trapped in a false vacuum [22]. Also, recently the ϕ^3 model has been used as an exactly solvable toy model for tachyon condensation in string field theory [23]. We have to stress that we must assume $A>0$ to guarantee the normalizability of $\psi_1(x)$. In Ref. [7], the authors considered the case $A=-1$ and obtained the Liouville model [24]. But in this case it is easy to see that the classical solutions that meet or leave the unique asymptotic vacuum have an infinite energy, that is, such solutions are not lumps.

2. $B\neq 0$

In this case $a\neq b$ and the integral given by Eq. (3.23) can be written as

$$\phi_c(x)=(b+a+2)\int^{\tanh(x)}\frac{y(1-y)^{a/2}(1+y)^{b/2}}{1-y^2}dy+(a-b)\int^{\tanh(x)}\frac{(1-y)^{a/2}(1+y)^{b/2}}{1-y^2}dy. \qquad (3.28)$$

The above integral can be performed for some particular values of a,b but the resulting expression is not invertible, and in this case we cannot recover the unstable field-theoretical models.

C. The $A\to\infty$ limit

The SE given by

$$\left[-\frac{d^2}{dx^2}+A+\frac{B^2}{A^3}-\frac{(A+1)}{\cosh^2(x/\sqrt{A})}+\frac{2B}{A}\tanh(x/\sqrt{A})\right]\psi_n(x)=\lambda_n^2\psi_n(x), \qquad (3.29)$$

can be put in the form given by Eq. (3.1) making $x=\sqrt{A}z$. In this case the eigenvalues λ_n^2 are given by

$$\lambda_n^2=\frac{1}{A}\left[A^2-(A-n)^2+\frac{B^2}{A^2}-\frac{B^2}{(A-n)^2}\right],\quad n=0,1,2,\ldots<A. \qquad (3.30)$$

Note that when $A\to\infty$ the number of eigenvalues goes to infinity. If we keep B fixed and take the limit $A\to\infty$ in the above equations we obtain

$$\left[-\frac{d^2}{dx^2}-1+2x^2\right]\psi_n(x)=\lambda_n^2\psi_n(x) \qquad (3.31)$$

and

$$\lambda_n^2=2n,\quad n=0,1,2 \qquad (3.32)$$

i.e. we obtain the SE for an harmonic oscillator. In order to obtain a stable field theory model we consider the ground state eigenfunction given by

$$\psi_0(x)=\mathrm{e}^{-x^2/2}, \qquad (3.33)$$

which allows us to obtain the kink-like solution

$$\phi_c(x)=\int^x \mathrm{e}^{-y^2/2}dy=\mathrm{erf}(x/\sqrt{2}). \qquad (3.34)$$

Since the error function cannot be inverted, it will be not possible to obtain the field theory model associated with this SE. We remark here that although Eq. (3.34) cannot be inverted it is possible to obtain the field theory model implicitly in terms of other fields. This was done in Ref. [25] where the field is redefined in terms of the so-called tachyon field. Note that in this case $V(\pm\infty)=\infty$. But from this one cannot conclude that the mass of the particles associated with the quantum fluctuations around the trivial vacua is infinity. In this case we cannot speak about particles since in this case we do not have continuous solutions, i.e. there are no solutions that behave asymptotically as plane waves.

In order to obtain the unstable model, we consider the first excited eigenfunction, given by

$$\psi_1(x)=x\mathrm{e}^{-x^2/2}, \qquad (3.35)$$

which implies that the unstable field theory model will be given by the density potential

$$U(\phi_c)=\frac{1}{2}\{\psi_1[x(\phi_c)]\}^2=[x(\phi_c)]^2\mathrm{e}^{-[x(\phi_c)]^2}. \qquad (3.36)$$

In this case the bounce profile is obtained easily,

$$\phi_c(x)=\int^x y\mathrm{e}^{-y^2}dy=\mathrm{e}^{-x^2/2}. \qquad (3.37)$$

Inverting the above equation and using the resulting expression in Eq. (3.36) we get

$$U(\phi)=-\phi^2\ln\phi^2, \qquad (3.38)$$

a field-theoretical unstable model that has been considered in Ref. [26] as a field-theoretical toy model for tachyon condensation in superstring field theory.

