Flow equation of quantum Einstein gravity in a higher-derivative truncation

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Motivated by recent evidence indicating that quantum Einstein gravity (QEG) might be nonperturbatively renormalizable, the exact renormalization group equation of QEG is evaluated in a truncation of theory space which generalizes the Einstein-Hilbert truncation by the inclusion of a higher-derivative term (R^2). The beta functions describing the renormalization group flow of the cosmological constant, Newton's constant, and the R^2 coupling are computed explicitly. The fixed point properties of the 3-dimensional flow are investigated, and they are confronted with those of the 2-dimensional Einstein-Hilbert flow. The non-Gaussian fixed point predicted by the latter is found to generalize to a fixed point on the enlarged theory space. In order to test the reliability of the R^2 truncation near this fixed point we analyze the residual scheme dependence of various universal quantities; it turns out to be very weak. The two truncations are compared in detail, and their numerical predictions are found to agree with a surprisingly high precision. Because of the consistency of the results it appears increasingly unlikely that the non-Gaussian fixed point is an artifact of the truncation. If it is present in the exact theory QEG is probably nonperturbatively renormalizable and "asymptotically safe." We discuss how the conformal factor problem of Euclidean gravity manifests itself in the exact renormalization group approach and show that, in the R^2 truncation, the investigation of the fixed point is not afflicted with this problem. Also the Gaussian fixed point of the Einstein-Hilbert truncation is analyzed; it turns out that it does not generalize to a corresponding fixed point on the enlarged theory space.

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I. INTRODUCTION

Recently a lot of work on quantum Einstein gravity (QEG) went into constructing an appropriate exact renormalization group (RG) equation [1,2], finding approximate solutions to it [3–7], and exploring their implications for black hole physics [8,9] and cosmology [10]. In particular, strong indications were found that QEG might be nonperturbatively renormalizable. If so, it could have the status of a fundamental, microscopic quantum theory of gravity.

The basic tool used in these investigations is the effective average action and its exact RG equation [11]. It is a continuum analogue of Wilson's lattice renormalization group of iterated block spin transformations [12]. Both in quantum field theory and statistical mechanics the idea is to integrate out all fluctuation modes which have momenta larger than a certain infrared (IR) cutoff k ("fast degrees of freedom"), and to take account of those modes in an implicit way by the modified dynamics which they induce for the remaining fluctuations with momenta smaller than k (the "slow degrees of freedom"). In field theory this "renormalized" dynamics is encoded in a scale dependent effective action, Γ_k, whose dependence on the cutoff scale k is governed by a functional differential equation referred to as the "exact RG equation" [13]. This equation gives rise to a flow on the space of all actions ("theory space"). The functional Γ_k defines an effective field theory valid near the scale k; evaluated at the tree level, it describes all loop effects due to the high-momentum modes. The effective average action can be thought of as a kind of microscope with a variable resolution. At large k, the physics at short distances $l=1/k$ can be read off directly from Γ_k; at small k we see a coarse-grained picture suitable for a simple description of structures with a large characteristic length scale $l=1/k$ [11].

The effective average action Γ_k, regarded as a function of k, interpolates between the ordinary effective action $\Gamma=\lim_{k\to 0}\Gamma_k$ and the bare (classical) action S which is approached for $k\to\infty$. The construction of Γ_k begins by adding a IR cutoff term $\Delta_k S$ to the classical action entering the standard Euclidean functional integral for the generating functional W of the connected Green's functions. The new piece $\Delta_k S$ introduces a momentum dependent $(\text{mass})^2$-term $\mathcal{R}_k(p^2)$ for each mode of the quantum field with momentum p. For $p^2 \gg k^2$, the cutoff function $\mathcal{R}_k(p^2)$ is assumed to vanish so that the high-momentum modes get integrated out unsuppressed. For $p^2 \ll k^2$, it behaves as $\mathcal{R}_k(p^2)\propto k^2$; hence the small-momentum modes are suppressed in the path integral by a mass term $\propto k^2$. Apart from a correction term which is known explicitly [11], the effective average action Γ_k is given by the Legendre transform of the modified generating functional W_k.

From this definition one can derive the exact RG equation obeyed by Γ_k. In a slightly symbolic notation it is of the form

$$k\partial_k\Gamma_k=\frac{1}{2}\mathrm{Tr}[(\Gamma_k^{(2)}+\mathcal{R}_k(-\Delta))^{-1}k\partial_k\mathcal{R}_k(-\Delta)]. \qquad (1.1)$$

The right-hand side (RHS) of this equation is a kind of "beta functional" which summarizes the beta functions for infinitely many running couplings. Geometrically, it defines a vector field on theory space, the corresponding flow lines being the RG trajectories $k\mapsto\Gamma_k$.

The functional Γ_k enters this vector field via its Hessian $\Gamma_k^{(2)}$, i.e. the infinite-dimensional matrix of all second functional derivatives of Γ_k with respect to the dynamical, i.e., non-background fields.

In Eq. (1.1) the c-number argument of $\mathcal{R}_k$ is replaced with the operator $-\Delta$. The discrimination of high-"momentum" vs low-"momentum" modes is performed according to the spectrum of this operator, i.e. p^2 is an eigenvalue of $-\Delta$. In simple theories where no gauge or diffeomorphism invariance needs to be respected, Δ is the free Laplacian, $\Delta=\partial_\mu\partial^\mu$, whose eigenmodes are momentum eigenstates in the usual sense of the word. In Yang-Mills theory [14,15] it has proven convenient to use the background field formalism [16,17] and to set $\Delta=\bar{D}_\mu\bar{D}^\mu$ where $\bar{D}_\mu$ is the covariant derivative in the background field. The background field technique plays a dual role in this context. Using a background gauge fixing term makes Γ_k a gauge invariant functional of its argument, and using $\bar{D}_\mu$ in the cutoff leads to a flow equation of the relatively simple type (1.1), similar to non-gauge theories. (The RG equation resulting from $\Delta=D_\mu D^\mu$ with D_μ constructed from the *dynamical* gauge field is quite unwieldy.)

Along similar lines an effective average action for d-dimensional Euclidean QEG has been constructed in Ref. [1], and the corresponding flow equation has been derived. Leaving the Faddeev-Popov ghosts aside for a moment, the gravitational effective average action, $\Gamma_k[g_{\mu\nu},\bar{g}_{\mu\nu}]$, is a functional of two different metrics, the "ordinary" dynamical metric $g_{\mu\nu}$ and the background metric $\bar{g}_{\mu\nu}$. The usual effective action $\Gamma[g_{\mu\nu}]$ is recovered by taking the limit $k\to 0$ of the functional $\bar{\Gamma}_k[g_{\mu\nu}]\equiv\Gamma_k[g_{\mu\nu},\bar{g}_{\mu\nu}=g_{\mu\nu}]$ in which the two metrics are taken equal. Thanks to the background gauge fixing condition this construction leads to a functional $\bar{\Gamma}_k[g_{\mu\nu}]$ which is invariant under general coordinate transformations.

One of the many advantages which the exact RG approach has in comparison to the standard canonical or path integral quantization is that it offers a very natural and intuitive nonperturbative approximation scheme. By truncating the theory space one can obtain approximate solutions to the RG equation which do not need a small expansion parameter. The idea is to project the RG flow from the "huge" infinite-dimensional space of all actions onto some smaller, typically finite dimensional subspace which is easier to handle. In this way the functional RG equation for Γ_k becomes a system of ordinary differential equations for a (finite) set of coupling constants which have the geometrical interpretation of coordinates on the subspace. It is clear that in applying this strategy the key problem is finding the "relevant" subspace which contains the essential physics.

In a first attempt at solving the gravitational RG equation [1] the flow has been projected onto the 2-dimensional subspace of theory space which is spanned by the invariants $\int d^dx\sqrt{g}$ and $\int d^dx\sqrt{g}R$. This is the so-called Einstein-Hilbert truncation defined by the ansatz

$$\Gamma_k[g,\bar{g}]=(16\pi G_k)^{-1}\int d^dx\sqrt{g}\{-R(g)+2\bar{\lambda}_k\} + \text{classical gauge fixing.} \tag{1.2}$$

The two running couplings involved are the running Newton constant G_k and the running cosmological constant $\bar{\lambda}_k$. The similarity of Eq. (1.2) to the action of classical general relativity is accidental in a sense; improved truncations would include both higher powers of the curvature and nonlocal terms [18,6]. In Eq. (1.2) only the gauge fixing term depends on $\bar{g}_{\mu\nu}$; it vanishes when we set $\bar{g}_{\mu\nu}=g_{\mu\nu}$.

The scale dependence of G_k and $\bar{\lambda}_k$ is most conveniently visualized as a flow in the λ-g plane where $g_k\equiv k^{d-2}G_k$ and $\lambda_k\equiv\bar{\lambda}_k/k^2$ are the dimensionless Newton constant and cosmological constant, respectively. Using the original cutoff of "type A" [1] the system of equations for g_k and λ_k was derived in [1] and solved numerically in [5]. In [2] a new cutoff of "type B" was introduced and the corresponding flow equations in the Einstein-Hilbert truncation were derived. (The "type B" cutoff is convenient if one uses the TT decomposition of the metric [19].) The fixed point properties of these equations were first discussed in [20] and [9], and analyzed in detail in [2] and [5]. One finds that the RG flow in the λ-g plane is governed by two fixed points (λ_*,g_*): a trivial or "Gaussian" fixed point at $(\lambda_*,g_*)=(0,0)$, and a non-Gaussian fixed point with $\lambda_*\neq 0$ and $g_*\neq 0$.

In order to appreciate the importance of the non-Gaussian fixed point we recall what it means to "quantize" a theory in the average action approach. One picks a bare action S and imposes the initial condition $\Gamma_{\hat{k}}=S$ at the ultraviolet (UV) cutoff scale $\hat{k}$, uses the RG equation to find Γ_k at all lower scales $k\leqslant\hat{k}$, and finally sends $k\to 0$ and $\hat{k}\to\infty$. A *fundamental* theory has the property that the "continuum" limit $\hat{k}\to\infty$ actually exists after redefining only finitely many parameters in the action. This is the case in perturbatively renormalizable theories [21], but there are also examples of perturbatively nonrenormalizable theories which possess a limit $\hat{k}\to\infty$ [22]. The continuum limit of those "nonperturbatively renormalizable" theories is taken at a non-Gaussian fixed point, i.e. the theory is defined by the set of RG trajectories which leave the fixed point when we lower k. These trajectories span the UV critical hypersurface of the fixed point, $\mathcal{S}_{\rm UV}$. If it is finite dimensional, the quantum theory thus constructed has only finitely many free parameters and therefore keeps its predictive power even at arbitrarily large momentum scales. This behavior is to be contrasted with an *effective* field theory which, at high energies, typically contains an increasing number of free parameters which must be taken from the experiment.

In his "asymptotic safety" scenario Weinberg [23,24] conjectured that a fundamental quantum field theory of gravity could perhaps be constructed nonperturbatively by taking the continuum limit at a non-Gaussian fixed point [25]. While originally this idea could be implemented in $d=2+\varepsilon$ dimensions only, the recent results coming from the effective average action strongly support the hypothesis that this fixed point exists also in 4 dimensions. Within the Einstein-Hilbert truncation, the existence of a suitable non-Gaussian fixed point is definitely established by now; the crucial question is whether it is the projection of a fixed point present in the exact theory or merely an artifact of the approximation.

Let us assume for a moment that the fixed point indeed exists in the exact 4-dimensional theory and that we define QEG by taking the $\hat{k}\to\infty$ limit there. Then, since $(\lambda_k, g_k, \ldots)$ approaches $(\lambda_*, g_*, \ldots)$ for $k\to\infty$, the dimensionful couplings behave as

$$G_k \approx g_*/k^2, \quad \bar{\lambda}_k \approx \lambda_* k^2, \ldots \tag{1.3}$$

for large k. Obviously G_k vanishes for $k\to\infty$. At least as far as this coupling is concerned QEG is asymptotically free similar to Yang-Mills theory. The $1/k^2$ dependence of the running Newton constant will lead to a characteristic momentum dependence of the cross sections for graviton-graviton scattering and graviton mediated matter-matter scattering. Because of this characteristic momentum dependence, QEG could be distinguished experimentally from alternative theories of quantum gravity such as string theory, at least in principle.

Let us generalize the standard definition of the Planck mass, $m_{\mathrm{Pl}} \equiv G^{-1/2}$, and introduce the *running Planck mass*

$$M_{\mathrm{Pl}}(k) \equiv 1/\sqrt{G_k}. \tag{1.4}$$

At the laboratory scale $M_{\mathrm{Pl}}(k)$ reduces to m_{Pl}, most probably, and its dependence on k is negligible. However, in the fixed point regime $k\to\infty$, the asymptotic freedom of G_k implies that $M_{\mathrm{Pl}}(k)$ is proportional to the scale k itself: $M_{\mathrm{Pl}}(k) = k/\sqrt{g_*}$. This shows that the running Planck mass is a rather elusive "barrier" which never can be jumped across in any experiment. If we analyze a system with a probe of increasing momentum k we will always push the running Planck mass ahead of us and never reach it.

Also the standard constant Planck mass, defined more precisely in terms of the IR value of G_k,

$$m_{\mathrm{Pl}} \equiv [G_{k=0}]^{-1/2}, \tag{1.5}$$

plays an important role in QEG, similar to that of Λ_{QCD} in QCD. According to the numerical solutions of the λ-g system [5], m_{Pl} marks the lower boundary of the asymptotic scaling region. Near $k = m_{\mathrm{Pl}}$ there is a crossover from the scaling laws (1.3) of the non-Gaussian fixed point to those of the Gaussian fixed point.[1]

According to the UV scaling laws (1.3) the dimensionful cosmological constant diverges for $k\to\infty$ proportional to k^2. This has an interesting geometrical interpretation. Let us consider the k-dependent, effective field equations implied by the truncation ansatz (1.2) with $\bar{g}_{\mu\nu} = g_{\mu\nu}$ for $d=4$. They happen to coincide with the familiar vacuum Einstein equations with the cosmological constant replaced by the scale-dependent quantity $\bar{\lambda}_k$:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\bar{\lambda}_k g_{\mu\nu}. \tag{1.6}$$

Since $\bar{\lambda}_k$ is the only quantity which sets a scale, every solution to Eq. (1.6) has a typical radius of curvature $r_c(k) \propto 1/\sqrt{\bar{\lambda}_k}$. (For instance, the maximally symmetric S^4 solution has the radius $r_c = r = \sqrt{3/\bar{\lambda}_k}$.) The k dependence of the solutions and in particular of r_c should be interpreted as follows. If we want to explore the spacetime structure at a fixed length scale $l \equiv 1/k$ it is most convenient to use the action $\Gamma_k[g_{\mu\nu}]$ at $k = 1/l$ because for this, and only this, functional a *tree level* analysis is sufficient to describe the essential physics at this scale, including all quantum effects. Hence, when we observe spacetime with a "microscope" of resolution l, we will see an average radius of curvature given by $r_c(l) \equiv r_c(k=1/l)$. Once l is smaller than the (standard) Planck length $l_{\mathrm{Pl}} \equiv m_{\mathrm{Pl}}^{-1}$ we are in the fixed point regime (1.3) so that $r_c(k) \propto 1/k$, or

$$r_c(l) \propto l. \tag{1.7}$$

Thus, when we look at the structure of spacetime with a microscope of resolution l, the average radius of curvature which we measure is proportional to the resolution itself. If we want to probe finer details and decrease l we automatically decrease r_c and hence *in*crease the average curvature. Spacetime seems to be more strongly curved at small distances than at larger ones. The scale-free relation (1.7) suggests that at distances below the Planck length quantum spacetime is a kind of fractal with a self-similar structure. It has no intrinsic scale.

Before we continue a remark might be in order on which theory precisely we refer to as "quantum Einstein gravity" or "QEG." While flow equations can also be used in the effective field theory approach to quantum gravity [27,1], in the present context "QEG" stands for the fundamental theory whose continuum limit $\hat{k}\to\infty$ is taken at the non-Gaussian fixed point. This theory has $\dim(\mathcal{S}_{\mathrm{UV}})$ free parameters, and fixing these parameters amounts to picking a specific trajectory $k \mapsto \Gamma_k$ in the full theory space. For $k\to\infty$ this trajectory hits the fixed point action Γ_*, regarded as the collection of its infinitely many dimensionless coordinates on theory space. The fixed point action Γ_* corresponds to the "bare" or "classical" action in conventional field theory. However, unlike the latter Γ_* is not put in by hand but is rather *derived* with the help of the RG equation. The usual canonical or path integral quantization is always based upon a "prejudice" about what the classical action is. In the asymptotic safety scenario the "classical" action is fixed instead by the condition of nonperturbative renormalizability; it cannot be guessed by simple power-counting, symmetry, or invariance arguments, but the effective average action provides a computational framework to determine it.

The really crucial property which defines QEG is the requirement of *diffeomorphism invariance*. Before we can write down a flow equation we must declare what the theory space is on which the renormalization group is supposed to operate. In the case of QEG it is defined to be the space of functionals $\Gamma[g_{\mu\nu}]$ depending on a nondegenerate, symmetric rank-2 tensor field in a diffeomorphism invariant way.

[1]A similar crossover was already known to occur in Liouville quantum gravity [26].

Leaving technical details aside, the theory space then fixes the flow equation which in turn determines the RG trajectories and the fixed points.

So far our discussion referred to the exact theory on the full theory space. If we project on the subspace spanned by the Einstein-Hilbert truncation it is clear that also the fixed point action Γ_* must be of the Einstein-Hilbert type. However, we emphasize that this is a trivial consequence of the simple truncation we have chosen, and we have no reason to believe that the exact Γ_* is of the Einstein-Hilbert type, too. In fact, within the more complicated truncation of the present paper Γ_* receives corrections which go beyond the Einstein-Hilbert form. Hence ‘‘quantum Einstein gravity’’ does *not* mean that the Einstein-Hilbert action is the bare action to be quantized. In this respect our approach is different from canonical quantum gravity, along the lines of Ashtekar’s program [28], for instance. (It is intriguing that also in this context remarkable finiteness properties have been proven recently [29].)

Clearly it is a highly attractive idea that there could be a nonperturbatively renormalizable field theory of the metric field so that there is no longer any conceptual need for leaving the framework of quantum field theory in order to arrive at a consistent microscopic theory of quantum gravity. Therefore every effort should be made to show that the non-Gaussian fixed point found in the Einstein-Hilbert truncation is not just an artifact of this approximation.

In Ref. [2] we therefore started an extensive analysis of the reliability of the Einstein-Hilbert truncation near the fixed point. There the strategy was to use the scheme dependence of universal quantities in order to get a first idea about the precision which can be achieved with this truncation. Here ‘‘scheme dependence’’ refers to the dependence on the details of the cutoff procedure, i.e. on the shape of the function $\mathcal{R}_k(p^2)$. By definition, universal quantities are exactly scheme independent in the exact theory, but they might acquire some scheme dependence once we make approximations. The level of this residual scheme dependence can serve as a measure for the quality of the approximation. Typical universal quantities are the critical exponents of fixed points and, as we argued, the product $g_*\lambda_*$. The upshot of our analysis was that the Einstein-Hilbert truncation seems to provide a description that is much more reliable and precise than originally hoped for, and that it would be very hard to understand the approximate scheme independence we found if the fixed point was just due to a misleading approximation.

These results are certainly very encouraging, but it is clear that the ultimate justification of a truncation ansatz consists of adding further terms to it and verifying that its predictions do not change much. In the present paper we take a first step in this direction and add one further invariant constructed from $g_{\mu\nu}$ to the ansatz.

Which invariant should we take? In standard renormalized perturbation theory where (at least implicitly) the $\hat{k}\to\infty$ limit is taken at the Gaussian fixed point, the relative importance or ‘‘relevance’’ of the various field monomials is measured by their scaling dimensions at the Gaussian fixed point, i.e. by their canonical dimensions simply. Since at the non-Gaussian fixed point the anomalous dimensions are large we have no similarly simple guide line at our disposal, and *a priori* all invariants are equally plausible. To get a first idea about what happens away from the Einstein-Hilbert subspace we shall include the higher-derivative invariant $\int d^dx\sqrt{g}R^2$ and study the RG flow in the ‘‘R^2 truncation’’[2]

$$\Gamma_k[g,\bar{g}]=\int d^dx\sqrt{g}\{(16\pi G_k)^{-1}[-R(g)+2\bar{\lambda}_k]+\bar{\beta}_k R^2(g)\}+\text{classical gauge fixing}. \tag{1.8}$$

Its truncation subspace is 3-dimensional, with coordinates G, $\bar{\lambda}$ and the new coupling $\bar{\beta}$.

It is well known that beyond $\int d^dx\sqrt{g}R^2$ there exist two[3] more $(\text{curvature})^2$ invariants: $\int d^dx\sqrt{g}R_{\mu\nu}R^{\mu\nu}$ and $\int d^dx\sqrt{g}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$. In a standard perturbative calculation near the Gaussian fixed point consistency would require us to include them along with the R^2 term because they all have the same canonical dimension. As for the non-Gaussian fixed point, we have no *a priori* information from general principles about the relative importance of the three terms. Since anyhow the best we can do is to take a ‘‘step into the dark,’’ without knowing whether we walk in the most ‘‘relevant’’ direction, we shall omit the other two invariants here. Including them would go far beyond the present calculational possibilities, in particular since it would require a much more complicated projection technique [2].

In this paper we shall derive the (extremely complicated) 3-dimensional RG equations of the R^2 truncation, and we shall use them in order to investigate the fixed points of the flow. Our main results will be the following: (a) The Gaussian fixed point of the Einstein-Hilbert truncation does *not* generalize to a fixed point of the R^2 truncation. (b) The non-Gaussian fixed point does indeed generalize to a fixed point of the R^2 truncation, and the λ-g projection of this fixed point is described almost perfectly by the Einstein-Hilbert truncation. Within the (weak) residual scheme dependence, the fixed point properties are almost insensitive to the inclusion of the R^2 invariant. This is further strong evidence against the theoretical possibility that the non-Gaussian fixed point is a truncation artifact.

In the second part of the paper we shall address a very important general problem which is of a more technical nature. It is related to a notorious disease of standard Euclidean quantum gravity: the conformal factor problem. In setting up the truncated RG equation the cutoff function $\mathcal{R}_k$ (actually a matrix in field space) is adapted to the truncation in such a way that, for $p^2\ll k^2$, the inverse propagator of every massless mode, p^2, is replaced by p^2+k^2. A problem arises if there are modes, such as those of the conformal factor in the Einstein-Hilbert truncation, which have a negative kinetic energy, i.e. their inverse propagator is $-p^2$. In [1] it has been argued that for these modes also the sign of $\mathcal{R}_k$ should be

[2]Our conventions are $R^\sigma{}_{\rho\mu\nu}=-\partial_\nu\Gamma^\sigma_{\mu\rho}+\cdots$, $R_{\mu\nu}=R^\sigma{}_{\mu\sigma\nu}$, $R=g^{\mu\nu}R_{\mu\nu}$.

[3]Except in $d=4$ where one invariant can be eliminated by virtue of the Gauss-Bonnet identity.

reversed so as to obtain the regularized inverse propagator $-(p^2+k^2)$. While there is little doubt that this procedure is correct for the Einstein-Hilbert truncation, it leads to the seemingly paradoxical situation that in the Euclidean path integral the modes of the conformal factor are enhanced rather than suppressed in the IR [1].

Contrary to the Einstein-Hilbert truncation, the R^2 truncation yields a functional $\Gamma_k[g_{\mu\nu}]$ which is bounded below and, as we shall see, gives positive kinetic energy to *all* modes, provided one stays close to the UV fixed point. Hence our investigation of the non-Gaussian fixed point is not plagued by the conformal factor problem, and the construction of the cutoff becomes straightforward. This advantage is an independent motivation for studying the R^2 truncation.

In the usual perturbative approach higher-derivative theories of (Lorentzian) gravity are notoriously problematic as far as causality and unitarity are concerned [30,31]. While in $d=4$ the most general (curvature)^2 theory, when expanded about flat space, is power-counting renormalizable it suffers from excitations with, classically, negative linearized energy and, quantum mechanically, a wrong-sign residue of the propagator leading to a state space containing negative-norm states [30]. These "ghosts" have masses of the order of the Planck mass. Correspondingly, if a truncation ansatz is of the (curvature)^2 type the k-dependent effective propagator $(\delta^2\Gamma_k/\delta g\,\delta g)^{-1}\equiv(\Gamma_k^{(2)})^{-1}$, evaluated for flat space, has similar ghosts with masses $\propto M_{\mathrm{Pl}}(k)$. By itself this does not indicate any real problem because generically flat space anyhow is not a solution of the effective equation of motion. Compared to the perturbative quantization the potential problem of ghost excitations manifests itself in the Euclidean RG approach in a conceptually different, more tractable manner. Here the linearization is performed about the backgrounds needed for the projection procedure, not about flat space. For a well-defined computation of the RG trajectories on a certain k interval it is sufficient that the (truncated) Γ_k gives positive linearized action to all modes contributing to the RG running in this interval. In our calculation this will indeed be the case for k large enough where the truncation is believed to be reliable [5]. The much more subtle issues related to a Lorentzian interpretation of the theory and its causality properties can be addressed only once a complete trajectory, valid down to the IR, and in particular the precise form of the fixed point action is known. From what we can tell now the exact QEG could very well be "causal" in an appropriate sense.

A brief summary of some of the results derived in the present paper appeared in [3], and an informal introduction to the older work can be found in [32]. In the present paper we focus on pure gravity. The gravitational average action with matter fields included was discussed in [7] and [33]. The gauge fixing dependence of the original formulation [1] was investigated in [34] and [35]. An incomplete higher-derivative calculation was begun in [36] where the running of the R^2 couplings was neglected, however, and no conclusions about the fixed point could be drawn.

The remaining sections of this paper are organized as follows. In Sec. II we review some general properties of the exact RG equation which will be needed later on. Section III is devoted to the construction of cutoffs which are adapted to a specific truncation; in particular the complications due to the conformal factor problem will be discussed there. In Sec. IV we derive the system of RG equations which results from the R^2 truncation, and in Sec. V we analyze its fixed point structure. We discuss the fate of the Gaussian fixed point which is present only in the Einstein-Hilbert truncation, and we reanalyze the non-Gaussian fixed point in the more general setting. In Sec. VI the positivity properties of the truncated action functional, its Hessian, and the cutoff operator are investigated; in particular we show that our analysis of the non-Gaussian fixed point is not affected by the conformal factor problem. The conclusions are contained in Sec. VII. Many important technical results, including the coefficients occurring in the rather complicated beta functions of the λ-g-β system, are tabulated in various Appendixes.

II. THE EXACT RG EQUATION

In this section we briefly review the construction of the "type B" RG equation for quantum gravity performed in Ref. [2] to which we refer for the details. We start from a scale-dependent modification of the generating functional for the connected Green's functions, W_k. It is defined by the following Euclidean functional integral:

$$\begin{aligned}\exp\{W_k[\text{sources}]\}=\int \mathcal{D}h_{\mu\nu}\mathcal{D}C^{\mu}\mathcal{D}\bar{C}_{\mu}\exp[&-S[\bar{g}+h]\\ &-S_{\mathrm{gf}}[h;\bar{g}]-S_{\mathrm{gh}}[h,C,\bar{C};\bar{g}]\\ &-\Delta_k S[h,C,\bar{C};\bar{g}]-S_{\mathrm{source}}].\end{aligned} \tag{2.1}$$

In Eq. (2.1) we use the background gauge fixing technique which necessitates the decomposition of the full quantum metric $\gamma_{\mu\nu}$ into a fixed background metric $\bar{g}_{\mu\nu}$ and a fluctuation variable $h_{\mu\nu}$: $\gamma_{\mu\nu}(x)=\bar{g}_{\mu\nu}(x)+h_{\mu\nu}(x)$. It allows us to replace the integration over $\gamma_{\mu\nu}$ by an integration over $h_{\mu\nu}$. Furthermore, $\bar{C}_\mu$ and C^μ are the Faddeev-Popov ghosts of the gravitational field.

The first term of the action, $S[\bar{g}+h]$, is the classical part, which is assumed to be invariant under general coordinate transformations. For the time being, we also assume that it is positive definite, $S>0$. The gauge fixing term is given by

$$S_{\mathrm{gf}}[h;\bar{g}]=\frac{1}{2\alpha}\int d^dx\,\sqrt{\bar{g}}\bar{g}^{\mu\nu}F_\mu[\bar{g},h]F_\nu[\bar{g},h]. \tag{2.2}$$

In the present paper we use the linear gauge condition $F_\mu[\bar{g},h]=\sqrt{2}\kappa\mathcal{F}_\mu^{\alpha\beta}[\bar{g}]h_{\alpha\beta}$ with $\mathcal{F}_\mu^{\alpha\beta}[\bar{g}]=\delta_\mu^\beta\bar{g}^{\alpha\gamma}\bar{D}_\gamma-\frac{1}{2}\bar{g}^{\alpha\beta}\bar{D}_\mu$, which amounts to a background version of the harmonic coordinate condition. Here we introduced the constant $\kappa\equiv(32\pi\bar{G})^{-1/2}$ where $\bar{G}$ is the bare Newton constant. Moreover, $\bar{D}_\mu$ denotes the covariant derivative constructed from the background metric $\bar{g}_{\mu\nu}$, while we shall write D_μ for the covariant derivative involving the complete metric $\gamma_{\mu\nu}$. S_{gh} is the Faddeev-Popov ghost action resulting from the above gauge fixing.

Furthermore, $\Delta_k S$ and $S_{\rm source}$ are the cutoff and the source action, respectively. $\Delta_k S$ provides an appropriate infrared cutoff for the integration variables and will be discussed in detail in a moment; $S_{\rm source}$ introduces sources for the fields $h_{\mu\nu}$, C^μ, and $\bar{C}_\mu$.

Next we decompose the gravitational field $h_{\mu\nu}$ according to (see e.g. [19])

$$h_{\mu\nu}=h^T_{\mu\nu}+\bar{D}_\mu\hat{\xi}_\nu+\bar{D}_\nu\hat{\xi}_\mu+\bar{D}_\mu\bar{D}_\nu\hat{\sigma}-\frac{1}{d}\bar{g}_{\mu\nu}\bar{D}^2\hat{\sigma}+\frac{1}{d}\bar{g}_{\mu\nu}\phi. \tag{2.3}$$

In order to obtain this "TT decomposition" one starts by writing $h_{\mu\nu}$ as a sum of its orthogonal parts: $h_{\mu\nu}=h^T_{\mu\nu}+h^L_{\mu\nu}+h^{Tr}_{\mu\nu}$. Here $h^T_{\mu\nu}$, $h^L_{\mu\nu}$ and $h^{Tr}_{\mu\nu}$ represent the transverse traceless, longitudinal traceless and pure trace part, respectively. Introducing two scalar fields ϕ and $\hat{\sigma}$, and a transverse vector field $\hat{\xi}_\mu$, the tensors $h^{Tr}_{\mu\nu}$ and $h^L_{\mu\nu}$ can be expressed by $h^{Tr}_{\mu\nu}\equiv\bar{g}_{\mu\nu}\phi/d$ and $h^L_{\mu\nu}=h^{LT}_{\mu\nu}+h^{LL}_{\mu\nu}$ with $h^{LT}_{\mu\nu}\equiv\bar{D}_\mu\hat{\xi}_\nu+\bar{D}_\nu\hat{\xi}_\mu$ and $h^{LL}_{\mu\nu}\equiv\bar{D}_\mu\bar{D}_\nu\hat{\sigma}-\bar{g}_{\mu\nu}\bar{D}^2\hat{\sigma}/d$. Thereby we end up with Eq. (2.3). In the following the components of $h_{\mu\nu}$ thus introduced will be referred to as the "component fields." They obey the relations

$$\bar{g}^{\mu\nu}h^T_{\mu\nu}=0, \quad \bar{D}^\mu h^T_{\mu\nu}=0,$$
$$\bar{D}^\mu\hat{\xi}_\mu=0, \quad \phi=\bar{g}_{\mu\nu}h^{\mu\nu}. \tag{2.4}$$

Obviously the complete field $h_{\mu\nu}$ receives no contribution from those $\hat{\xi}_\mu$ and $\hat{\sigma}$ modes which satisfy the Killing equation

$$\bar{D}_\mu\hat{\xi}_\nu+\bar{D}_\nu\hat{\xi}_\mu=0 \tag{2.5}$$

and the scalar equation

$$\bar{D}_\mu\bar{D}_\nu\hat{\sigma}-\frac{1}{d}\bar{g}_{\mu\nu}\bar{D}^2\hat{\sigma}=0, \tag{2.6}$$

respectively. Such "unphysical" $\hat{\xi}_\mu$ and $\hat{\sigma}$ modes have to be excluded from the functional integral and all subsequent calculations [2]. Having a closer look at the scalar equation (2.6), one recognizes that there is a one-to-one correspondence between the nonconstant solutions of Eq. (2.6) and the purely longitudinal, or proper, conformal Killing vectors (PCKV) $\mathcal{C}_\mu$. They are related via $\mathcal{C}_\mu=\bar{D}_\mu\hat{\sigma}$.

Likewise we decompose the ghost and the antighost into their orthogonal components:

$$\bar{C}_\mu=\bar{C}^T_\mu+\bar{D}_\mu\hat{\bar{\eta}}, \quad C^\mu=C^{T\mu}+\bar{D}^\mu\hat{\eta}. \tag{2.7}$$

Here $\bar{C}^T_\mu$ and $C^{T\mu}$ are the transverse components of $\bar{C}_\mu$ and C^μ: $\bar{D}^\mu\bar{C}^T_\mu=0$, $\bar{D}_\mu C^{T\mu}=0$. Furthermore, the scalars $\hat{\bar{\eta}}$ and $\hat{\eta}$ parametrize the longitudinal part of $\bar{C}_\mu$ and C^μ, respectively. The constant $\hat{\bar{\eta}}$ and $\hat{\eta}$ modes represent unphysical modes which have to be excluded.

For calculational convenience we now introduce new variables $\{\xi_\mu,\sigma,\bar{\eta},\eta\}$ replacing $\{\hat{\xi}_\mu,\hat{\sigma},\hat{\bar{\eta}},\hat{\eta}\}$, by means of the momentum dependent (nonlocal) redefinitions

$$\xi^\mu\equiv\sqrt{-\bar{D}^2-\overline{\rm Ric}}\,\hat{\xi}^\mu$$
$$\sigma\equiv\sqrt{(\bar{D}^2)^2+\frac{d}{d-1}\bar{D}_\mu\bar{R}^{\mu\nu}\bar{D}_\nu}\,\hat{\sigma}$$
$$\bar{\eta}\equiv\sqrt{-\bar{D}^2}\,\hat{\bar{\eta}}, \quad \eta\equiv\sqrt{-\bar{D}^2}\,\hat{\eta}. \tag{2.8}$$

Here the operator $\overline{\rm Ric}$ maps vectors onto vectors according to $(\overline{\rm Ric}\,v)^\mu=\bar{R}^{\mu\nu}v_\nu$. In accordance with the decompositions (2.3), (2.7) and the redefinitions (2.8) we then perform the combined transformation of integration variables $h_{\mu\nu}\to\{h^T_{\mu\nu},\xi_\mu,\sigma,\phi\}$, $\bar{C}_\mu\to\{\bar{C}^T_\mu,\bar{\eta}\}$, $C^\mu\to\{C^{T\mu},\eta\}$ in the functional integral (2.1). The Jacobian induced by this change of variables is such that it boils down to an unimportant constant if Einstein backgrounds, characterized by $\bar{R}_{\mu\nu}=C\bar{g}_{\mu\nu}$ with C a constant, are inserted into Eq. (2.1). (For the general case see Ref. [2].)

Let us now come to the "type B" cutoff term $\Delta_k S$. At the component field level, it is a sum of inner products,

$$\Delta_k S[h,C,\bar{C};\bar{g}]=\frac{1}{2}\sum_{\zeta_1,\zeta_2\in I_1}\langle\zeta_1,(\mathcal{R}_k)_{\zeta_1\zeta_2}\zeta_2\rangle+\frac{1}{2}\sum_{\psi_1,\psi_2\in I_2}\langle\psi_1,(\mathcal{R}_k)_{\psi_1\psi_2}\psi_2\rangle \tag{2.9}$$

with the index sets $I_1\equiv\{h^T,\xi,\sigma,\phi\}$, $I_2\equiv\{\bar{C}^T,C^T,\bar{\eta},\eta\}$. At this stage of the discussion it is not necessary to specify the explicit structure of the cutoff operators $\mathcal{R}_k$ acting on the component fields. In order to provide the desired suppression of low-momentum modes, these operators must vanish for $p^2/k^2\to\infty$ (in particular for $k\to 0$) and must behave as $\mathcal{R}_k\to\mathcal{Z}_k k^2$ for $p^2/k^2\to 0$. (The meaning of the constant $\mathcal{Z}_k$ will be explained later.) Furthermore, they have to satisfy certain Hermiticity conditions [2].

Now we are in a position to construct the effective average action Γ_k. It is defined as the difference between the Legendre transform of W_k at fixed $\bar{g}_{\mu\nu}$, denoted $\tilde{\Gamma}_k[\bar{h},v,\bar{v};\bar{g}]$, and the cutoff action with the classical fields inserted [37,14]:

$$\Gamma_k[g,\bar{g},v,\bar{v}]\equiv\tilde{\Gamma}_k[g-\bar{g},v,\bar{v};\bar{g}]-\Delta_k S[g-\bar{g},v,\bar{v};\bar{g}]. \tag{2.10}$$

Here the classical fields represent the (k-dependent) expectation values of the quantum fluctuations: $\bar{h}_{\mu\nu}\equiv\langle h_{\mu\nu}\rangle$, $\bar{v}_\mu\equiv\langle\bar{C}_\mu\rangle$, $v^\mu\equiv\langle C^\mu\rangle$. They are obtained in the usual way as functional derivatives of W_k with respect to the sources. In Eq. (2.10) we expressed $\bar{h}_{\mu\nu}$ in terms of the classical counterpart $g_{\mu\nu}$ of the quantum metric $\gamma_{\mu\nu} \equiv \bar{g}_{\mu\nu} + h_{\mu\nu}$ which, by definition, is given by $g_{\mu\nu} \equiv \bar{g}_{\mu\nu} + \bar{h}_{\mu\nu}$. The classical analogs of the components $(h^T, \xi, \sigma, \phi, \bar{C}^T, C^T, \bar{\eta}, \eta)$ will be denoted $(\bar{h}^T, \bar{\xi}, \bar{\sigma}, \bar{\phi}, \bar{v}^T, v^T, \bar{\varrho}, \varrho)$.

The exact RG equation for the effective average action describes the change of Γ_k induced by an infinitesimal change of the scale k. Introducing the RG "time" $t \equiv \ln k$, it can be derived from the t derivative of the functional integral (2.1). It takes the form

$$\partial_t \Gamma_k[g, \bar{g}, v, \bar{v}] = \frac{1}{2} \mathrm{Tr}' \left[\sum_{\zeta_1, \zeta_2 \in \bar{I}_1} (\Gamma_k^{(2)}[g, \bar{g}, v, \bar{v}] + \mathcal{R}_k)^{-1}_{\zeta_1 \zeta_2} \times \partial_t (\mathcal{R}_k)_{\zeta_2 \zeta_1} \right] + \frac{1}{2} \mathrm{Tr}' \left[\sum_{\psi_1, \psi_2 \in \bar{I}_2} (\Gamma_k^{(2)}[g, \bar{g}, v, \bar{v}] + \mathcal{R}_k)^{-1}_{\psi_1 \psi_2} \times \partial_t (\mathcal{R}_k)_{\psi_2 \psi_1} \right]. \quad (2.11)$$

Here $\Gamma_k^{(2)}$ denotes the Hessian of Γ_k with respect to the component fields. Furthermore, we wrote $(\mathcal{R}_k)_{\zeta_1 \zeta_2} \equiv (\mathcal{R}_k)_{\langle \zeta_1 \rangle \langle \zeta_2 \rangle}$, $(\mathcal{R}_k)_{\psi_1 \psi_2} \equiv (\mathcal{R}_k)_{\langle \psi_1 \rangle \langle \psi_2 \rangle}$ and introduced the index sets $\bar{I}_1 \equiv \{\bar{h}^T, \bar{\xi}, \bar{\sigma}, \bar{\phi}\}$, $\bar{I}_2 \equiv \{\bar{v}^T, v^T, \bar{\varrho}, \varrho\}$. Furthermore, the primes at the traces indicate that all unphysical $\bar{\xi}_\mu$ and $\bar{\sigma}$ modes, characterized by Eqs. (2.5) and (2.6), are to be excluded from the calculation of the traces.

III. TRUNCATIONS AND THEIR ADAPTED CUTOFFS

A. Truncating the ghost sector

In concrete applications of the exact RG equation one encounters the problem of dealing with an infinite system of coupled differential equations. Usually it is impossible to find an exact solution so that we are forced to rely upon approximations. A powerful nonperturbative approximation scheme is the truncation of theory space, which means that only a finite number of couplings is considered and the RG flow is projected onto a finite-dimensional subspace of theory space. In practice one proceeds as follows. One makes an ansatz for Γ_k that comprises only a few couplings and inserts it on both sides of Eq. (2.11). By projecting the RHS of this equation onto the space of operators appearing on the LHS one obtains a finite set of coupled differential equations for the couplings taken into account.

Given an arbitrary truncation it is not clear *a priori* whether it is sensible and leads to at least approximately correct results. In this respect the modified BRS Ward identities satisfied by the exact Γ_k [1] are of special importance, since only those truncations which are (approximately) consistent with them can be reliable. In [1] it was shown that under certain conditions truncations of the form

$$\Gamma_k[g, \bar{g}, v, \bar{v}] = \bar{\Gamma}_k[g] + \hat{\Gamma}_k[g, \bar{g}] + S_{\mathrm{gf}}[g - \bar{g}; \bar{g}] + S_{\mathrm{gh}}[g - \bar{g}, v, \bar{v}; \bar{g}] \quad (3.1)$$

which neglect the RG running in the ghost sector are approximate solutions to the Ward identities for the exact Γ_k. Here $\bar{\Gamma}_k[g]$ is defined as

$$\bar{\Gamma}_k[g] \equiv \Gamma_k[g, g, 0, 0] \quad (3.2)$$

and $\hat{\Gamma}_k[g, \bar{g}]$ encodes the quantum corrections of the gauge fixing term. (For the details we refer to [1,2].) Inserting the ansatz (3.1) into the exact evolution equation (2.11) leads to a truncated RG equation which describes the RG flow of Γ_k in the subspace of action functionals spanned by Eq. (3.1). The equation governing the evolution of the purely gravitational action

$$\Gamma_k[g, \bar{g}] \equiv \Gamma_k[g, \bar{g}, 0, 0] = \bar{\Gamma}_k[g] + S_{\mathrm{gf}}[g - \bar{g}; \bar{g}] + \hat{\Gamma}_k[g, \bar{g}] \quad (3.3)$$

takes the form

$$\partial_t \Gamma_k[g, \bar{g}] = \frac{1}{2} \mathrm{Tr}' \left[\sum_{\zeta_1, \zeta_2 \in \bar{I}_1} (\Gamma_k^{(2)}[g, \bar{g}] + \mathcal{R}_k)^{-1}_{\zeta_1 \zeta_2} \times \partial_t (\mathcal{R}_k)_{\zeta_2 \zeta_1} \right] + \frac{1}{2} \mathrm{Tr}' \left[\sum_{\psi_1, \psi_2 \in \bar{I}_2} (S_{\mathrm{gh}}^{(2)}[g, \bar{g}] + \mathcal{R}_k)^{-1}_{\psi_1 \psi_2} \times \partial_t (\mathcal{R}_k)_{\psi_2 \psi_1} \right]. \quad (3.4)$$

Here $\Gamma_k^{(2)}$ and $S_{\mathrm{gh}}^{(2)}$ are the Hessians of $\Gamma_k[g, \bar{g}]$ and $S_{\mathrm{gh}}[\bar{h}, v, \bar{v}; \bar{g}]$ with respect to the gravitational and the ghost component fields, respectively. They are taken at fixed $\bar{g}_{\mu\nu}$.

B. Construction of the cutoff, and the conformal factor problem

In order to obtain a tractable evolution equation for a given truncation it is necessary to use a cutoff which is adapted to this truncation but still has the desired suppression properties for a class of backgrounds which is as large as possible.

A convenient cutoff which is adapted to the truncation ansatz can be found in the following way [1,7]. Given a truncation, we assume that for $\bar{g} = g$ the kinetic operators of all modes with a definite helicity can be brought to the form $(\Gamma_k^{(2)})_{ij} = f_{ij}(-\bar{D}^2, k, \dots)$ where $\{f_{ij}\}$ is a set of c-number functions and the indices i, j refer to the different types of fields. Then we choose the cutoff operator $\mathcal{R}_k$ in such a way that the structure

$$(\Gamma_k^{(2)} + \mathcal{R}_k)_{ij} = f_{ij}(-\bar{D}^2 + k^2 R^{(0)}(-\bar{D}^2/k^2), k, \dots) \quad (3.5)$$

is achieved. Here the function $R^{(0)}(y)$, $y=-\bar{D}^2/k^2$, describes the details of the mode suppression; it is required to satisfy the boundary conditions $R^{(0)}(0)=1$ and $\lim_{y\to\infty}R^{(0)}(y)=0$, but is arbitrary otherwise. By virtue of Eq. (3.5), the inverse propagator of a massless field mode with covariant momentum square $p^2\equiv-\bar{D}^2$ is proportional to $p^2+k^2R^{(0)}(p^2/k^2)$ which equals p^2 for $p^2\gg k^2$ and p^2+k^2 for $p^2\ll k^2$. This means that the small-p^2 modes, and only those, have acquired a mass $\propto k$ which leads to the desired suppression.

In order to see the potential problems of the rule (3.5) let us be more specific and assume that the functions f_{ij} are linear in $\bar{D}^2$ and contain no constant term. Then, after diagonalizing f_{ij} with respect to the field indices, $\Gamma_k^{(2)}$ decomposes into a set of (massless, by assumption) inverse propagators $z_k p^2$ with a running wave function normalization z_k. In this diagonal basis $\mathcal{R}_k$ is diagonal, too. It is of the form

$$\mathcal{R}_k(p^2)=\mathcal{Z}_k k^2 R^{(0)}\left(\frac{p^2}{k^2}\right). \tag{3.6}$$

A priori $\mathcal{Z}_k$ is a free constant, but when we apply the rule (3.5) we are forced to set $\mathcal{Z}_k=z_k$ for each mode. Only then the propagator and the cutoff combine in the right way, leading to the modified inverse propagator $z_k[p^2+k^2R^{(0)}(p^2/k^2)]$.

The choice $\mathcal{Z}_k=z_k$ is certainly the correct one if z_k is positive. This is indeed the case in the familiar unitary theories on flat spacetime. In QEG, however, there are truncations, the Einstein-Hilbert truncation, for instance, which give a negative kinetic energy to certain modes φ of the metric. In particular, in the Einstein-Hilbert truncation, the conformal factor $\varphi\equiv\phi$ has $z_k<0$.

The important question is how $\mathcal{Z}_k$ should be chosen when z_k is negative. If we continue to use $\mathcal{Z}_k=z_k$, the RG equation is still well-defined because the inverse propagator $-|z_k|[p^2+k^2R^{(0)}(p^2/k^2)]$ never vanishes so that the functional traces on the RHS of Eq. (3.4) are not suffering from any IR problem. In fact, if we write down the perturbative expansion of the traces, for instance, we see that all propagators are correctly cut off in the IR, and that loop momenta smaller than *k* are suppressed correctly. This would not have been the case if we had insisted on a positive $\mathcal{Z}_k$, setting $\mathcal{Z}_k=-z_k>0$. In this case the modified inverse propagator $-|z_k|[p^2-k^2R^{(0)}(p^2/k^2)]$, because of the relative minus sign between p^2 and the $\mathcal{R}_k$ term, fails to suppress the IR modes. Even worse, it can introduce a spurious singularity at the value of p^2 for which $p^2-k^2R^{(0)}(p^2/k^2)=0$.

At first sight the choice $\mathcal{Z}_k=-z_k>0$ might have appeared to be the more natural one because only if $\mathcal{Z}_k>0$ the factor $\exp(-\Delta_k S)\propto\exp(-\int\mathcal{R}_k\varphi^2)$ is a damped exponential which suppresses the low-momentum modes under the path integral. Nevertheless it was argued in [1] that the "$\mathcal{Z}_k=z_k$ rule" is the correct choice both for $z_k>0$ and $z_k<0$. The calculations in [1] and all subsequent papers [2,7,34,33] were based upon this rule. On the one hand, this rule guarantees that the RG equation is well-defined and consistent. On the other hand, it is difficult to give a meaning to the Euclidean functional integral from which this RG equation was derived at the formal level. In the case $\mathcal{Z}_k=z_k<0$ the factor $\exp(+\int|\mathcal{R}_k|\varphi^2)$ is a growing exponential which seems to enhance rather than suppress the low-momentum modes. However, as suggested by the perturbative argument above, this conclusion is too naive probably.

Let us now come to the case where the functions f_{ij} are not linear in $\bar{D}^2$ but, say, of the form $z_k^{(1)}(\bar{D}^2)^2+z_k^{(2)}\bar{D}^2$. In this case the rule (3.5) demands that we choose the corresponding operators $(\mathcal{R}_k)_{ij}$ in such a way that they contain cutoff terms adjusted to both the $(\bar{D}^2)^2$ and the $\bar{D}^2$ terms of the kinetic operator. For the above example, which is relevant to the R^2 truncation, $(\mathcal{R}_k)_{ij}$ assumes the general form

$$\mathcal{R}_k(p^2)=\mathcal{Z}_k^{(1)}\left[2p^2k^2R^{(0)}\left(\frac{p^2}{k^2}\right)+k^4\left(R^{(0)}\left(\frac{p^2}{k^2}\right)\right)^2\right]+\mathcal{Z}_k^{(2)}k^2R^{(0)}\left(\frac{p^2}{k^2}\right) \tag{3.7}$$

where we omitted the indices referring to the types of fields. Obviously we have to set $\mathcal{Z}_k^{(1)}=z_k^{(1)}$ and $\mathcal{Z}_k^{(2)}=z_k^{(2)}$ in order to achieve that the propagator and the cutoff combine as prescribed by the rule (3.5). This leads to the modified inverse propagator $z_k^{(1)}[p^2+k^2R^{(0)}(p^2/k^2)]^2+z_k^{(2)}[p^2+k^2R^{(0)}(p^2/k^2)]$. For brevity we refer to this prescription, too, as the $\mathcal{Z}_k=z_k$ rule.

We believe that the $\mathcal{Z}_k=z_k$ rule is correct also for $z_k<0$, and that it is the relation between the manifestly well-defined flow equation and the formal path integral that needs to be understood better. Various attitudes are possible here. For instance, one could postulate that the fundamental definition of the theory is in terms of the flow equation rather than the path integral. Since the former is much better defined than the latter (in particular also with respect to the usual UV and IR problems) one would simply discard the path integral then. Another way out is to adopt the usual, albeit rather *ad hoc*, prescription of Wick rotating the conformal factor ($\varphi\to i\varphi$) which turns the growing exponential into a decaying one.

A much more attractive and less radical possibility is the following. Presumably it will be possible to construct an effective average action for *Lorentzian* quantum gravity by invoking a kind of stationary phase argument for the mode suppression. Then one deals with oscillating exponentials $\exp(i\,S)\exp(i\Delta_k S)$, and apart from the trivial substitutions $\Gamma_k\to-i\Gamma_k$, $\mathcal{R}_k\to-i\mathcal{R}_k$ the flow equation remains the same as in Euclidean gravity. For $\mathcal{Z}_k=z_k$ it has all the desired features, and $z_k<0$ poses no special problem for the path integral. It is interesting that there are also recent indications [38] coming from the dynamical triangulation approach to quantum gravity [39] which suggest that the Lorentzian path integral might have a better chance of being well-defined than the Euclidean one.

As the last possibility we mention the best of all situations, namely that in an exact treatment there are simply no factors $z_k<0$. If this is actually the case, the conformal factor problem which we encounter in the Einstein-Hilbert and similar truncations would have the status of an unphysical truncation artifact. If so, the "$\mathcal{Z}_k = z_k$ rule" could be interpreted as a device which helps in approximating as well as possible the exact RG flow by a truncated flow.

It is one of the main results of the present paper that this scenario is indeed realized to some extent. We shall see that within the R^2 truncation those terms of f_{ij} which dominate at sufficiently large momenta have $z_k > 0$ at least for large enough values of k ($k \gg m_{\rm Pl}$). For too low scales ($k \lesssim m_{\rm Pl}$) some of the z_k's might turn negative, but at these scales the R^2 truncation becomes unreliable probably [5] so that the negative z_k's might be due to an insufficient truncation. It is not excluded that in the exact theory the dominating z_k's are positive down to $k=0$.[4]

As all those z_k's which determine the sign of the dominating contributions to f_{ij} are positive for large values of k, the cutoff has the standard suppression properties and is not plagued by any conformal factor problem for $k \to \infty$. This, then, allows for an unambiguous investigation of the UV fixed point and its properties. It is quite remarkable that, as we shall see, all results concerning the fixed point are basically the same for the R^2 truncation (where $\mathcal{Z}_k = z_k > 0$ at least for the dominating terms) and the Einstein-Hilbert truncation with both positive and negative factors $\mathcal{Z}_k = z_k$. This is certainly a quite impressive confirmation of the $\mathcal{Z}_k = z_k$ rule.

C. The cutoff adapted to the R^2 truncation

In the next section we shall see in detail that for the truncation studied in this paper we can comply with the $\mathcal{Z}_k = z_k$ rule by using the following cutoff operators for the component fields:

$$
\begin{aligned}
(\mathcal{R}_k)_{\bar h^T h^T}^{\mu\nu\alpha\beta} &= \frac{1}{4}(\bar g^{\mu\alpha}\bar g^{\nu\beta} + \bar g^{\mu\beta}\bar g^{\nu\alpha})\{\mathcal{Z}_k^{\bar h^T h^T}\kappa^2 + \mathcal{Y}_k^{\bar h^T h^T}\bar R, k^2 R^{(0)}(-D^2/k^2)\},\\
(\mathcal{R}_k)_{\bar\xi\xi}^{\mu\nu} &= \mathcal{Z}_k^{\bar\xi\xi}\kappa^2 \bar g^{\mu\nu} k^2 R^{(0)}(-\bar D^2/k^2),\\
(\mathcal{R}_k)_{\bar\sigma\sigma} &= \mathcal{X}_k^{\bar\sigma\sigma}(-2\bar D^2 k^2 R^{(0)}(-\bar D^2/k^2) + k^4 R^{(0)}(-\bar D^2/k^2)^2) + \frac{1}{2}\{\mathcal{Z}_k^{\bar\sigma\sigma}\kappa^2 + \mathcal{Y}_k^{\bar\sigma\sigma}\bar R, k^2 R^{(0)}(-\bar D^2/k^2)\},\\
(\mathcal{R}_k)_{\bar\phi\sigma} &= (\mathcal{R}_k)^{\dagger}_{\bar\sigma\phi} = \mathcal{X}_k^{\bar\phi\sigma}\Bigg[\bar P_k\sqrt{\left(\bar P_k + \frac{d}{d-1}\bar D_\mu \bar R^{\mu\nu}\bar D_\nu(-\bar D^2)^{-1}\right)\bar P_k}\\
&\quad + \bar D^2\sqrt{(\bar D^2)^2 + \frac{d}{d-1}\bar D_\mu \bar R^{\mu\nu}\bar D_\nu}\Bigg] + (\mathcal{Y}_k^{\bar\phi\sigma}\bar R + \mathcal{Z}_k^{\bar\phi\sigma}\kappa^2)\\
&\quad \times\Bigg[\sqrt{\left(\bar P_k + \frac{d}{d-1}\bar D_\mu \bar R^{\mu\nu}\bar D_\nu(-\bar D^2)^{-1}\right)\bar P_k} - \sqrt{(\bar D^2)^2 + \frac{d}{d-1}\bar D_\mu \bar R^{\mu\nu}\bar D_\nu}\Bigg],\\
(\mathcal{R}_k)_{\bar\phi\phi} &= \mathcal{X}_k^{\bar\phi\phi}(-2\bar D^2 k^2 R^{(0)}(-\bar D^2/k^2) + k^4 R^{(0)}(-\bar D^2/k^2)^2) + \frac{1}{2}\{\mathcal{Z}_k^{\bar\phi\phi}\kappa^2 + \mathcal{Y}_k^{\bar\phi\phi}\bar R, k^2 R^{(0)}(-\bar D^2/k^2)\},\\
(\mathcal{R}_k)_{\bar\upsilon^T\upsilon^T}^{\mu\nu} &= -(\mathcal{R}_k)_{\upsilon^T\bar\upsilon^T}^{\mu\nu} = \mathcal{Z}_k^{\bar\upsilon^T\upsilon^T}\bar g^{\mu\nu} k^2 R^{(0)}(-\bar D^2/k^2),\\
(\mathcal{R}_k)_{\bar\varrho\varrho} &= -(\mathcal{R}_k)_{\varrho\bar\varrho} = \mathcal{Z}_k^{\bar\varrho\varrho} k^2 R^{(0)}(-\bar D^2/k^2).
\end{aligned}
\tag{3.8}
$$

Here $\bar P_k$ is defined as

$$\bar P_k \equiv -\bar D^2 + k^2 R^{(0)}(-\bar D^2/k^2) \tag{3.9}$$

and the curly brackets denote the anticommutator, i.e. $\{A,B\} = AB + BA$ for arbitrary operators A, B. The remaining cutoff operators which appear in Eq. (2.9) but are not listed in Eq. (3.8) are set to zero.

The constants $\mathcal{X}_k$, $\mathcal{Y}_k$, and $\mathcal{Z}_k$ will be adjusted later. It should be noted that the terms proportional to the $\mathcal{X}_k$'s and $\mathcal{Y}_k$'s provide the cutoff for those contributions to $\Gamma_k^{(2)}$ which come from the higher-derivative terms. For $\mathcal{Y}_k^{\bar h^T h^T} = \mathcal{Y}_k^{\bar\sigma\sigma} = \mathcal{Y}_k^{\bar\phi\sigma} = \mathcal{Y}_k^{\bar\phi\phi} = \mathcal{X}_k^{\bar\sigma\sigma} = \mathcal{X}_k^{\bar\phi\sigma} = \mathcal{X}_k^{\bar\phi\phi} = 0$, Eq. (3.8) actually boils down to the cutoff of type B used in [2] in the context of the Einstein-Hilbert truncation. (This cutoff type has to be

[4] Recent investigations in a scalar toy model [40] indeed suggest that the conformal factor problem could be solved dynamically by strong instability-driven renormalization effects.

distinguished from the cutoff of type A used in the original paper [1], which is formulated in terms of the complete fields and does not involve the component fields.)

Each cutoff contains some "shape function" $R^{(0)}$. A particularly suitable choice is the exponential shape function

$$R^{(0)}(y)=y[\exp(y)-1]^{-1}. \tag{3.10}$$

In order to check the scheme independence of universal quantities we employ a one-parameter generalization of Eq. (3.10), the class of exponential shape functions,

$$R^{(0)}(y;s)=sy[\exp(sy)-1]^{-1}, \tag{3.11}$$

with the "shape parameter" $s>0$ parametrizing the profile of $R^{(0)}$ [35]. Another admissible choice we are going to use is the following class of shape functions with compact support:

$$R^{(0)}(y;b)=\begin{cases}1, & y\leqslant b,\\ \exp[(y-1.5)^{-1}\exp[(b-y)^{-1}]], & b<y<1.5,\\ 0, & y\geqslant 1.5.\end{cases} \tag{3.12}$$

Here it is the shape parameter $b\in[0,1.5)$ which parametrizes the profile of $R^{(0)}$ [41].

IV. THE R^2 TRUNCATION

A. The ansatz

In all previous papers [1,20,35,34,2,5] the flow equation of QEG was used in the Einstein-Hilbert truncation. In this section we generalize this truncation by taking also an R^2 term with associated running coupling $\bar{\beta}_k$ into account and we derive the RG flow within this "R^2 truncation." We assume that, at the UV scale $\hat{k}\to\infty$, gravity in d dimensions is described by the action

$$\bar{\Gamma}_{\hat{k}}[g]=S[g]=\int d^dx\,\sqrt{g}\left\{\frac{1}{16\pi\bar{G}}(-R(g)+2\bar{\lambda})+\bar{\beta}R^2(g)\right\}. \tag{4.1}$$

It consists of the conventional Einstein-Hilbert action and a higher-derivative term with bare coupling $\bar{\beta}$. In order to study the RG flow of $\Gamma_k[g,\bar{g}]$ towards smaller scales $k<\hat{k}$ we employ a truncated action functional of the following form:

$$\Gamma_k[g,\bar{g}]=\int d^dx\,\sqrt{g}\{2\kappa^2 Z_{Nk}(-R(g)+2\bar{\lambda}_k)+\bar{\beta}_k R^2(g)\}+\kappa^2\frac{Z_{Nk}}{\alpha}\int d^dx\,\sqrt{\bar{g}}\bar{g}^{\mu\nu}(\mathcal{F}_\mu^{\alpha\beta}g_{\alpha\beta})(\mathcal{F}_\nu^{\rho\sigma}g_{\rho\sigma}). \tag{4.2}$$

The ansatz (4.2) is obtained from $S+S_{\mathrm{gf}}$ by replacing

$$\bar{G}\to G_k\equiv Z_{Nk}^{-1}\bar{G},\quad \bar{\lambda}\to\bar{\lambda}_k,\quad \bar{\beta}\to\bar{\beta}_k,\quad \alpha\to Z_{Nk}^{-1}\alpha \tag{4.3}$$

so that its form agrees with that of the gravitational sector in the ansatz (3.1) with

$$\hat{\Gamma}_k[g,\bar{g}]=\kappa^2\frac{Z_{Nk}-1}{\alpha}\int d^dx\,\sqrt{\bar{g}}\bar{g}^{\mu\nu}\times(\mathcal{F}_\mu^{\alpha\beta}g_{\alpha\beta})(\mathcal{F}_\nu^{\rho\sigma}g_{\rho\sigma}). \tag{4.4}$$

In principle, also the gauge fixing parameter α should be treated as a scale-dependent quantity: $\alpha\to\alpha_k$. Its evolution is neglected here for simplicity. However, setting $\alpha=0$ by hand mimics a dynamical treatment of the gauge fixing parameter since $\alpha=0$ can be argued to be a RG fixed point [42,2].

B. Projecting the flow equation

The ansatz (4.2) comprises three k-dependent couplings. They satisfy the initial conditions $\bar{\lambda}_{\hat{k}}=\bar{\lambda}$, $Z_{N\hat{k}}=1$ which implies $G_{\hat{k}}=\bar{G}$, and $\bar{\beta}_{\hat{k}}=\bar{\beta}$. Here the UV scale $\hat{k}$ is taken to be large but finite. In order to determine the evolution of $\bar{\lambda}_k$, Z_{Nk} and $\bar{\beta}_k$ towards smaller scales we have to project the flow equation onto the space spanned by the operators $\int d^dx\sqrt{g}$, $\int d^dx\sqrt{g}R$ and $\int d^dx\sqrt{g}R^2$. After having inserted the ansatz (4.2) into both sides of the flow equation and having performed the $g_{\mu\nu}$ derivatives implicit in $\Gamma_k^{(2)}$ we may set $g_{\mu\nu}=\bar{g}_{\mu\nu}$. As a consequence, the gauge fixing term drops out from the LHS which then reads

$$\partial_t\Gamma_k[\bar{g},\bar{g}]=\int d^dx\,\sqrt{\bar{g}}\{2\kappa^2[-\bar{R}(\bar{g})\partial_t Z_{Nk}+2\partial_t(Z_{Nk}\bar{\lambda}_k)]+\bar{R}^2(\bar{g})\partial_t\bar{\beta}_k\}. \tag{4.5}$$

Obviously the LHS is spanned by the operators $\int d^dx\sqrt{g}$, $\int d^dx\sqrt{g}R$ and $\int d^dx\sqrt{g}R^2$. This means that we have to perform a derivative expansion on the RHS in order to extract precisely those contributions from the traces which are proportional to these operators. By equating the result to Eq. (4.5) and comparing the coefficients we can read off the system of coupled differential equations for $\bar{\lambda}_k$, Z_{Nk}, and $\bar{\beta}_k$.

In order to make these technically rather involved calculations feasible we may insert any metric $\bar{g}_{\mu\nu}$ that is general enough to admit a unique identification of the operators spanning the truncated theory space. We exploit this freedom by assuming that $\bar{g}_{\mu\nu}$ corresponds to a maximally symmetric space. Such spaces form a special class of Einstein spaces and are characterized by

$$\bar{R}_{\mu\nu\rho\sigma}=\frac{\bar{R}}{d(d-1)}(\bar{g}_{\mu\rho}\bar{g}_{\nu\sigma}-\bar{g}_{\mu\sigma}\bar{g}_{\nu\rho}),$$

$$\bar{R}_{\mu\nu}=\frac{\bar{R}}{d}\bar{g}_{\mu\nu} \tag{4.6}$$

with the curvature scalar $\bar{R}$ considered a constant number rather than a functional of the metric. It is sufficient to employ spaces with $\bar{R}>0$, i.e. d spheres S^d. They are parametrized by their radius r which is related to the curvature scalar and the volume in the usual way,

$$\bar{R}=\frac{d(d-1)}{r^2}, \quad \int d^dx\sqrt{\bar{g}}=\frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma(d)}(4\pi r^2)^{d/2}. \tag{4.7}$$

We emphasize that the beta functions of $\bar{\lambda}_k$, Z_{Nk}, and $\bar{\beta}_k$ do not depend on this choice for $\bar{g}_{\mu\nu}$; it is simply a technical trick without any physical meaning. In principle the beta-functions could be computed without any specification of $\bar{g}_{\mu\nu}$.

While this projection technique is capable of distinguishing $\int d^dx\sqrt{g}\propto r^d$ from both $\int d^dx\sqrt{g}R\propto r^{d-2}$ and $\int d^dx\sqrt{g}R^2\propto r^{d-4}$, it cannot disentangle the three (curvature)^2 invariants $\int d^dx\sqrt{g}R^2$, $\int d^dx\sqrt{g}R^2_{\mu\nu}$, and $\int d^dx\sqrt{g}R^2_{\mu\nu\rho\sigma}$ which are all proportional to r^{d-4}. If one wants to project them out individually one has to insert non-maximally symmetric spaces, but then the evaluation of the functional traces on the RHS of Eq. (3.4) is a rather formidable problem with the present technology. In fact, this is one of the reasons for omitting the other two (curvature)^2 invariants from our truncation ansatz.

In Ref. [2] we discussed already the expansion of fields defined on spherical backgrounds. Both the classical and the quantum TT-component fields can be expanded in terms of transverse-traceless tensor harmonics $T^{lm}_{\mu\nu}$, transverse vector harmonics T^{lm}_{μ}, and scalar harmonics T^{lm}. They form complete sets of orthogonal eigenfunctions with respect to the corresponding covariant Laplacians. We summarize the main results of [2] in Appendix C. In particular, the expansions of $h^T_{\mu\nu}$, ϕ, C^μ, $\bar{C}_\mu$ and their classical counterparts can be read off from Eq. (C3), while the remaining component fields are expanded according to

$$\xi_\mu(x)=\sum_{l=2}^{\infty}\sum_{m=1}^{D_l(d,1)}\xi_{lm}T^{lm}_\mu(x),$$

$$\sigma(x)=\sum_{l=2}^{\infty}\sum_{m=1}^{D_l(d,0)}\sigma_{lm}T^{lm}(x),$$

$$\eta(x)=\sum_{l=1}^{\infty}\sum_{m=1}^{D_l(d,0)}\eta_{lm}T^{lm}(x),$$

$$\bar{\eta}(x)=\sum_{l=1}^{\infty}\sum_{m=1}^{D_l(d,0)}\bar{\eta}_{lm}T^{lm}(x). \tag{4.8}$$

Similar expansions hold for the associated classical fields (expectation values).

Contrary to Eq. (C3), the summations in Eq. (4.8) do *not* start at $l=1$ for vectors and at $l=0$ for scalars, but at $l=2$ for ξ_μ and σ, and at $l=1$ for the scalar ghost fields. The modes omitted here are the Killing vectors ($T^{l=1,m}_\mu$), the solutions of the scalar equation (2.6) ($T^{l=1,m}$), and the constants ($T^{l=0,m=1}$). As we mentioned in Sec. II, these modes do not correspond to fluctuations of $h_{\mu\nu}$ or the Faddeev-Popov ghosts.

As in [2], we decompose the quantum field ϕ into a part ϕ_1 spanned by the same set of eigenfunctions as σ, and a part ϕ_0 containing the contributions from the remaining modes:

$$\phi(x)=\phi_0(x)+\phi_1(x),$$

$$\phi_0(x)=\sum_{l=0}^{1}\sum_{m=1}^{D_l(d,0)}\phi_{lm}T^{lm}(x), \tag{4.9}$$

$$\phi_1(x)=\sum_{l=2}^{\infty}\sum_{m=1}^{D_l(d,0)}\phi_{lm}T^{lm}(x).$$

Due to the orthogonality of the spherical harmonics, ϕ_0 is orthogonal to ϕ_1 and σ: $\langle\phi_1,\phi_0\rangle=\langle\sigma,\phi_0\rangle=0$. This implies $\langle\phi,\phi\rangle=\langle\phi_0,\phi_0\rangle+\langle\phi_1,\phi_1\rangle$ and $\langle\sigma,\phi\rangle=\langle\sigma,\phi_1\rangle$. As a consequence, splitting ϕ according to Eq. (4.9) ensures that any nonzero bilinear cross term of the scalar fields is such that the scalars involved can be expanded in the same set of eigenfunctions. Of course, the same holds for the corresponding classical fields $\bar{\phi}_0$ and $\bar{\phi}_1$.

C. Inserting the ansatz into the RHS of the RG equation

Let us now start with the evaluation of the RHS of Eq. (3.4). After having inserted the truncation ansatz (4.2) we identify the two metrics $g_{\mu\nu}$ and $\bar{g}_{\mu\nu}$. Therefore it is sufficient to calculate the operators $(\Gamma^{(2)}_k[g,\bar{g}]+\mathcal{R}_k[\bar{g}])^{-1}$ and $(S^{(2)}_{\text{gh}}[g,\bar{g}]+\mathcal{R}_k[\bar{g}])^{-1}$ at $g_{\mu\nu}=\bar{g}_{\mu\nu}$. To this end we first expand the ansatz (4.2) according to

$$\Gamma_k[\bar{g}+\bar{h},\bar{g}]=\Gamma_k[\bar{g},\bar{g}]+\mathcal{O}(\bar{h})+\Gamma^{\text{quad}}_k[\bar{h};\bar{g}]+\mathcal{O}(\bar{h}^3) \tag{4.10}$$

and concentrate on the part quadratic in $\bar{h}_{\mu\nu}$, i.e. $\Gamma^{\text{quad}}_k[\bar{h};\bar{g}]$. This leads to

$$\Gamma_k^{\rm quad}[\bar{h};\bar{g}]=\int d^dx\,\sqrt{\bar{g}}\bar{h}_{\mu\nu}\Bigg\{\kappa^2 Z_{Nk}\Bigg[-\left(\frac{1}{2}\delta^\mu_\rho\delta^\nu_\sigma+\frac{1-2\alpha}{4\alpha}\bar{g}^{\mu\nu}\bar{g}_{\rho\sigma}\right)\bar{D}^2+\frac{1}{4}(2\delta^\mu_\rho\delta^\nu_\sigma-\bar{g}^{\mu\nu}\bar{g}_{\rho\sigma})(\bar{R}-2\bar{\lambda}_k)+\bar{g}^{\mu\nu}\bar{R}_{\rho\sigma}-\delta^\mu_\sigma\bar{R}^\nu{}_\rho$$
$$-\bar{R}^\nu{}_\rho{}^\mu{}_\sigma+\frac{1-\alpha}{\alpha}(\bar{g}^{\mu\nu}\bar{D}_\rho\bar{D}_\sigma-\delta^\mu_\sigma\bar{D}^\nu\bar{D}_\rho)\Bigg]+\frac{1}{2}\bar{\beta}_k\Bigg[\frac{1}{2}\left(\frac{1}{2}\bar{g}^{\mu\nu}\bar{g}_{\rho\sigma}-\delta^\mu_\rho\delta^\nu_\sigma\right)\bar{R}^2+2\bar{g}^{\mu\nu}\bar{R}(-\bar{R}_{\rho\sigma}+\bar{D}_\rho\bar{D}_\sigma-\bar{g}_{\rho\sigma}\bar{D}^2)$$
$$+2R(\delta^\nu_\sigma\bar{R}^\mu{}_\rho-\bar{R}^\mu{}_{\rho\sigma}{}^\nu-3\delta^\nu_\sigma\bar{D}^\mu\bar{D}_\rho+2\bar{g}_{\rho\sigma}\bar{D}^\mu\bar{D}^\nu+2\delta^\mu_\rho\delta^\nu_\sigma\bar{D}^2-\delta^\nu_\sigma\bar{D}_\rho\bar{D}^\mu)+2\delta^\nu_\sigma\bar{D}_\rho\overline{RD}^\mu-3\delta^\mu_\rho\delta^\nu_\sigma\bar{D}^\lambda\overline{RD}_\lambda$$
$$+4\delta^\nu_\sigma\bar{D}^\mu\overline{RD}_\rho-4\bar{g}_{\rho\sigma}\bar{D}^\mu\overline{RD}^\nu+\bar{g}^{\mu\nu}\bar{g}_{\rho\sigma}\bar{D}^\lambda\overline{RD}_\lambda+2\bar{R}^{\mu\nu}\bar{R}_{\rho\sigma}-4\bar{R}^{\mu\nu}(\bar{D}_\rho\bar{D}_\sigma-\bar{g}_{\rho\sigma}\bar{D}^2)+2\bar{D}^\mu\bar{D}^\nu\bar{D}_\rho\bar{D}_\sigma$$
$$-4\bar{g}^{\mu\nu}\bar{D}^2\bar{D}_\rho\bar{D}_\sigma+2\bar{g}^{\mu\nu}\bar{g}_{\rho\sigma}(\bar{D}^2)^2\Bigg]\Bigg\}\bar{h}^{\rho\sigma}. \tag{4.11}$$

At this stage $\bar{g}_{\mu\nu}$ is still arbitrary. In order to (partially) diagonalize this quadratic form we insert the family of S^d background metrics into Eq. (4.11) and decompose $\bar{h}_{\mu\nu}$ according to Eq. (2.3). Then we apply Eq. (4.9) to the classical field $\bar{\phi}$ to decompose it as well. This yields

$$\Gamma_k^{\rm quad}[\bar{h};\bar{g}]=\int d^dx\,\sqrt{\bar{g}}\frac{1}{2}\Bigg\{\bar{h}^T_{\mu\nu}[Z_{Nk}\kappa^2(-\bar{D}^2+A_T(d)\bar{R}-2\bar{\lambda}_k)+\bar{\beta}_k(\overline{RD}^2+G_T(d)\bar{R}^2)]\bar{h}^{T\mu\nu}+\bar{\xi}_\mu\Bigg[\frac{2}{\alpha}Z_{Nk}\kappa^2(-\bar{D}^2+A_V(d,\alpha)\bar{R}$$
$$-2\alpha\bar{\lambda}_k)+G_V(d)\bar{\beta}_k\bar{R}^2\Bigg]\bar{\xi}^\mu+\bar{\sigma}[C_{S2}(d,\alpha)Z_{Nk}\kappa^2(-\bar{D}^2+A_{S2}(d,\alpha)\bar{R}+B_{S2}(d,\alpha)\bar{\lambda}_k)$$
$$+\bar{\beta}_k(H_S(d)(\bar{D}^2)^2-G_{S1}(d)(2\overline{RD}^2+\bar{R}^2))]\bar{\sigma}+2\bar{\phi}_1[C_{S2}(d,\alpha)C_{S3}(d,\alpha)Z_{Nk}\kappa^2+\bar{\beta}_k(-H_S(d)\bar{D}^2+2G_{S1}(d)\bar{R})]$$
$$\times\sqrt{-\bar{D}^2}\sqrt{-\bar{D}^2-\frac{\bar{R}}{d-1}}\bar{\sigma}+\sum_{\bar{\phi}\in\{\bar{\phi}_0,\bar{\phi}_1\}}\bar{\phi}[C_{S2}(d,\alpha)C_{S1}(d,\alpha)Z_{Nk}\kappa^2(-\bar{D}^2+A_{S1}(d,\alpha)\bar{R}+B_{S1}(d,\alpha)\bar{\lambda}_k)$$
$$+\bar{\beta}_k(H_S(d)(\bar{D}^2)^2-G_{S2}(d)\overline{RD}^2+G_{S3}(d)\bar{R}^2)]\bar{\phi}\Bigg\}. \tag{4.12}$$

Here the various A's, B's, C's and G's and H_S are functions of the dimensionality d and the gauge fixing parameter α. The explicit expressions for these coefficients are given in Appendix D 1.