IV. THE MORSE POTENTIAL: THE $\phi^2\ln^2(\phi^2)$ MODEL

The Morse potential is given by

$$V(x)=A^2+B^2\exp(2x)-B(2A+1)\exp(x). \qquad (4.1)$$

This potential is showed in Fig. 3. Also in this case the SE has discrete and continuous solutions. The discrete eigenfunctions and eigenvalues are given, respectively, by

$$\psi_n(x)=y^{A-n}e^{-y/2}L_n^{2A-2n}(y),\qquad y=2B\mathrm{e}^x, \qquad (4.2)$$

and

$$\lambda_n^2=A^2-(A-n)^2,\qquad n=0,1,2,\ldots<A, \qquad (4.3)$$

while the continuous (scattering) solutions behave asymptotically as

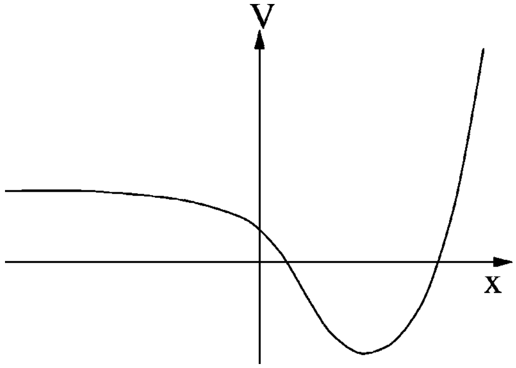

FIG. 3. The Morse potential.

$$\psi_k(x)=\begin{cases} \mathrm{e}^{ikx}+a_k e^{-ikx}, & x\to-\infty, \\ b_k e^{-\kappa x}, & x\to\infty, \end{cases} \tag{4.4}$$

where the continuous eigenvalues λ_k^2 are expressed in terms of k and κ, respectively, by

$$\lambda_k^2=k^2+A^2, \tag{4.5}$$

$$\lambda_k^2=-\kappa^2+A^2. \tag{4.6}$$

Also in this case the coefficients a_k, b_k can be computed exactly [17].

In order to obtain the stable field-theoretical models we work with the ground state eigenfunction ψ_0 that can be obtained from Eq. (4.2) taking $n=0$,

$$\psi_0(x)=y^A\mathrm{e}^{-y/2}, \tag{4.7}$$

from which we can obtain the kink

$$\phi_c(x)=\int^{2B\mathrm{e}^x} y^{A-1}\mathrm{e}^{-y/2}dy. \tag{4.8}$$

The stable field-theoretic models are now given by

$$U(\phi)=y^{2A}\mathrm{e}^{-y}. \tag{4.9}$$

The integral in Eq. (4.8) can be performed only for integer A. For $A=1$ we obtain

$$\phi_c(x)=\exp(-B\mathrm{e}^x), \tag{4.10}$$

from which, solving for $y=2B\mathrm{e}^x$ and replacing in Eq. (4.9) we obtain

$$U(\phi)=\phi^2\ln^2(\phi^2). \tag{4.11}$$

We have plotted the above density potential in Fig. 4, where we see that there are three minima. One of which is located at $\phi=0$, i.e. we have $U'(0)=0$. At this point we have $U''(0)=\infty$ and also all the higher derivatives are infinity. Although the integral in Eq. (4.8) can be done for $A=2,3,\ldots$ it is not possible to invert the resulting expression to obtain the field-theoretical models in this cases. Also, in this case it is not possible to construct unstable models.

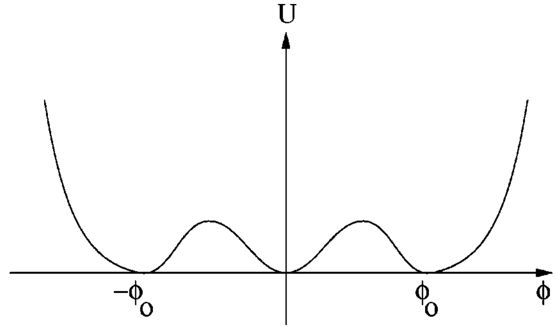

FIG. 4. The density potential $U(\phi)$ given by Eq. (4.11).

In the present case the scattering of particles by the soliton quantum state as given by Eq. (4.4) is totally different than that in the case of the sine-Gordon and ϕ^4 kink models. Particles with squared mass $V(-\infty)$ incoming from $x\to-\infty$ are totally reflected by the soliton quantum state into particles of the same mass. In this case the soliton quantum state acts as an impenetrable barrier. Since for $x\to\infty$ the solutions do not behave as plane waves, we cannot interpret $V(\infty)=\infty$ as the squared mass of particles associated with the trivial vacuum $\phi_c(\infty)=0$, as given by Eq. (4.10). This also can be concluded from Eq. (4.6), where we see that there is no Einstein-type dispersion relation. We have to remark that this phenomenon also occurs for the models constructed in Sec. III A with $B\neq 0$ (for example, in the ϕ^6 and ϕ^8 models). Particles incoming from $x\to-\infty$ with energy such that $m_-^2<\omega_k^2<m_+^2$ are totally reflected by the soliton quantum state. For the model with the density potential given by Eq. (4.11) this happens for all energies. Then, in the presence of a soliton quantum state, particles can exist only for $x\to-\infty$ (of course, if we consider the anti-soliton quantum state particles can exist only for $x\to\infty$). Also we remark that particles associated with the trivial vacua are defined only perturbatively, i.e. we expand the density potential around one of the trivial vacua, $\phi_{\rm v}$, and identify the squared mass with $U''(\phi_{\rm v})$ and when this is infinity the perturbative expansion is not possible and then in this case we cannot speak about particles.