This partial diagonalization is performed in order to simplify the inversion of the operator $\Gamma_k^{(2)}[g,\bar{g}]+\mathcal{R}_k[\bar{g}]$. In fact, this is the main reason for using the TT decomposition (2.3) and specifying a concrete background. Note that in the pure Einstein-Hilbert truncation it is only the term in Eq. (4.11) which is proportional to $1-\alpha$ that gives rise to mixings between the traceless part of $\bar{h}_{\mu\nu}$ and $\bar{\phi}$ and therefore necessitates the complete decomposition (2.3). For $\alpha=1$, a complete diagonalization can be achieved by merely splitting off the trace part [1,2]. This has to be contrasted with the R^2 truncation where the higher-derivative term introduces additional mixings between the traceless part and $\bar{\phi}$. These cross terms do *not* vanish for $\alpha=1$. Hence the complete decomposition of $\bar{h}_{\mu\nu}$ is necessary for a partial diagonalization even in the case $\alpha=1$.

At the component field level the cross terms boil down to a purely scalar $\bar{\sigma}$-$\bar{\phi}$ mixing term that vanishes for the spherical harmonics $T^{l=0,m=1}$ and $T^{l=1,m}$. Since these modes contribute to $\bar{\phi}$, but not to $\bar{\sigma}$, we cannot directly invert the associated matrix differential operator $((\Gamma_k^{(2)})_{ij})_{i,j\in\{\bar{h}^T,\bar{\xi},\bar{\sigma},\bar{\phi}\}}$. As a way out, we split $\bar{\phi}$ according to Eq. (4.9) into $\bar{\phi}_0$ and $\bar{\phi}_1$. This has the effect that only mixings between the scalars $\bar{\sigma}$ and $\bar{\phi}_1$ survive, which can be expanded in the same set of eigenfunctions T^{lm} starting at $l=2$. Hence the resulting matrix differential operator $((\Gamma_k^{(2)})_{ij})_{i,j\in\{\bar{h}^T,\bar{\xi},\bar{\phi}_0,\bar{\sigma},\bar{\phi}_1\}}$ is invertible. However, it should be noted that this additional split of $\bar{\phi}$ leads to a slightly modified flow equation since it affects the matrix structure of this operator. In fact, the summation in the gravitational sector of Eq. (3.4) now runs over the set of fields $\{\bar{h}^T,\bar{\xi},\bar{\phi}_0,\bar{\sigma},\bar{\phi}_1\}$, with $(\mathcal{R}_k)_{\bar{\phi}_0\bar{\phi}_0}\equiv(\mathcal{R}_k)_{\bar{\phi}_1\bar{\phi}_1}\equiv(\mathcal{R}_k)_{\bar{\phi}\bar{\phi}}$ and $(\mathcal{R}_k)_{\bar{\sigma}\bar{\phi}_1}\equiv(\mathcal{R}_k)_{\bar{\sigma}\bar{\phi}}$.

As a next step we calculate the contributions from the ghost fields appearing on the RHS of Eq. (3.4). For this purpose we insert the family of spherical background spaces S^d into $S_{\rm gh}$ and set $g_{\mu\nu}=\bar{g}_{\mu\nu}$. Then we use Eq. (2.7) to decompose the ghost fields. This yields

$$S_{\rm gh}[0,v,\bar{v};g]=\sqrt{2}\int d^dx\,\sqrt{g}\Bigg\{\bar{v}^T_\mu\left[-D^2-\frac{R}{d}\right]v^{T\mu}+\bar{\varrho}\left[-D^2-2\frac{R}{d}\right]\varrho\Bigg\}. \tag{4.13}$$

From now on the bars are omitted from the metric, the curvature and the operators D^2 and P_k.

Before we can continue with our evaluation we have to specify the precise form of the cutoff operators. Adapting them to $\Gamma_k^{(2)}$ and $S_{\rm gh}^{(2)}$ of Eqs. (4.12), (4.13) by applying the rule (3.5) leads precisely to the structure (3.8) with the following choices for the $\mathcal{X}_k$'s, $\mathcal{Y}_k$'s, and $\mathcal{Z}_k$'s:

$$\begin{gathered}\mathcal{X}_k^{\bar\phi_1\bar\sigma}=\mathcal{X}_k^{\bar\sigma\bar\sigma}=\mathcal{X}_k^{\bar\phi_0\bar\phi_0}=\mathcal{X}_k^{\bar\phi_1\bar\phi_1}=H_S(d)\bar\beta_k, \quad \mathcal{Y}_k^{\bar h^T\bar h^T}=-\bar\beta_k,\\ \mathcal{Y}_k^{\bar\phi_1\bar\sigma}=\mathcal{Y}_k^{\bar\sigma\bar\sigma}=2G_{S1}(d)\bar\beta_k, \quad \mathcal{Y}_k^{\bar\phi_0\bar\phi_0}=\mathcal{Y}_k^{\bar\phi_1\bar\phi_1}=G_{S2}(d)\bar\beta_k, \quad \mathcal{Z}_k^{\bar h^T\bar h^T}=Z_{Nk},\\ \mathcal{Z}_k^{\bar\xi\bar\xi}=\frac{2}{\alpha}Z_{Nk}, \quad \mathcal{Z}_k^{\bar\phi_1\bar\sigma}=C_{S2}(d,\alpha)C_{S3}(d,\alpha)Z_{Nk}, \quad \mathcal{Z}_k^{\bar\sigma\bar\sigma}=C_{S2}(d,\alpha)Z_{Nk},\\ \mathcal{Z}_k^{\bar\phi_0\bar\phi_0}=\mathcal{Z}_k^{\bar\phi_1\bar\phi_1}=C_{S2}(d,\alpha)C_{S1}(d,\alpha)Z_{Nk}, \quad \mathcal{Z}_k^{\bar v^T v^T}=\mathcal{Z}_k^{\bar\varrho\varrho}=\sqrt{2}.\end{gathered} \tag{4.14}$$

Thus, for $g_{\mu\nu}=\bar g_{\mu\nu}$, the nonvanishing entries of the matrix differential operators $\Gamma_k^{(2)}+\mathcal{R}_k$ and $S_{\rm gh}^{(2)}+\mathcal{R}_k$ take the form

$$\begin{aligned}&(\Gamma_k^{(2)}[g,g]+\mathcal{R}_k)_{\bar h^T\bar h^T}=Z_{Nk}\kappa^2(P_k+A_T(d)R-2\bar\lambda_k)+\bar\beta_k(-RP_k+G_T(d)R^2),\\ &(\Gamma_k^{(2)}[g,g]+\mathcal{R}_k)_{\bar\xi\bar\xi}=\frac{2}{\alpha}Z_{Nk}\kappa^2(P_k+A_V(d,\alpha)R-2\alpha\bar\lambda_k)+G_V(d)\bar\beta_kR^2,\\ &(\Gamma_k^{(2)}[g,g]+\mathcal{R}_k)_{\bar\sigma\bar\sigma}=C_{S2}(d,\alpha)Z_{Nk}\kappa^2(P_k+A_{S2}(d,\alpha)R+B_{S2}(d,\alpha)\bar\lambda_k)+\bar\beta_k(H_S(d)P_k^2+G_{S1}(d)(2RP_k-R^2)),\\ &(\Gamma_k^{(2)}[g,g]+\mathcal{R}_k)_{\bar\phi_1\bar\sigma}=(\Gamma_k^{(2)}[g,g]+\mathcal{R}_k)_{\bar\sigma\bar\phi_1}\\ &\qquad=[C_{S2}(d,\alpha)C_{S3}(d,\alpha)Z_{Nk}\kappa^2+\bar\beta_k(H_S(d)P_k+2G_{S1}(d)R)]\sqrt{P_k}\sqrt{P_k-\frac{R}{d-1}},\\ &(\Gamma_k^{(2)}[g,g]+\mathcal{R}_k)_{\bar\phi_0\bar\phi_0}=(\Gamma_k^{(2)}[g,g]+\mathcal{R}_k)_{\bar\phi_1\bar\phi_1}\\ &\qquad=C_{S2}(d,\alpha)C_{S1}(d,\alpha)Z_{Nk}\kappa^2(P_k+A_{S1}(d,\alpha)R+B_{S1}(d,\alpha)\bar\lambda_k)\\ &\qquad\quad+\bar\beta_k(H_S(d)P_k^2+G_{S2}(d)RP_k+G_{S3}(d)R^2),\\ &(S_{\rm gh}^{(2)}[g,g]+\mathcal{R}_k)_{\bar v^T v^T}=-(S_{\rm gh}^{(2)}[g,g]+\mathcal{R}_k)_{v^T\bar v^T}=\sqrt{2}\left[P_k-\frac{R}{d}\right],\\ &(S_{\rm gh}^{(2)}[g,g]+\mathcal{R}_k)_{\bar\varrho\varrho}=-(S_{\rm gh}^{(2)}[g,g]+\mathcal{R}_k)_{\varrho\bar\varrho}=\sqrt{2}\left[P_k-2\frac{R}{d}\right].\end{aligned} \tag{4.15}$$

For notational simplicity we set $(S_{\rm gh}^{(2)}[0,v,\bar v;g])_{\psi_1\psi_2}\equiv(S_{\rm gh}^{(2)}[g,g])_{\psi_1\psi_2}$ with $\psi_1,\psi_2\in\bar I_2$.

Now we are in a position to write down the RHS of the flow equation with $g_{\mu\nu}=\bar g_{\mu\nu}$. We shall denote it $\mathcal{S}_k(R)$ in the following. Obviously we need the inverse operators $(\Gamma_k^{(2)}+\mathcal{R}_k)^{-1}$ and $(S_{\rm gh}^{(2)}+\mathcal{R}_k)^{-1}$. This inversion is carried out in Appendix A 1. Inserting the inverse operators into $\mathcal{S}_k(R)$ leads to the somewhat complicated result

$$
\begin{aligned}
\mathcal{S}_k(R)=&\mathrm{Tr}_{(2ST^2)}[(P_k+A_T(d)R-2\overline{\lambda}_k+(Z_{Nk}\kappa^2)^{-1}\overline{\beta}_k[-RP_k+G_T(d)R^2])^{-1}(\mathcal{N}-(Z_{Nk}\kappa^2)^{-1}\overline{\beta}_kR\mathcal{T}_1)]\\
&+\mathrm{Tr}'_{(1T)}[(P_k+A_V(d,\alpha)R-2\alpha\overline{\lambda}_k+G_V(d)(Z_{Nk}\kappa^2)^{-1}\overline{\beta}_kR^2)^{-1}\mathcal{N}]\\
&+\mathrm{Tr}''_{(0)}\Bigg[\Bigg\{(C_{S2}(d,\alpha)[P_k+A_{S2}(d,\alpha)R+B_{S2}(d,\alpha)\overline{\lambda}_k]\\
&+(Z_{Nk}\kappa^2)^{-1}\overline{\beta}_k[H_S(d)P_k^2+G_{S1}(d)(2RP_k-R^2)])(C_{S2}(d,\alpha)C_{S1}(d,\alpha)[P_k+A_{S1}(d,\alpha)R+B_{S1}(d,\alpha)\overline{\lambda}_k]\\
&+(Z_{Nk}\kappa^2)^{-1}\overline{\beta}_k[H_S(d)P_k^2+G_{S2}(d)RP_k+G_{S3}(d)R^2])-(C_{S2}(d,\alpha)C_{S3}(d,\alpha)+(Z_{Nk}\kappa^2)^{-1}\overline{\beta}_k[H_S(d)P_k\\
&+2G_{S1}(d)R])^2P_k\left(P_k-\frac{R}{d-1}\right)\Bigg\}^{-1}\Bigg\{(C_{S2}(d,\alpha)[P_k+A_{S2}(d,\alpha)R+B_{S2}(d,\alpha)\overline{\lambda}_k]+(Z_{Nk}\kappa^2)^{-1}\overline{\beta}_k[H_S(d)P_k^2+G_{S1}(d)\\
&\times(2RP_k-R^2)])(C_{S2}(d,\alpha)C_{S1}(d,\alpha)\mathcal{N}+(Z_{Nk}\kappa^2)^{-1}\overline{\beta}_k[H_S(d)\mathcal{T}_2+G_{S2}(d)R\mathcal{T}_1])\\
&+(C_{S2}(d,\alpha)C_{S1}(d,\alpha)[P_k+A_{S1}(d,\alpha)R+B_{S1}(d,\alpha)\overline{\lambda}_k]+(Z_{Nk}\kappa^2)^{-1}\overline{\beta}_k[H_S(d)P_k^2+G_{S2}(d)RP_k+G_{S3}(d)R^2])\\
&\times(C_{S2}(d,\alpha)\mathcal{N}+(Z_{Nk}\kappa^2)^{-1}\overline{\beta}_k[H_S(d)\mathcal{T}_2+2G_{S1}(d)R\mathcal{T}_1])+2(C_{S2}(d,\alpha)C_{S3}(d,\alpha)+(Z_{Nk}\kappa^2)^{-1}\overline{\beta}_k[H_S(d)P_k\\
&+2G_{S1}(d)R])\sqrt{P_k}\sqrt{P_k-\frac{R}{d-1}}\,\frac{1}{2Z_{Nk}\kappa^2}\partial_t\Bigg(H_S(d)\overline{\beta}_k\Bigg[P_k\sqrt{P_k}\sqrt{P_k-\frac{R}{d-1}}\\
&+D^2\sqrt{-D^2}\sqrt{-D^2-\frac{R}{d-1}}\Bigg]+(2G_{S1}(d)\overline{\beta}_kR+C_{S2}(d,\alpha)C_{S3}(d,\alpha)Z_{Nk}\kappa^2)\\
&\times\Bigg[\sqrt{P_k}\sqrt{P_k-\frac{R}{d-1}}-\sqrt{-D^2}\sqrt{-D^2-\frac{R}{d-1}}\Bigg]\Bigg)\Bigg\}\Bigg]-2\mathrm{Tr}_{(1T)}\Bigg[\left(P_k-\frac{R}{d}\right)^{-1}\mathcal{N}_0\Bigg]\\
&-2\mathrm{Tr}'_{(0)}\Bigg[\left(P_k-2\frac{R}{d}\right)^{-1}\mathcal{N}_0\Bigg]+\frac{1}{2Z_{Nk}\kappa^2}\sum_{l=0}^{1}\{D_l(d,0)\{C_{S2}(d,\alpha)C_{S1}(d,\alpha)(\Lambda_l(d,0)+k^2R^{(0)}(\Lambda_l(d,0)/k^2)+A_{S1}(d,\alpha)R\\
&+B_{S1}(d,\alpha)\overline{\lambda}_k)+(Z_{Nk}\kappa^2)^{-1}\overline{\beta}_k[H_S(d)(\Lambda_l(d,0)+k^2R^{(0)}(\Lambda_l(d,0)/k^2))^2+G_{S2}(d)R(\Lambda_l(d,0)+k^2R^{(0)}(\Lambda_l(d,0)/k^2))\\
&+G_{S3}(d)R^2]\}^{-1}\partial_t\{C_{S2}(d,\alpha)C_{S1}(d,\alpha)Z_{Nk}\kappa^2k^2R^{(0)}(\Lambda_l(d,0)/k^2)+\overline{\beta}_k[H_S(d)(2\Lambda_l(d,0)k^2R^{(0)}(\Lambda_l(d,0)/k^2)\\
&+k^4R^{(0)}(\Lambda_l(d,0)/k^2)^2)+G_{S2}(d)Rk^2R^{(0)}(\Lambda_l(d,0)/k^2)]\}\}. \qquad (4.16)
\end{aligned}
$$

The new quantities $\mathcal{N}$, $\mathcal{N}_0$, $\mathcal{T}_1$, and $\mathcal{T}_2$ introduced in Eq. (4.16) are defined as

$$
\begin{aligned}
\mathcal{N}&\equiv(2Z_{Nk})^{-1}\partial_t[Z_{Nk}k^2R^{(0)}(-D^2/k^2)]\\
&=\left[1-\frac{1}{2}\,\eta_N(k)\right]k^2R^{(0)}(-D^2/k^2)+D^2R^{(0)\prime}(-D^2/k^2),\\
\mathcal{N}_0&\equiv2^{-1}\partial_t[k^2R^{(0)}(-D^2/k^2)]=k^2R^{(0)}(-D^2/k^2)+D^2R^{(0)\prime}(-D^2/k^2),\\
\mathcal{T}_1&\equiv(2\overline{\beta}_k)^{-1}\partial_t[\overline{\beta}_kk^2R^{(0)}(-D^2/k^2)]\\
&=\left[1-\frac{1}{2}\,\eta_\beta(k)\right]k^2R^{(0)}(-D^2/k^2)+D^2R^{(0)\prime}(-D^2/k^2),\\
\mathcal{T}_2&\equiv(2\overline{\beta}_k)^{-1}\partial_t[\overline{\beta}_k(-2D^2k^2R^{(0)}(-D^2/k^2)+k^4R^{(0)}(-D^2/k^2)^2)]\\
&=2P_k[k^2R^{(0)}(-D^2/k^2)+D^2R^{(0)\prime}(-D^2/k^2)]-\frac{1}{2}\,\eta_\beta(k)(-D^2k^2R^{(0)}(-D^2/k^2)+k^4R^{(0)}(-D^2/k^2)^2). \qquad (4.17)
\end{aligned}
$$

Here

$$\eta_N(k) \equiv -\partial_t \ln Z_{Nk} \tag{4.18}$$

and

$$\eta_\beta(k) \equiv -\partial_t \ln \bar{\beta}_k \tag{4.19}$$

are the anomalous dimensions of the operators $\int d^d x \sqrt{g} R$ and $\int d^d x \sqrt{g} R^2$, respectively. Furthermore, the prime at $R^{(0)}$ denotes the derivative with respect to the argument.

In Eq. (4.16) we refined our notation concerning the primes at the traces. From now on one prime indicates that the mode corresponding to the lowest eigenvalue has to be excluded, while two primes indicate the subtraction of the contributions from the lowest two eigenvalues. The subscripts at the traces describe on which kind of field the operators under the traces act. We use the subscripts (0), $(1T)$ and $(2ST^2)$ for spin-0 fields, transverse spin-1 fields, and symmetric transverse traceless spin-2 fields, respectively.

D. The system of flow equations for λ_k, g_k, and β_k

Next we derive the flow equations for the couplings. In order to make the rather complicated calculations feasible we are forced to work from now on in the technically convenient gauge $\alpha=1$. Here we merely summarize the main steps, the details of the calculation can be found in Appendix A 2.

By expanding $\mathcal{S}_k(R)$ where $R \propto r^{-2}$ with respect to r and evaluating the traces by means of heat kernel techniques we extract those pieces from the RHS of the flow equation, Eq. (4.16), which are proportional to the appropriate powers of the radius, i.e. $r^d \propto \int d^d x \sqrt{g}$, $r^{d-2} \propto \int d^d x \sqrt{g} R$, and $r^{d-4} \propto \int d^d x \sqrt{g} R^2$. Then we equate the result to the LHS, Eq. (4.5), and compare the coefficients of the various powers of r. This leads to a system of coupled differential equations for $\bar{\lambda}_k$, Z_{Nk} and $\bar{\beta}_k$.

In order to present it in a transparent manner we introduce the dimensionless running cosmological constant

$$\lambda_k \equiv k^{-2} \bar{\lambda}_k, \tag{4.20}$$

the dimensionless running Newton constant

$$g_k \equiv k^{d-2} G_k \equiv k^{d-2} Z_{Nk}^{-1} \bar{G}, \tag{4.21}$$

and the dimensionless running R^2 coupling

$$\beta_k \equiv k^{4-d} \bar{\beta}_k. \tag{4.22}$$

$G_k \equiv \bar{G}/Z_{Nk}$ denotes the dimensionful running Newton constant.

In terms of the couplings λ_k, g_k, and β_k, our final result for the 3-dimensional flow equation reads

$$\begin{aligned}\partial_t \lambda_k = {}& \boldsymbol{\beta}_\lambda(\lambda_k, g_k, \beta_k; d) \\ \equiv {}& A_1(\lambda_k, g_k, \beta_k; d) + \eta_N(k) A_2(\lambda_k, g_k, \beta_k; d) \\ & + \eta_\beta(k) A_3(\lambda_k, g_k, \beta_k; d),\end{aligned} \tag{4.23}$$

$$\begin{aligned}\partial_t g_k = {}& \boldsymbol{\beta}_g(\lambda_k, g_k, \beta_k; d) \\ \equiv {}& [d-2+\eta_N(k)] g_k,\end{aligned} \tag{4.24}$$

$$\begin{aligned}\partial_t \beta_k = {}& \boldsymbol{\beta}_\beta(\lambda_k, g_k, \beta_k; d) \\ \equiv {}& [4-d-\eta_\beta(k)] \beta_k.\end{aligned} \tag{4.25}$$

The anomalous dimensions are explicitly given by

$$\eta_N(k) \equiv \eta_N(\lambda_k, g_k, \beta_k; d) = g_k \frac{B_1(\lambda_k, g_k, \beta_k; d)[\beta_k + C_3(\lambda_k, g_k, \beta_k; d)] - C_1(\lambda_k, g_k, \beta_k; d) B_3(\lambda_k, g_k, \beta_k; d)}{[1 - g_k B_2(\lambda_k, g_k, \beta_k; d)][\beta_k + C_3(\lambda_k, g_k, \beta_k; d)] + g_k C_2(\lambda_k, g_k, \beta_k; d) B_3(\lambda_k, g_k, \beta_k; d)} \tag{4.26}$$

and

$$\eta_\beta(k) \equiv \eta_\beta(\lambda_k, g_k, \beta_k; d) = -\frac{C_1(\lambda_k, g_k, \beta_k; d)[1 - g_k B_2(\lambda_k, g_k, \beta_k; d)] + g_k B_1(\lambda_k, g_k, \beta_k; d) C_2(\lambda_k, g_k, \beta_k; d)}{[1 - g_k B_2(\lambda_k, g_k, \beta_k; d)][\beta_k + C_3(\lambda_k, g_k, \beta_k; d)] + g_k C_2(\lambda_k, g_k, \beta_k; d) B_3(\lambda_k, g_k, \beta_k; d)}. \tag{4.27}$$

The three $\boldsymbol{\beta}$ functions $\boldsymbol{\beta}_\lambda$, $\boldsymbol{\beta}_g$ and $\boldsymbol{\beta}_\beta$ contain the quantities A_i, B_i, C_i, $i=1,2,3$, which are extremely complicated functions of the couplings and the dimensionality d. The explicit expressions for these coefficient functions can be found in Appendix B. They contain the new threshold functions Ψ and $\tilde{\Psi}$ of Eqs. (A29) and (A30) which functionally depend on $R^{(0)}$. They generalize the familiar threshold functions Φ and $\tilde{\Phi}$ which occur in the Einstein-Hilbert truncation. The three $\boldsymbol{\beta}$ functions are one of the main results of this paper.

V. THE FIXED POINTS

A. Fixed points, critical exponents, and nonperturbative renormalizability

Because of its complexity it is clearly impossible to solve the system of flow equations for λ_k, g_k and β_k, Eqs. (4.23), (4.24) and (4.25), exactly. Even a numerical solution would be a formidable task. However, it is possible to gain important information about the general structure of the RG flow by looking at its fixed point structure.

Given a set of $\boldsymbol{\beta}$ functions corresponding to an arbitrary set of dimensionless essential couplings $\mathrm{g}_i(k)$, it is often possible to predict their scale dependence for very small and/or very large scales k by investigating their fixed points. They are those points in the space spanned by the g_i where all $\boldsymbol{\beta}$ functions vanish. Fixed points are characterized by their stability properties. A given eigendirection of the linearized flow is said to be UV or IR attractive (or stable) if, for $k \to \infty$ or $k \to 0$, respectively, the trajectories are attracted towards the fixed point along this particular direction. The UV (IR) critical hypersurface $\mathcal{S}_{\mathrm{UV}}$ ($\mathcal{S}_{\mathrm{IR}}$) in the space of all couplings is defined to consist of all trajectories that run into a given fixed point for $k \to \infty$ ($k \to 0$).

In quantum field theory, fixed points play an important role in the modern approach to renormalization theory [12]. At a UV fixed point the infinite cutoff limit can be taken in a controlled way, the theory can be renormalized nonperturbatively there. As for gravity, Weinberg [23] argued that a theory described by a RG trajectory lying on a *finite-dimensional* UV critical hypersurface of some fixed point is presumably free from unphysical singularities. It is predictive since it depends only on a *finite* number of free (essential) parameters. In Weinberg's words, such a theory is *asymptotically safe*. Asymptotic safety has to be regarded as a generalized, nonperturbative version of renormalizability. It covers the class of perturbatively renormalizable theories, whose infinite cutoff limit is taken at the Gaussian fixed point $\mathrm{g}_{*i} = 0$, as well as those perturbatively nonrenormalizable theories which are described by a RG trajectory on a finite-dimensional UV critical hypersurface of a non-Gaussian fixed point $\mathrm{g}_{*i} \neq 0$ and are nonperturbatively renormalizable therefore [23].

Let us now consider the component form of the exact RG equation, i.e. the system of differential equations

$$k \partial_k \mathrm{g}_i(k) = \boldsymbol{\beta}_i(\mathrm{g}) \tag{5.1}$$

for a set of dimensionless essential couplings $\mathrm{g}(k) \equiv (\mathrm{g}_1(k), \ldots, \mathrm{g}_n(k))$. In an exact treatment the number n is infinite; in a specific truncation it might be finite. We assume that g_* is a fixed point of Eq. (5.1), i.e. $\boldsymbol{\beta}_i(\mathrm{g}_*) = 0$ for all $i = 1, \ldots, n$. We linearize the RG flow about g_* which leads to

$$k \partial_k \mathrm{g}_i(k) = \sum_{j=1}^{n} B_{ij} (\mathrm{g}_j(k) - \mathrm{g}_{*j}) \tag{5.2}$$

where $B_{ij} \equiv \partial_j \boldsymbol{\beta}_i(\mathrm{g}_*)$ are the entries of the stability matrix $\mathbf{B} = (B_{ij})$. Diagonalizing $\mathbf{B}$ according to $S^{-1}\mathbf{B}S = -\mathrm{diag}(\theta_1, \ldots, \theta_n)$, $S = (V^1, \ldots, V^n)$, where V^I is the right-eigenvector of $\mathbf{B}$ with eigenvalue $-\theta_I$ we have

$$\sum_{j=1}^{n} B_{ij} V_j^I = -\theta_I V_i^I, \quad I = 1, \ldots, n. \tag{5.3}$$

The general solution to Eq. (5.2) may be written as

$$\mathrm{g}_i(k) = \mathrm{g}_{*i} + \sum_{I=1}^{n} C_I V_i^I \left(\frac{k_0}{k} \right)^{\theta_I}. \tag{5.4}$$

Here

$$C_I \equiv \sum_{j=1}^{n} (S^{-1})_{Ij} \mathrm{g}_j(k_0) \tag{5.5}$$

are arbitrary real parameters and k_0 is a reference scale.

Obviously a fixed point g_* is UV attractive for a given trajectory (i.e. attractive for $k \to \infty$) only if all its C_I corresponding to negative $\theta_I < 0$ are set to zero. Therefore the dimensionality $\Delta_{\mathrm{UV}} \equiv \dim(\mathcal{S}_{\mathrm{UV}})$ of the UV critical hypersurface belonging to this particular fixed point equals the number of positive θ_I's. Conversely, for a trajectory where all C_I corresponding to positive θ_I are set to zero, g_* is an IR attractive fixed point (approached in the limit $k \to 0$). As a consequence, the IR critical hypersurface $\mathcal{S}_{\mathrm{IR}}$ of a fixed point has a dimensionality $\Delta_{\mathrm{IR}} \equiv \dim(\mathcal{S}_{\mathrm{IR}})$ which equals the number of negative θ_I's.

In a slight abuse of language we shall refer to the θ_I's as the *critical exponents*.

Strictly speaking, the solution (5.4) and its above interpretation is valid only in such cases where all eigenvalues $-\theta_I$ are real, which is not guaranteed since the matrix $\mathbf{B}$ is not symmetric in general. If complex eigenvalues occur one has to consider complex C_I's and to take the real part of Eq. (5.4), see below. Then the real parts of the critical exponents determine which directions in coupling constant space are attractive or repulsive.

At this point it is necessary to discuss the impact a change of the cutoff scheme has on the scaling behavior. Since the path integral for Γ_k depends on the cutoff scheme, i.e. on the $\Delta_k S$ chosen, it is clear that generically the k-dependent couplings and their fixed point values are scheme dependent. Hence a variation of the cutoff scheme, i.e. of $\mathcal{R}_k$, induces a change in the corresponding $\mathbf{B}$ matrix. So one might naively expect that also its eigenvalues, the critical exponents, are scheme dependent. In fact, this is not the case. According to the general theory of critical phenomena and a recent reanalysis in the framework of the exact RG equations [43] any variation of the cutoff scheme can be generated by a specific coordinate transformation in the space of couplings, with the cutoff held fixed. Such transformations leave the eigenvalues of the $\mathbf{B}$ matrix invariant, so that the critical behavior near the corresponding fixed point is universal. The positions of fixed points are scheme dependent but their (non)existence and the qualitative structure of the RG flow are $\mathcal{R}_k$-independent features. Quantities like the θ_I's which are $\mathcal{R}_k$ independent are called *universal*. Their residual scheme dependence present in an approximate treatment (truncation, etc.) can be used in order to judge the quality of the approximation. A truncation can be considered reliable only if it predicts the same fixed point structure for all admissible choices of $\mathcal{R}_k$.

In the context of the R^2 truncation the space of couplings is parametrized by $\mathrm{g}_1 = \lambda$, $\mathrm{g}_2 = g$, and $\mathrm{g}_3 = \beta$. The $\boldsymbol{\beta}$ functions occurring in the three flow equations

$$\partial_t \lambda_k = \boldsymbol{\beta}_\lambda(\lambda_k, g_k, \beta_k), \quad \partial_t g_k = \boldsymbol{\beta}_g(\lambda_k, g_k, \beta_k),$$
$$\partial_t \beta_k = \boldsymbol{\beta}_\beta(\lambda_k, g_k, \beta_k) \tag{5.6}$$

are given in Eqs. (4.23), (4.24) and (4.25), respectively.

In Ref. [2] the fixed point structure of the pure Einstein-Hilbert truncation was investigated. In this case the $\boldsymbol{\beta}$ functions were found to have both a trivial zero at $\lambda_* = g_* = 0$, referred to as the Gaussian fixed point, and a non-Gaussian fixed point at $\lambda_* \neq 0$, $g_* \neq 0$. As we will see in Sec. V B, the Gaussian fixed point is *not* present any more in the generalized truncation. This has to be contrasted with the non-Gaussian fixed point which is found with the R^2 truncation, too. In Sec. V C we study its cutoff dependence and the cutoff dependence of the associated critical exponents employing the above $\boldsymbol{\beta}$ functions with the families of shape functions (3.11) or (3.12) inserted.

B. The fate of the Gaussian fixed point

In this subsection we study the fate of the Gaussian fixed point found in the context of the pure Einstein-Hilbert truncation. In [2] we investigated the 2-dimensional RG flow near this fixed point $(\lambda_*, g_*) = (0,0)$ and discussed its stability properties. It is an important question how the situation changes by enlarging the parameter space.

Quite remarkably, we find that in the 3-dimensional λ-g-β space of the R^2 truncation there is no Gaussian fixed point, i.e. $(\lambda, g, \beta) = (0,0,0)$ is not a simultaneous zero of all three $\boldsymbol{\beta}$ functions. While $\boldsymbol{\beta}_\lambda$ and $\boldsymbol{\beta}_g$ vanish at the origin $(\lambda, g, \beta) = (0,0,0)$, setting $\lambda_k = g_k = 0$ in $\boldsymbol{\beta}_\beta$ leads to

$$\boldsymbol{\beta}_\beta(0,0,\beta_k; d) = \boldsymbol{\beta}_\beta(0,0,0; d) = \gamma_d \quad \forall \beta_k. \tag{5.7}$$

The nonzero constant γ_d is given by

$$\gamma_d = (4\pi)^{-d/2}\{h_{31}(d)\Phi^1_{d/2-2}(0) + h_{32}(d)\Phi^2_{d/2-1}(0)$$
$$+ h_{33}(d)\Phi^3_{d/2}(0)\}. \tag{5.8}$$

Here the h_i's are defined as in Appendix D 2. In $d=4$ dimensions

$$\gamma_4 = \frac{419}{1080}(4\pi)^{-2} \tag{5.9}$$

is a universal quantity since $\Phi^1_0(0) = \Phi^2_1(0) = 2\Phi^3_2(0) = 1$ independently of the cutoff; see Appendix F.

Although there is no fixed point at the origin of the parameter space it is nevertheless very interesting to study the RG flow in the vicinity of $(\lambda, g, \beta) = (0,0,0)$. For simplicity we restrict our considerations to the case $d > 2$. Expanding $(\boldsymbol{\beta}_\lambda, \boldsymbol{\beta}_g, \boldsymbol{\beta}_\beta)$ about the origin we obtain instead of Eq. (5.2) the *inhomogeneous* system

$$k\partial_k \mathrm{g}_i(k) = \gamma_d \delta_{i,3} + \sum_{j=1}^{3} M_{ij}\mathrm{g}_j(k). \tag{5.10}$$

The linearized renormalization group flow is governed by the Jacobi-matrix $\mathbf{M} = (M_{ij})$, $M_{ij} \equiv \partial_j \boldsymbol{\beta}_i(0,0,0;d)$, which takes the form

$$\mathbf{M} = \begin{pmatrix} -2 & \nu_d d & 0 \\ 0 & d-2 & 0 \\ \varsigma_d & \tau_d & 4-d \end{pmatrix}. \tag{5.11}$$

Its entries follow from the expanded $\boldsymbol{\beta}$ functions (E3) of Appendix E. Here ν_d, ς_d and τ_d are d-dependent parameters defined as

$$\nu_d \equiv (d-3)(4\pi)^{1-d/2}\Phi^1_{d/2}(0), \tag{5.12}$$

$$\varsigma_d \equiv (4\pi)^{-d/2}\{h_{34}(d)\Phi^2_{d/2-2}(0) + h_{35}(d)\Phi^3_{d/2-1}(0) + h_{36}(d)\Phi^4_{d/2}(0)\}, \tag{5.13}$$

$$\tau_d \equiv -(4\pi)^{1-d}\Big\{[h_{37}(d)\Phi^1_{d/2-1}(0) + h_{38}(d)\Phi^2_{d/2}(0)]\Big[\frac{1}{4}h_{34}(d)\tilde{\Phi}^1_{d/2-2}(0) + \frac{1}{8}h_{35}(d)\tilde{\Phi}^2_{d/2-1}(0)$$
$$+ \frac{1}{12}h_{36}(d)\tilde{\Phi}^3_{d/2}(0)\Big] + [h_{31}(d)\Phi^1_{d/2-2}(0) + h_{32}(d)\Phi^2_{d/2-1}(0) + h_{33}(d)\Phi^3_{d/2}(0)]$$
$$\times\Big[h_{39}(d)\tilde{\Phi}^0_{d/2-2}(0) + h_{40}(d)\tilde{\Phi}^1_{d/2-1}(0)$$
$$+ h_{41}(d)\tilde{\Phi}^2_{d/2}(0) + h_{42}(d)\tilde{\Phi}^3_{d/2+1}(0) + \frac{3}{2}\Big]\Big\}. \tag{5.14}$$

At this point it should be noted that the submatrix $(M_{ij})_{i,j\in\{1,2\}}$ coincides precisely with the stability matrix of the Gaussian fixed point which was calculated in [2] in the Einstein-Hilbert truncation.