The density potential given by Eq. (4.11) can be redefined in such a way that there will appear coupling constants in the model. Since we have constructed this field theory model starting from a SE with a free parameter ($A=1$ and B arbitrary), and since we can rescale field and coordinates in the Lagrangian (thus eliminating two coupling constants), we conclude that the density potential given by Eq. (4.11) can be redefined with no more than three coupling constants. We redefine Eq. (4.11) with two coupling constants as

$$U(\phi)=\frac{m^2}{8}\phi^2\ln^2\left(\frac{\alpha^2\phi^2}{9m^4}\right), \tag{4.12}$$

where we have chosen the numerical factors adequately. Solving $U'(\phi)=0$ we obtain that the three absolute minima are located at the points $\phi=0$ and $\phi=\pm\phi_0$, where

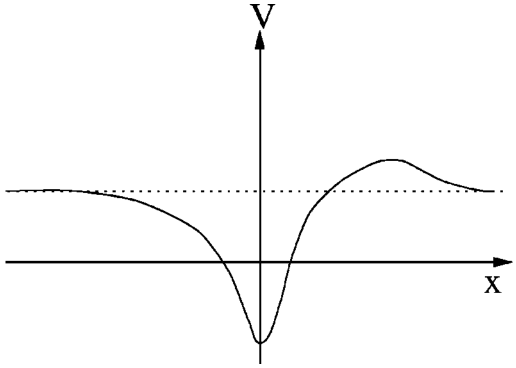

FIG. 5. The Scarf II hyperbolic potential.

$$\phi_0=\frac{3m^2}{\alpha}. \tag{4.13}$$

Using Eq. (4.12) in Eq. (2.2) we obtain

$$\phi_c(x)=\pm\frac{3m^2}{\alpha}\exp(-\mathrm{e}^{\pm m(x-x_0)}). \tag{4.14}$$

We have two pairs of kink and anti-kink solutions that can link $\phi=-\phi_0$ and $\phi=0$ or $\phi=0$ and $\phi=\phi_0$. One can see easily that $U''(\pm\phi_0)=m^2$, i.e. m^2 is the squared mass of the particles associated with the quantum fluctuations around the trivial vacua at the points $\pm\phi_0$. As was mentioned for the vacuum at $\phi=0$ we cannot speak about particles. Using Eq. (4.12) we can see that the masses of the kinks (anti-kinks) solutions given by Eq. (4.14) are the same and given (classically) by

$$H[\phi_c]=\int_{-\infty}^{\infty}dx\left\{\frac{1}{2}\left(\frac{d\phi_c}{dx}\right)^2+\frac{m^2}{8}\phi_c^2\left[\ln\left(\frac{\alpha^2\phi_c^2}{9m^4}\right)\right]^2\right\}$$

$$=\frac{9m^6}{2\alpha^2}\int_{-\infty}^{\infty}dx\quad \mathrm{e}^{\pm mx}\exp(-2\mathrm{e}^{\pm mx})$$

$$=\frac{9m^5}{4\alpha^2}. \tag{4.15}$$

V. THE SCARF II HYPERBOLIC POTENTIAL: THE $U(\phi)=\phi^2\cos^2\ln(\phi^2)$ MODEL

The Scarf II hyperbolic potential is given by

$$V(x)=A^2+\frac{(B^2-A^2-A)}{\cosh^2(x)}+B(2A+1)\frac{\tanh(x)}{\cosh(x)}. \tag{5.1}$$

This potential is showed in Fig. 5. In this case the discrete eigenfunctions and eigenvalues are given, respectively, by

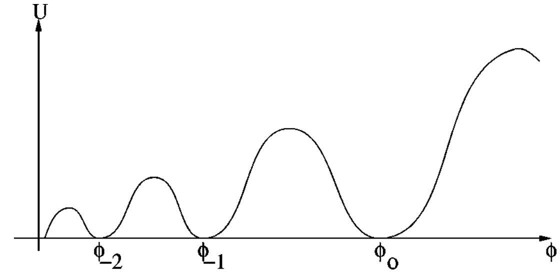

FIG. 6. The density potential $U(\phi)$ given by Eq. (5.10).

$$\psi_n(x)=(i)^n(1+y^2)^{-A/2}\mathrm{e}^{-B\tan^{-1}(y)}\times P_n^{(iB-A-1/2,-iB-A-1/2)}(y),$$
$$y=\sinh(x), \tag{5.2}$$

and

$$\lambda_n^2=A^2-(A-n)^2,\qquad n=0,1,2,\ldots<A, \tag{5.3}$$

while the (scattering) continuous eigenfunctions behave asymptotically as

$$\psi_k(x)=\begin{cases}\mathrm{e}^{ikx}+a_k\mathrm{e}^{-ikx}, & x\to-\infty,\\ b_k\mathrm{e}^{ikx}, & x\to\infty,\end{cases} \tag{5.4}$$

where the continuous eigenvalues are given by

$$\lambda_k^2=k^2+A^2. \tag{5.5}$$

Also in this case a_k and b_k can be computed exactly [17]. They are equal, respectively, to the reflection and transmission coefficient amplitudes [in Eqs. (6.12) and (6.13) we give them with a mass scale factor m].

In order to obtain the stable field-theoretical models we work with the zero node eigenfunction, obtained from Eq. (5.2) taking $n=0$

$$\psi_0(x)=(1+y^2)^{-A/2}\mathrm{e}^{-B\tan^{-1}(y)}. \tag{5.6}$$

Then the field-theoretic models are given by

$$U(\phi)=(1+y^2)^{-A}\mathrm{e}^{-2B\tan^{-1}(y)}. \tag{5.7}$$

Integrating Eq. (5.6) we obtain, for ϕ_c,

$$\phi_c(x)=\int^{\sinh(x)}(1+y^2)^{-(A+1)/2}\mathrm{e}^{-B\tan^{-1}(y)}dy. \tag{5.8}$$

The above integral can be performed analytically only when $A=1$. In this case we obtain

$$\phi_c(x)=\mathrm{e}^{-B\tan^{-1}\sinh(x)}, \tag{5.9}$$

solving for $y=\sinh(x)$ and replacing in Eq. (5.7) we obtain

$$U(\phi)=\phi^2\cos^2\left(\frac{1}{2B}\ln\phi^2\right). \tag{5.10}$$

In Fig. 6 we have plotted this density potential for $\phi>0$ and under the assumption $B>0$. As in the case of the Morse potential in this case it is not possible to construct unstable models.

For the model with density potential given by Eq. (5.10) we have the following picture about the scattering of the particles by the soliton quantum state: From Eq. (5.4) we see that the particles are scattered by the soliton quantum state into reflected and transmitted particles of the same mass, $V(-\infty)=V(\infty)$. Equation (5.4) only describes the scattering of particles incoming from the left. For particles incoming from the right the picture is qualitatively the same. But since the potential is nonsymmetric the scattering is different quantitatively [30].

Redefining the density potential given by Eq. (5.10) with adequate coupling constants we can write

$$U(\phi)=\frac{1}{2}m^2B^2\phi^2\cos^2\left[\frac{1}{2B}\ln\left(\frac{\alpha^2\phi^2}{9m^4}\right)\right]. \tag{5.11}$$

This model has infinitely degenerate trivial vacua at the points $\phi=\pm\phi_n$ with ϕ_n given by

$$\phi_n=\frac{3m^2}{\alpha}\exp\left(\frac{2n+1}{2}\pi B\right),\quad n=0,\pm 1,\pm 2,\ldots . \tag{5.12}$$

The kinks and anti-kinks are obtained using Eq. (2.2),

$$\phi_c(x)=\pm\frac{3m^2}{\alpha}\exp[n\pi B\pm B\tan^{-1}\sinh(mx)],\quad n=0,\pm 1,\pm 2,\ldots, \tag{5.13}$$

where the solutions with $(\pm)$ signs in the exponents correspond to the kinks–anti-kinks solutions, respectively, for each value of n and for each sign that appears in front. We have an infinite number of kinks and anti-kinks that link the infinite number of trivial vacua. This reminds us in some sense of the sine-Gordon model. But contrary to the sine-Gordon model, where all the (one) solitonic sectors describe the same physics, in the present model it is not the case. For example, if we compute the (classical) masses of the kinks (or anti-kinks) we obtain

$$\begin{aligned}H[\phi_c]&=\int_{-\infty}^{\infty}dx\left\{\frac{1}{2}\left(\frac{d\phi_c}{dx}\right)^2+\frac{1}{2}m^2B^2\phi^2\cos^2\left[\ln\left(\frac{\alpha^2\phi^2}{9m^4}\right)\right]\right\}\\&=\frac{9m^6B^2e^{2Bn\pi}}{\alpha^2}\times\int_{-\infty}^{\infty}dx\frac{\exp[\pm 2B\tan^{-1}\sinh(mx)]}{\cosh^2(mx)}\\&=\frac{9m^5B^2e^{2Bn\pi}}{\alpha^2}I(B),\end{aligned} \tag{5.14}$$

where $I(B)$ is given by

$$I(B)=\int_{-1}^{1}ds\ \ e^{2B\sin^{-1}(s)}. \tag{5.15}$$

From Eq. (5.14) we see clearly that the masses are different for different values of n.