Diagonalizing the matrix (5.11) yields the (obviously universal) eigenvalues $\vartheta_1=-2$, $\vartheta_2=d-2$ and $\vartheta_3=4-d$ which are associated with the eigenvectors

$$V^1=(1,0,\varsigma_d/(d-6))^{\mathbf{T}},$$
$$V^2=(\nu_d,1,(\varsigma_d\nu_d+\tau_d)/[2(d-3)])^{\mathbf{T}}, \tag{5.15}$$
$$V^3=(0,0,1)^{\mathbf{T}}.$$

Equation (5.15) is valid only for $d\neq 3,6$. In $d=3$ we obtain $V^1=(1,0,-\varsigma_3/3)$, $V^2=V^3=(0,0,1)$, and in the 6-dimensional case the eigenvectors are $V^1=V^3=(0,0,1)$, $V^2=(\nu_6,1,(\varsigma_6\nu_6+\tau_6)/6)$. Thus in both cases the space spanned by the eigenvectors in only 2-dimensional, i.e. they do not form a complete system. For all values of d, including $d=3$ and $d=6$, the solutions for λ_k and g_k obtained from the linearized system (5.10) assume the following form:

$$\lambda_k=(\lambda_{k_0}-\nu_d g_{k_0})\left(\frac{k_0}{k}\right)^2+\nu_d g_{k_0}\left(\frac{k}{k_0}\right)^{d-2},$$
$$g_k=g_{k_0}\left(\frac{k}{k_0}\right)^{d-2}. \tag{5.16}$$

Since the expanded $\boldsymbol{\beta}$ function $\boldsymbol{\beta}_g$ of Eq. (E3) does not depend on λ_k and β_k up to terms of third order in the couplings we can easily calculate also the next-to-leading approximation for g_k near the origin. In terms of the dimensionful quantity G_k this improved solution reads

$$G_k=G_{k_0}[1-\omega_d G_{k_0}(k_0^{d-2}-k^{d-2})]^{-1} \tag{5.17}$$

with

$$\omega_d\equiv-\frac{1}{d-2}B_1(0,0,0;d)$$
$$=(4\pi)^{1-d/2}\{h_{43}(d)\Phi^1_{d/2-1}(0)+h_{44}(d)\Phi^2_{d/2}(0)\} \tag{5.18}$$

a d-dependent parameter. It agrees with the ω_d defined in [2] in the context of the pure Einstein-Hilbert truncation. For $k\ll|\omega_d G_{k_0}|^{-1/(d-2)}$ and with the reference scale $k_0=0$ (which is admissible only for specific initial conditions of the cosmological constant), Eq. (5.17) yields

$$G_k=G_0[1-\omega_d G_0 k^{d-2}+\mathcal{O}(G_0^2k^{2(d-2)})]. \tag{5.19}$$

For the dimensionful cosmological constant we obtain, from Eq. (5.16),

$$\bar{\lambda}_k=\bar{\lambda}_{k_0}+\nu_d G_{k_0}(k^d-k_0^d). \tag{5.20}$$

Equations (5.19) and (5.20) agree completely with the corresponding results from the Einstein-Hilbert truncation.

Let us now discuss the solution for β_k. In order to derive it we start by picking $i=3$ in Eq. (5.10) and rewrite the corresponding equation as

$$\partial_k[k^{d-4}\beta_k]=k^{d-5}[\gamma_d+\varsigma_d\lambda_k+\tau_d g_k]. \tag{5.21}$$

Then we insert the solutions for g_k and λ_k of Eq. (5.16) into Eq. (5.21). The resulting differential equation may easily be solved. For $d\neq 3,4,6$, the solution reads

$$\beta_k=\frac{\gamma_d}{d-4}+\frac{(\lambda_{k_0}-\nu_d g_{k_0})\varsigma_d}{d-6}\left(\frac{k_0}{k}\right)^2+\frac{\nu_d\varsigma_d+\tau_d}{2(d-3)}g_{k_0}\left(\frac{k}{k_0}\right)^{d-2}+\left[\beta_{k_0}-\frac{\gamma_d}{d-4}-\frac{(\lambda_{k_0}-\nu_d g_{k_0})\varsigma_d}{d-6}-\frac{\nu_d\varsigma_d+\tau_d}{2(d-3)}g_{k_0}\right]\left(\frac{k_0}{k}\right)^{d-4}. \tag{5.22}$$

The solutions in $d=3$, $d=4$, and $d=6$ can be obtained from Eq. (5.22) by a careful evaluation of the limits $d\to 3$, $d\to 4$, and $d\to 6$, respectively. In the most interesting case of $d=4$ dimensions this leads to the following solution:

$$\beta_k=\beta_{k_0}+\frac{(\lambda_{k_0}-\nu_4 g_{k_0})\varsigma_4}{2}-\frac{\nu_4\varsigma_4+\tau_4}{2}g_{k_0}+\frac{419}{1080}(4\pi)^{-2}\ln\left(\frac{k}{k_0}\right)-\frac{(\lambda_{k_0}-\nu_4 g_{k_0})\varsigma_4}{2}\left(\frac{k_0}{k}\right)^2+\frac{\nu_4\varsigma_4+\tau_4}{2}g_{k_0}\left(\frac{k}{k_0}\right)^2. \tag{5.23}$$

The parameters appearing in Eq. (5.23) are

$$\nu_4=\frac{1}{4\pi}\Phi^1_2(0)$$
$$\varsigma_4=(4\pi)^{-2}\left\{-\frac{559}{432}+\frac{71}{36}\Phi^3_1(0)+\frac{347}{24}\Phi^4_2(0)\right\}$$
$$\tau_4=(4\pi)^{-3}\left\{\left[\frac{13}{3}\Phi^1_1(0)+\frac{79}{3}\Phi^2_2(0)\right]\left[-\frac{559}{1728}+\frac{71}{288}\tilde{\Phi}^2_1(0)+\frac{347}{288}\tilde{\Phi}^3_2(0)\right]+\frac{419}{1080}\left[-\frac{299}{180}+\frac{13}{3}\tilde{\Phi}^1_1(0)+\frac{40}{3}\tilde{\Phi}^2_2(0)\right]\right\}. \tag{5.24}$$

Employing the exponential shape function with $s=1$ and inserting the corresponding values of the $\Phi^p_n(0)$ and $\tilde{\Phi}^p_n(0)$ integrals given in Appendix F we obtain

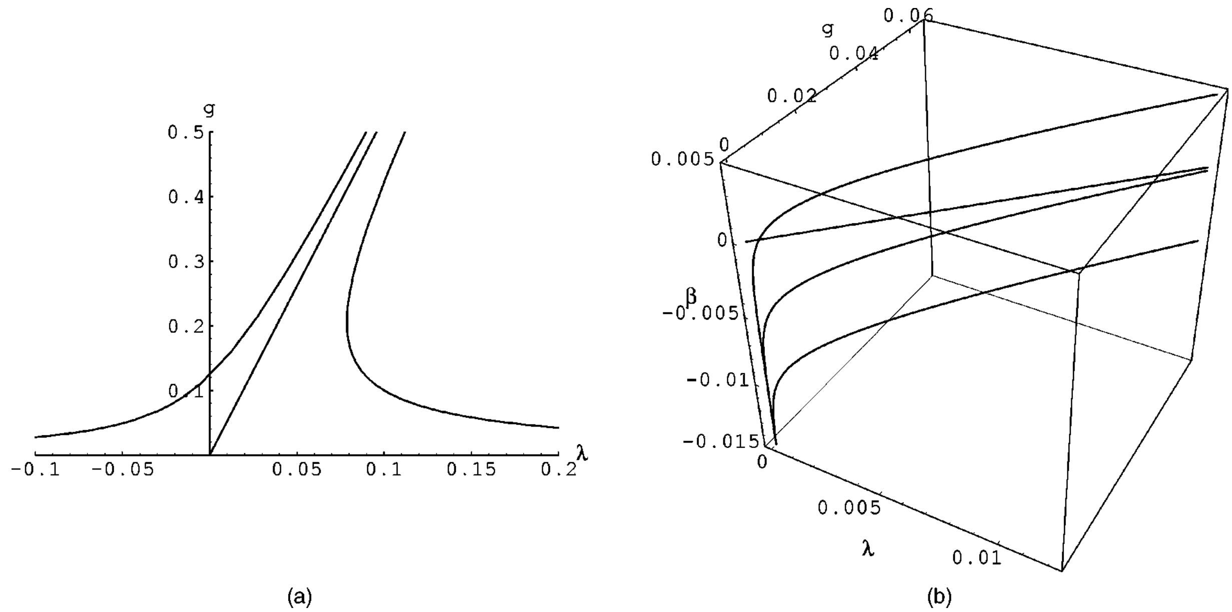


FIG. 1. The case $d=4$: (a) Type Ia, type IIa, and type IIIa trajectories (from left to right) obtained from the λ-g projection (5.16). They coincide precisely with the corresponding trajectories of the Einstein-Hilbert truncation. The type IIa trajectory is the separatrix with $\lambda_{k=0}=0$, which separates the region of trajectories with $\lambda_{k\to 0}\to -\infty$ (type Ia) from those running towards more positive λ's (type IIIa). (b) Three typical trajectories of the linearized λ-g-β equation. They correspond to different values of β_{k_0}, but all of them satisfy Eq. (5.26). We also depict their projection onto the λ-g plane. It coincides with the separatrix of the pure Einstein-Hilbert truncation.

$$\nu_4=\zeta(3)/(2\pi)\approx 0.19, \qquad \varsigma_4=(4\pi)^{-2}\left\{-\frac{559}{432}+\frac{278}{4}\ln(2)-\frac{347}{24}\ln(3)\right\}\approx 0.20,$$
$$\tau_4=\frac{2817356+25(474+13\pi^2)(-559+4590\ln(2)-2082\ln(3))}{49766400\pi^3}\approx 0.0051. \tag{5.25}$$

Let us now analyze the RG flow near the origin of the parameter space. Strictly speaking our analysis even extends to all points of the λ-g-β space which satisfy $|\lambda|,|g|\ll 1$ and $|\beta|\ll 1/|g|$. This is because in any of the three $\boldsymbol{\beta}$ functions all terms of second and higher orders in β_k appear as products $g_k^n\beta_k^m$ with $n\geqslant m$.

Since $\vartheta_1=-2<0$ and, for $d>2$, $\vartheta_2=d-2>0$, λ_k of Eq. (5.16) starts growing as soon as k falls below k_0 and the linearization breaks down, unless the couplings run along a trajectory which satisfies

$$\lambda_k=\nu_d g_k \Leftrightarrow \overline{\lambda}_k=\nu_d G_k k^d \tag{5.26}$$

for sufficiently small values of k [5], with G_k given by Eq. (5.17). In this case both λ_k and g_k approach $\lambda=g=0$ in the limit $k\to 0$ as long as $|\beta_k|\ll 1/|g_k|$ is satisfied as well. Since $\Phi^1_{d/2}(0)$ depends on the shape function $R^{(0)}$, ν_d is not a universal quantity. Therefore the slope of the distinguished trajectories characterized by Eq. (5.26) is not fixed in a universal manner.

Equation (5.26) is exactly the condition for the "separatrix" found in [5] in the context of the 4-dimensional Einstein-Hilbert truncation. In the terminology of [5], the separatrix is the "type IIa trajectory" that interpolates between the Gaussian and the non-Gaussian fixed point of the Einstein-Hilbert truncation, thereby separating the region of trajectories with $\lambda_{k\to 0}\to -\infty$ (type Ia) from those which hit the boundary of parameter space ($\lambda=1/2$) for some finite value of k (type IIIa). In Fig. 1(a) we depict the separatrix, a type Ia and a type IIIa trajectory in the vicinity of $(\lambda,g)=(0,0)$. The plot should be thought of as a projection from λ-g-β space onto its $\beta=0$ plane.

For the separatrix, Eq. (5.19) for the running of G_k is valid down to $k=0$ because g_k stays near the origin. For $d\neq 2$ the parameters $\Phi^1_{d/2-1}(0)$ and $\Phi^2_{d/2}(0)$ appearing in ω_d are scheme dependent, and ω_d is nonuniversal. In the most interesting case of $d=4$,

$$\omega_4=\frac{1}{24\pi}[13\Phi^1_1(0)+79\Phi^2_2(0)]. \tag{5.27}$$

Since $\Phi^1_1(0)$ and $\Phi^2_2(0)$ are positive for any admissible shape function[5] we can infer from Eq. (5.27) that ω_4 is positive. Thus, if we define QEG with vanishing renormalized

[5]Using the exponential shape function $R^{(0)}$ with $s=1$, for instance, we have $\Phi^1_1(0)=\pi^2/6$, $\Phi^2_2(0)=1$ so that $\omega_4\approx 1.33$. Furthermore, we have $\Phi^1_2(0)=2\zeta(3)$ where ζ denotes the Riemann zeta function, and thus $\nu_4\approx 0.19$.

cosmological constant to be the theory described by a trajectory in λ-g-β space, whose λ and g coordinates follow the separatrix in the limit $k\to 0$, Eq. (5.17) implies that QEG is antiscreening in the IR, i.e. G_k decreases as k increases.

Up to now we investigated the $\beta=0$ projection of the flow on λ-g-β space. Next we discuss the linearized RG flow of the β component.

1. $d=4$

In $d=4$ the solution for β_k, Eq. (5.23), diverges in the limit $k\to 0$. If $\lambda_k\neq \nu_d g_k$ as $k\to 0$, the leading divergence is quadratic in k. However, in this case the linearization cannot be trusted down to arbitrarily small values of k anyhow since trajectories with $\lambda_k\neq \nu_d g_k$ ultimately run away from $(\lambda,g)=(0,0)$ for $k\to 0$.

For the distinguished trajectories which satisfy the separatrix condition (5.26) for $k\to 0$, the coefficient of the term $\propto(k_0/k)^2$ in Eq. (5.23) vanishes and only a logarithmic running with a *universal* coefficient remains:

$$\beta_k=\beta_{k_0}-\frac{\nu_4 s_4+\tau_4}{2}g_{k_0}+\frac{419}{1080}(4\pi)^{-2}\ln\left(\frac{k}{k_0}\right)+\mathcal{O}(k^2). \tag{5.28}$$

Since higher orders in β_k appear exclusively as products $g_k^n\beta_k^m$ with $n\geqslant m$, the vanishing of $g_k^n(\ln(k/k_0))^m\propto(k/k_0)^{2n}(\ln(k/k_0))^m$ in the limit $k\to 0$ then implies that terms of order β_k^2 remain negligible as $k\to 0$. As a consequence, the linearization does not break down for $k\to 0$ although β_k diverges in this limit.

According to Eqs. (5.16), (5.26), λ_k and g_k quickly approach $\lambda=g=0$ so that the corresponding trajectories run almost along the β axis for $k\to 0$, and the RG flow becomes essentially one-dimensional. This logarithmic running of β_k was expected on the basis of conventional perturbation theory [31]. We observe that $|\beta_k|$ *de*creases logarithmically with *in*creasing k. This is what is usually referred to as the "asymptotic freedom" of the (curvature)^2 coupling. We emphasize, however, that according to our results this logarithmic running occurs only close to $g=\lambda=0$ and does not represent the true short-distance behavior of the theory. In fact, we shall find that β_k runs towards a fixed point value β_* for $k\to\infty$.

Figure 1(b) shows three typical trajectories of the R^2 truncation close to $(\lambda,g,\beta)=(0,0,0)$; all of them satisfy the separatrix condition (5.26). Their β component diverges logarithmically towards $-\infty$ as k goes to zero, which is due to the positive coefficient in front of the $\ln(k/k_0)$ term in Eq. (5.28). In this figure we also depict the common projection of the trajectories onto the λ-g plane. It coincides precisely with the separatrix of the Einstein-Hilbert truncation [5], i.e. the curve in Fig. 1(a) that hits $(\lambda,g)=(0,0)$. Conversely, all trajectories of the R^2 truncation satisfying Eq. (5.26) represent specific "lifts" of the separatrix with nonvanishing β components; they are distinguished by their β_{k_0} and g_{k_0} values.

2. $d\neq 4$

Let us now discuss the β evolution for $2<d\neq 4$. Again, the linearization breaks down for trajectories which do not satisfy Eq. (5.26) for $k\to 0$ since $|\lambda_{k\to 0}|\to\infty$ in this case. Therefore we restrict our considerations to the trajectories with $\lambda_k=\nu_d g_k$ for sufficiently small k. In this case the second, quadratically divergent term of Eq. (5.22) drops out, and the only powers of k which occur in β_k are k^{d-2} and k^{4-d}.

In $2<d<4$, both $\vartheta_2=d-2$ and $\vartheta_3=4-d$ are positive which implies that the RG trajectories considered are attracted towards $(\lambda,g,\beta)=(0,0,\gamma_d/(d-4))$ as k is sent to zero.

For $d>4$, ϑ_3 is negative and thus β_k contains a divergent term $\propto k^{4-d}$. As a consequence, the coefficient of this term must vanish, if a trajectory is to hit the point $(\lambda,g,\beta)=(0,0,\gamma_d/(d-4))$ in the limit $k\to 0$. The distinguished trajectory which runs into this point as $k\to 0$ satisfies, for sufficiently small values of k,

$$\beta_k=\frac{\nu_d s_d+\tau_d}{2(d-3)}g_k=\frac{\nu_d s_d+\tau_d}{2(d-3)\nu_d}\lambda_k$$

$$\Leftrightarrow\bar{\beta}_k=\frac{\nu_d s_d+\tau_d}{2(d-3)}G_k k^{2(d-3)}$$

$$=\frac{\nu_d s_d+\tau_d}{2(d-3)\nu_d}\bar{\lambda}_k k^{d-6}. \tag{5.29}$$

For all other trajectories the β component diverges for $k\to 0$. However, higher orders of β_k are again suppressed by powers of g_k and may therefore be neglected. (Note that $\lim_{k\to 0}g_k^n\beta_k^m\propto\lim_{k\to 0}k^{n(d-2)-m(d-4)}=0$ for $n\geqslant m$.) As a consequence, the linearization can be trusted down to arbitrarily small scales k even in this case. The shape of the corresponding trajectories resembles the one found in $d=4$. While $|\beta_k|\to\infty$, the λ and g components approach $\lambda=g=0$ in the limit $k\to 0$. Thus, for sufficiently small k, the trajectories are almost straight lines which virtually coincide with the β axis.

Having a closer look at the $\boldsymbol{\beta}$ functions one recognizes that the IR scaling behavior in $d\neq 4$ dimensions is actually governed by a "quasi-Gaussian" fixed point at

$$(\lambda_*,g_*,\beta_*)=(0,0,\gamma_d/(d-4)). \tag{5.30}$$

The quasi-Gaussian fixed point is not present in $d=4$. Linearizing the RG flow about this fixed point yields essentially the same results as our expansion about $(0,0,0)$ above. The linearized $\boldsymbol{\beta}$ functions with stability matrix $\mathbf{B}$, and the linear

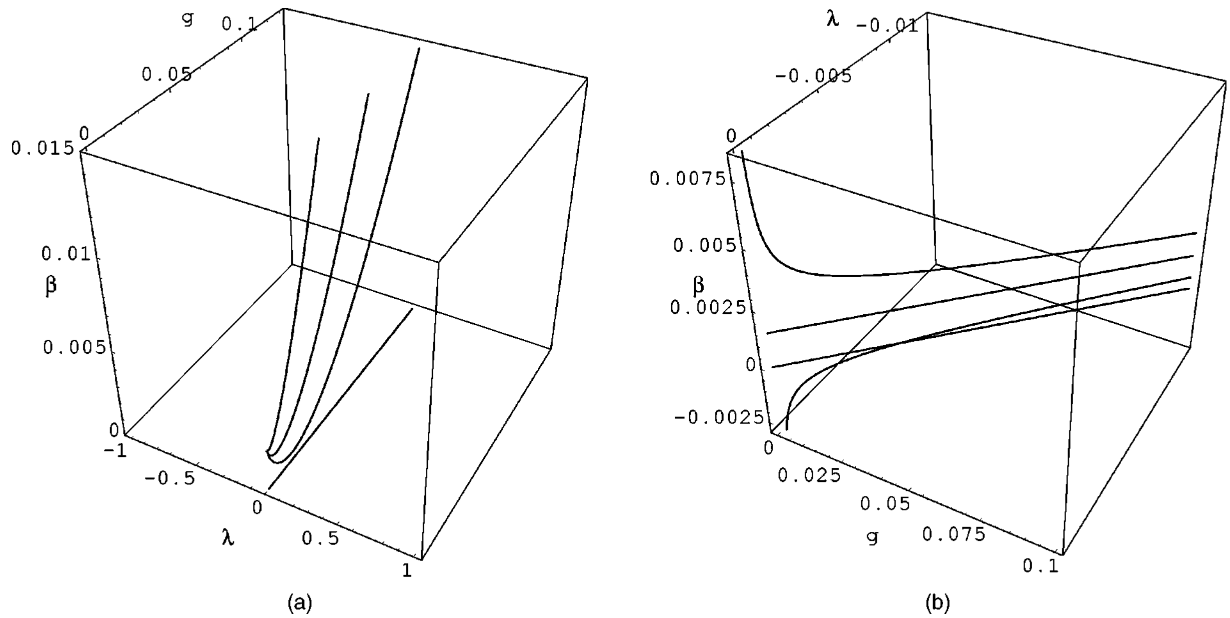


FIG. 2. (a) The case $d=3$: three typical trajectories of the linearized flow. They correspond to different values of β_{k_0}, but all of them satisfy Eq. (5.26). As a consequence, all 3 trajectories hit the quasi-Gaussian fixed point for $k\rightarrow 0$. (b) The case $d=5$: three typical trajectories of the linearized flow corresponding to different values of β_{k_0}. All of them satisfy Eq. (5.26), but in contrast to the other two curves, the one in the middle satisfies also Eq. (5.29). As a consequence, it hits the quasi-Gaussian fixed point for $k\rightarrow 0$. In both (a) and (b) we also depict the projection of the curves onto the $\beta=0$ plane.

solutions associated with this fixed point, may be obtained from Eqs. (5.10), (5.11), (5.16), (5.22), and (5.29) simply by replacing τ_d with

$$\hat{\tau}_d \equiv \tau_d - \frac{2(4\pi)^{1-d}}{d-4}\left(h_{39}(d)+h_{45}(d)\Phi^2_{d/2}(0)\right). \tag{5.31}$$

In particular, $\mathbf{B}=\mathbf{M}(\tau_d \rightarrow \hat{\tau}_d)$. The constants $\theta_I=-\vartheta_I$ assume the meaning of critical exponents now, and their signs determine the dimensionality $\Delta_{\rm IR}$ of the (truncated) IR critical hypersurface $\mathcal{S}_{\rm IR}$ of the quasi-Gaussian fixed point.

In $2<d<4$ we have one positive critical exponent $\theta_1>0$ and two negative critical exponents $\theta_2,\theta_3<0$. Therefore, within the truncation, $\Delta_{\rm IR}=2$, as suggested by the corresponding solutions discussed above.

In $d>4$, θ_1 and θ_3 are positive and θ_2 is negative. Hence, in this case $\Delta_{\rm IR}=1$, i.e. $\mathcal{S}_{\rm IR}$ consists of a single trajectory. For sufficiently small values of k this IR critical trajectory is given by Eq. (5.29) with τ_d replaced with $\hat{\tau}_d$. Since the parameters ν_d, ς_d and $\hat{\tau}_d$ contain $R^{(0)}$-dependent integrals $\Phi^p_n(0)$, $\tilde{\Phi}^p_n(0)$, they are not universal. Therefore the slopes in both directions of the distinguished trajectory (5.29) are not fixed in a universal manner. This is in accordance with the general expectation that the eigenvalues of **B** should be universal, but not its eigenvectors.

We illustrate our results for $d\neq 4$ in Fig. 2. In Fig. 2(a) we consider $d=3$ and in Fig. 2(b) the 5-dimensional case. Each figure shows three typical trajectories in the vicinity of the quasi-Gaussian fixed point. All of them satisfy Eq. (5.26), so that in both $d=3$ and $d=5$ the projections of the 3 trajectories onto the $\beta=0$ plane coincide with the separatrix.

In $d=3$, Fig. 2(a), all 3 trajectories hit the fixed point, independently of their β_{k_0} value. As we already pointed out above, the quasi-Gaussian fixed point in $d<4$ is IR attractive for *all* trajectories satisfying Eq. (5.26). In $d>4$ this is no longer the case. This is confirmed by Fig. 2(b) for $d=5$. Here only one of the trajectories hits the quasi-Gaussian fixed point for $k\rightarrow 0$, and this is precisely the one which satisfies the additional condition (5.29). The other two trajectories shown in Fig. 2(b) correspond to β_{k_0} values which are different from the one in Eq. (5.29) and thus their β component diverges in the limit $k\rightarrow 0$. Depending on the β_{k_0} value, β_k runs towards $+\infty$ or $-\infty$.

C. The non-Gaussian fixed point

Now we turn to the nontrivial simultaneous zeros of the set of $\boldsymbol{\beta}$ functions $\{\boldsymbol{\beta}_\lambda,\boldsymbol{\beta}_g,\boldsymbol{\beta}_\beta\}$ given by Eqs. (4.23), (4.24), (4.25). Such non-Gaussian fixed points with λ_*, g_*, β_* all different from zero have the anomalous dimensions

$$\eta_{N*}=2-d, \quad \eta_{\beta*}=d-4 \tag{5.32}$$

which follow immediately from Eqs. (4.24) and (4.25).

1. Results obtained from the pure Einstein-Hilbert truncation

In $d=4$ dimensions, and for the cutoff of the type A introduced in [2], the non-Gaussian fixed point of the pure Einstein-Hilbert truncation was first discussed in [20,9], and in Ref. [35] the α and $R^{(0)}$ dependence of its projection $(0,g_*)$ onto the g direction has been investigated. However, since for $\alpha\neq 1$ the cutoff of type A was defined in [34] by an *ad hoc* modification of the standard one-loop determinants it is not clear whether it can be derived from an action $\Delta_k S$, except for the case $\alpha=1$ [1]. Since a specification of $\Delta_k S$ is indispensable for the actual construction of Γ_k, the status of the results derived in [35] is somewhat unclear. In Ref. [2] we performed a comprehensive analysis of the fixed point properties using different cutoffs of type B, for which a $\Delta_k S$ is known to exist. In particular, we investigated the cutoff scheme dependence of various universal quantities of interest, both by looking at their dependence on the shape function $R^{(0)}$ and by comparing the "type A" and "type B" results.

In this respect universal quantities are of special importance because, by definition, they are strictly cutoff scheme independent in the exact theory. Any truncation leads to a scheme dependence of these quantities whose magnitude is a measure for the reliability of the truncation [43]. Typical examples of universal quantities are the critical exponents θ_I. The existence or nonexistence of a fixed point is also a universal, scheme independent feature, but its precise location in parameter space is scheme dependent. Nevertheless it can be argued that, in $d=4$, the product $g_*\lambda_*$ is universal [44,2] while g_* and λ_* separately are not.

For later comparison with the R^2 truncation, let us briefly list some of the results we obtained in [2] with the pure Einstein-Hilbert truncation:

($1_{\text{E.H.}}$) Universal Existence: Both for type A and type B cutoffs the non-Gaussian fixed point exists for all shape functions $R^{(0)}$ we considered. This result is highly nontrivial since in higher dimensions ($d\gtrsim 5$) the fixed point exists for some but does not exist for other cutoffs [5].

($2_{\text{E.H.}}$) Positive Newton Constant: While the position of the fixed point is scheme dependent, all cutoffs yield *positive* values of g_* and λ_*. A negative g_* might be problematic for stability reasons, but there is no mechanism in the flow equation which would exclude it on general grounds.

($3_{\text{E.H.}}$) Stability: For any cutoff employed, the non-Gaussian fixed point is found to be UV attractive in both directions of the λ-g plane. Linearizing the flow equation according to Eq. (5.2) we obtain a pair of complex conjugate critical exponents $\theta_1=\theta_2^*$ with positive real part θ' and imaginary parts $\pm\theta''$. Due to the positivity of θ', all trajectories in its basin of attraction hit the fixed point as k is sent to infinity. Because of the nonvanishing imaginary part θ'' the trajectories spiral into the fixed point for $k\to\infty$.

Solving the full, nonlinear flow equations [5] shows that the asymptotic scaling region where the linearization is valid extends from k"="∞ down to about $k\approx m_{\text{Pl}}$ with the Planck mass defined as $m_{\text{Pl}}\equiv G_0^{-1/2}$. It is the regime above the Planck scale where the asymptotic freedom of G_k sets in.

($4_{\text{E.H.}}$) Scheme and Gauge Dependence: The critical exponents are reasonably constant within about a factor of 2. For the gauges $\alpha=1$ and $\alpha=0$, for instance, they assume values in the ranges $1.4\lesssim\theta'\lesssim 1.8$, $2.3\lesssim\theta''\lesssim 4$ and $1.7\lesssim\theta'\lesssim 2.1$, $2.5\lesssim\theta''\lesssim 5$, respectively. The universality properties of the product $g_*\lambda_*$ are much more impressive though. Despite the rather strong scheme dependence of g_* and λ_* separately, their product exhibits almost no visible $R^{(0)}$ dependence. Its value is $g_*\lambda_*\approx 0.12$ for $\alpha=1$ and $g_*\lambda_*\approx 0.14$ for $\alpha=0$. The differences between the "physical" (fixed point) value of the gauge parameter, $\alpha=0$, and the technically more convenient $\alpha=1$ are at the level of about 10 to 20 percent.

The above results suggest that the UV attractive non-Gaussian fixed point occurring in the Einstein-Hilbert truncation is very unlikely to be an artifact of this truncation but should rather be the projection of a fixed point in the exact theory. We interpreted them as nontrivial indications supporting the conjecture that 4-dimensional QEG is "asymptotically safe" in Weinberg's sense.

2. Results obtained from the R^2 truncation

The actual justification of a truncation is that when one adds further terms to it its physical predictions do not change significantly any more. In order to test the stability of the Einstein-Hilbert truncation against the inclusion of other invariants we shall now reanalyze the non-Gaussian fixed point in the generalized truncation (4.2) including the R^2 term. Starting from the $\boldsymbol{\beta}$ functions of the R^2 truncation, Eqs. (4.23), (4.24) and (4.25), we determine the location of the fixed point in λ-g-β space and the linearized flow in its vicinity. Then we investigate the residual cutoff scheme dependence of the associated universal quantities, and we compare our results to those obtained from the pure Einstein-Hilbert truncation.

Note that, contrary to the pure Einstein-Hilbert truncation, only a cutoff of type B is used in the context of the generalized truncation. Therefore we omit the specification of the cutoff type when we refer to results obtained from the R^2 truncation.

Location of the fixed point ($d=4$). In a first attempt at finding the non-Gaussian fixed point in the R^2 truncation we neglect the cosmological constant and the coupling of the R^2 invariant. We approximate $\lambda_k\equiv\lambda_*=0$, $\beta_k\equiv\beta_*=0$, thereby projecting the renormalization group flow onto the one-dimensional space parametrized by g. In this case the non-Gaussian fixed point is obtained as the nontrivial solution of $\boldsymbol{\beta}_g(0,g_*,0;d)=0$. It is determined in Appendix E with the result given by Eq. (E2). For any d, this solution coincides precisely with the analogous approximate solution (H2) of Ref. [2] with $\alpha=1$, obtained in the pure Einstein-Hilbert truncation. In order to get a numerical value for the fixed point we have to specify $R^{(0)}$. Inserting the exponential shape function with $s=1$ into Eq. (E2) and setting $d=4$ leads to $g_*\approx 0.590$.

Assuming that for the combined λ-g-β system the numbers λ_*, g_* and β_* are of the same order of magnitude as g_* above we expand the $\boldsymbol{\beta}$ functions about $(\lambda_k,g_k,\beta_k)=(0,0,0)$ and neglect terms of higher orders in the couplings. Again in Appendix E we determine the non-Gaussian fixed point from the corresponding system of differential equations. Inserting the shape function (3.10) and setting $d=4$, we find $(\lambda_*,g_*,\beta_*)\approx(0.287,0.751,0.002)$. Quite remarkably, for any cutoff λ_* and g_* agree perfectly with the corresponding values obtained in [2] by the same approximation applied to the pure Einstein-Hilbert truncation.

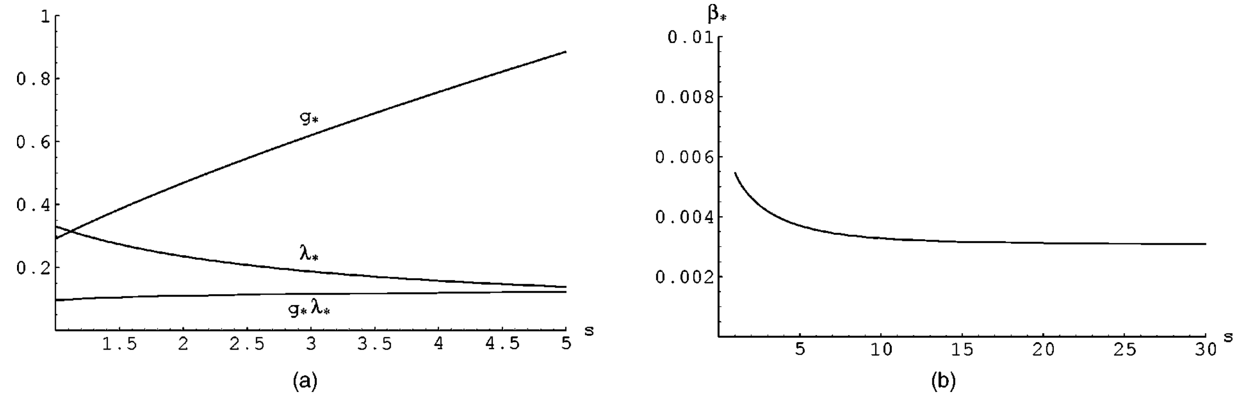


FIG. 3. (a) g_*, λ_*, and $g_*\lambda_*$ as functions of s for $1 \leqslant s \leqslant 5$, and (b) β_* as a function of s for $1 \leqslant s \leqslant 30$, using the family of exponential shape functions.

In order to determine the *exact* position of the non-Gaussian fixed point $(\lambda_*, g_*, \beta_*)$ we have to resort to numerical methods. Given a starting value for the fixed point, for instance one of the approximate solutions above, the program we use determines a numerical solution which is exact up to an arbitrary degree of accuracy. Under the same conditions as above, i.e. for $s=1$ and $d=4$, we obtain

$$(\lambda_*, g_*, \beta_*) = (0.330, 0.292, 0.005). \tag{5.33}$$

In the pure Einstein-Hilbert truncation the corresponding coordinates of the fixed point are $(\lambda_*, g_*) = (0.348, 0.272)$ [2]. Obviously the values of λ_* and g_* are almost the same in both cases. While λ_* and g_* are of the same order of magnitude, we find that β_* is significantly smaller than λ_* and g_*.

In order to test whether these properties of the fixed point coordinates are universal we study their scheme dependence by looking at the s or b dependence introduced via the one-parameter families of shape functions (3.11) or (3.12), respectively. Here s parametrizes the family of exponential shape functions (3.11), while the shape parameter b allows us to change the profile of the shape functions with compact support (3.12).

As for the family of exponential shape functions, we are forced to restrict our considerations to shape parameters $s \geqslant 1$. This is because for $s<1$ the numerical integrations are plagued by convergence problems. They are due to the fact that in $d=4$ some of the threshold functions appearing in $\boldsymbol{\beta}_\lambda$, $\boldsymbol{\beta}_g$ and $\boldsymbol{\beta}_\beta$ diverge in the limit $s \to 0$, see also [35]. As for the family of shape functions with compact support, we have to restrict ourselves to $b \leqslant 1.2$ for similar reasons. Here $R^{(0)}(y;b)$ approaches a sharp cutoff as $b \to 1.5$, which introduces discontinuities into the integrands of the threshold functions Φ_n^p and $\tilde{\Phi}_n^p$. Already for $b \gtrsim 1.2$ the $\boldsymbol{\beta}$ functions start to "feel" the sharp cutoff limit, which leads to convergence problems.

As in the case of the pure Einstein-Hilbert truncation [2] our results establish the existence of the non-Gaussian fixed point in a wide range of s and b values. As expected, the position of the fixed point turns out to depend on s or b, i.e. on the cutoff scheme, but the crucial point is that it exists for any of the cutoffs employed. Figures 3 and 4 show its coordinates $(\lambda_*, g_*, \beta_*)$ as well as the product $g_*\lambda_*$ for the shape functions (3.11) and (3.12), respectively. In Fig. 3(a) we plotted the various quantities in the range $1 \leqslant s \leqslant 5$ where the largest changes in λ_* and g_* occur, but we calculated them for $1 \leqslant s \leqslant 30$. For every shape parameter s or b, the values of λ_* and g_* are almost the same as those obtained with the Einstein-Hilbert truncation [2]. As a consequence, the product $g_*\lambda_*$ is again almost constant and its value differs only slightly from the one in [2] for the same gauge $\alpha = 1$. Both Figs. 3(a) and 4(a) suggest the universal value

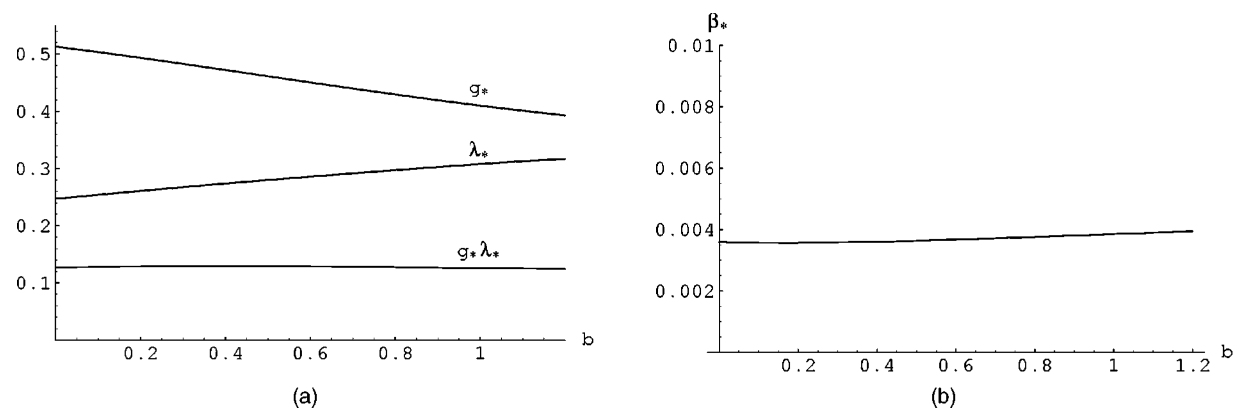
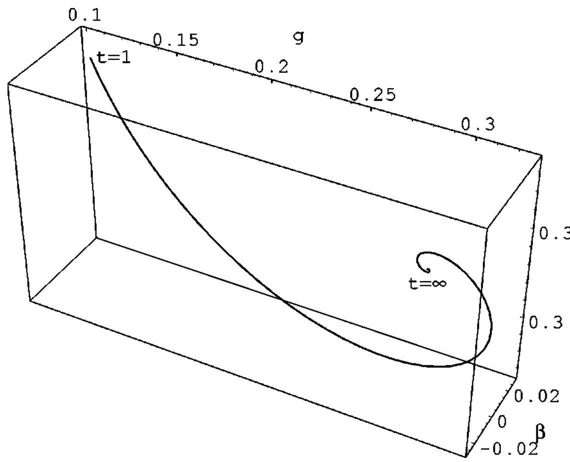


FIG. 4. (a) g_*, λ_* and $g_*\lambda_*$, and (b) β_* as functions of b for $0 \leqslant b \leqslant 1.2$, using the family of shape functions with compact support.