Before leaving this section we would like to make the following remark: Expanding the density potential given by Eq. (5.11) (for simplicity we choose $B=1/2$) around one of the trivial vacua we obtain

$$\begin{aligned}U(\varphi)=&\frac{m^2}{2}\varphi^2\pm\frac{\alpha}{6}e^{-(1/4)(2n+1)\pi}\varphi^3-\frac{1}{4!}\frac{\alpha^2}{m^2}e^{-(1/2)(2n+1)\pi}\varphi^4\\&+\frac{(256)}{(81)6!}\frac{\alpha^4}{m^6}e^{-(2n+1)\pi}\varphi^6+\mathcal{O}(e^{-4(2n+1)\pi}),\end{aligned} \tag{5.16}$$

where $\varphi=(\phi\pm\phi_n)$. From this equation we see that for $n\to\infty$, $U(\varphi)=(m^2/2)\varphi^2$, i.e. in the perturbative sector (for $n\to\infty$) the model becomes free.

VI. KINK QUANTUM MASS CORRECTIONS IN THE $U(\phi)=\phi^2\cos^2\ln(\phi^2)$ MODEL

As we have said in the Introduction, we have constructed scalar field theory models starting from solvable SE's since in this case we have a chance to compute quantum corrections. In this section we give a closed expression, Eq. (6.15), for the kink quantum mass correction in the model constructed in the preceding section. All the steps (with the exception of a final integration) are done analytically and only the final integration is evaluated numerically.

The first (bare) quantum corrections for the kink mass is given by

$$\Delta M_{bare}=\frac{1}{2}\sum_n\ \omega_n-\frac{1}{2}\sum_k\ \omega_k^0, \tag{6.1}$$

where ω_n is given from Eq. (2.6), while $\omega_k^0=\sqrt{k^2+m^2}$ are the free soliton modes and given from Eq. (2.6) with $U''[\phi_c(x)]$ replaced by $U''(\pm\phi_n)=m^2$. Equation (2.6) has discrete and continuous eigenvalues. The number of discrete modes are finite and we denote them as ω_j. The continuous modes are given by $\omega(k)=\sqrt{k^2+m^2}$. We note that these continuous modes are equal to ω_k^0, but from this one cannot conclude that Eq. (6.1) is equal to the sum over the discrete modes only. The reason is that ω_k^0 and $\omega(k)$ have different densities of states. We can divide Eq. (6.1) into a sum over discrete modes and an integral over a continuum, representing the latter in terms of the phase shift,

$$\Delta M_{bare}=\frac{1}{2}\sum_j \omega_j+\int_0^\infty \frac{dk}{2\pi}\omega(k)\frac{d}{dk}\delta(k), \qquad (6.2)$$

where $\delta(k)$, the phase shift, is given by

$$\delta(k)=\frac{1}{2i}\ln \det S(k). \qquad (6.3)$$

In the above equation $S(k)$, the S matrix is given by [30]

$$S(k)=\begin{pmatrix} T(k) & -R^*(k)T(k)/T^*(k) \\ R(k) & T(k)\end{pmatrix}, \qquad (6.4)$$

where $R(k)$ and $T(k)$ are, respectively, the reflection and transmission coefficient amplitudes associated with the one-dimensional scattering problem described by the continuous solutions of Eq. (2.6). Since the phase shift in general behaves like $1/k$ for $k\to\infty$ we can see that the kink quantum mass corrections given by Eq. (6.2) are logarithmically divergent. Then we have to renormalize such expression in order to obtain a finite physically meaningful quantity. In the present paper we use a method developed in Refs. [27–29] to give a finite renormalized expression for the kink quantum mass corrections. In order to write a finite renormalized expression for Eq. (6.2) the basic idea is to subtract from the phase shift its first Born approximation. Since the first Born approximation for the phase shift behaves like $1/k$ this subtraction cancels the ultraviolet divergence. But the Born approximation is singular for $k=0$, and in order to overcome this infrared divergence, before making such subtraction one uses the one-dimensional Levinson theorem [30]. In the case of potentials that are finite and such that $V(-\infty)=V(\infty)$ the one-dimensional Levinson theorem states that [30]

$$\delta(0)=n\pi-\frac{\pi}{2}, \qquad (6.5)$$

where n is the number of bound states. This includes the half bound state $(\omega^2=m^2)$ that corresponds to $k=0$, counted with a factor $\frac{1}{2}$. For example, in the free case we have that $\delta(k)=0$ everywhere and in this case we have a half bound state with the wave function equal to a constant. We can rewrite Eq. (6.2) by adding and subtracting $m/4$, obtaining

$$\Delta M_{bare}=\frac{1}{2}\sum_j \omega_j-\frac{m}{4}+\int_0^\infty \frac{dk}{2\pi}\omega(k)\frac{d}{dk}\delta(k), \qquad (6.6)$$

where now in the sum over j we are including the half bound state with a $\frac{1}{2}$ contribution. We can rewrite Eq. (6.5) as

$$0=\sum_j 1+\int_0^\infty \frac{dk}{\pi}\frac{d}{dk}\delta(k)-\frac{1}{2}. \qquad (6.7)$$

Subtracting $m/2$ times this equation from Eq. (6.6) we obtain

$$\Delta M_{bare}=\frac{1}{2}\sum_j (\omega_j-m)+\int_0^\infty \frac{dk}{2\pi}[\omega(k)-m]\frac{d}{dk}\delta(k). \qquad (6.8)$$

Now we subtract from the phase shift its first Born approximation (this is equivalent to subtracting the tadpole graph, see [28]). We must then add it back and we adopt the simple renormalization condition that the counterterms (which will be present in the bare classical mass) cancel the tadpole graph and perform no additional finite renormalization beyond this cancellation. This renormalization prescription in equivalent to a normal ordering prescription for the field operators (see, for example, Ref. [18]). With this choice there is nothing to add back and it is understood that the parameters that will appear are the renormalized ones and then we can drop out the subscript *bare*. The first Born approximation $\delta^1(k)$ for the phase shift is given by

$$\delta^1(k)=-\frac{1}{2k}\int_{-\infty}^{\infty} dx\{U''[\phi_c(x)]-m^2\}. \qquad (6.9)$$

Then subtracting this first Born approximation from Eq. (6.8) we obtain

$$\Delta M=\frac{1}{2}\sum_j (\omega_j-m) +\int_0^\infty \frac{dk}{2\pi}[\omega(k)-m]\frac{d}{dk}[\delta(k)-\delta^1(k)]. \qquad (6.10)$$

Note that the method described here is restricted to models for which the associated Schrödinger potentials are finite and such that $V(-\infty)=V(\infty)$ since for such cases the Levinson theorem is valid and also the integration in Eq. (6.9) is finite. This is accomplished in the cases of the sine-Gordon and ϕ^4 kink models and also for the model constructed in the preceding section. Since the quantum mass correction for the kinks of sine-Gordon and ϕ^4 kink models are already know, in the present paper we compute only the quantum mass corrections for the kinks of the model constructed in the preceding section.

Using Eqs. (5.11) and (5.13) we find that

$$U''[\phi_c(x)]=m^2+m^2\left[\frac{(B^2-2)}{\cosh^2(mx)}\mp 3B\frac{\tanh(mx)}{\cosh(mx)}\right], \qquad (6.11)$$

Note that as expected this potential is the case $A=1$ of Eq. (5.1) with a mass scale factor. For this potential the reflection and transmission coefficient amplitudes are given, respectively, by [17]

$$R(k)=\pm T(k)\frac{\sinh(\pi B)}{\cosh(\pi k/m)}, \qquad (6.12)$$

and

$$T(k)=\frac{\Gamma(-1-ik/m)\Gamma(2-ik/m)\Gamma\left(\frac{1}{2}\mp iB-ik/m\right)\Gamma\left(\frac{1}{2}\pm iB-ik/m\right)}{\Gamma(-ik/M)\Gamma(1-ik/m)\Gamma^2\left(\frac{1}{2}-ik/m\right)}, \tag{6.13}$$

where the signs $\pm$ in Eq. (6.12) refer to the kink and anti-kink-like solutions, respectively. Using Eq. (6.11) in Eq. (6.9) we obtain

$$\delta^1(k)=-\frac{m^2}{k}(B^2-2). \tag{6.14}$$

Now, since by construction the above model possesses only one discrete eigenvalue equal to zero, and using Eqs. (6.12)–(6.14) in Eq. (6.10) we obtain after some algebraic manipulations,

$$\frac{\Delta M}{m}=-\frac{1}{\pi}-\int_0^\infty\frac{dq}{2\pi}\frac{q}{\sqrt{q^2+1}}\left\{\frac{1}{2i}\ln\left[\frac{\Gamma(1/2-iB-iq)\Gamma(1/2+iB-iq)\Gamma^2(1/2+iq)}{\Gamma(1/2+iB+iq)\Gamma(1/2-iB+iq)\Gamma^2(1/2-iq)}\right]+\frac{B^2}{q}\right\}. \tag{6.15}$$

The integration in the above equation cannot be done analytically but, for $B=0$ it vanishes, and in this case we obtain

$$\Delta M(B=0)=-\frac{m}{\pi}, \tag{6.16}$$

a result equal to the first quantum corrections for the kink mass in the sine-Gordon model. This result is expected since in Sec. III we have shown that for such a case we recover the sine-Gordon model starting from the potential given by Eq. (6.11) with $B=0$. For other values of B we can perform the integration only numerically. We have performed the integration given by Eq. (6.15) numerically and plotted the result. This is shown in Fig. 7. From our numerical calculation we conclude that the quantum correction for the kink mass is negative. We observe from Fig. 7 that the quantum correction for the kink mass becomes more negative for increasing B. Also since ΔM is real we conclude that the kink remains stable (as expected by construction) when quantum corrections are taken into account.