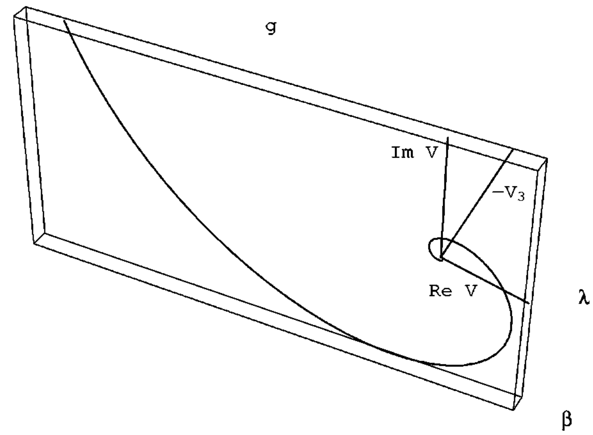


FIG. 5. Trajectory of the linearized flow equation obtained from the R^2 truncation for $1 \leqslant t = \ln(k/k_0) < \infty$. In (b) we depict the eigendirections and the ''box'' to which the trajectory is confined.

$g_*\lambda_* \approx 0.14$ while we obtained $g_*\lambda_* \approx 0.12$ from the pure Einstein-Hilbert truncation. Thus we may expect that our $g_*\lambda_*$-value is precise at the 10 to 20 percent level. Presumably this degree of precision is the best we can achieve in the present calculation because we saw already that the error due to using $\alpha = 1$ instead of the ''correct'' $\alpha = 0$ leads to an uncertainty of the same size.

Furthermore, our results show that β_* is always significantly smaller than g_* and λ_* for both families of shape functions, which is quite remarkable. Within the limited precision of our calculation this means that in the three-dimensional λ-g-β space the fixed point practically lies in the λ-g plane with $\beta = 0$, i.e. on the parameter space of the pure Einstein-Hilbert truncation.

It is also interesting to note that the scheme dependence of β_* is unexpectedly small. As for the family of exponential shape functions (3.11), the function $\beta_*(s)$ depicted in Fig. 3(b) develops a plateau-like shape for not too small values of s. Employing the family of shape functions with compact support, the scheme dependence of β_* is even weaker. The function $\beta_*(b)$ plotted in Fig. 4(b) is almost constant in the range $0 \leqslant b \leqslant 1.2$. Moreover, the positions of the two plateaus are nearly identical. While Fig. 3(b) suggests the value $\beta_* \approx 0.0031$, we obtain $\beta_* \approx 0.0036$ from Fig. 4(b). This indicates that in $d = 4$ dimensions also β_* might be a universal quantity.

The linearized flow (d=4). Let us now analyze the critical behavior near the non-Gaussian fixed point. Quite remarkably, the non-Gaussian fixed point of the R^2 truncation proves to be UV attractive in any of the three directions of λ-g-β space, for all cutoffs used. The linearized flow in its vicinity is always governed by a pair of complex conjugate critical exponents $\theta_1 = \theta' + \mathrm{i}\theta'' = \theta_2^*$ with $\theta' > 0$ and a single real, positive critical exponent $\theta_3 > 0$. (We define θ_1 as the critical exponent with the positive imaginary part so that $\theta'' > 0$.) The general solution to the linearized flow equations is obtained by taking the real part of Eq. (5.4). Introducing the RG time $t \equiv \ln(k/k_0)$ it may be written as

$$\begin{aligned}(\lambda_k, g_k, \beta_k)^{\mathbf{T}} &= (\lambda_*, g_*, \beta_*)^{\mathbf{T}} + 2\{[\mathrm{Re}\, C \cos(\theta'' t) \\ &\quad + \mathrm{Im}\, C \sin(\theta'' t)]\mathrm{Re}\, V + [\mathrm{Re}\, C \sin(\theta'' t) \\ &\quad - \mathrm{Im}\, C \cos(\theta'' t)]\mathrm{Im}\, V\} e^{-\theta' t} + C_3 V^3 e^{-\theta_3 t}\end{aligned} \tag{5.34}$$

with arbitrary complex $C \equiv C_1 = (C_2)^*$ and arbitrary real C_3. Furthermore, $V \equiv V^1 = (V^2)^*$ and V^3 are the right eigenvectors of the stability matrix $(B_{ij})_{i,j \in \{\lambda, g, \beta\}}$ with eigenvalues $-\theta_1 = -\theta_2^*$ and $-\theta_3$, respectively. Obviously the conditions for UV stability are $\theta' > 0$ and $\theta_3 > 0$. They are indeed satisfied for all cutoffs. As a consequence, all RG trajectories which reach its basin of attraction hit the fixed point as t is sent to infinity. The trajectories (5.34) comprise three independent normal modes with amplitudes proportional to $\mathrm{Re}\, C$, $\mathrm{Im}\, C$ and C_3, respectively. The first two are of the spiral type, the third one is a straight line.

Let us illustrate these features by means of an example. For the exponential shape function (3.11) with $s = 1$, for instance, we have $(\lambda_*, g_*, \beta_*) = (0.330, 0.292, 0.005)$. The corresponding stability matrix $\mathbf{B}$ takes the form

$$\mathbf{B} = -\begin{pmatrix} 8.83 & 2.61 & 401.75 \\ 6.18 & 4.46 & 89.24 \\ 0.29 & 0.32 & 19.82 \end{pmatrix}. \tag{5.35}$$

It leads to the pair of complex critical exponents $\theta_1 = \theta_2^*$ with $\theta' = 2.15$, $\theta'' = 3.79$, and to the real critical exponent $\theta_3 = 28.8$. For the associated right eigenvectors we find

$$\begin{aligned}\mathrm{Re}\, V &= (-0.164, 0.753, -0.008)^{\mathbf{T}}, \\ \mathrm{Im}\, V &= (0.64, 0, -0.01)^{\mathbf{T}}, \\ V^3 &= -(0.92, 0.39, 0.04)^{\mathbf{T}}.\end{aligned} \tag{5.36}$$

(The vectors are normalized such that $||V|| = ||V^3|| = 1$.) In Fig. 5 we show a typical trajectory which has all three nor-

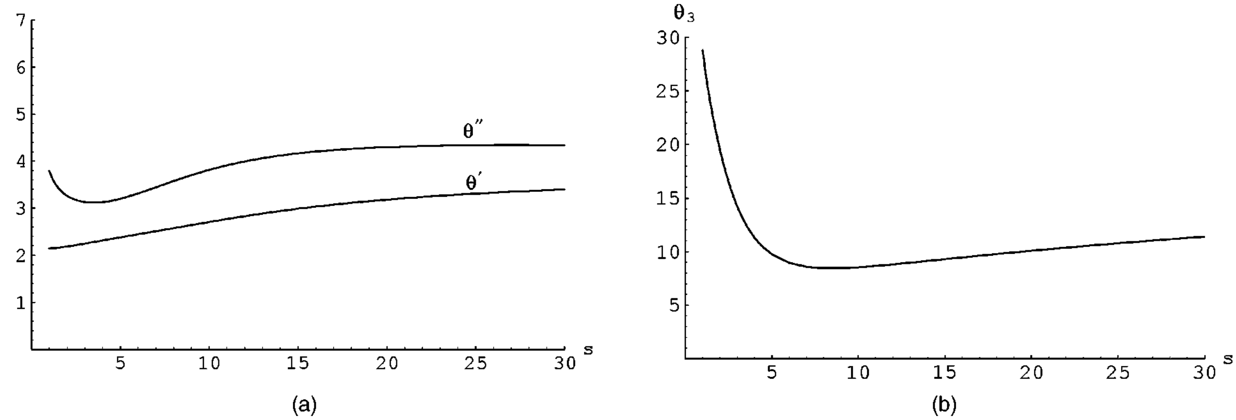


FIG. 6. (a) $\theta'=\mathrm{Re}\,\theta_1$ and $\theta''=\mathrm{Im}\,\theta_1$, and (b) θ_3 as functions of s, using the family of exponential shape functions.

mal modes excited with equal strength ($\mathrm{Re}\,C=\mathrm{Im}\,C=1/\sqrt{2}$, $C_3=1$). All its way down from k"$=$"∞ to about $k=m_{\mathrm{Pl}}$ it is confined to a very thin box surrounding the $\beta=0$ plane, i.e. the parameter space of the Einstein-Hilbert truncation.

In fact, the linearized flow is characterized by the following quite remarkable properties, independently of the cutoff. They all indicate that, close to the non-Gaussian fixed point, the RG flow is rather well approximated by the pure Einstein-Hilbert truncation.

(a) The β components of $\mathrm{Re}\,V$ and $\mathrm{Im}\,V$ are very tiny. Hence these two vectors span a plane which virtually coincides with the λ-g subspace at $\beta=0$, i.e. with the parameter space of the Einstein-Hilbert truncation. As a consequence, the $\mathrm{Re}\,C$ and $\mathrm{Im}\,C$ normal modes are essentially the same trajectories as the "old" normal modes already found without the R^2 term. Also the corresponding θ' and θ'' values coincide within the scheme dependence; see below.

(b) For all cutoffs employed, the new eigenvalue θ_3 introduced by the R^2 term is significantly larger than θ'; see below. When a trajectory approaches the fixed point from below ($t\to\infty$), the "old" normal modes $\propto \mathrm{Re}\,C, \mathrm{Im}\,C$ are proportional to $\exp(-\theta' t)$, but the new one is proportional to $\exp(-\theta_3 t)$, so that it decays much more quickly. For every trajectory running into the fixed point, i.e. for every set of constants $(\mathrm{Re}\,C, \mathrm{Im}\,C, C_3)$, we find therefore that, once t is sufficiently large, the trajectory lies entirely in the $\mathrm{Re}\,V$-$\mathrm{Im}\,V$ subspace, i.e. the $\beta=0$ plane practically.

Due to the large value of θ_3, the new scaling field is very "relevant." However, when we start at the fixed point (t"$=$"∞) and lower t it is only at the low energy scale $k\approx m_{\mathrm{Pl}}$ ($t\approx 0$) that $\exp(-\theta_3 t)$ reaches unity, and only then, i.e. far away from the fixed point, the new scaling field starts growing rapidly.

(c) Since the matrix **B** is not symmetric its eigenvectors have no reason to be orthogonal. In fact, we find that V^3 lies almost in the $\mathrm{Re}\,V$-$\mathrm{Im}\,V$ plane. For the angles between the eigenvectors given above we obtain $\sphericalangle(\mathrm{Re}\,V,\mathrm{Im}\,V)=102.3°$, $\sphericalangle(\mathrm{Re}\,V,V^3)=100.7°$, $\sphericalangle(\mathrm{Im}\,V,V^3)=156.7°$. Their sum is 359.7° which confirms that $\mathrm{Re}\,V$, $\mathrm{Im}\,V$ and V^3 are almost coplanar. This implies that when we lower t and move away from the fixed point so that the V^3-scaling field starts growing, it is again predominantly the $\int d^dx\sqrt{g}$ and $\int d^dx\sqrt{g}R$ invariants which get excited, but not $\int d^dx\sqrt{g}R^2$ in the first place.

Summarizing the three points above we can say that very close to the fixed point the RG flow seems to be essentially two-dimensional, and that this two-dimensional flow is well approximated by the RG equations of the Einstein-Hilbert truncation.

Scheme dependence of the critical exponents ($d=4$). As we pointed out already the critical exponents are universal in an exact treatment, but in a truncated parameter space a scheme dependence is expected to occur as an artifact of the truncation. We may use it to judge the quality of our truncation. Also in this respect the R^2 truncation yields satisfactory results, which we display in Figs. 6 and 7. Figures 6(a) and 7(a) show the real and the imaginary part θ' and θ'' of the complex conjugate pair $\theta_1=\theta_2^*$ while θ_3 is depicted in Figs. 6(b) and 7(b). The plots in Fig. 6 are based on the family of exponential shape functions (3.11) and those in Fig. 7 are obtained by employing the family of shape functions with compact support (3.12). They display the s and the b dependence of the critical exponents, respectively.

As for the complex conjugate pair of critical exponents, the scheme dependence is of the same order of magnitude as in the case of the Einstein-Hilbert truncation [2]. While the scheme dependence of θ'' is weaker than that found in [2] we see that it is slightly larger for θ'. For the exponential shape functions with $1\leqslant s\leqslant 30$, θ' and θ'' assume values in the ranges $2.1\lesssim\theta'(s)\lesssim 3.4$ and $3.1\lesssim\theta''(s)\lesssim 4.3$, respectively. Employing the shape functions with compact support leads to a weaker dependence on the shape parameter b. However, the corresponding values $\theta'(b)$ and $\theta''(b)$ are in good agreement with those obtained with the exponential cutoffs. In fact, they all lie in the $\theta'(s)$ and $\theta''(s)$ intervals given above. The average values of θ' and θ'' are slightly larger than those obtained from the pure Einstein-Hilbert truncation. The difference between the corresponding average values is approximately 1 for both θ' and θ''.

Let us now come to the new critical exponent θ_3 which was not present in the Einstein-Hilbert truncation. Using the exponential shape functions (3.11) it suffers from relatively strong variations as the shape parameter s is changed. It as-

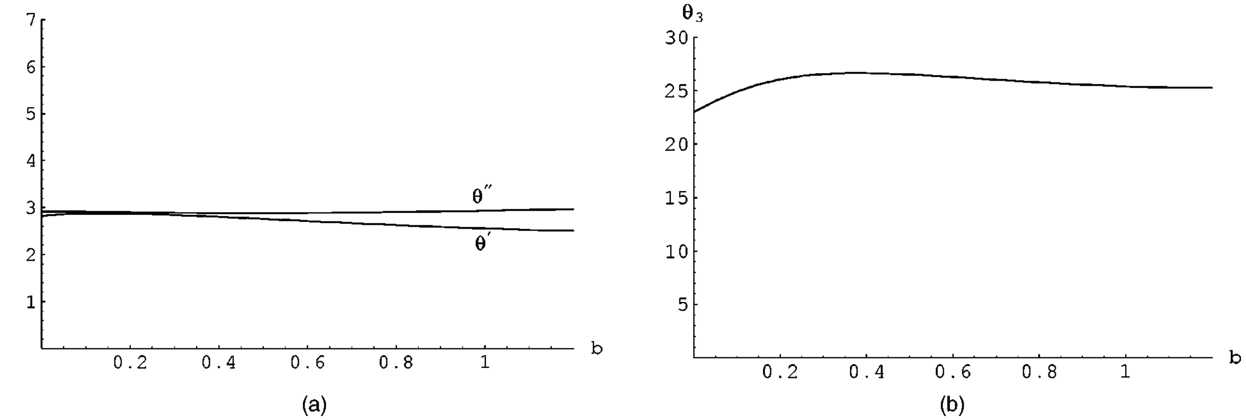


FIG. 7. (a) $\theta' = \mathrm{Re}\theta_1$ and $\theta'' = \mathrm{Im}\theta_1$, and (b) θ_3 as functions of b, using the family of shape functions with compact support.

sumes values in the range $8.4 \lesssim \theta_3(s) \lesssim 28.8$. As compared to the exponential cutoffs, the cutoffs with compact support lead to a much weaker scheme dependence. For $b \in [0,1.2]$ we have $23.0 \lesssim \theta_3(b) \lesssim 26.7$. However, the results obtained with the two families of shape functions agree within the scheme dependence. Moreover, θ_3 is always systematically larger than θ' (and θ'') with both families of cutoffs. As a consequence, the hierarchy of critical exponents which was mentioned in (b) above and which squeezes the trajectories into a thin box is a universal feature.

Obviously the critical exponents, in particular θ_3, exhibit a much stronger scheme dependence than $g_* \lambda_*$. This is most probably due to neglecting further relevant operators in the truncation so that the **B** matrix we are diagonalizing is still too small.

In 2+ε dimensions. The above results and their mutual consistency strongly suggest that 4-dimensional quantum Einstein gravity indeed possesses a RG fixed point with precisely the properties needed for its nonperturbative renormalizability or "asymptotic safety." However, with the present approach it is clearly not possible to determine the dimensionality Δ_{UV} of the UV critical hypersurface, which coincides with the number of invariants relevant at the non-Gaussian fixed point. According to the canonical dimensional analysis, the $(\text{curvature})^n$ invariants in 4 dimensions are *classically* marginal for $n=2$ and irrelevant for $n>2$. The results for θ_3 indicate that there are large *nonclassical* contributions so that there might be relevant operators perhaps even beyond $n=2$. However, as it is hardly conceivable that the quantum effects change the signs of arbitrarily large (negative) classical scaling dimensions, Δ_{UV} should be finite [23].

A first confirmation of this picture comes from our R^2 calculation in $d=2+\varepsilon$ where the dimensional count is shifted by two units. In this case we find indeed that the third scaling field is *irrelevant* for any cutoff employed, $\theta_3 < 0$.

For our analysis of the R^2 truncation in $d=2+\varepsilon$ dimensions with $0<\varepsilon \ll 1$ we had to resort to numerical methods. Using the ε expansion we calculated the fixed point coordinates and the critical exponents for selected values of the shape parameter s. For all quantities only the leading nontrivial order of the ε expansion was retained. In Table I we present the corresponding numerical results.

For all cutoffs used we obtain three *real* critical exponents, the first two are positive and the third is negative. Thus, the corresponding V^3 direction is UV repulsive. This suggests that the dimensionality of $\mathcal{S}_{\mathrm{UV}}$ could be as small as $\Delta_{\mathrm{UV}}=2$, but this is not a proof, of course. If so, the quantum theory would be characterized by only two free parameters, the renormalized Newton constant G_0 and the renormalized cosmological constant $\bar{\lambda}_0$, for instance.

Let us now compare the results to those from the Einstein-Hilbert truncation [1,2]. The λ and g coordinates of the fixed point and the critical exponents θ_1 and θ_2 are found to be similar to those in [1,2]. However, in the Einstein-Hilbert truncation the leading-order results $g_* = 3/38\varepsilon + \mathcal{O}(\varepsilon^2) \approx 0.079\varepsilon + \mathcal{O}(\varepsilon^2)$ and $\theta_2 = \varepsilon + \mathcal{O}(\varepsilon^2)$ are scheme independent, which is not quite true for the results above. Both truncations agree on $\theta_1 = 2 + \mathcal{O}(\varepsilon)$.

Summary. Our main results concerning the non-Gaussian fixed point in the R^2 truncation are:

(1_{R^2}) Position of the fixed point: The fixed point is found to exist for all cutoffs used. This result is highly nontrivial since the example of the Gaussian fixed point clearly shows that a fixed point of the Einstein-Hilbert truncation does not necessarily generalize to a fixed point of the R^2 truncation. For every shape parameter the fixed point practically lies on

TABLE I. Fixed point coordinates and critical exponents.

s	$\lambda_*(+\mathcal{O}(\varepsilon^2))$	$g_*(+\mathcal{O}(\varepsilon^2))$	$\beta_*(+\mathcal{O}(\varepsilon))$	$\theta_1(+\mathcal{O}(\varepsilon))$	$\theta_2(+\mathcal{O}(\varepsilon^2))$	$\theta_3(+\mathcal{O}(\varepsilon))$
1	-0.131ε	0.087ε	-0.083	2	0.963ε	-1.968
5	-0.055ε	0.092ε	-0.312	2	0.955ε	-1.955
10	-0.035ε	0.095ε	-0.592	2	0.955ε	-1.956

the λ-g plane, and its position almost exactly coincides with that from the Einstein-Hilbert truncation.

(2_{R^2}) Eigenvalues and eigenvectors: The fixed point is UV attractive in any of the three directions of the λ-g-β space for all cutoffs employed. The linearized flow in its vicinity is always governed by a pair of complex conjugate critical exponents $\theta_1=\theta'+\mathrm{i}\theta''=\theta_2^*$ with $\theta'>0$ and a single real, positive critical exponent $\theta_3>0$. It is essentially two-dimensional, and this two-dimensional flow is well described by the RG equations of the Einstein-Hilbert truncation.

(3_{R^2}) Scheme dependence: The scheme dependence of the critical exponents and of the product $g_*\lambda_*$ is of the same order of magnitude as in the case of the Einstein-Hilbert truncation. While the scheme dependence of θ'' is weaker than in the case of the Einstein-Hilbert truncation we find that it is slightly larger for θ'. The exponent θ_3 shows a relatively strong dependence on the cutoff. The product $g_*\lambda_*$ again exhibits an impressively weak scheme dependence.

(4_{R^2}) Dimensionality of $\mathcal{S}_{\rm UV}$: The dimensionality $\Delta_{\rm UV}$ of the UV critical hypersurface cannot be determined within the present approach. However, the results from our R^2 calculation in $2+\varepsilon$ dimensions suggest that $\Delta_{\rm UV}$ should be finite also in 4 dimensions.

On the basis of the above results we believe that the non-Gaussian fixed point occurring in the Einstein-Hilbert truncation is very unlikely to be an artifact of this truncation but rather should be the projection of a fixed point in the exact theory. We demonstrated explicitly that the fixed point and all its qualitative properties are stable against the inclusion of a further invariant in the truncation. These results strongly support the hypothesis that 4-dimensional QEG is indeed nonperturbatively renormalizable.

VI. POSITIVITY OF ACTION, HESSIAN, AND CUTOFF

A. Positivity of the action

It is a well known problem that in $d>2$ dimensions the Euclidean Einstein-Hilbert action

$$S_{\rm EH}[g]=\frac{1}{16\pi\bar{G}}\int d^dx\,\sqrt{g}\{-R(g)+2\bar{\lambda}\} \tag{6.1}$$

is not bounded below. In fact, decomposing the metric as $g_{\mu\nu}=\exp(2\chi)\bar{g}_{\mu\nu}$ where $\bar{g}_{\mu\nu}$ is a fixed reference metric we obtain

$$S_{\rm EH}[g]=\frac{1}{16\pi\bar{G}}\int d^dx\,\sqrt{\bar{g}}e^{(d-2)\chi}[-\bar{R}+2\bar{\lambda}e^{2\chi} -(d-1)(d-2)\bar{g}^{\mu\nu}(\bar{D}_\mu\chi)(\bar{D}_\nu\chi)]. \tag{6.2}$$

This shows that $S_{\rm EH}$ can become arbitrarily negative if the conformal factor $\chi(x)$ varies rapidly enough so that $(\bar{D}_\mu\chi)^2$ is large. Therefore it seems difficult to define a path integral $Z=\int\mathcal{D}g_{\mu\nu}\exp(-S_{\rm EH})$ for Euclidean quantum gravity.

The situation improves by including the term $\int d^dx\,\sqrt{g}R^2$ with a positive coefficient since the resulting action is *bounded below* [18]. While the Einstein-Hilbert term $\int d^dx\,\sqrt{g}R$ leads to a negative contribution to the kinetic term of the conformal factor, which dominates at small momenta, the R^2 term gives rise to a positive contribution dominating at large momenta. As a consequence, both the truncated action functional $\Gamma_k[g,\bar{g}]$ of Eq. (4.2) and the bare action $S[g]=\Gamma_{\hat{k}\to\infty}[g,g]$ possess an absolute minimum. Moreover, rewriting the truncation ansatz (4.2) with $\bar{g}_{\mu\nu}=g_{\mu\nu}$ as

$$\Gamma_k[g,g]=\int d^dx\,\sqrt{g}\left\{\bar{\beta}_k\left(R-\frac{Z_{Nk}\kappa^2}{\bar{\beta}_k}\right)^2 +Z_{Nk}\kappa^2\left(4\bar{\lambda}_k-\frac{Z_{Nk}\kappa^2}{\bar{\beta}_k}\right)\right\} \tag{6.3}$$

one can easily determine a sufficient condition for a manifestly *positive* action $\Gamma_k[g,g]>0$. In terms of the dimensionless couplings it reads $g_k>0$, $\beta_k>0$, and

$$128\pi g_k\lambda_k\beta_k>1. \tag{6.4}$$

B. Positivity of the Hessian

At the level of the flow equation, Γ_k appears on the RHS in terms of its Hessian $\Gamma_k^{(2)}$ to which the cutoff operator $\mathcal{R}_k$ is adapted by the rule (3.5). Thus, only if $\Gamma_k^{(2)}$ is a positive definite operator can we obtain a cutoff which leads to a "correct" mode suppression. Since we expect the R^2 truncation anyhow to be reliable only for large k, it is actually sufficient if $\Gamma_k^{(2)}$ and $\mathcal{R}_k$ are positive definite for sufficiently large momenta $p^2\equiv-\bar{D}^2$. The reason is that, due to the factor $\partial_t\mathcal{R}_k(p^2)$ which emphasizes the region $p^2\approx k^2$, the traces on the RHS of the RG equation (3.4) receive the dominant contributions from modes whose p^2 is close to k^2.

In general $\Gamma_k^{(2)}[g,\bar{g}]$ depends on both $g_{\mu\nu}$ and the background metric $\bar{g}_{\mu\nu}$. Here we concentrate on $\Gamma_k^{(2)}[g,g]\equiv\Gamma_k^{(2)}$ with the two metrics identified. Furthermore, we assume that $g_{\mu\nu}=\bar{g}_{\mu\nu}$ is the metric of a d sphere with radius r since our projection technique requires these backgrounds only. In this case the eigenvalues $p^2=\Lambda_l(d,s)$ depend on the discrete quantum number l. The explicit expressions for $\Lambda_l(d,s)$ are tabulated in Appendix C. They are strictly monotonically increasing functions of l with $\lim_{l\to\infty}\Lambda_l(d,s)=\infty$.

In the following we show that the operator $\Gamma_k^{(2)}$ with k very large indeed becomes positive definite if it is restricted to the subspace spanned by the $-D^2$ eigenfunctions with sufficiently large eigenvalues, certain assumptions on the couplings being made. The spherical harmonics $T^{lm}_{\mu\nu}$, T^{lm}_{μ}, and T^{lm} with l larger than a certain minimum value $l_{\rm min}$ provide a basis of this subspace. We shall concentrate on the conditions implied by the leading large-l behavior.

The Hessian $\Gamma_k^{(2)}[g,g]$ as given by the quadratic form (4.12) is a symmetric block diagonal matrix. Therefore, according to the Jacobi criterion, the condition for positivity takes the simple form $(\Gamma_k^{(2)})_{\overline{h^T}\overline{h^T}}>0$, $(\Gamma_k^{(2)})_{\overline{\xi}\overline{\xi}}>0$, $(\Gamma_k^{(2)})_{\overline{\phi}\overline{\phi}}>0$, and $(\Gamma_k^{(2)})_{\overline{\sigma}\overline{\sigma}}(\Gamma_k^{(2)})_{\overline{\phi}\overline{\phi}}-(\Gamma_k^{(2)})^2_{\overline{\sigma}\overline{\phi}}>0$. For sufficiently large values of l the leading l powers of $\Lambda_l(d,s)$ are the dominating contributions to the entries of $\Gamma_k^{(2)}$ in Eq. (4.12) so that, in this limit, the above condition boils down to

$$\begin{aligned}0<(\Gamma_k^{(2)})_{\overline{h^T}\overline{h^T}}&\xrightarrow{l\to\infty}(Z_{Nk}\kappa^2-\overline{\beta}_kR)\Lambda_l(d,2)\\ &\Rightarrow Z_{Nk}\kappa^2-\overline{\beta}_kR>0\end{aligned}\tag{6.5}$$

$$\begin{aligned}0<(\Gamma_k^{(2)})_{\overline{\xi}\overline{\xi}}&\xrightarrow{l\to\infty}2\frac{Z_{Nk}\kappa^2}{\alpha}\Lambda_l(d,1)\\ &\Rightarrow \frac{Z_{Nk}\kappa^2}{\alpha}>0\end{aligned}\tag{6.6}$$

$$\begin{aligned}0<(\Gamma_k^{(2)})_{\overline{\phi}\overline{\phi}}&\xrightarrow{l\to\infty}2\left(\frac{d-1}{d}\right)^2\overline{\beta}_k(\Lambda_l(d,0))^2\\ &\Rightarrow \overline{\beta}_k>0\end{aligned}\tag{6.7}$$

$$\begin{aligned}0<(\Gamma_k^{(2)})_{\overline{\sigma}\overline{\sigma}}&(\Gamma_k^{(2)})_{\overline{\phi}\overline{\phi}}-(\Gamma_k^{(2)})^2_{\overline{\sigma}\overline{\phi}}\\ &\xrightarrow{l\to\infty}\left(\frac{d-1}{d}\right)^2\frac{Z_{Nk}\kappa^2\overline{\beta}_k}{\alpha}(\Lambda_l(d,0))^3>0\\ &\Rightarrow \frac{Z_{Nk}\kappa^2\overline{\beta}_k}{\alpha}>0.\end{aligned}\tag{6.8}$$

For non-negative values of the gauge parameter, $\alpha\geqslant 0$, this leads to the following restrictions on the dimensionless couplings:

$$g_k>0,\quad \beta_k>0,\quad k^2/(32\pi g_k\beta_k)>R.\tag{6.9}$$

In the UV fixed point regime of the $(d=4)$-dimensional case we have $g_k\approx g_*$ and $\beta_k\approx\beta_*$ with $g_*,\beta_*>0$. Hence, close to the non-Gaussian fixed point, the first two conditions of Eq. (6.9) are obviously satisfied. Furthermore, the third condition then takes the form $R<k^2/(32\pi g_*\beta_*)$. For R fixed this condition is satisfied as well provided k is sufficiently large. Thus, for k large and on modes with large eigenvalues of $-D^2$, the restricted operator $\Gamma_k^{(2)}$ is positive. The cutoff should have the desired suppression properties therefore.

The above argument treats R as a constant parameter. Recalling the derivation of the projected flow equation where we compared powers of the radius $r\propto R^{-1/2}$ it is indeed clear that in this context r and R should be regarded as fixed, k-independent quantities.

It is instructive to look also at the operator $\Gamma_k^{(2)}[g^{\rm os}(k),g^{\rm os}(k)]$ where $g^{\rm os}(k)$ is the k-dependent "on-shell" S^d metric which solves the equation of motion $\delta\Gamma_k/\delta g_{\mu\nu}=0$ for $g_{\mu\nu}=\overline{g}_{\mu\nu}$. The difference to the situation discussed before is that R is a function of k now, to be computed from $g^{\rm os}(k)$. (The operator $\Gamma_k^{(2)}[g^{\rm os}(k),g^{\rm os}(k)]$ would appear in a standard one-loop (saddle point) calculation based upon the "classical" action Γ_k.)

For the truncated action functional Γ_k of Eq. (4.2) with $\overline{g}_{\mu\nu}=g_{\mu\nu}$ the field equation takes the form

$$\begin{aligned}2Z_{Nk}\kappa^2[G_{\mu\nu}+g_{\mu\nu}\overline{\lambda}_k]+\overline{\beta}_k[-(G_{\mu\nu}+R_{\mu\nu})R\\ +2D_\mu D_\nu R-2g_{\mu\nu}D^2R]=0\end{aligned}\tag{6.10}$$

with $G_{\mu\nu}=R_{\mu\nu}-g_{\mu\nu}R/2$ the Einstein tensor. Precisely for $d=4$, the maximally symmetric solutions to Eq. (6.10) satisfy Einstein's equation $G_{\mu\nu}=-g_{\mu\nu}\overline{\lambda}_k$, they are not affected by the R^2 term. Inserting the contracted equation $R=4\overline{\lambda}_k$ into the third condition of Eq. (6.9) leads to

$$128\pi g_k\lambda_k\beta_k<1.\tag{6.11}$$

Remarkably, this condition is satisfied precisely if $\Gamma_k[g,g]$ is *not* a manifestly positive functional of $g_{\mu\nu}$, as follows from Eq. (6.4). This implies that for $d=4$ the S^4 solution of Eq. (6.10) cannot correspond to the absolute minimum of $\Gamma_k[g,g]$ if this functional is manifestly positive.

In the UV fixed point regime, the condition (6.11) becomes $128\pi g_*\lambda_*\beta_*<1$. For all cutoffs employed we found that $0.17\lesssim 128\pi g_*\lambda_*\beta_*\lesssim 0.22$ so that this condition is indeed satisfied. It is reassuring that also, upon inserting the S^4 solution of Eq. (6.10), the Hessian $\Gamma_k^{(2)}$ becomes a positive operator for sufficiently large values of l and k, independently of the cutoff.

Furthermore, the concomitant violation of Eq. (6.4) implies that in the vicinity of the fixed point the functional $\Gamma_k[g,g]$ is bounded below but not positive. By adding an appropriate constant it is trivial though to turn it into a manifestly positive functional.

C. Positivity of the cutoff

The cutoff Δ_kS is expected to be positive definite under the same conditions as found for the Hessian $\Gamma_k^{(2)}$ in the previous subsection. In order to obtain more quantitative information about $l_{\rm min}$ and the momentum regime where Δ_kS is positive we continue our analysis with an explicit investigation of the cutoff operator $\mathcal{R}_k$. For simplicity we again restrict our considerations to the most interesting case of $d=4$ and to spherical backgrounds.

After setting $(\mathcal{R}_k)^{\mu\nu\alpha\beta}_{\overline{h^T}\overline{h^T}}\equiv 1/2(g^{\mu\alpha}g^{\nu\beta}+g^{\mu\beta}g^{\nu\alpha})(\mathcal{R}_k)_{\overline{h^T}\overline{h^T}}$ and $(\mathcal{R}_k)^{\mu\nu}_{\overline{\xi}\overline{\xi}}\equiv g^{\mu\nu}(\mathcal{R}_k)_{\overline{\xi}\overline{\xi}}$, and inserting the eigenvalues of the covariant Laplacians, the entries of the cutoff matrix (3.8) assume the form

$$
\begin{aligned}
(\mathcal{R}_k)_{\overline{h^T}\overline{h^T}} &= \frac{k^4 R^{(0)}(\Lambda_l(4,2)/k^2)}{32\pi g_k}\left\{1-32\pi g_k\beta_k\frac{R}{k^2}\right\},\\
(\mathcal{R}_k)_{\overline{\xi}\overline{\xi}} &= \frac{k^4 R^{(0)}(\Lambda_l(4,1)/k^2)}{16\pi g_k\alpha},\\
(\mathcal{R}_k)_{\overline{\sigma}\overline{\sigma}} &= \frac{k^4 R^{(0)}(\Lambda_l(4,0)/k^2)}{32\pi g_k}\left\{36\pi g_k\beta_k(2\Lambda_l(4,0)/k^2+R^{(0)}(\Lambda_l(4,0)/k^2))+\frac{3}{4}\right\},\\
(\mathcal{R}_k)_{\overline{\phi}\overline{\sigma}} &= (\mathcal{R}_k)^{\dagger}_{\sigma\phi_1}=\frac{9}{8}\beta_k k^4\Bigg\{[\Lambda_l(4,0)/k^2+R^{(0)}(\Lambda_l(4,0)/k^2)]^{3/2}\sqrt{\Lambda_l(4,0)/k^2+R^{(0)}(\Lambda_l(4,0)/k^2)-\frac{R}{3k^2}}\\
&\qquad -[\Lambda_l(4,0)/k^2]^{3/2}\sqrt{\Lambda_l(4,0)/k^2-\frac{R}{3k^2}}\Bigg\},\\
(\mathcal{R}_k)_{\overline{\phi}\overline{\phi}} &= \frac{k^4 R^{(0)}(\Lambda_l(4,0)/k^2)}{32\pi g_k}\left\{36\pi g_k\beta_k(2\Lambda_l(4,0)/k^2+R^{(0)}(\Lambda_l(4,0)/k^2))-\frac{1}{4}\right\}.
\end{aligned}
\tag{6.12}
$$

As compared to Eq. (3.8) which was written in terms of the dimensionful quantities $Z_{Nk}\kappa^2$, $\bar{\lambda}_k$, and $\bar{\beta}_k$, we switched here to a description in terms of the dimensionless couplings.

In analogy with the Hessian $\Gamma_k^{(2)}$ the condition for the cutoff matrix $\mathcal{R}_k$ to be positive definite reads $(\mathcal{R}_k)_{\overline{h^T}\overline{h^T}}>0$, $(\mathcal{R}_k)_{\overline{\xi}\overline{\xi}}>0$, $(\mathcal{R}_k)_{\overline{\phi}\overline{\phi}}>0$, and $(\mathcal{R}_k)_{\overline{\phi}\overline{\phi}}(\mathcal{R}_k)_{\overline{\sigma}\overline{\sigma}}-(\mathcal{R}_k)^2_{\overline{\phi}\overline{\sigma}}>0$. These conditions indeed reproduce the restrictions on the couplings obtained from $\Gamma_k^{(2)}$ for sufficiently large momenta $p^2=\Lambda_l(d,s)$. Provided that $\alpha\geqslant 0$, they take the form $g_k>0$, $\beta_k>0$ and $k^2/(32\pi g_k\beta_k)>R$, which coincides with Eq. (6.9).