VII. CONCLUSIONS

In this paper we have obtained stable and unstable $(1+1)$ scalar models starting from exactly solvable SE's. In this way we have obtained two (to our knowledge) new stable models that permit kink-like solutions. Starting from the Morse potential we have obtained the model $\phi^2\ln^2(\phi^2)$ and starting from the Scarf II hyperbolic potential we have have obtained the model $U(\phi)=\phi^2\cos^2\ln(\phi^2)$. Note that we have analyzed only SE's that reduce to hypergeometric or confluent hypergeometric equations [15]. It will be interesting to analyze other (more complicated) differential equations, for example the Heun or Lame equations and search for possible interesting field-theoretical models.

Note that the model constructed starting from the Morse potential does not depend on the parameter B, Eq. (4.11). This can be easily understood from Eq. (4.1), where we can eliminate the parameter B making a translation $x\to x-\ln B$. For this model we have a behavior for the scattering of particles by the soliton quantum state totally different from the scattering of the particles by the solitons of the sine-Gordon and ϕ^4 kink models: particles are totally reflected by the soliton quantum state. Although for this model we have not computed the quantum corrections for the kink mass we believe that the transmissionless property of the Morse potential will be of utility in solving this problem. Regarding physical applications, for example, this field-theoretical model could be used to describe some physical situation for tachyon condensation in superstring field theory, since this model possesses one of the known properties of the tachyon effective action: Absence of plane wave solutions around the minima [31].

In the case of the model $\phi^2\cos^2\ln(\phi^2)$ constructed in Sec. V we have been able to compute the kink quantum mass corrections. Note that in deriving Eq. (6.10) we used the Levinson theorem stated in terms of the total phase shift, $\delta(k)$. In Ref. [27] the authors refer to the Levinson theorem separately for the symmetric and antisymmetric scattering channels. In the present case this is not possible, since the decomposition of the phase shift in terms of symmetric and antisymmetric scattering channels is possible only when the Schrödinger potential is symmetric [30]. As we have previ-

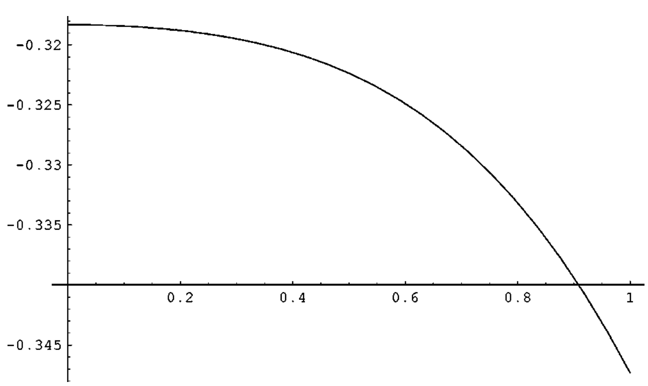

FIG. 7. $\Delta M/m$ as function of B.

ously mentioned, in the case $B=0$ the kink quantum mass correction is equal to the kink quantum mass correction in the sine-Gordon model. From this one can expect to recover the sine-Gordon model from $\phi^2\cos^2\ln(\phi^2)$ taking the limit $B\rightarrow 0$ adequately. Actually this is not the case. The reason is that in deducing Eq. (5.10) we have multiplied B by Eq. (5.9), introducing in this way (when $B\rightarrow 0$) an ambiguity. Then, in this model, the case $B=0$ only make sense as an approximation for the kink quantum mass correction for small B. Also from Eq. (6.14) we can see that when $B=\sqrt{2}$ we have $\delta^1(k)=0$ and in this case $\Delta M_{bare}=\Delta M$, i.e. at order $\hbar$ the quantum correction for the kink mass is free of divergences. Note that from Eq. (6.9) this will happen for any model where $U''[\phi_c(x)]-m^2$ is antisymmetric in x. Also we would like to call attention to the following fact: note that Eq. (6.11) is independent of n (the index that labels the locations of the perturbative vacua). Remembering that the one-dimensional scattering problem described by the continuous solutions of Eq. (2.6) is physically interpreted as the scattering of the usual particles by the soliton quantum state (see [3]), we see that usual particles interact in the same way with all the solitonic sectors (these solitonic sectors are indexed with n). But in the final part of Sec. V we remarked that for $n\rightarrow\infty$ the theory (around the trivial vacuum with $n\rightarrow\infty$) becomes free in the perturbative sector. Then we see an interesting peculiarity in the model described by Eq. (5.11): in the perturbative sector (around the vacuum with $n\rightarrow\infty$) the theory is free but in the solitonic sector it is not, the particles are scattered by the soliton quantum state. We do not know any other model in which this fact also occurs. Also we would like to comment about the utility of this method in testing some approximate and numerical methods that have been developed in order to compute quantum corrections around static field configurations. See for example [32]. Note that our computation is analytical; only a final integration is done numerically and this numerical integration can be done easily. Although our calculation is not exact it is precise, and then the present model can be used to test the efficiency of numerical or approximate methods. Frequently, the sine-Gordon and ϕ^4 kink models are used to test these methods, but as the authors claim, in many cases these approximate or numerical methods pass the test because of the peculiarity of these models, i.e. the reflectionless property of the Schrödinger potentials [32] associated with these models.