Given an arbitrary set of parameters (R,k,g_k,β_k) satisfying these three inequalities, we have $(\mathcal{R}_k)_{\overline{h^T}\overline{h^T}}>0$ and $(\mathcal{R}_k)_{\overline{\xi}\overline{\xi}}>0$ [and also $(\mathcal{R}_k)_{\overline{\sigma}\overline{\sigma}}>0$] for *any* allowed value of l. This is not the case for the other two conditions which stem from the scalar sector of the cutoff. Clearly $(\mathcal{R}_k)_{\overline{\phi}\overline{\phi}}$ of Eq. (6.12) can assume negative values for sufficiently small values of l, provided g_k, β_k and R/k^2 are small enough. Since $(\mathcal{R}_k)_{\overline{\sigma}\overline{\sigma}}>0$, a negative $(\mathcal{R}_k)_{\overline{\phi}\overline{\phi}}$ implies that also $(\mathcal{R}_k)_{\overline{\phi}\overline{\phi}}(\mathcal{R}_k)_{\overline{\sigma}\overline{\sigma}}-(\mathcal{R}_k)^2_{\overline{\phi}\overline{\sigma}}<0$.

The l values for which $(\mathcal{R}_k)_{\overline{\phi}\overline{\phi}}>0$ satisfy

$$2\Lambda_l(4,0)/k^2+R^{(0)}(\Lambda_l(4,0)/k^2)>\frac{1}{144\pi g_k\beta_k}. \tag{6.13}$$

In Appendix G we derive a similar inequality involving the $\bar{\phi}$-$\bar{\sigma}$ cross term. There we find that $(\mathcal{R}_k)_{\overline{\phi}\overline{\phi}}(\mathcal{R}_k)_{\overline{\sigma}\overline{\sigma}}-(\mathcal{R}_k)^2_{\overline{\phi}\overline{\sigma}}>0$ at least for all values of l satisfying

$$2\Lambda_l(4,0)/k^2+R^{(0)}(\Lambda_l(4,0)/k^2)>\frac{1}{96\pi g_k\beta_k}. \tag{6.14}$$

Both inequalities, Eqs. (6.13) and (6.14), depend on $g_k\beta_k$, on p^2/k^2 with $p^2=\Lambda_l(4,0)=l(l+3)R/12$, and on the shape function $R^{(0)}$. Given a specific set $(R,k,g_k,\beta_k;R^{(0)};T^{lm})$ with a certain scalar eigenmode T^{lm}, they tell us whether the contribution from this mode is suppressed correctly or not. The more restrictive inequality (6.14) applies to the scalar eigenmodes $\{T^{lm}\}$ with $l\geqslant 2$, while Eq. (6.13) can be used for the constant mode $T^{l=0,m=1}$ and the PCKV's $\{T^{l=1,m}\}$ only.

Let us now focus on RG trajectories which run into the non-Gaussian fixed point as $k\to\infty$. Furthermore, we assume that R is either kept fixed or that $R=a_k k^2$ with a constant $a_k<(32\pi g_k\beta_k)^{-1}$. Then, for large enough values of k, we have $g_k\approx g_*>0$, $\beta_k\approx\beta_*>0$ and $k^2/(32\pi g_*\beta_*)>R$ so that the conditions (6.9) for the positivity of $\Gamma_k^{(2)}$ and $\mathcal{R}_k$ are satisfied. Moreover, the RHS of Eqs. (6.13) and (6.14) may be expressed as $(144\pi g_*\beta_*)^{-1}$ and $(96\pi g_*\beta_*)^{-1}$, respectively.

Now we are in a position to determine the l regime for which the cutoff is manifestly positive definite. Using the family of exponential shape functions (3.11) with $1\leqslant s\leqslant 30$, a numerical analysis reveals that *any* value of the ratio $x\equiv p^2/k^2=\Lambda_l(4,0)/k^2$ satisfies Eq. (6.13) or (6.14) provided $s\gtrsim 2$ or $s\gtrsim 7$, respectively. Hence, under the above conditions, all cutoffs employing an exponential shape function with $s\gtrsim 7$ are manifestly positive definite for *all* momenta, i.e. for all quantum numbers l.

This is a rather intriguing result. It might indicate that cutoffs with $s>7$ are particularly reliable.

Conversely, for any $s\lesssim 7$ there exists a specific value $x_0(s)$ such that all x with $x\leqslant x_0(s)$ violate Eq. (6.14). Furthermore, there exists a specific $x_1(s)$ for any $s\lesssim 2$ such that any $x\leqslant x_1(s)$ leads to a violation of Eq. (6.13). In Fig. 8 we show $x_0(s)$ and $x_1(s)$ in the ranges $1\leqslant s\leqslant 6$ and $1\leqslant s\leqslant 2$, respectively. It is important to note that $x_0(s)<0.7$ and $x_1(s)<0.26$ for any value of s considered. This implies that in the UV fixed point regime the cutoff has the desired suppression properties for all modes with momenta ranging from infinity down to values well below k.

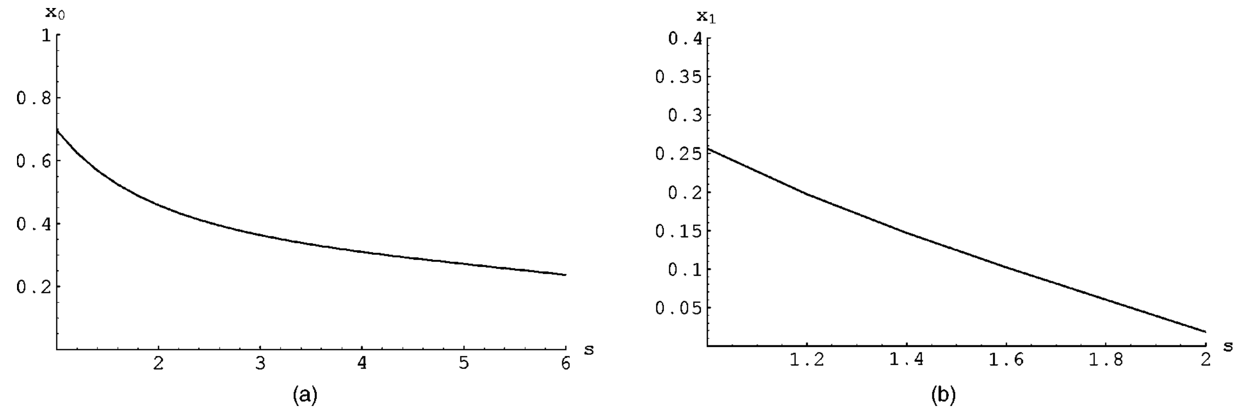


FIG. 8. (a) x_0, and (b) x_1 as functions of s, using the family of exponential shape functions.

To complete the analysis let us study the inequalities (6.13) and (6.14) also at the spherically symmetric stationary point $g^{\rm os}(k)$ of $\Gamma_k[g,g]$ which we discussed in the previous subsection. In the vicinity of the fixed point the on-shell value of the curvature is $R\approx 4\lambda_* k^2$. Hence, we obtain from Eqs. (6.13) and (6.14), respectively,

$$f_1[R^{(0)};l]\equiv\frac{2l(l+3)\lambda_*}{3}+R^{(0)}(l(l+3)\lambda_*/3)-\frac{1}{144\pi g_*\beta_*}>0,\quad l=0,1 \tag{6.15}$$

and

$$f_0[R^{(0)};l]\equiv\frac{2l(l+3)\lambda_*}{3}+R^{(0)}(l(l+3)\lambda_*/3)-\frac{1}{96\pi g_*\beta_*}>0,\quad l\geqslant 2. \tag{6.16}$$

The first inequality stems from the scalar eigenmodes T^{lm} with $l=0,1$, and the second from those with $l\geqslant 2$. Both Eqs. (6.15) and (6.16) depend on l and $R^{(0)}$.

Again we restrict our investigation to the family of exponential shape functions with $1\leqslant s\leqslant 30$. Then the LHS of Eqs. (6.15) and (6.16) are functions of s and l: $f_1[R^{(0)};l]\equiv f_1(s,l)$, $f_0[R^{(0)};l]\equiv f_0(s,l)$. For $l\geqslant 2$ we have $f_0(s;l)\geqslant f_0(s;l=2)$ independently of the shape parameter. Numerically we find that $f_0(s;l=2)$ is always positive. Hence, *any* momentum with $l\geqslant 2$ satisfies the condition (6.16) for all values of s considered. Furthermore, our numerical analysis shows that also $f_1(s,l=1)>0$ for all cutoffs employed. However, $f_1(s,l=0)$ is not always positive. We obtain $f_1(s,l=0)>0$ for $s\gtrsim 2$ and $f_1(s,l=0)<0$ for $s\lesssim 2$. Our results are illustrated in Fig. 9.

The above results have to be interpreted as follows. Assume that k lies in the UV scaling regime, and consider the cutoff operator at the spherically symmetric stationary point, $\mathcal{R}_k[g^{\rm os}(k)]$. Then this operator is strictly positive on the space spanned by *all* spherical harmonics and would correctly suppress all modes in a path integral containing this "on-shell" cutoff, provided we choose an exponential shape function with $s\gtrsim 2$. For $s\lesssim 2$, only the contributions from the constant mode with $p^2=0$ are not suppressed correctly. For any other mode the cutoff term is positive even in this case.

To summarize. In this section we found that at least in the asymptotic domain relevant in our investigation of the UV fixed point the cutoff which is adapted to the R^2 truncation is positive definite and therefore has all the required mode suppression properties. No conformal factor problem and no growing exponentials produced by $\exp(-\Delta_k S)$ are encountered.

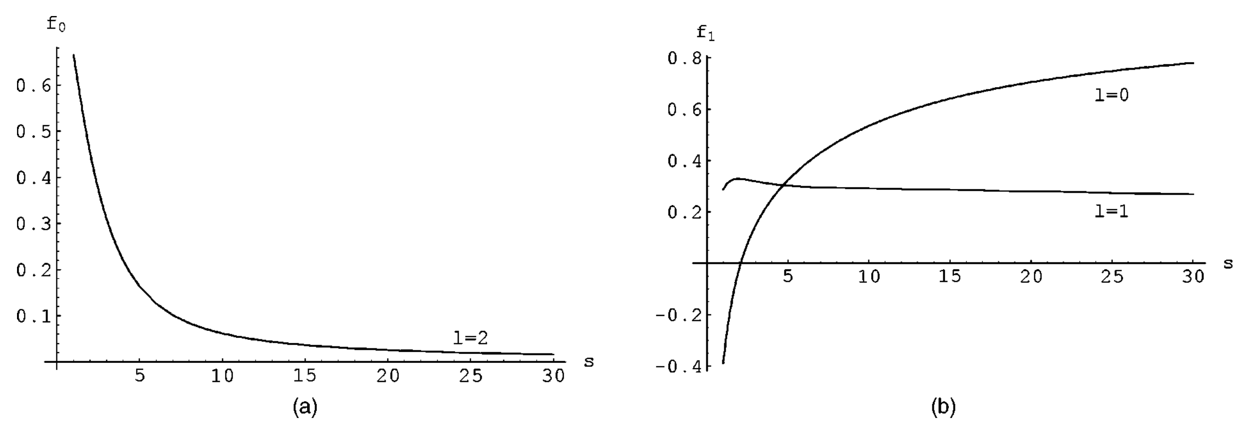


FIG. 9. (a) $f_0(s,l=2)$, and (b) $f_1(s,l=0)$ and $f_1(s,l=1)$ as functions of s, using the family of exponential shape functions.

VII. SUMMARY AND CONCLUSION

In this paper we evaluated the exact RG equation of quantum Einstein gravity in a truncation which generalizes the Einstein-Hilbert approximation used so far by the inclusion of a higher-derivative term. We derived the beta-functions of the resulting λ-g-β system which turned out to be by far more complicated than the old λ-g system. We used these beta functions in order to investigate how the two fixed points known to exist in the Einstein-Hilbert truncation manifest themselves in the enlarged theory space.

We found that the Gaussian fixed point of the Einstein-Hilbert truncation does not generalize to a corresponding fixed point of the R^2 truncation. Nevertheless, the 2-dimensional projection of the λ-g-β flow onto the λ-g plane at $\beta=0$, near the origin $\lambda=g=\beta=0$, is well approximated by the flow resulting from the Einstein-Hilbert truncation. The projected flow does indeed have a fixed point at $\lambda_*=g_*=0$. In the Einstein-Hilbert truncation there exists a distinguished RG trajectory, the "separatrix" [5], which gives rise to a vanishing renormalized cosmological constant, $\lim_{k\to 0}\bar{\lambda}_k=0$. In $d=4$, it turned out that this trajectory possesses a 3-dimensional "lift" which is characterized by a logarithmic running of the R^2 coupling β_k. For $d\neq 4$ its running is power-like, and there exists a "quasi-Gaussian" fixed point at $\lambda_*=g_*=0$, $\beta_*\neq 0$. This picture puts the older perturbative calculations in R^2 gravity into a broader context.

Quite differently, the non-Gaussian fixed point $(\lambda_*,g_*)\neq(0,0)$ implied by the Einstein-Hilbert truncation does "lift" to a corresponding fixed point of the 3-dimensional flow, with a tiny but nonzero third component $\beta_*>0$. It is UV attractive in all 3 directions of λ-g-β space. We demonstrated in detail that close to the fixed point the flow on the extended theory space is essentially 2-dimensional, and that the 2-dimensional projected flow is very well approximated by the Einstein-Hilbert flow. For the λ_* and g_* coordinates both truncations yield virtually identical values, and the same is true for the critical exponents pertaining to the 2-dimensional subspace. For universal quantities the differences between the two truncations are typically smaller than their weak residual scheme dependence.

This stability of the Einstein-Hilbert truncation against the inclusion of a further invariant, together with the other pieces of evidence which we summarized in Secs. V C 1 and V C 2 strongly support the conjecture that this approximation is at least qualitatively reliable in the UV. Hence it appears increasingly unlikely that the very existence of the non-Gaussian fixed point is an artifact of the truncation. We believe that QEG has indeed very good chances of being nonperturbatively renormalizable.

A notorious difficulty of Euclidean quantum gravity is the conformal factor problem. In the exact RG approach, it appears in the Einstein-Hilbert truncation, but not in the R^2 truncation provided k is large enough. When this complication occurs the construction of an appropriate cutoff operator is rather subtle. However, it was possible to show that our investigation of the non-Gaussian fixed point in the R^2 truncation is not affected by this problem, and that a straightforward positive definite cutoff can be employed. The numerical agreement of the results with those from the Einstein-Hilbert truncation indicates that the rule for constructing an adapted cutoff in the presence of the conformal factor problem which was proposed in [1] ("$\mathcal{Z}_k=z_k$ rule") should indeed be correct.

ACKNOWLEDGMENTS

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APPENDIX A: EVALUATING THE RHS OF THE TRUNCATED FLOW EQUATION

In this section we present several rather lengthy calculations needed for the discussion of the R^2 truncation in Sec. IV. In the following, all calculations are performed with $g_{\mu\nu}=\bar{g}_{\mu\nu}$ where $\bar{g}_{\mu\nu}$ is assumed to correspond to a S^d background. For simplicity the bars are omitted from the metric, the curvature and the operators.

1. Computation of $(\Gamma_k^{(2)}+\mathcal{R}_k)^{-1}$ and $(S_{\mathrm{gh}}^{(2)}+\mathcal{R}_k)^{-1}$

In Sec. IV we derived explicit expressions for the kinetic operators $\tilde{\Gamma}_k^{(2)}\equiv\Gamma_k^{(2)}+\mathcal{R}_k$ and $\tilde{S}_{\mathrm{gh}}^{(2)}\equiv S_{\mathrm{gh}}^{(2)}+\mathcal{R}_k$. They may be represented as matrix differential operators acting on the column vectors $(\bar{h}^T,\bar{\xi},\bar{\phi}_0,\bar{\sigma},\bar{\phi}_1)^{\mathbf{T}}$ and $(\bar{v}^T,v^T,\bar{\varrho},\varrho)^{\mathbf{T}}$, respectively. In this representation they take the form

$$\tilde{\Gamma}_k^{(2)}[g,g]=\begin{pmatrix} (\tilde{\Gamma}_k^{(2)}[g,g])_{\bar{h}^T\bar{h}^T} & 0 & 0 & 0_{1\times 2} \\ 0 & (\tilde{\Gamma}_k^{(2)}[g,g])_{\bar{\xi}\bar{\xi}} & 0 & 0_{1\times 2} \\ 0 & 0 & (\tilde{\Gamma}_k^{(2)}[g,g])_{\bar{\phi}_0\bar{\phi}_0} & 0_{1\times 2} \\ 0_{2\times 1} & 0_{2\times 1} & 0_{2\times 1} & \mathcal{Q}_k \end{pmatrix} \tag{A1}$$

and

$$\widetilde{S}^{(2)}_{\rm gh}[g,g]=\begin{pmatrix} 0 & (\widetilde{S}^{(2)}_{\rm gh}[g,g])_{\overline{v}^T v^T} & 0 & 0 \\ (\widetilde{S}^{(2)}_{\rm gh}[g,g])_{v^T\overline{v}^T} & 0 & 0 & 0 \\ 0 & 0 & 0 & (\widetilde{S}^{(2)}_{\rm gh}[g,g])_{\overline{\varrho}\varrho} \\ 0 & 0 & (\widetilde{S}^{(2)}_{\rm gh}[g,g])_{\varrho\overline{\varrho}} & 0 \end{pmatrix} \tag{A2}$$

where

$$\mathcal{Q}_k\equiv\begin{pmatrix} (\widetilde{\Gamma}^{(2)}_k[g,g])_{\overline{\sigma\sigma}} & (\widetilde{\Gamma}^{(2)}_k[g,g])_{\overline{\phi}_1\overline{\sigma}} \\ (\widetilde{\Gamma}^{(2)}_k[g,g])_{\overline{\phi}_1\overline{\sigma}} & (\widetilde{\Gamma}^{(2)}_k[g,g])_{\overline{\phi\phi}} \end{pmatrix}. \tag{A3}$$

The entries of these matrices are given in Eq. (4.15). On the RHS of the flow equation (3.4) the inverse operators $[\widetilde{\Gamma}^{(2)}_k]^{-1}$ and $[\widetilde{S}^{(2)}_{\rm gh}]^{-1}$ appear which we determine in the following. At this point it is important to note that, because of the maximally symmetric background, all covariant derivatives contained in the operators (A1) and (A2) appear as covariant Laplacians, and that the various entries are x independent otherwise. This implies that these entries are *commuting* differential operators which allows for particularly simple manipulations. Therefore it is not difficult to verify that the inverse operators assume the form

$$(\widetilde{\Gamma}^{(2)}_k[g,g])^{-1}=\begin{pmatrix} [(\widetilde{\Gamma}^{(2)}_k[g,g])_{\overline{h}^T\overline{h}^T}]^{-1} & 0 & 0 & 0_{1\times 2} \\ 0 & [(\widetilde{\Gamma}^{(2)}_k[g,g])_{\overline{\xi\xi}}]^{-1} & 0 & 0_{1\times 2} \\ 0 & 0 & [(\widetilde{\Gamma}^{(2)}_k[g,g])_{\overline{\phi}_0\overline{\phi}_0}]^{-1} & 0_{1\times 2} \\ 0_{2\times 1} & 0_{2\times 1} & 0_{2\times 1} & \mathcal{Q}_k^{-1} \end{pmatrix} \tag{A4}$$

and

$$(\widetilde{S}^{(2)}_{\rm gh}[g,g])^{-1}=\begin{pmatrix} 0 & [(\widetilde{S}^{(2)}_{\rm gh}[g,g])_{\overline{v}^T v^T}]^{-1} & 0 & 0 \\ [(\widetilde{S}^{(2)}_{\rm gh}[g,g])_{v^T\overline{v}^T}]^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & [(\widetilde{S}^{(2)}_{\rm gh}[g,g])_{\overline{\varrho}\varrho}]^{-1} \\ 0 & 0 & [(\widetilde{S}^{(2)}_{\rm gh}[g,g])_{\varrho\overline{\varrho}}]^{-1} & 0 \end{pmatrix} \tag{A5}$$

with

$$\mathcal{Q}_k^{-1}=[(\widetilde{\Gamma}^{(2)}_k[g,g])_{\overline{\sigma\sigma}}(\widetilde{\Gamma}^{(2)}_k[g,g])_{\overline{\phi}_1\overline{\phi}_1}-(\widetilde{\Gamma}^{(2)}_k[g,g])^2_{\overline{\phi}_1\overline{\sigma}}]^{-1}\begin{pmatrix} (\widetilde{\Gamma}^{(2)}_k[g,g])_{\overline{\phi}_1\overline{\phi}_1} & -(\widetilde{\Gamma}^{(2)}_k[g,g])_{\overline{\phi}_1\overline{\sigma}} \\ -(\widetilde{\Gamma}^{(2)}_k[g,g])_{\overline{\phi}_1\overline{\sigma}} & (\widetilde{\Gamma}^{(2)}_k[g,g])_{\overline{\sigma\sigma}} \end{pmatrix}. \tag{A6}$$

Inserting these expressions into the RHS of the flow equation (3.4) leads to

$$\begin{aligned}\mathcal{S}_k(R)=&\frac{1}{2}\mathrm{Tr}'\Bigg[\sum_{\zeta\in\{\overline{h}^T,\overline{\xi},\overline{\phi}_0\}}[(\Gamma^{(2)}_k[g,g]+\mathcal{R}_k[g])_{\zeta\zeta}]^{-1}\partial_t(\mathcal{R}_k[g])_{\zeta\zeta}\Bigg]+\frac{1}{2}\mathrm{Tr}'[\{(\widetilde{\Gamma}^{(2)}_k[g,g])_{\overline{\sigma\sigma}}(\widetilde{\Gamma}^{(2)}_k[g,g])_{\overline{\phi}_1\overline{\phi}_1}\\ &-(\widetilde{\Gamma}^{(2)}_k[g,g])^2_{\overline{\phi}_1\overline{\sigma}}\}^{-1}\{(\Gamma^{(2)}_k[g,g]+\mathcal{R}_k[g])_{\overline{\sigma\sigma}}\partial_t(\mathcal{R}_k[g])_{\overline{\phi}_1\overline{\phi}_1}+(\Gamma^{(2)}_k[g,g]+\mathcal{R}_k[g])_{\overline{\phi}_1\overline{\phi}_1}\partial_t(\mathcal{R}_k[g])_{\overline{\sigma\sigma}}\\ &-2(\Gamma^{(2)}_k[g,g]+\mathcal{R}_k[g])_{\overline{\phi}_1\overline{\sigma}}\partial_t(\mathcal{R}_k[g])_{\overline{\phi}_1\overline{\sigma}}\}]-\mathrm{Tr}'\Bigg[\sum_{\psi\in\{v^T,\varrho\}}[(S^{(2)}_{\rm gh}[g,g]+\mathcal{R}_k[g])_{\overline{\psi}\psi}]^{-1}\partial_t(\mathcal{R}_k[g])_{\overline{\psi}\psi}\Bigg]\end{aligned} \tag{A7}$$

where we used the relations

$$\begin{aligned}{}[(S^{(2)}_{\rm gh})_{\overline{v}^T v^T}]^{\mu x}{}_{\nu y}=-[(S^{(2)}_{\rm gh})_{v^T\overline{v}^T}]_{\nu y}{}^{\mu x}=\frac{1}{\sqrt{g(y)}}\frac{\delta}{\delta v^{T\nu}(y)}\frac{1}{\sqrt{g(x)}}\frac{\delta S_{\rm gh}}{\delta\overline{v}^T_\mu(x)}\\ [(S^{(2)}_{\rm gh})_{\overline{\varrho}\varrho}]^x{}_y=-[(S^{(2)}_{\rm gh})_{\varrho\overline{\varrho}}]_y{}^x=\frac{1}{\sqrt{g(y)}}\frac{\delta}{\delta\varrho(y)}\frac{1}{\sqrt{g(x)}}\frac{\delta S_{\rm gh}}{\delta\overline{\varrho}(x)}.\end{aligned} \tag{A8}$$

The trace of the ϕ_0 term appearing in the first line of Eq. (A7) may be evaluated easily since only the scalar eigenmodes T^{01} and T^{1m} contribute. We obtain

$$\frac{1}{2}\mathrm{Tr}'[[(\Gamma_k^{(2)}[g,g]+\mathcal{R}_k[g])_{\phi_0\phi_0}]^{-1}\partial_t(\mathcal{R}_k[g])_{\phi_0\phi_0}]$$
$$=\frac{1}{2Z_{Nk}\kappa^2}\sum_{l=0}^{1}\sum_{m=1}^{D_l(d,0)}\int d^dx\,\sqrt{g(x)}T^{lm}(x)[C_{S2}(d,\alpha)C_{S1}(d,\alpha)(P_k+A_{S1}(d,\alpha)R+B_{S1}(d,\alpha)\overline{\lambda}_k)$$
$$+(Z_{Nk}\kappa^2)^{-1}\overline{\beta}_k(H_S(d)P_k^2+G_{S2}(d)RP_k+G_{S3}(d)R^2)]^{-1}\partial_t[C_{S2}(d,\alpha)C_{S1}(d,\alpha)Z_{Nk}\kappa^2k^2R^{(0)}(-D^2/k^2)$$
$$+\overline{\beta}_k(H_S(d)[-2D^2R^{(0)}(-D^2/k^2)+k^4R^{(0)}(-D^2/k^2)^2]+G_{S2}(d)Rk^2R^{(0)}(-D^2/k^2))]T^{lm}(x)$$
$$=\frac{1}{2Z_{Nk}\kappa^2}\sum_{l=0}^{1}\{D_l(d,0)\{C_{S2}(d,\alpha)C_{S1}(d,\alpha)(\Lambda_l(d,0)+k^2R^{(0)}(\Lambda_l(d,0)/k^2)+A_{S1}(d,\alpha)R+B_{S1}(d,\alpha)\overline{\lambda}_k)$$
$$+(Z_{Nk}\kappa^2)^{-1}\overline{\beta}_k[H_S(d)(\Lambda_l(d,0)+k^2R^{(0)}(\Lambda_l(d,0)/k^2))^2+G_{S2}(d)R(\Lambda_l(d,0)+k^2R^{(0)}(\Lambda_l(d,0)/k^2))+G_{S3}(d)R^2]\}^{-1}$$
$$\times\partial_t\{C_{S2}(d,\alpha)C_{S1}(d,\alpha)Z_{Nk}\kappa^2k^2R^{(0)}(\Lambda_l(d,0)/k^2)+\overline{\beta}_k[H_S(d)(2\Lambda_l(d,0)k^2R^{(0)}(\Lambda_l(d,0)/k^2)+k^4R^{(0)}(\Lambda_l(d,0)/k^2)^2)$$
$$+G_{S2}(d)Rk^2R^{(0)}(\Lambda_l(d,0)/k^2)]\}\}. \tag{A9}$$

Here $\Lambda_l(d,0)$ is the eigenvalue with respect to $-D^2$ corresponding to T^{lm}. Inserting also the remaining entries given in Eq. (4.15) into Eq. (A7) finally leads to Eq. (4.16).

2. Heat kernel expansion and evaluation of the traces

In this part of the appendix we expand $\mathcal{S}_k(R)$ of Eq. (4.16) with respect to r and evaluate the traces appearing in the resulting equation (A10) below by applying the heat kernel expansion. Thereby we extract the contributions proportional to $\int d^dx\sqrt{g}$, $\int d^dx\sqrt{g}R$ and $\int d^dx\sqrt{g}R^2$. This puts us in a position to read off the RG equations for the three couplings. For technical convenience we restrict our considerations to the gauge $\alpha=1$.

We start our evaluation of $\mathcal{S}_k(R)$, Eq. (4.16), by expanding it with respect to $R\propto r^{-2}$. Since we are only interested in the contributions proportional to $\int d^dx\sqrt{g}\propto r^d$, $\int d^dx\sqrt{g}R\propto r^{d-2}$ and $\int d^dx\sqrt{g}R^2\propto r^{d-4}$, only terms of order r^d, r^{d-2} and r^{d-4} are needed. This leads to

$$\mathcal{S}_k(R)=\mathrm{Tr}_{(2ST^2)}[\mathcal{A}_1^{-1}\mathcal{N}]+\mathrm{Tr}'_{(1T)}[\mathcal{A}_1^{-1}\mathcal{N}]-h_1(d)\mathrm{Tr}''_{(0)}[\mathcal{A}_2^{-1}\mathcal{N}]+a_k\mathrm{Tr}''_{(0)}[\mathcal{A}_2^{-1}\mathcal{T}_2]+a_k\mathrm{Tr}''_{(0)}[(\mathcal{A}_1\mathcal{A}_2)^{-1}P_k^2\mathcal{N}]$$
$$-2\mathrm{Tr}_{(1T)}[P_k^{-1}\mathcal{N}_0]-2\mathrm{Tr}'_{(0)}[\mathrm{P_k^{-1}}\mathcal{N}_0]+\Big\{-a_k\mathrm{Tr}_{(2ST^2)}[\mathcal{A}_1^{-1}\mathcal{T}_1]+a_k\mathrm{Tr}_{(2ST^2)}[\mathcal{A}_1^{-2}P_k\mathcal{N}]-A_T(d)\mathrm{Tr}_{(2ST^2)}[\mathcal{A}_1^{-2}\mathcal{N}]$$
$$-A_V(d,1)\mathrm{Tr}'_{(1T)}[\mathcal{A}_1^{-2}\mathcal{N}]+h_2(d)a_k\mathrm{Tr}''_{(0)}[(\mathcal{A}_1\mathcal{A}_2)^{-1}\mathcal{T}_2]+h_3(d)a_k\mathrm{Tr}''_{(0)}[(\mathcal{A}_1\mathcal{A}_2)^{-1}P_k\mathcal{N}]+h_4(d)\mathrm{Tr}''_{(0)}[(\mathcal{A}_1\mathcal{A}_2)^{-1}\mathcal{N}]$$
$$+h_3(d)a_k\mathrm{Tr}''_{(0)}[\mathcal{A}_2^{-1}\mathcal{T}_1]-h_2(d)a_k^2\mathrm{Tr}''_{(0)}[\mathcal{A}_1^{-1}\mathcal{A}_2^{-2}P_k^2\mathcal{T}_2]-h_3(d)a_k^2\mathrm{Tr}''_{(0)}[\mathcal{A}_2^{-2}P_k\mathcal{T}_2]-h_4(d)a_k\mathrm{Tr}''_{(0)}[\mathcal{A}_2^{-2}\mathcal{T}_2]$$
$$-h_2(d)a_k^2\mathrm{Tr}''_{(0)}[(\mathcal{A}_1\mathcal{A}_2)^{-2}P_k^4\mathcal{N}]-h_3(d)a_k^2\mathrm{Tr}''_{(0)}[\mathcal{A}_1^{-1}\mathcal{A}_2^{-2}P_k^3\mathcal{N}]-2h_4(d)a_k\mathrm{Tr}''_{(0)}[\mathcal{A}_1^{-1}\mathcal{A}_2^{-2}P_k^2\mathcal{N}]$$
$$+h_1(d)h_3(d)a_k\mathrm{Tr}''_{(0)}[\mathcal{A}_2^{-2}P_k\mathcal{N}]+h_1(d)h_4(d)\mathrm{Tr}''_{(0)}[\mathcal{A}_2^{-2}\mathcal{N}]-\frac{2}{d}\mathrm{Tr}_{(1T)}[P_k^{-2}\mathcal{N}_0]-\frac{4}{d}\mathrm{Tr}'_{(0)}[P_k^{-2}\mathcal{N}_0]$$
$$+\frac{\delta_{d,2}}{4\pi}[(\overline{\beta}_kk^4)^{-1}\partial_t(\overline{\beta}_kk^4)]\int d^dx\sqrt{g}\Big\}R+\Big\{-a_k^2\mathrm{Tr}_{(2ST^2)}[\mathcal{A}_1^{-2}P_k\mathcal{T}_1]+a_kA_T(d)\mathrm{Tr}_{(2ST^2)}[\mathcal{A}_1^{-2}\mathcal{T}_1]$$
$$-G_T(d)a_k\mathrm{Tr}_{(2ST^2)}[\mathcal{A}_1^{-2}\mathcal{N}]+a_k^2\mathrm{Tr}_{(2ST^2)}[\mathcal{A}_1^{-3}P_k^2\mathcal{N}]-2A_T(d)a_k\mathrm{Tr}_{(2ST^2)}[\mathcal{A}_1^{-3}P_k\mathcal{N}]+A_T(d)^2\mathrm{Tr}_{(2ST^2)}[\mathcal{A}_1^{-3}\mathcal{N}]$$
$$-G_V(d)a_k\mathrm{Tr}'_{(1T)}[\mathcal{A}_1^{-2}\mathcal{N}]+A_V(d,1)^2\mathrm{Tr}'_{(1T)}[\mathcal{A}_1^{-3}\mathcal{N}]-h_5(d)a_k^2\mathrm{Tr}''_{(0)}[(\mathcal{A}_1\mathcal{A}_2)^{-1}\mathcal{T}_2]+h_6(d)a_k\mathrm{Tr}''_{(0)}[(\mathcal{A}_1\mathcal{A}_2)^{-1}\mathcal{N}]$$
$$-\frac{2}{d^2}a_k^2\mathrm{Tr}''_{(0)}[(\mathcal{A}_1\mathcal{A}_2)^{-1}P_k\mathcal{T}_1]+h_2(d)h_3(d)a_k\mathrm{Tr}''_{(0)}[(\mathcal{A}_1\mathcal{A}_2)^{-1}\mathcal{T}_1]-h_2(d)^2a_k^2\mathrm{Tr}''_{(0)}[(\mathcal{A}_1\mathcal{A}_2)^{-2}P_k^2\mathcal{T}_2]$$
$$+\frac{1}{2}h_2(d)a_k^3\mathrm{Tr}''_{(0)}[\mathcal{A}_1^{-1}\mathcal{A}_2^{-2}P_k^2\mathcal{T}_2]-2h_2(d)h_3(d)a_k^2\mathrm{Tr}''_{(0)}[\mathcal{A}_1^{-1}\mathcal{A}_2^{-2}P_k\mathcal{T}_2]-\frac{3}{2}h_2(d)h_4(d)a_k\mathrm{Tr}''_{(0)}[\mathcal{A}_1^{-1}\mathcal{A}_2^{-2}\mathcal{T}_2]$$

$$-h_6(d)a_k^2\mathrm{Tr}''_{(0)}[\mathcal{A}_2^{-2}\mathcal{T}_2]+\frac{1}{2}h_2(d)a_k^3\mathrm{Tr}''_{(0)}[(\mathcal{A}_1\mathcal{A}_2)^{-2}P_k^4\mathcal{N}]-2h_2(d)h_3(d)a_k^2\mathrm{Tr}''_{(0)}[(\mathcal{A}_1\mathcal{A}_2)^{-2}P_k^3\mathcal{N}]$$
$$-\frac{3}{2}h_2(d)h_4(d)a_k\mathrm{Tr}''_{(0)}[(\mathcal{A}_1\mathcal{A}_2)^{-2}P_k^2\mathcal{N}]-h_7(d)a_k^2\mathrm{Tr}''_{(0)}[\mathcal{A}_1^{-1}\mathcal{A}_2^{-2}P_k^2\mathcal{N}]-3h_3(d)h_4(d)a_k\mathrm{Tr}''_{(0)}[\mathcal{A}_1^{-1}\mathcal{A}_2^{-2}P_k\mathcal{N}]$$
$$-\frac{3}{2}h_4(d)^2\mathrm{Tr}''_{(0)}[\mathcal{A}_1^{-1}\mathcal{A}_2^{-2}\mathcal{N}]+h_1(d)h_6(d)a_k\mathrm{Tr}''_{(0)}[\mathcal{A}_2^{-2}\mathcal{N}]-h_2(d)h_3(d)a_k^2\mathrm{Tr}''_{(0)}[\mathcal{A}_1^{-1}\mathcal{A}_2^{-2}P_k^2\mathcal{T}_1]$$
$$-h_3(d)^2a_k^2\mathrm{Tr}''_{(0)}[\mathcal{A}_2^{-2}P_k\mathcal{T}_1]-h_3(d)h_4(d)a_k\mathrm{Tr}''_{(0)}[\mathcal{A}_2^{-2}\mathcal{T}_1]+h_2(d)^2a_k^3\mathrm{Tr}''_{(0)}[\mathcal{A}_1^{-2}\mathcal{A}_2^{-3}P_k^4\mathcal{T}_2]$$
$$+2h_2(d)h_3(d)a_k^3\mathrm{Tr}''_{(0)}[\mathcal{A}_1^{-1}\mathcal{A}_2^{-3}P_k^3\mathcal{T}_2]+2h_2(d)h_4(d)a_k^2\mathrm{Tr}''_{(0)}[\mathcal{A}_1^{-1}\mathcal{A}_2^{-3}P_k^2\mathcal{T}_2]+h_3(d)^2a_k^3\mathrm{Tr}''_{(0)}[\mathcal{A}_2^{-3}P_k^2\mathcal{T}_2]$$
$$+2h_3(d)h_4(d)a_k^2\mathrm{Tr}''_{(0)}[\mathcal{A}_2^{-3}P_k\mathcal{T}_2]+h_4(d)^2a_k\mathrm{Tr}''_{(0)}[\mathcal{A}_2^{-3}\mathcal{T}_2]+h_2(d)^2a_k^3\mathrm{Tr}''_{(0)}[(\mathcal{A}_1\mathcal{A}_2)^{-3}P_k^6\mathcal{N}]$$
$$+2h_2(d)h_3(d)a_k^3\mathrm{Tr}''_{(0)}[\mathcal{A}_1^{-2}\mathcal{A}_2^{-3}P_k^5\mathcal{N}]+3h_2(d)h_4(d)a_k^2\mathrm{Tr}''_{(0)}[\mathcal{A}_1^{-2}\mathcal{A}_2^{-3}P_k^4\mathcal{N}]+h_3(d)^2a_k^3\mathrm{Tr}''_{(0)}[\mathcal{A}_1^{-1}\mathcal{A}_2^{-3}P_k^4\mathcal{N}]$$
$$+4h_3(d)h_4(d)a_k^2\mathrm{Tr}''_{(0)}[\mathcal{A}_1^{-1}\mathcal{A}_2^{-3}P_k^3\mathcal{N}]+3h_4(d)^2a_k\mathrm{Tr}''_{(0)}[\mathcal{A}_1^{-1}\mathcal{A}_2^{-3}P_k^2\mathcal{N}]-h_1(d)h_3(d)^2a_k^2\mathrm{Tr}''_{(0)}[\mathcal{A}_2^{-3}P_k^2\mathcal{N}]$$
$$-2h_1(d)h_3(d)h_4(d)a_k\mathrm{Tr}''_{(0)}[\mathcal{A}_2^{-3}P_k\mathcal{N}]-h_1(d)h_4(d)^2\mathrm{Tr}''_{(0)}[\mathcal{A}_2^{-3}\mathcal{N}]-\frac{2}{d^2}\mathrm{Tr}_{(1T)}[P_k^{-3}\mathcal{N}_0]-\frac{8}{d^2}\mathrm{Tr}'_{(0)}[P_k^{-3}\mathcal{N}_0]$$
$$+\int d^dx\sqrt{g}\Bigg[\frac{\delta_{d,2}}{8\pi k^4}(1-3R^{(0)\prime}(0))((\overline{\beta}_k k^2)^{-1}\partial_t(\overline{\beta}_k k^4)-\overline{\beta}_k^{-1}\partial_t(\overline{\beta}_k k^2))+\frac{\delta_{d,4}}{8(4\pi)^2}(9a_kk^4-2(k^2-2\overline{\lambda}_k))^{-1}$$
$$\times(9a_k\overline{\beta}_k^{-1}\partial_t(\overline{\beta}_kk^4)-2Z_{Nk}^{-1}\partial_t(Z_{Nk}k^2))\Bigg]\Bigg\}R^2+\mathcal{O}(r^{<d-4}). \tag{A10}$$

Here we set

$$\mathcal{A}_1\equiv P_k-2\overline{\lambda}_k,$$
$$\mathcal{A}_2\equiv a_kP_k^2-\frac{1}{2}h_1(d)(P_k-2\overline{\lambda}_k) \tag{A11}$$

and

$$a_k\equiv(Z_{Nk}\kappa^2)^{-1}\overline{\beta}_k. \tag{A12}$$

The quantities $\mathcal{N}$, $\mathcal{N}_0$, $\mathcal{T}_1$ and $\mathcal{T}_2$ are defined as in Eq. (4.17). Furthermore, $\mathcal{O}(r^{<d-4})$ means that terms $\propto r^n$ with powers $n<d-4$ are neglected.

The terms in Eq. (A10) proportional to $\delta_{d,2}$ and $\delta_{d,4}$ arise from the last term in Eq. (4.16). Contrary to the other terms of Eq. (4.16), its expansion does not contain d-dependent powers of r, but is of the form $\Sigma_{m=0}^{\infty}b_{2m}r^{-2m}$ with $\{b_{2m}\}$ a set of r-independent coefficients. As for comparing powers of r, this has the following consequence. Since, for all $m\geqslant 0$ and $d>0$, $-2m=d-4$ or $-2m=d-2$ are satisfied only if $(m,d)\in\{(0,2),(1,2),(0,4)\}$, and since $-2m=d$ cannot be satisfied at all, this term contributes to the evolution equation only in the two- and the four-dimensional case. Using Eq. (4.7) the pieces contributing, i.e. $b_{2m=0}r^0$ in $d=2$ and $d=4$, and $b_{2m=2}r^{-2}$ in $d=2$, may be expressed in terms of the operators $\int d^2x\sqrt{g}R$, $\int d^2x\sqrt{g}R^2$ or $\int d^4x\sqrt{g}R^2$. This yields the terms in Eq. (A10) which are proportional to the δ's.

As the next step we evaluate the r expansion of the traces appearing in Eq. (A10) by applying the heat kernel expansion. In its original form it has often been used to compute traces of operators acting on unconstrained fields. For our purposes we need the heat kernel expansions for operators acting on constrained fields, i.e., fields satisfying appropriate transversality conditions. In Ref. [2] these expansions are derived in detail for Laplacians D^2 on S^d backgrounds acting on symmetric transverse traceless tensors, on transverse vectors and on scalars, with the following results:

$$\mathrm{Tr}_{(2ST^2)}[e^{-(\mathrm{i}s-\varepsilon)D^2}]=\left(\frac{\mathrm{i}}{4\pi(s+\mathrm{i}\varepsilon)}\right)^{d/2}\int d^dx\sqrt{g}\Bigg\{\frac{1}{2}(d-2)(d+1)-\frac{(d+1)(d+2)(d-5+3\delta_{d,2})}{12(d-1)}(\mathrm{i}s-\varepsilon)R$$
$$-\frac{(d+1)(5d^4-22d^3-83d^2-392d-228+1440\delta_{d,2}+3240\delta_{d,4})}{720d(d-1)^2}(s+\mathrm{i}\varepsilon)^2R^2+\mathcal{O}(R^3)\Bigg\}, \tag{A13}$$

$$\mathrm{Tr}_{(1T)}[e^{-(\mathrm{i}s-\varepsilon)D^2}]=\left(\frac{\mathrm{i}}{4\pi(s+\mathrm{i}\varepsilon)}\right)^{d/2}\int d^dx\sqrt{g}\left\{d-1-\frac{(d+2)(d-3)+6\delta_{d,2}}{6d}(\mathrm{i}s-\varepsilon)R\right.$$
$$\left.-\frac{5d^4-12d^3-47d^2-186d+180+360\delta_{d,2}+720\delta_{d,4}}{360d^2(d-1)}(s+\mathrm{i}\varepsilon)^2R^2+\mathcal{O}(R^3)\right\}, \tag{A14}$$

$$\mathrm{Tr}_{(0)}[e^{-(\mathrm{i}s-\varepsilon)D^2}]=\left(\frac{\mathrm{i}}{4\pi(s+\mathrm{i}\varepsilon)}\right)^{d/2}\int d^dx\sqrt{g}\left\{1-\frac{1}{6}(\mathrm{i}s-\varepsilon)R-\frac{5d^2-7d+6}{360d(d-1)}(s+\mathrm{i}\varepsilon)^2R^2+\mathcal{O}(R^3)\right\}. \tag{A15}$$

Here the terms proportional to the δ's arise from the exclusion of unphysical modes of the type discussed in Sec. II.

Let us now consider an arbitrary function $W(z)$ with a Fourier transform $\widetilde{W}(s)$. For such functions W, we may express the trace of the operator $W(-D^2)$ that results from replacing the argument of W with $-D^2$ in terms of $\widetilde{W}(s)$:

$$\mathrm{Tr}[W(-D^2)]=\lim_{\varepsilon\searrow 0}\int_{-\infty}^{\infty}ds\,\widetilde{W}(s)\mathrm{Tr}[e^{-(\mathrm{i}s-\varepsilon)D^2}]. \tag{A16}$$

We obtain the asymptotic expansion of $\mathrm{Tr}[W(-D^2)]$ by inserting the heat kernel expansion for $\mathrm{Tr}[e^{-(\mathrm{i}s-\varepsilon)D^2}]$ into Eq. (A16). For Laplacians acting on the constrained fields considered here they read as follows:

$$\mathrm{Tr}_{(2ST^2)}[W(-D^2)]=(4\pi)^{-d/2}\left\{\frac{1}{2}(d-2)(d+1)Q_{d/2}[W]\int d^dx\sqrt{g}+\frac{(d+1)(d+2)(d-5+3\delta_{d,2})}{12(d-1)}Q_{d/2-1}[W]\int d^dx\sqrt{g}R\right.$$
$$+\frac{(d+1)(5d^4-22d^3-83d^2-392d-228+1440\delta_{d,2}+3240\delta_{d,4})}{720d(d-1)^2}Q_{d/2-2}[W]$$
$$\left.\times\int d^dx\sqrt{g}R^2+\mathcal{O}(r^{<d-4})\right\}, \tag{A17}$$

$$\mathrm{Tr}_{(1T)}[W(-D^2)]=(4\pi)^{-d/2}\left\{(d-1)Q_{d/2}[W]\int d^dx\sqrt{g}+\frac{(d+2)(d-3)+6\delta_{d,2}}{6d}Q_{d/2-1}[W]\int d^dx\sqrt{g}R\right.$$
$$\left.+\frac{5d^4-12d^3-47d^2-186d+180+360\delta_{d,2}+720\delta_{d,4}}{360d^2(d-1)}Q_{d/2-2}[W]\int d^dx\sqrt{g}R^2+\mathcal{O}(r^{<d-4})\right\}, \tag{A18}$$

$$\mathrm{Tr}_{(0)}[W(-D^2)]=(4\pi)^{-d/2}\left\{Q_{d/2}[W]\int d^dx\sqrt{g}+\frac{1}{6}Q_{d/2-1}[W]\int d^dx\sqrt{g}R\right.$$
$$\left.+\frac{5d^2-7d+6}{360d(d-1)}Q_{d/2-2}[W]\int d^dx\sqrt{g}R^2+\mathcal{O}(r^{<d-4})\right\}. \tag{A19}$$

Here the set of functionals $Q_n[W]$ is defined as

$$Q_n[W]\equiv\lim_{\varepsilon\searrow 0}\int_{-\infty}^{\infty}ds(-\mathrm{i}s+\varepsilon)^{-n}\widetilde{W}(s). \tag{A20}$$

By virtue of the Mellin transformation we may now reexpress Q_n in terms of W so that

$$Q_n[W]=\frac{(-1)^i}{\Gamma(n+i)}\int_0^{\infty}dz\,z^{n+i-1}\frac{d^iW(z)}{dz^i},\quad i>-n,\quad i\in\mathbb{N}\cup\{0\}\ \text{ arbitrary}. \tag{A21}$$

In particular we obtain $Q_0[W]=W(0)$. Furthermore, if $n>0$ we may choose $i=0$ for simplicity. As can be seen by an appropriate integration by parts, $Q_n[W]$ does not depend on i.

At this point it is necessary to discuss the case where isolated eigenvalues have to be excluded from $\mathrm{Tr}[W(-D^2)]$. As we showed in [2], such traces can be expressed as

$$\mathrm{Tr}'^{\cdots\prime}[W(-D^2)]=\mathrm{Tr}[W(-D^2)]-\sum_{l\in\{l_1,\ldots,l_n\}} D_l(d,s)W(\Lambda_l(d,s)). \tag{A22}$$

Here n primes at $\mathrm{Tr}'^{\cdots\prime}$ symbolize the exclusion of all eigenmodes T^{lm} with $l\in\{l_1,\ldots,l_n\}$, and $\Lambda_l(d,s)$ and $D_l(d,s)$ denote the corresponding eigenvalues of $-D^2$ and their degrees of degeneracy, respectively. Since $\Lambda_l(d,s)\propto R$ we may view $W(\Lambda_l(d,s))$ as a function of R. As outlined above, such a function contributes to the evolution of $\overline{\lambda}_k$, Z_{Nk} and $\overline{\beta}_k$ only for $d=2$ and for $d=4$, with the contributions given by $W(0)+W'(0)\Lambda_l(2,s)$ and $W(0)$, respectively. Using the explicit expressions for $\Lambda_l(d,s)$ and $D_l(d,s)$ (see Table II in Appendix C) and applying Eq. (4.7) we therefore obtain for the traces relevant to the flow equation:

$$\mathrm{Tr}'_{(1T)}[W(-D^2)]=\mathrm{Tr}_{(1T)}[W(-D^2)]-\frac{\delta_{d,2}}{16\pi}\left\{6W(0)\int d^2x\sqrt{g}R+3W'(0)\int d^2x\sqrt{g}R^2\right\}-\frac{5\,\delta_{d,4}}{12(4\pi)^2}W(0)\int d^4x\sqrt{g}R^2 +\mathcal{O}(r^{<d-4}), \tag{A23}$$

$$\mathrm{Tr}''_{(0)}[W(-D^2)]=\mathrm{Tr}_{(0)}[W(-D^2)]-\frac{\delta_{d,2}}{16\pi}\left\{8W(0)\int d^2x\sqrt{g}R+3W'(0)\int d^2x\sqrt{g}R^2\right\}-\frac{\delta_{d,4}}{4(4\pi)^2}W(0)\int d^4x\sqrt{g}R^2 +\mathcal{O}(r^{<d-4}), \tag{A24}$$

$$\mathrm{Tr}'_{(0)}[W(-D^2)]=\mathrm{Tr}_{(0)}[W(-D^2)]-\frac{\delta_{d,2}}{8\pi}W(0)\int d^2x\sqrt{g}R-\frac{\delta_{d,4}}{24(4\pi)^2}W(0)\int d^4x\sqrt{g}R^2+\mathcal{O}(r^{<d-4}). \tag{A25}$$

Here W' denotes the derivative with respect to the argument: $W'(z)=dW(z)/dz$ with $z=\Lambda_l(d,s)$.

The next step is to insert the expansions of the traces into $\mathcal{S}_k(R)$, Eq. (A10), and to compare the coefficients of the operators $\int d^dx\sqrt{g}$, $\int d^dx\sqrt{g}R$ and $\int d^dx\sqrt{g}R^2$ with those on the LHS, Eq. (4.5). This leads to the following differential equations:

$$\partial_t(Z_{Nk}\overline{\lambda}_k)=(4\kappa^2)^{-1}(4\pi)^{-d/2}\{h_8(d)Q_{d/2}[\mathcal{A}_1^{-1}\mathcal{N}]-h_1(d)Q_{d/2}[\mathcal{A}_2^{-1}\mathcal{N}]+a_kQ_{d/2}[\mathcal{A}_2^{-1}\mathcal{T}_2]+a_kQ_{d/2}[(\mathcal{A}_1\mathcal{A}_2)^{-1}P_k^2\mathcal{N}] -2dQ_{d/2}[P_k^{-1}\mathcal{N}_0]\}, \tag{A26}$$

$$\begin{aligned}\partial_tZ_{Nk}=-(2\kappa^2)^{-1}(4\pi)^{-d/2}\Big\{&h_9(d)Q_{d/2-1}[\mathcal{A}_1^{-1}\mathcal{N}]-\frac{1}{6}h_1(d)Q_{d/2-1}[\mathcal{A}_2^{-1}\mathcal{N}]+\frac{1}{6}a_kQ_{d/2-1}[\mathcal{A}_2^{-1}\mathcal{T}_2]\\
&+\frac{1}{6}a_kQ_{d/2-1}[(\mathcal{A}_1\mathcal{A}_2)^{-1}P_k^2\mathcal{N}]+h_{10}(d)Q_{d/2-1}[P_k^{-1}\mathcal{N}_0]-h_{11}(d)a_kQ_{d/2}[\mathcal{A}_1^{-1}\mathcal{T}_1]+h_{11}(d)a_kQ_{d/2}[\mathcal{A}_1^{-2}P_k\mathcal{N}]\\
&+h_{12}(d)Q_{d/2}[\mathcal{A}_1^{-2}\mathcal{N}]+h_2(d)a_kQ_{d/2}[(\mathcal{A}_1\mathcal{A}_2)^{-1}\mathcal{T}_2]+h_3(d)a_kQ_{d/2}[(\mathcal{A}_1\mathcal{A}_2)^{-1}P_k\mathcal{N}]\\
&+h_4(d)Q_{d/2}[(\mathcal{A}_1\mathcal{A}_2)^{-1}\mathcal{N}]+h_3(d)a_kQ_{d/2}[\mathcal{A}_2^{-1}\mathcal{T}_1]-h_2(d)a_k^2Q_{d/2}[\mathcal{A}_1^{-1}\mathcal{A}_2^{-2}P_k^2\mathcal{T}_2]\\
&-h_3(d)a_k^2Q_{d/2}[\mathcal{A}_2^{-2}P_k\mathcal{T}_2]-h_4(d)a_kQ_{d/2}[\mathcal{A}_2^{-2}\mathcal{T}_2]-h_2(d)a_k^2Q_{d/2}[(\mathcal{A}_1\mathcal{A}_2)^{-2}P_k^4\mathcal{N}]\\
&-h_3(d)a_k^2Q_{d/2}[\mathcal{A}_1^{-1}\mathcal{A}_2^{-2}P_k^3\mathcal{N}]-2h_4(d)a_kQ_{d/2}[\mathcal{A}_1^{-1}\mathcal{A}_2^{-2}P_k^2\mathcal{N}]+h_1(d)h_3(d)a_kQ_{d/2}[\mathcal{A}_2^{-2}P_k\mathcal{N}]\\
&+h_1(d)h_4(d)Q_{d/2}[\mathcal{A}_2^{-2}\mathcal{N}]+h_{13}(d)Q_{d/2}[P_k^{-2}\mathcal{N}_0]\Big\},\end{aligned} \tag{A27}$$

$$\begin{aligned}\partial_t\overline{\beta}_k=(4\pi)^{-d/2}\Big\{&h_{14}(d)Q_{d/2-2}[\mathcal{A}_1^{-1}\mathcal{N}]-h_1(d)h_{15}(d)Q_{d/2-2}[\mathcal{A}_2^{-1}\mathcal{N}]+h_{15}(d)a_kQ_{d/2-2}[\mathcal{A}_2^{-1}\mathcal{T}_2]\\
&+h_{15}(d)a_kQ_{d/2-2}[(\mathcal{A}_1\mathcal{A}_2)^{-1}P_k^2\mathcal{N}]-h_{16}(d)Q_{d/2-2}[P_k^{-1}\mathcal{N}_0]-h_{17}(d)a_kQ_{d/2-1}[\mathcal{A}_1^{-1}\mathcal{T}_1]\\
&+h_{17}(d)a_kQ_{d/2-1}[\mathcal{A}_1^{-2}P_k\mathcal{N}]-h_{18}(d)Q_{d/2-1}[\mathcal{A}_1^{-2}\mathcal{N}]+\frac{1}{6}h_2(d)a_kQ_{d/2-1}[(\mathcal{A}_1\mathcal{A}_2)^{-1}\mathcal{T}_2]\\
&+\frac{1}{6}h_3(d)a_kQ_{d/2-1}[(\mathcal{A}_1\mathcal{A}_2)^{-1}P_k\mathcal{N}]+\frac{1}{6}h_4(d)Q_{d/2-1}[(\mathcal{A}_1\mathcal{A}_2)^{-1}\mathcal{N}]+\frac{1}{6}h_3(d)a_kQ_{d/2-1}[\mathcal{A}_2^{-1}\mathcal{T}_1]\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{6}h_2(d)a_k^2Q_{d/2-1}[\mathcal{A}_1^{-1}\mathcal{A}_2^{-2}P_k^2\mathcal{T}_2]-\frac{1}{6}h_3(d)a_k^2Q_{d/2-1}[\mathcal{A}_2^{-2}P_k\mathcal{T}_2]-\frac{1}{6}h_4(d)a_kQ_{d/2-1}[\mathcal{A}_2^{-2}\mathcal{T}_2]\\
&-\frac{1}{6}h_2(d)a_k^2Q_{d/2-1}[(\mathcal{A}_1\mathcal{A}_2)^{-2}P_k^4\mathcal{N}]-\frac{1}{6}h_3(d)a_k^2Q_{d/2-1}[\mathcal{A}_1^{-1}\mathcal{A}_2^{-2}P_k^3\mathcal{N}]\\
&-\frac{1}{3}h_4(d)a_kQ_{d/2-1}[\mathcal{A}_1^{-1}\mathcal{A}_2^{-2}P_k^2\mathcal{N}]+\frac{1}{6}h_1(d)h_3(d)a_kQ_{d/2-1}[\mathcal{A}_2^{-2}P_k\mathcal{N}]+\frac{1}{6}h_1(d)h_4(d)Q_{d/2-1}[\mathcal{A}_2^{-2}\mathcal{N}]\\
&-h_{20}(d)Q_{d/2-1}[P_k^{-2}\mathcal{N}_0]-h_{11}(d)a_k^2Q_{d/2}[\mathcal{A}_1^{-2}P_k\mathcal{T}_1]+\frac{1}{2}h_{11}(d)h_{21}(d)a_kQ_{d/2}[\mathcal{A}_1^{-2}\mathcal{T}_1]\\
&+h_{22}(d)a_kQ_{d/2}[\mathcal{A}_1^{-2}\mathcal{N}]+h_{11}(d)a_k^2Q_{d/2}[\mathcal{A}_1^{-3}P_k^2\mathcal{N}]-h_{11}(d)h_{21}(d)a_kQ_{d/2}[\mathcal{A}_1^{-3}P_k\mathcal{N}]+h_{23}(d)Q_{d/2}[\mathcal{A}_1^{-3}\mathcal{N}]\\
&-h_5(d)a_k^2Q_{d/2}[(\mathcal{A}_1\mathcal{A}_2)^{-1}\mathcal{T}_2]+h_6(d)a_kQ_{d/2}[(\mathcal{A}_1\mathcal{A}_2)^{-1}\mathcal{N}]-\frac{2}{d^2}a_k^2Q_{d/2}[(\mathcal{A}_1\mathcal{A}_2)^{-1}P_k\mathcal{T}_1]\\
&+h_2(d)h_3(d)a_kQ_{d/2}[(\mathcal{A}_1\mathcal{A}_2)^{-1}\mathcal{T}_1]-h_2(d)^2a_k^2Q_{d/2}[(\mathcal{A}_1\mathcal{A}_2)^{-2}P_k^2\mathcal{T}_2]+\frac{1}{2}h_2(d)a_k^3Q_{d/2}[\mathcal{A}_1^{-1}\mathcal{A}_2^{-2}P_k^2\mathcal{T}_2]\\
&-2h_2(d)h_3(d)a_k^2Q_{d/2}[\mathcal{A}_1^{-1}\mathcal{A}_2^{-2}P_k\mathcal{T}_2]-\frac{3}{2}h_2(d)h_4(d)a_kQ_{d/2}[\mathcal{A}_1^{-1}\mathcal{A}_2^{-2}\mathcal{T}_2]-h_6(d)a_k^2Q_{d/2}[\mathcal{A}_2^{-2}\mathcal{T}_2]\\
&+\frac{1}{2}h_2(d)a_k^3Q_{d/2}[(\mathcal{A}_1\mathcal{A}_2)^{-2}P_k^4\mathcal{N}]-2h_2(d)h_3(d)a_k^2Q_{d/2}[(\mathcal{A}_1\mathcal{A}_2)^{-2}P_k^3\mathcal{N}]\\
&-\frac{3}{2}h_2(d)h_4(d)a_kQ_{d/2}[(\mathcal{A}_1\mathcal{A}_2)^{-2}P_k^2\mathcal{N}]-h_7(d)a_k^2Q_{d/2}[\mathcal{A}_1^{-1}\mathcal{A}_2^{-2}P_k^2\mathcal{N}]\\
&-3h_3(d)h_4(d)a_kQ_{d/2}[\mathcal{A}_1^{-1}\mathcal{A}_2^{-2}P_k\mathcal{N}]-\frac{3}{2}h_4(d)^2Q_{d/2}[\mathcal{A}_1^{-1}\mathcal{A}_2^{-2}\mathcal{N}]+h_1(d)h_6(d)a_kQ_{d/2}[\mathcal{A}_2^{-2}\mathcal{N}]\\
&-h_2(d)h_3(d)a_k^2Q_{d/2}[\mathcal{A}_1^{-1}\mathcal{A}_2^{-2}P_k^2\mathcal{T}_1]-h_3(d)^2a_k^2Q_{d/2}[\mathcal{A}_2^{-2}P_k\mathcal{T}_1]-h_3(d)h_4(d)a_kQ_{d/2}[\mathcal{A}_2^{-2}\mathcal{T}_1]\\
&+h_2(d)^2a_k^3Q_{d/2}[\mathcal{A}_1^{-2}\mathcal{A}_2^{-3}P_k^4\mathcal{T}_2]+2h_2(d)h_3(d)a_k^3Q_{d/2}[\mathcal{A}_1^{-1}\mathcal{A}_2^{-3}P_k^3\mathcal{T}_2]\\
&+2h_2(d)h_4(d)a_k^2Q_{d/2}[\mathcal{A}_1^{-1}\mathcal{A}_2^{-3}P_k^2\mathcal{T}_2]+h_3(d)^2a_k^3Q_{d/2}[\mathcal{A}_2^{-3}P_k^2\mathcal{T}_2]+2h_3(d)h_4(d)a_k^2Q_{d/2}[\mathcal{A}_2^{-3}P_k\mathcal{T}_2]\\
&+h_4(d)^2a_kQ_{d/2}[\mathcal{A}_2^{-3}\mathcal{T}_2]+h_2(d)^2a_k^3Q_{d/2}[(\mathcal{A}_1\mathcal{A}_2)^{-3}P_k^6\mathcal{N}]+2h_2(d)h_3(d)a_k^3Q_{d/2}[\mathcal{A}_1^{-2}\mathcal{A}_2^{-3}P_k^5\mathcal{N}]\\
&+3h_2(d)h_4(d)a_k^2Q_{d/2}[\mathcal{A}_1^{-2}\mathcal{A}_2^{-3}P_k^4\mathcal{N}]+h_3(d)^2a_k^3Q_{d/2}[\mathcal{A}_1^{-1}\mathcal{A}_2^{-3}P_k^4\mathcal{N}]\\
&+4h_3(d)h_4(d)a_k^2Q_{d/2}[\mathcal{A}_1^{-1}\mathcal{A}_2^{-3}P_k^3\mathcal{N}]+3h_4(d)^2a_kQ_{d/2}[\mathcal{A}_1^{-1}\mathcal{A}_2^{-3}P_k^2\mathcal{N}]-h_1(d)h_3(d)^2a_k^2Q_{d/2}[\mathcal{A}_2^{-3}P_k^2\mathcal{N}]\\
&-2h_1(d)h_3(d)h_4(d)a_kQ_{d/2}[\mathcal{A}_2^{-3}P_k\mathcal{N}]-h_1(d)h_4(d)^2Q_{d/2}[\mathcal{A}_2^{-3}\mathcal{N}]+h_{24}(d)Q_{d/2}[P_k^{-3}\mathcal{N}_0]\\
&+\delta_{d,2}\Bigg[-\frac{3}{2}a_k\frac{\partial_t[\overline{\beta}_kk^2]}{\overline{\beta}_k(k^2-2\overline{\lambda}_k)}+\left(\frac{11}{4}R^{(0)\prime}(0)+\frac{3}{2}a_kk^2\right)\frac{\partial_t[Z_{Nk}k^2]}{Z_{Nk}(k^2-2\overline{\lambda}_k)^2}-\frac{11}{4}R^{(0)\prime}(0)\frac{\partial_tZ_{Nk}}{Z_{Nk}(k^2-2\overline{\lambda}_k)}\\
&-\frac{1}{2k^2}R^{(0)\prime}(0)\Bigg]+\delta_{d,4}\Bigg[\frac{1}{8}\frac{\partial_t[Z_{Nk}k^2]}{Z_{Nk}(k^2-2\overline{\lambda}_k)}+\frac{1}{4}\frac{\partial_t[Z_{Nk}k^2]}{Z_{Nk}(3a_kk^4-(k^2-2\overline{\lambda}_k))}\\
&-\frac{3}{8}a_kk^4\frac{\partial_t[Z_{Nk}k^2]}{Z_{Nk}(k^2-2\overline{\lambda}_k)(3a_kk^4-(k^2-2\overline{\lambda}_k))}-\frac{1}{4}\frac{\partial_t[Z_{Nk}k^2]}{Z_{Nk}(9a_kk^4-2(k^2-2\overline{\lambda}_k))}-\frac{3}{8}a_k\frac{\partial_t[\overline{\beta}_kk^4]}{\overline{\beta}_k(3a_kk^4-(k^2-2\overline{\lambda}_k))}\\
&+\frac{9}{8}a_k\frac{\partial_t[\overline{\beta}_kk^4]}{\overline{\beta}_k(9a_kk^4-2(k^2-2\overline{\lambda}_k))}\Bigg]\Bigg\}.
\end{aligned}
\tag{A28}
$$

Here the coefficients h_i are functions of the dimensionality d. They are tabulated in Eq. (D2) of Appendix D 2.

Now we introduce the cutoff-dependent generalized threshold functions

$$\Psi_{n;m}^{p;q}(v,w;d)\equiv\frac{(-1)^i}{\Gamma(n+i)}\int_0^\infty dy\, y^{n+i-1}\frac{\partial^i}{\partial y^i}\Big[(y+R^{(0)}(y))^m(R^{(0)}(y)-yR^{(0)\prime}(y))(y+R^{(0)}(y)+w)^{-p}$$
$$\times\left(32\pi v(y+R^{(0)}(y))^2-\frac{d-2}{2(d-1)}(y+R^{(0)}(y)+w)\right)^{-q}\Big] \tag{A29}$$

and

$$\widetilde{\Psi}_{n;m;l}^{p;q}(v,w;d)\equiv\frac{(-1)^i}{\Gamma(n+i)}\int_0^\infty dy\, y^{n+i-1}\frac{\partial^i}{\partial y^i}\Big[(y+R^{(0)}(y))^m(2y+R^{(0)}(y))^l R^{(0)}(y)(y+R^{(0)}(y)+w)^{-p}$$
$$\times\left(32\pi v(y+R^{(0)}(y))^2-\frac{d-2}{2(d-1)}(y+R^{(0)}(y)+w)\right)^{-q}\Big]. \tag{A30}$$

In Eqs. (A29) and (A30), i is a non-negative integer which satisfies $i>-n$, but which is arbitrary otherwise. The functions Ψ and $\widetilde{\Psi}$ are independent of i which can be seen by an integration by parts. Again, we may set $i=0$ if $n>0$. Furthermore, noting that

$$\Phi_n^p(w)\equiv\Psi_{n;0}^{p;0}(v,w;d), \quad \widetilde{\Phi}_n^p(w)\equiv\widetilde{\Psi}_{n;0;0}^{p;0}(v,w;d) \quad \forall v \;\; \forall d \tag{A31}$$

we recover the threshold functions $\Phi_n^p(w)$ and $\widetilde{\Phi}_n^p(w)$ originally defined in [1] in the context of the pure Einstein-Hilbert truncation.

Using the relations

$$\begin{aligned}
Q_n[\mathcal{A}_1^{-p}\mathcal{A}_2^{-q}P_k^m\mathcal{T}_1]&=k^{2(m+n-p-q+1)}\Psi_{n;m}^{p;q}(a_k k^2/(32\pi),\overline{\lambda}_k/k^2;d)\\
&\quad-\frac{1}{2}\eta_\beta(k)k^{2(m+n-p-q+1)}\widetilde{\Psi}_{n;m;0}^{p;q}(a_k k^2/(32\pi),\overline{\lambda}_k/k^2;d)\\
Q_n[\mathcal{A}_1^{-p}\mathcal{A}_2^{-q}P_k^m\mathcal{T}_2]&=2k^{2(m+n-p-q+2)}\Psi_{n;m+1}^{p;q}(a_k k^2/(32\pi),\overline{\lambda}_k/k^2;d)\\
&\quad-\frac{1}{2}\eta_\beta(k)k^{2(m+n-p-q+2)}\widetilde{\Psi}_{n;m;1}^{p;q}(a_k k^2/(32\pi),\overline{\lambda}_k/k^2;d)\\
Q_n[\mathcal{A}_1^{-p}\mathcal{A}_2^{-q}P_k^m\mathcal{N}]&=k^{2(m+n-p-q+1)}\Psi_{n;m}^{p;q}(a_k k^2/(32\pi),\overline{\lambda}_k/k^2;d)\\
&\quad-\frac{1}{2}\eta_N(k)k^{2(m+n-p-q+1)}\widetilde{\Psi}_{n;m;0}^{p;q}(a_k k^2/(32\pi),\overline{\lambda}_k/k^2;d)\\
Q_n[P_k^{-p}\mathcal{N}_0]&=k^{2(n-p+1)}\Phi_n^p(0)
\end{aligned} \tag{A32}$$

the differential equations (A26), (A27) and (A28) may be rewritten in terms of the threshold functions $\Psi_{n;m}^{p;q}(v,w;d)$ and $\widetilde{\Psi}_{n;m;l}^{p;q}(v,w;d)$ instead of the Q_n.

In order to make the integrals in Eq. (A20) convergent we have to demand that $R^{(0)}(y)$ decreases rapidly as $y\rightarrow\pm\infty$. However, since from now on its form for $y<0$ does not play a role any more we identify $R^{(0)}(y)$ with its part for non-negative arguments and assume that $R^{(0)}(y)$ is a smooth function defined only for $y\geqslant 0$ and endowed with the properties stated in Sec. III B.

Next we introduce the dimensionless couplings λ_k, g_k and β_k of Eqs. (4.20)–(4.22). Inserting Eq. (4.20) into $\partial_t(Z_{Nk}\overline{\lambda}_k)$ leads to the relation

$$\partial_t\lambda_k=-(2-\eta_N(k))\lambda_k+32\pi g_k\kappa^2k^{-d}\partial_t(Z_{Nk}\overline{\lambda}_k). \tag{A33}$$

Then, by using Eq. (A26), we obtain the differential equation (4.23) for the dimensionless cosmological constant. The corresponding differential equations for g_k and β_k may be determined as follows. Taking the scale derivative of Eqs. (4.21) and (4.22) leads to Eqs. (4.24) and (4.25), respectively. For the anomalous dimensions η_N and η_β we obtain from Eqs. (A27) and (A28), respectively,

$$\eta_N=g_kB_1(\lambda_k,g_k,\beta_k;d)+\eta_N g_kB_2(\lambda_k,g_k,\beta_k;d)+\eta_\beta g_kB_3(\lambda_k,g_k,\beta_k;d) \tag{A34}$$

and

$$\eta_\beta = -\beta_k^{-1} C_1(\lambda_k, g_k, \beta_k; d) - \eta_N \beta_k^{-1} C_2(\lambda_k, g_k, \beta_k; d) - \eta_\beta \beta_k^{-1} C_3(\lambda_k, g_k, \beta_k; d). \tag{A35}$$

The set of equations (A34) and (A35) may now be solved for the anomalous dimensions η_N and η_β in terms of λ_k, g_k, β_k and d which eventually leads to the expressions (4.26) and (4.27).