ACKNOWLEDGMENTS

G.F.H. would like to thank N. Graham for interesting discussions. G.F.H. and N.F.S. are supported by Conselho Nacional de Desenvolvimento Cientifico e Tecnológico CNPq of Brazil.

[1] R. Rajaraman, *Solitons and Instantons* (North-Holland, Amsterdam, 1982).
[2] C. Rebbi and G. Soliani, *Solitons and Particles* (World Scientific, Singapore, 1984).
[3] R. Jackiw, Rev. Mod. Phys. **49**, 681 (1977).
[4] G.H. Flores, R.O. Ramos, and N.F. Svaiter, Int. J. Mod. Phys. A **14**, 3715 (1999).
[5] N.H. Christ and T.D. Lee, Phys. Rev. D **12**, 1606 (1975).
[6] J. Casahorran and S. Nam, Int. J. Mod. Phys. A **30**, 5467 (1991).
[7] L.J. Boya and J. Casahorran, Ann. Phys. (N.Y.) **196**, 361 (1989).
[8] R. de Lima Rodrigues, Mod. Phys. Lett. A **10**, 1309 (1995).
[9] P.B. da Silva Filho, R. de Lima Rodrigues, and A.N. Vaidya, J. Phys. A **32**, 2395 (1999).
[10] R. de Lima Rodrigues, P.B. da Silva Filho, and A.N. Vaidya, Phys. Rev. D **58**, 125023 (1998).
[11] C.N. Kumar, J. Phys. A **20**, 5397 (1987).
[12] E.B. Bogomol'nyi, Sov. J. Nucl. Phys. **24**, 449 (1976).
[13] S. Coleman, Phys. Rev. D **15**, 2929 (1977); C. Callan and S. Coleman, *ibid.* **16**, 1762 (1977).
[14] C. Bonato, M.T. Thomaz, and A.P.C. Malbouisson, Phys. Rev. D **41**, 1939 (1990).
[15] F. Cooper, A. Khare, and U. Sukhatme, Phys. Rep. **251**, 267 (1995).
[16] *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. Stegun (Dover, New York, 1964).
[17] A. Khare and U. Sukhatme, J. Phys. A **21**, L501 (1988).
[18] K. Cahill, A. Comtet, and R. Glauber, Phys. Lett. **64B**, 283 (1976).
[19] L. Boya and J. Casahorran, Phys. Rev. D **41**, 1342 (1990).
[20] M.A. Lohe, Phys. Rev. D **20**, 3120 (1979).
[21] J.S. Langer, Ann. Phys. (N.Y.) **41**, 108 (1967).
[22] H.J. Muller-Kirsten, Z. Jian-Zu, and D.H. Tchrakian, Int. J. Mod. Phys. A **5**, 1319 (1990).
[23] B. Zwiebach, J. High Energy Phys. **09**, 028 (2000).
[24] J.D. Brown, *Lower Dimensional Gravity* (World Scientific, Singapore, 1988).
[25] J.A. Minahan and B. Zwiebach, J. High Energy Phys. **03**, 038 (2001).
[26] J.A. Minahan and B. Zwiebach, J. High Energy Phys. **09**, 029 (2000).
[27] N. Graham and R.L. Jaffe, Phys. Lett. B **435**, 145 (1998).
[28] E. Farhi, N. Graham, R.L. Jaffe, and H. Weigel, Nucl. Phys. **B585**, 443 (2000).
[29] E. Farhi, N. Graham, P. Haagensen, and R.L. Jaffe, Phys. Lett. B **427**, 334 (1998).
[30] G. Barton, J. Phys. A **18**, 479 (1985).
[31] A. Sen, hep-th/0204143.
[32] C. Barnes and N. Turok, hep-th/9711071; T. Weidig, hep-th/9912005.