APPENDIX B: COEFFICIENT FUNCTIONS APPEARING IN THE β FUNCTIONS

In the following we list the coefficient functions A_i, B_i, C_i, $i=1,2,3$, which appear in the β functions (4.23), (4.24) and (4.25). The coefficients $h_i(d)$ contained in these expressions are defined in Appendix D 2, and the generalized threshold functions Ψ and $\tilde{\Psi}$ were introduced in Eq. (A29) and Eq. (A30), respectively. The other threshold functions, Φ and $\tilde{\Phi}$, are those already introduced in the context of the pure Einstein-Hilbert truncation [1]:

$$\begin{aligned} A_1(\lambda_k, g_k, \beta_k; d) \equiv & -2\lambda_k + 2(4\pi)^{1-d/2} g_k \big(h_8(d) \Phi^1_{d/2}(-2\lambda_k) - 2d\Phi^1_{d/2}(0) + 64\pi g_k \beta_k \Psi^{0;1}_{d/2;1}(g_k\beta_k, -2\lambda_k; d) \\ & - h_1(d) \Psi^{0;1}_{d/2;0}(g_k\beta_k, -2\lambda_k; d) + 32\pi g_k \beta_k \Psi^{1;1}_{d/2;2}(g_k\beta_k, -2\lambda_k; d) \big) \end{aligned} \tag{B1}$$

$$\begin{aligned} A_2(\lambda_k, g_k, \beta_k; d) \equiv & \lambda_k - (4\pi)^{1-d/2} g_k \big(h_8(d) \tilde{\Phi}^1_{d/2}(-2\lambda_k) - h_1(d) \tilde{\Psi}^{0;1}_{d/2;0;0}(g_k\beta_k, -2\lambda_k; d) \\ & + 32\pi g_k \beta_k \tilde{\Psi}^{1;1}_{d/2;2;0}(g_k\beta_k, -2\lambda_k; d) \big) \end{aligned} \tag{B2}$$

$$A_3(\lambda_k, g_k, \beta_k; d) \equiv -8(4\pi)^{2-d/2} g_k^2 \beta_k \tilde{\Psi}^{0;1}_{d/2;0;1}(g_k\beta_k, -2\lambda_k; d) \tag{B3}$$

$$\begin{aligned} B_1(\lambda_k, g_k, \beta_k; d) \equiv & 4(4\pi)^{1-d/2} \Big\{ h_9(d) \Phi^1_{d/2-1}(-2\lambda_k) - h_{10}(d) \Phi^1_{d/2-1}(0) + \frac{32}{3}\pi g_k \beta_k \Psi^{0;1}_{d/2-1;1}(g_k\beta_k, -2\lambda_k; d) \\ & - \frac{1}{6} h_1(d) \tilde{\Psi}^{0;1}_{d/2-1;0}(g_k\beta_k, -2\lambda_k; d) + \frac{16}{3}\pi g_k \beta_k \Psi^{1;1}_{d/2-1;2}(g_k\beta_k, -2\lambda_k; d) - 32 h_{11}(d) \pi g_k \beta_k \Phi^1_{d/2}(-2\lambda_k) \\ & + 32 h_{11}(d) \pi g_k \beta_k \Psi^{2;0}_{d/2;1}(g_k\beta_k, -2\lambda_k; d) + h_{12}(d) \Phi^2_{d/2}(-2\lambda_k) + h_{13}(d) \Phi^2_{d/2}(0) \\ & + 32 h_{19}(d) \pi g_k \beta_k \Psi^{1;1}_{d/2;1}(g_k\beta_k, -2\lambda_k; d) + h_4(d) \Psi^{1;1}_{d/2;0}(g_k\beta_k, -2\lambda_k; d) \\ & + 32 h_3(d) \pi g_k \beta_k \Psi^{0;1}_{d/2;0}(g_k\beta_k, -2\lambda_k; d) - h_{19}(d)(32\pi g_k\beta_k)^2 \Psi^{1;2}_{d/2;3}(g_k\beta_k, -2\lambda_k; d) \\ & - 64 h_4(d) \pi g_k \beta_k \Psi^{1;2}_{d/2;2}(g_k\beta_k, -2\lambda_k; d) - 2h_3(d)(32\pi g_k\beta_k)^2 \Psi^{0;2}_{d/2;2}(g_k\beta_k, -2\lambda_k; d) \\ & + 32 h_1(d) h_{19}(d) \pi g_k \beta_k \Psi^{0;2}_{d/2;1}(g_k\beta_k, -2\lambda_k; d) + h_1(d) h_4(d) \Psi^{0;2}_{d/2;0}(g_k\beta_k, -2\lambda_k; d) \\ & - h_2(d)(32\pi g_k\beta_k)^2 \Psi^{2;2}_{d/2;4}(g_k\beta_k, -2\lambda_k; d) \Big\} \end{aligned} \tag{B4}$$

$$\begin{aligned} B_2(\lambda_k, g_k, \beta_k; d) \equiv & -2(4\pi)^{1-d/2} \Big\{ h_9(d) \tilde{\Phi}^1_{d/2-1}(-2\lambda_k) - \frac{1}{6} h_1(d) \tilde{\Psi}^{0;1}_{d/2-1;0;0}(g_k\beta_k, -2\lambda_k; d) \\ & + \frac{16}{3}\pi g_k \beta_k \tilde{\Psi}^{1;1}_{d/2-1;2;0}(g_k\beta_k, -2\lambda_k; d) + 32 h_{11}(d) \pi g_k \beta_k \tilde{\Psi}^{2;0}_{d/2;1;0}(g_k\beta_k, -2\lambda_k; d) \\ & + h_{12}(d) \tilde{\Phi}^2_{d/2}(-2\lambda_k) + 32 h_3(d) \pi g_k \beta_k \tilde{\Psi}^{1;1}_{d/2;1;0}(g_k\beta_k, -2\lambda_k; d) + h_4(d) \tilde{\Psi}^{1;1}_{d/2;0;0}(g_k\beta_k, -2\lambda_k; d) \\ & - h_3(d)(32\pi g_k\beta_k)^2 \tilde{\Psi}^{1;2}_{d/2;3;0}(g_k\beta_k, -2\lambda_k; d) - 64 h_4(d) \pi g_k \beta_k \tilde{\Psi}^{1;2}_{d/2;2;0}(g_k\beta_k, -2\lambda_k; d) \\ & + 32 h_1(d) h_3(d) \pi g_k \beta_k \tilde{\Psi}^{0;2}_{d/2;1;0}(g_k\beta_k, -2\lambda_k; d) + h_1(d) h_4(d) \tilde{\Psi}^{0;2}_{d/2;0;0}(g_k\beta_k, -2\lambda_k; d) \\ & - h_2(d)(32\pi g_k\beta_k)^2 \tilde{\Psi}^{2;2}_{d/2;4;0}(g_k\beta_k, -2\lambda_k; d) \Big\} \end{aligned} \tag{B5}$$

$$B_3(\lambda_k,g_k,\beta_k;d)\equiv-2(4\pi)^{1-d/2}\Big\{\frac{16}{3}\pi g_k\beta_k\tilde{\Psi}^{0;1}_{d/2-1;0;1}(g_k\beta_k,-2\lambda_k;d)-32h_{11}(d)\pi g_k\beta_k\tilde{\Phi}^{1}_{d/2}(-2\lambda_k)$$
$$+32h_2(d)\pi g_k\beta_k\tilde{\Psi}^{1;1}_{d/2;0;1}(g_k\beta_k,-2\lambda_k;d)+32h_3(d)\pi g_k\beta_k\tilde{\Psi}^{0;1}_{d/2;0;0}(g_k\beta_k,-2\lambda_k;d)$$
$$-h_2(d)(32\pi g_k\beta_k)^2\tilde{\Psi}^{1;2}_{d/2;2;1}(g_k\beta_k,-2\lambda_k;d)-h_3(d)(32\pi g_k\beta_k)^2\tilde{\Psi}^{0;2}_{d/2;1;1}(g_k\beta_k,-2\lambda_k;d)$$
$$-32h_4(d)\pi g_k\beta_k\tilde{\Psi}^{0;2}_{d/2;0;1}(g_k\beta_k,-2\lambda_k;d)\Big\} \tag{B6}$$

$$C_1(\lambda_k,g_k,\beta_k;d)\equiv(4\pi)^{-d/2}\Big\{h_{14}(d)\Phi^1_{d/2-2}(-2\lambda_k)-h_{16}(d)\Phi^1_{d/2-2}(0)+64h_{15}(d)\pi g_k\beta_k\Psi^{0;1}_{d/2-2;1}(g_k\beta_k,-2\lambda_k;d)$$
$$-h_1(d)h_{15}(d)\Psi^{0;1}_{d/2-2;0}(g_k\beta_k,-2\lambda_k;d)+32h_{15}(d)\pi g_k\beta_k\Psi^{1;1}_{d/2-2;2}(g_k\beta_k,-2\lambda_k;d)$$
$$-32h_{17}(d)\pi g_k\beta_k\Phi^1_{d/2-1}(-2\lambda_k)+\frac{16}{3}h_3(d)\pi g_k\beta_k\Psi^{0;1}_{d/2-1;0}(g_k\beta_k,-2\lambda_k;d)$$
$$+\frac{16}{3}h_{19}(d)\pi g_k\beta_k\Psi^{1;1}_{d/2-1;1}(g_k\beta_k,-2\lambda_k;d)+\frac{1}{6}h_4(d)\Psi^{1;1}_{d/2-1;0}(g_k\beta_k,-2\lambda_k;d)$$
$$+32h_{17}(d)\pi g_k\beta_k\Psi^{2;0}_{d/2-1;1}(g_k\beta_k,-2\lambda_k;d)-h_{18}(d)\Phi^2_{d/2-1}(-2\lambda_k)-h_{20}(d)\Phi^2_{d/2-1}(0)-\frac{1}{6}h_{19}(d)$$
$$\times(32\pi g_k\beta_k)^2\Psi^{1;2}_{d/2-1;3}(g_k\beta_k,-2\lambda_k;d)-\frac{32}{3}h_4(d)\pi g_k\beta_k\Psi^{1;2}_{d/2-1;2}(g_k\beta_k,-2\lambda_k;d)-\frac{1}{3}h_3(d)$$
$$\times(32\pi g_k\beta_k)^2\Psi^{0;2}_{d/2-1;2}(g_k\beta_k,-2\lambda_k;d)+\frac{16}{3}h_1(d)h_{19}(d)\pi g_k\beta_k$$
$$\times\Psi^{0;2}_{d/2-1;1}(g_k\beta_k,-2\lambda_k;d)+\frac{1}{6}h_1(d)h_4(d)\Psi^{0;2}_{d/2-1;0}(g_k\beta_k,-2\lambda_k;d)-\frac{1}{6}h_2(d)(32\pi g_k\beta_k)^2$$
$$\times\Psi^{2;2}_{d/2-1;4}(g_k\beta_k,-2\lambda_k;d)-h_{11}(d)(32\pi g_k\beta_k)^2\Psi^{2;0}_{d/2;1}(g_k\beta_k,-2\lambda_k;d)+32h_{25}(d)\pi g_k\beta_k\Phi^2_{d/2}(-2\lambda_k)$$
$$+h_{11}(d)(32\pi g_k\beta_k)^2\Psi^{3;0}_{d/2;2}(g_k\beta_k,-2\lambda_k;d)-32h_{11}(d)h_{21}(d)\pi g_k\beta_k\Psi^{3;0}_{d/2;1}(g_k\beta_k,-2\lambda_k;d)$$
$$+h_{23}(d)\Phi^3_{d/2}(-2\lambda_k)+h_{24}(d)\Phi^3_{d/2}(0)-h_2(d)(32\pi g_k\beta_k)^2\Psi^{1;1}_{d/2;1}(g_k\beta_k,-2\lambda_k;d)$$
$$+32h_2(d)h_{26}(d)\pi g_k\beta_k\Psi^{1;1}_{d/2;0}(g_k\beta_k,-2\lambda_k;d)+h_2(d)(32\pi g_k\beta_k)^3\Psi^{1;2}_{d/2;3}(g_k\beta_k,-2\lambda_k;d)-h_{27}(d)$$
$$\times(32\pi g_k\beta_k)^2\Psi^{1;2}_{d/2;2}(g_k\beta_k,-2\lambda_k;d)+32h_4(d)h_{28}(d)\pi g_k\beta_k\Psi^{1;2}_{d/2;1}(g_k\beta_k,-2\lambda_k;d)$$
$$-\frac{3}{2}h_4(d)^2\Psi^{1;2}_{d/2;0}(g_k\beta_k,-2\lambda_k;d)-h_{29}(d)(32\pi g_k\beta_k)^2\Psi^{0;2}_{d/2;1}(g_k\beta_k,-2\lambda_k;d)$$
$$-32h_4(d)h_{26}(d)\pi g_k\beta_k\Psi^{0;2}_{d/2;0}(g_k\beta_k,-2\lambda_k;d)+\frac{1}{2}h_2(d)(32\pi g_k\beta_k)^3$$
$$\times\Psi^{2;2}_{d/2;4}(g_k\beta_k,-2\lambda_k;d)+\frac{2}{3}h_2(d)h_{28}(d)(32\pi g_k\beta_k)^2\Psi^{2;2}_{d/2;3}(g_k\beta_k,-2\lambda_k;d)$$
$$-48h_2(d)h_4(d)\pi g_k\beta_k\Psi^{2;2}_{d/2;2}(g_k\beta_k,-2\lambda_k;d)+2h_3(d)^2(32\pi g_k\beta_k)^3\Psi^{0;3}_{d/2;3}(g_k\beta_k,-2\lambda_k;d)$$
$$-3h_1(d)h_3(d)h_{30}(d)(32\pi g_k\beta_k)^2\Psi^{0;3}_{d/2;2}(g_k\beta_k,-2\lambda_k;d)+\frac{64}{3}h_1(d)h_4(d)h_{28}(d)\pi g_k\beta_k\Psi^{0;3}_{d/2;1}(g_k\beta_k,$$
$$-2\lambda_k;d)-h_1(d)h_4(d)^2\Psi^{0;3}_{d/2;0}(g_k\beta_k,-2\lambda_k;d)+3h_3(d)h_{30}(d)(32\pi g_k\beta_k)^3\Psi^{1;3}_{d/2;4}(g_k\beta_k,-2\lambda_k;d)$$

$$-\frac{4}{3}h_4(d)h_{28}(d)(32\pi g_k\beta_k)^2\Psi^{1;3}_{d/2;3}(g_k\beta_k,-2\lambda_k;d)+96h_4(d)^2\pi g_k\beta_k\Psi^{1;3}_{d/2;2}(g_k\beta_k,-2\lambda_k;d)$$

$$-\frac{2}{3}h_2(d)h_{28}(d)(32\pi g_k\beta_k)^3\Psi^{2;3}_{d/2;5}(g_k\beta_k,-2\lambda_k;d)+3h_2(d)h_4(d)(32\pi g_k\beta_k)^2$$

$$\times\Psi^{2;3}_{d/2;4}(g_k\beta_k,-2\lambda_k;d)+h_2(d)^2(32\pi g_k\beta_k)^3\Psi^{3;3}_{d/2;6}(g_k\beta_k,-2\lambda_k;d)+\bigg[-\frac{1}{2}R^{(0)\prime}(0)-\frac{96\pi g_k\beta_k}{1-2\lambda_k}$$

$$+\frac{11R^{(0)\prime}(0)+192\pi g_k\beta_k}{2(1-2\lambda_k)^2}\bigg]\delta_{d,2}+\bigg[\frac{288\pi g_k\beta_k-1}{4(144\pi g_k\beta_k-(1-2\lambda_k))}+\frac{1}{4(1-2\lambda_k)}-\frac{96\pi g_k\beta_k-1}{2(96\pi g_k\beta_k-(1-2\lambda_k))}$$

$$-\frac{24\pi g_k\beta_k}{(1-2\lambda_k)(96\pi g_k\beta_k-(1-2\lambda_k))}\bigg]\delta_{d,4}\bigg\} \tag{B7}$$

$$C_2(\lambda_k,g_k,\beta_k;d)\equiv-\frac{1}{2}(4\pi)^{-d/2}\bigg\{h_{14}(d)\tilde{\Phi}^1_{d/2-2}(-2\lambda_k)-h_1(d)h_{15}(d)\tilde{\Psi}^{0;1}_{d/2-2;0;0}(g_k\beta_k,-2\lambda_k;d)$$

$$+32h_{15}(d)\pi g_k\beta_k\tilde{\Psi}^{1;1}_{d/2-2;2;0}(g_k\beta_k,-2\lambda_k;d)+\frac{16}{3}h_3(d)\pi g_k\beta_k\tilde{\Psi}^{1;1}_{d/2-1;1;0}(g_k\beta_k,-2\lambda_k;d)$$

$$+\frac{1}{6}h_4(d)\tilde{\Psi}^{1;1}_{d/2-1;0;0}(g_k\beta_k,-2\lambda_k;d)+32h_{17}(d)\pi g_k\beta_k\tilde{\Psi}^{2;0}_{d/2-1;1;0}(g_k\beta_k,-2\lambda_k;d)$$

$$-h_{18}(d)\tilde{\Phi}^2_{d/2-1}(-2\lambda_k)-\frac{1}{6}h_3(d)(32\pi g_k\beta_k)^2\tilde{\Psi}^{1;2}_{d/2-1;3;0}(g_k\beta_k,-2\lambda_k;d)$$

$$-\frac{32}{3}h_4(d)\pi g_k\beta_k\tilde{\Psi}^{1;2}_{d/2-1;2;0}(g_k\beta_k,-2\lambda_k;d)+\frac{16}{3}h_1(d)h_3(d)\pi g_k\beta_k\tilde{\Psi}^{0;2}_{d/2-1;1;0}(g_k\beta_k,-2\lambda_k;d)$$

$$+\frac{1}{6}h_1(d)h_4(d)\tilde{\Psi}^{0;2}_{d/2-1;0;0}(g_k\beta_k,-2\lambda_k;d)-\frac{1}{6}h_2(d)(32\pi g_k\beta_k)^2\tilde{\Psi}^{2;2}_{d/2-1;4;0}(g_k\beta_k,-2\lambda_k;d)$$

$$+32h_{22}(d)\pi g_k\beta_k\tilde{\Phi}^2_{d/2}(-2\lambda_k)+h_{11}(d)(32\pi g_k\beta_k)^2\tilde{\Psi}^{3;0}_{d/2;2;0}(g_k\beta_k,-2\lambda_k;d)$$

$$-32h_{11}(d)h_{21}(d)\pi g_k\beta_k\tilde{\Psi}^{3;0}_{d/2;1;0}(g_k\beta_k,-2\lambda_k;d)+h_{23}(d)\tilde{\Phi}^3_{d/2}(-2\lambda_k)$$

$$+32h_6(d)\pi g_k\beta_k\tilde{\Psi}^{1;1}_{d/2;0;0}(g_k\beta_k,-2\lambda_k;d)-h_7(d)(32\pi g_k\beta_k)^2\tilde{\Psi}^{1;2}_{d/2;2;0}(g_k\beta_k,-2\lambda_k;d)$$

$$-96h_3(d)h_4(d)\pi g_k\beta_k\tilde{\Psi}^{1;2}_{d/2;1;0}(g_k\beta_k,-2\lambda_k;d)-\frac{3}{2}h_4(d)^2\tilde{\Psi}^{1;2}_{d/2;0;0}(g_k\beta_k,-2\lambda_k;d)$$

$$+32h_1(d)h_6(d)\pi g_k\beta_k\tilde{\Psi}^{0;2}_{d/2;0;0}(g_k\beta_k,-2\lambda_k;d)+\frac{1}{2}h_2(d)(32\pi g_k\beta_k)^3\tilde{\Psi}^{2;2}_{d/2;4;0}(g_k\beta_k,-2\lambda_k;d)$$

$$-2h_2(d)h_3(d)(32\pi g_k\beta_k)^2\tilde{\Psi}^{2;2}_{d/2;3;0}(g_k\beta_k,-2\lambda_k;d)-48h_2(d)h_4(d)\pi g_k\beta_k\tilde{\Psi}^{2;2}_{d/2;2;0}(g_k\beta_k,-2\lambda_k;d)$$

$$-h_1(d)h_3(d)^2(32\pi g_k\beta_k)^2\tilde{\Psi}^{0;3}_{d/2;2;0}(g_k\beta_k,-2\lambda_k;d)-64h_1(d)h_3(d)h_4(d)\pi g_k\beta_k\tilde{\Psi}^{0;3}_{d/2;1;0}(g_k\beta_k,-2\lambda_k;d)$$

$$-h_1(d)h_4(d)^2\tilde{\Psi}^{0;3}_{d/2;0;0}(g_k\beta_k,-2\lambda_k;d)+h_3(d)^2(32\pi g_k\beta_k)^3\tilde{\Psi}^{1;3}_{d/2;4;0}(g_k\beta_k,-2\lambda_k;d)$$

$$+4h_3(d)h_4(d)(32\pi g_k\beta_k)^2\tilde{\Psi}^{1;3}_{d/2;3;0}(g_k\beta_k,-2\lambda_k;d)+96h_4(d)^2\pi g_k\beta_k\tilde{\Psi}^{1;3}_{d/2;2;0}(g_k\beta_k,-2\lambda_k;d)$$

$$+2h_2(d)h_3(d)(32\pi g_k\beta_k)^3\tilde{\Psi}^{2;3}_{d/2;5;0}(g_k\beta_k,-2\lambda_k;d)+3h_2(d)h_4(d)(32\pi g_k\beta_k)^2\tilde{\Psi}^{2;3}_{d/2;4;0}(g_k\beta_k,-2\lambda_k;d)$$

$$+h_2(d)^2(32\pi g_k\beta_k)^3\tilde{\Psi}^{3;3}_{d/2;6;0}(g_k\beta_k,-2\lambda_k;d)+\bigg[-\frac{11R^{(0)\prime}(0)}{2(1-2\lambda_k)}+\frac{192\pi g_k\beta_k+11R^{(0)\prime}(0)}{2(1-2\lambda_k)^2}\bigg]\delta_{d,2}$$

$$+\left[-\frac{1}{4(144\pi g_k\beta_k-(1-2\lambda_k))}+\frac{1}{4(1-2\lambda_k)}+\frac{1}{2(96\pi g_k\beta_k-(1-2\lambda_k))}\right.$$
$$\left.\left.-\frac{24\pi g_k\beta_k}{(1-2\lambda_k)(96\pi g_k\beta_k-(1-2\lambda_k))}\right]\delta_{d,4}\right\} \tag{B8}$$

$$C_3(\lambda_k,g_k,\beta_k;d)\equiv-\frac{1}{2}(4\pi)^{-d/2}\Big\{32h_{15}(d)\pi g_k\beta_k\tilde{\Psi}^{0;1}_{d/2-2;0;1}(g_k\beta_k,-2\lambda_k;d)-32h_{17}(d)\pi g_k\beta_k\tilde{\Phi}^{1}_{d/2-1}(-2\lambda_k)$$
$$+\frac{16}{3}h_3(d)\pi g_k\beta_k\tilde{\Psi}^{0;1}_{d/2-1;0;0}(g_k\beta_k,-2\lambda_k;d)+\frac{16}{3}h_2(d)\pi g_k\beta_k\tilde{\Psi}^{1;1}_{d/2-1;0;1}(g_k\beta_k,-2\lambda_k;d)-\frac{1}{6}h_2(d)$$
$$\times(32\pi g_k\beta_k)^2\tilde{\Psi}^{1;2}_{d/2-1;2;1}(g_k\beta_k,-2\lambda_k;d)-\frac{1}{6}h_3(d)(32\pi g_k\beta_k)^2$$
$$\times\tilde{\Psi}^{0;2}_{d/2-1;1;1}(g_k\beta_k,-2\lambda_k;d)-\frac{16}{3}h_4(d)\pi g_k\beta_k\tilde{\Psi}^{0;2}_{d/2-1;0;1}(g_k\beta_k,-2\lambda_k;d)-h_{11}(d)(32\pi g_k\beta_k)^2$$
$$\times\tilde{\Psi}^{2;0}_{d/2;1;0}(g_k\beta_k,-2\lambda_k;d)+16h_{11}(d)h_{21}(d)\pi g_k\beta_k\tilde{\Phi}^{2}_{d/2}(-2\lambda_k)-h_5(d)(32\pi g_k\beta_k)^2\tilde{\Psi}^{1;1}_{d/2;0;1}(g_k\beta_k,$$
$$-2\lambda_k;d)-\frac{2}{d^2}(32\pi g_k\beta_k)^2\tilde{\Psi}^{1;1}_{d/2;1;0}(g_k\beta_k,-2\lambda_k;d)+32h_2(d)h_3(d)\pi g_k\beta_k\tilde{\Psi}^{1;1}_{d/2;0;0}(g_k\beta_k,-2\lambda_k;d)$$
$$+\frac{1}{2}h_2(d)(32\pi g_k\beta_k)^3\tilde{\Psi}^{1;2}_{d/2;2;1}(g_k\beta_k,-2\lambda_k;d)-2h_2(d)h_3(d)(32\pi g_k\beta_k)^2$$
$$\times\tilde{\Psi}^{1;2}_{d/2;1;1}(g_k\beta_k,-2\lambda_k;d)-48h_2(d)h_4(d)\pi g_k\beta_k\tilde{\Psi}^{1;2}_{d/2;0;1}(g_k\beta_k,-2\lambda_k;d)$$
$$-h_2(d)h_3(d)(32\pi g_k\beta_k)^2\tilde{\Psi}^{1;2}_{d/2;2;0}(g_k\beta_k,-2\lambda_k;d)-h_6(d)(32\pi g_k\beta_k)^2\tilde{\Psi}^{0;2}_{d/2;0;1}(g_k\beta_k,-2\lambda_k;d)$$
$$-h_3(d)^2(32\pi g_k\beta_k)^2\tilde{\Psi}^{0;2}_{d/2;1;0}(g_k\beta_k,-2\lambda_k;d)-32h_3(d)h_4(d)\pi g_k\beta_k\tilde{\Psi}^{0;2}_{d/2;0;0}(g_k\beta_k,-2\lambda_k;d)$$
$$-h_2(d)^2(32\pi g_k\beta_k)^2\tilde{\Psi}^{2;2}_{d/2;2;1}(g_k\beta_k,-2\lambda_k;d)+h_3(d)^2(32\pi g_k\beta_k)^3\tilde{\Psi}^{0;3}_{d/2;2;1}(g_k\beta_k,-2\lambda_k;d)$$
$$+2h_3(d)h_4(d)(32\pi g_k\beta_k)^2\tilde{\Psi}^{0;3}_{d/2;1;1}(g_k\beta_k,-2\lambda_k;d)+32h_4(d)^2\pi g_k\beta_k\tilde{\Psi}^{0;3}_{d/2;0;1}(g_k\beta_k,-2\lambda_k;d)$$
$$+2h_2(d)h_3(d)(32\pi g_k\beta_k)^3\tilde{\Psi}^{1;3}_{d/2;3;1}(g_k\beta_k,-2\lambda_k;d)+2h_2(d)h_4(d)(32\pi g_k\beta_k)^2$$
$$\times\tilde{\Psi}^{1;3}_{d/2;2;1}(g_k\beta_k,-2\lambda_k;d)+h_2(d)^2(32\pi g_k\beta_k)^3\tilde{\Psi}^{2;3}_{d/2;4;1}(g_k\beta_k,-2\lambda_k;d)-\frac{96\pi g_k\beta_k}{1-2\lambda_k}\delta_{d,2}$$
$$\left.+\left[\frac{36\pi g_k\beta_k}{144\pi g_k\beta_k-(1-2\lambda_k)}-\frac{24\pi g_k\beta_k}{96\pi g_k\beta_k-(1-2\lambda_k)}\right]\delta_{d,4}\right\}. \tag{B9}$$

In Eqs. (B7), (B8) and (B9) the terms proportional to $\delta_{d,2}$ or $\delta_{d,4}$ arise not only from the δ-terms of Eq. (A10), but also by evaluating the "primed" traces, i.e., by subtracting the contributions coming from unphysical modes; see Appendix A 2 for details. All these contributions are obtained by expanding various functions $f(R)$ with respect to R and retaining only the terms $f(0)+f'(0)R$ in $d=2$ and $f(0)$ in $d=4$. As we explained above, these are the only pieces of f which may contribute to the evolution in the truncated parameter space. Furthermore, the heat kernel expansions of the traces corresponding to differentially constrained fields introduce additional contributions proportional to $\delta_{d,2}$ or $\delta_{d,4}$ into Eqs. (B7)–(B9).

APPENDIX C: TENSOR SPHERICAL HARMONICS ON S^d

The spherical harmonics $T^{lm}_{\mu\nu}$, T^{lm}_{μ} and T^{lm} for symmetric transverse traceless (ST^2) tensors $h^T_{\mu\nu}$, transverse (T) vectors ξ_μ, and scalars ϕ on S^d form complete sets of orthogonal eigenfunctions with respect to the covariant Laplacians. They satisfy

$$\begin{aligned}-\bar{D}^2T^{lm}_{\mu\nu}(x)&=\Lambda_l(d,2)T^{lm}_{\mu\nu}(x),\\ -\bar{D}^2T^{lm}_{\mu}(x)&=\Lambda_l(d,1)T^{lm}_{\mu}(x),\\ -\bar{D}^2T^{lm}(x)&=\Lambda_l(d,0)T^{lm}(x)\end{aligned} \tag{C1}$$

TABLE II. Eigenvalues of $-\overline{D}^2$ and their degeneracies on the d sphere.

Eigenfunction	Spin s	Eigenvalue $\Lambda_l(d,s)$	Degeneracy $D_l(d,s)$	l
$T^{lm}_{\mu\nu}(x)$	2	$\frac{l(l+d-1)-2}{d(d-1)}\overline{R}$	$\frac{(d+1)(d-2)(l+d)(l-1)(2l+d-1)(l+d-3)!}{2(d-1)!(l+1)!}$	2,3, . . .
$T^{lm}_{\mu}(x)$	1	$\frac{l(l+d-1)-1}{d(d-1)}\overline{R}$	$\frac{l(l+d-1)(2l+d-1)(l+d-3)!}{(d-2)!(l+1)!}$	1,2, . . .
$T^{lm}(x)$	0	$\frac{l(l+d-1)}{d(d-1)}\overline{R}$	$\frac{(2l+d-1)(l+d-2)!}{l!(d-1)!}$	0,1, . . .

and, after proper normalization,

$$\begin{aligned}\delta^{lk}\delta^{mn}&=\int d^dx\,\sqrt{\overline{g}}(\mathbf{1}_{(2ST^2)})^{\mu\nu\rho\sigma}T^{lm}_{\mu\nu}T^{kn}_{\rho\sigma}\\&=\int d^dx\,\sqrt{\overline{g}}(\mathbf{1}_{(1T)})^{\mu\nu}T^{lm}_{\mu}T^{kn}_{\nu}\\&=\int d^dx\,\sqrt{\overline{g}}T^{lm}T^{kn}.\end{aligned}\tag{C2}$$

Here $(\mathbf{1}_{(2ST^2)})^{\mu\nu\rho\sigma}=(d-2)/(2d)(\overline{g}^{\mu\rho}\overline{g}^{\nu\sigma}+\overline{g}^{\mu\sigma}\overline{g}^{\nu\rho})$ and $(\mathbf{1}_{(1T)})^{\mu\nu}=(d-1)/d\overline{g}^{\mu\nu}$ are the unit matrices in the spaces of ST^2 tensors and transverse vectors, respectively. The $\Lambda_l(d,s)$'s denote the eigenvalues of $-\overline{D}^2$ where s is the spin of the field under consideration and l takes the values $s,s+1,s+2,\ldots$. The index $m=1,\ldots,D_l(d,s)$ is a degeneracy index.

In Ref. [45] explicit expressions for $\Lambda_l(d,s)$ and the degeneracies $D_l(d,s)$ were derived which are summarized in Table II. The eigenvalues are expressed in terms of the curvature scalar $\overline{R}=d(d-1)/r^2$ of the sphere with radius r.

The spherical harmonics $T^{lm}_{\mu\nu}$, T^{lm}_{μ} and T^{lm} span the spaces of ST^2 tensors, T vectors, and scalars so that we may expand arbitrary functions $h^T_{\mu\nu}$, ξ_μ and ϕ according to

$$h^T_{\mu\nu}(x)=\sum_{l=2}^{\infty}\sum_{m=1}^{D_l(d,2)}h^T_{lm}T^{lm}_{\mu\nu}(x),$$

$$\xi_\mu(x)=\sum_{l=1}^{\infty}\sum_{m=1}^{D_l(d,1)}\xi_{lm}T^{lm}_{\mu}(x),\tag{C3}$$

$$\phi(x)=\sum_{l=0}^{\infty}\sum_{m=1}^{D_l(d,0)}\phi_{lm}T^{lm}(x).$$

Equations (C3) may now be used to expand also any symmetric non-T^2 tensor and nontransverse vector in terms of spherical harmonics since they may be expressed in terms of ST^2 tensors, T vectors and scalars by using the decompositions (2.3), (2.7); see e.g., [45–48].

Note that the $D_1(d,1)=d(d+1)/2$ modes $\{T^{1,m}_{\mu}\}$ and the $D_1(d,0)=d+1$ modes $\{T^{1,m}\}$ satisfy the Killing equation (2.5) and the scalar equation (2.6), respectively, and that $T^{0,1}=\text{const}$. Arbitrary symmetric rank-2 tensors receive no contribution from these modes. In the case of arbitrary vectors the constant scalar mode does not contribute. Such modes have no physical meaning and have to be omitted therefore.

APPENDIX D: TABLES OF COEFFICIENT FUNCTIONS

1. Coefficients introduced in $\Gamma^{(2)}_k[g,g]$

In this subsection we define the various A's, B's, C's and G's and $H_S(d)$ which appear in Eqs. (4.12)–(4.16) of Sec. IV C and in Eqs. (A9) and (A10) of Appendix A:

$$A_T(d)\equiv\frac{d(d-3)+4}{d(d-1)},\quad G_T(d)\equiv-\frac{d(d-5)+8}{2d(d-1)},\quad A_V(d,\alpha)\equiv\frac{\alpha(d-2)-1}{d},$$

$$G_V(d)\equiv-\frac{d-4}{2d},\quad A_{S1}(d,\alpha)\equiv\frac{\alpha(d-4)}{2\alpha(d-1)-(d-2)},$$

$$A_{S2}(d,\alpha)\equiv-\frac{\alpha(d-2)-2}{\alpha(d-2)-2(d-1)},\quad B_{S1}(d,\alpha)\equiv-\frac{2\alpha d}{2\alpha(d-1)-(d-2)},$$

$$B_{S2}(d,\alpha)\equiv\frac{2\alpha d}{\alpha(d-2)-2(d-1)},\quad C_{S1}(d,\alpha)\equiv-\frac{2\alpha(d-1)-(d-2)}{4(d-1)-2\alpha(d-2)}\,\frac{d-2}{d-1},\tag{D1}$$

$$C_{S2}(d,\alpha)\equiv\frac{d-1}{d^2}\,\frac{2(d-1)-\alpha(d-2)}{\alpha},\quad C_{S3}(d,\alpha)\equiv\frac{(d-2)(\alpha-1)}{\alpha(d-2)-2(d-1)},$$

$$G_{S1}(d)\equiv\frac{(d-1)(d-4)}{2d^2},\quad G_{S2}(d)\equiv\frac{(d-1)(d-6)}{d^2},$$

$$G_{S3}(d)\equiv\frac{(d-4)(d-6)}{4d^2},\quad H_S(d)\equiv 2\left(\frac{d-1}{d}\right)^2.$$

2. Coefficients appearing in the $\boldsymbol{\beta}$ functions

Next we define the coefficients $h_i(d)$ contained in the $\boldsymbol{\beta}$ functions (4.23), (4.24) and (4.25) via the coefficient functions A_i, B_i, C_i, $i=1,2,3$, given in Appendix B. They also appear in the approximate solutions for the non-Gaussian fixed point of Appendix E:

$$h_1(d)\equiv\frac{d-2}{d-1},\quad h_2(d)\equiv\frac{d-4}{d},\quad h_3(d)\equiv\frac{d^2-8d+4}{2d(d-1)},\quad h_4(d)\equiv-\frac{(d-2)(d-4)}{d(d-1)},$$

$$h_5(d)\equiv\frac{d^2-4d-2}{2d^2},\quad h_6(d)\equiv\frac{(d-4)^2}{2d(d-1)},\quad h_7(d)\equiv\frac{5d^4-48d^3+148d^2-112d+16}{4d^2(d-1)^2},$$

$$h_8(d)\equiv\frac{d^2+d-4}{2},\quad h_9(d)\equiv\frac{(d+3)(d+2)(d^2-5d+2)}{12d(d-1)},\quad h_{10}(d)\equiv-\frac{d^2-6}{3d},$$

$$h_{11}(d)\equiv\frac{(d+1)(d-2)}{2},\quad h_{12}(d)\equiv-\frac{d^4-2d^3-5d^2+16d-14}{2d(d-1)},\tag{D2}$$

$$h_{13}(d)\equiv-\frac{2(d+1)}{d},\quad h_{14}(d)\equiv\frac{5d^6-7d^5-139d^4-545d^3-898d^2+504d-360}{720d^2(d-1)^2},$$

$$h_{15}(d)\equiv\frac{5d^2-7d+6}{360d(d-1)},\quad h_{16}(d)\equiv\frac{5d^4-7d^3-54d^2-180d+180}{180d^2(d-1)},$$

$$h_{17}(d)\equiv\frac{(d+2)(d+1)(d-5)}{12(d-1)},\quad h_{18}(d)\equiv\frac{(d+2)(d^5-5d^4-5d^3+43d^2-68d+18)}{12d^2(d-1)^2},$$

$$h_{19}(d)\equiv\frac{5d^2-28d+20}{2d(d-1)},\quad h_{20}(d)\equiv\frac{(d+3)(d-2)}{3d^2},$$

$$h_{21}(d)\equiv 2\frac{d^2-3d+4}{d(d-1)},\quad h_{22}(d)\equiv\frac{(d-3)(d^3-d^2-4d+8)}{4d(d-1)},$$

$$h_{23}(d)\equiv\frac{d^6-5d^5+3d^4+31d^3-86d^2+98d-50}{2d^2(d-1)^2},\quad h_{24}(d)\equiv-2\frac{d+3}{d^2},$$

$$h_{25}(d)\equiv\frac{3d^4-12d^3+9d^2+24d-40}{4d(d-1)},\quad h_{26}(d)\equiv\frac{d^2-6d+2}{d(d-1)},$$

$$h_{27}(d)\equiv\frac{15d^4-178d^3+628d^2-632d+176}{4d^2(d-1)^2},\quad h_{28}(d)\equiv-\frac{9(d^2-6d+4)}{2d(d-1)},$$

$$h_{29}(d)\equiv\frac{5d^4-52d^3+168d^2-128d+16}{4d^2(d-1)^2},\quad h_{30}(d)\equiv\frac{3d^2-16d+12}{2d(d-1)},$$

$$h_{31}(d)\equiv\frac{5d^6-27d^5-71d^4-405d^3-342d^2-960d+360}{720d^2(d-1)^2},$$

$$h_{32}(d)\equiv-\frac{d^6-3d^5-7d^4+5d^3+26d^2-82d+12}{12d^2(d-1)^2},$$

$$h_{33}(d)\equiv\frac{d^6-5d^5+7d^4-13d^3+42d^2-42d+2}{2d^2(d-1)^2},$$

$$h_{34}(d)\equiv\frac{5d^6-7d^5-119d^4-593d^3-846d^2+480d-360}{360d^2(d-1)^2},$$

$$h_{35}(d)\equiv-\frac{d^6-3d^5-11d^4+9d^3+54d^2-134d+36}{3d^2(d-1)^2},$$

$$h_{36}(d)\equiv3\frac{d^6-5d^5+7d^4-9d^3+46d^2-62d+14}{d^2(d-1)^2},$$

$$h_{37}(d)\equiv\frac{(d+2)(d^3-6d^2+3d-6)}{3d(d-1)},\quad h_{38}(d)\equiv-2\frac{d^4-2d^3+3d^2-4d-2}{d(d-1)},$$

$$h_{39}(d)\equiv\frac{5d^2-7d+6}{45d(d-2)},\quad h_{40}(d)\equiv\frac{30d^5-115d^4-362d^3+721d^2+182d+264}{90d(d-1)(d-2)},$$

$$h_{41}(d)\equiv-2\frac{3d^6-17d^5+25d^4+39d^3-166d^2+224d-96}{3d^2(d-1)(d-2)},$$

$$h_{42}(d)\equiv4\frac{(d-1)(d-4)^2}{d(d-2)},\quad h_{43}(d)\equiv-\frac{(d+2)(d^3-6d^2+3d-6)}{3d(d-1)(d-2)},$$

$$h_{44}(d)\equiv2\frac{d^4-2d^3+3d^2-4d-2}{d(d-1)(d-2)},\quad h_{45}(d)\equiv\frac{d^4-3d^3+32d-32}{d^2},$$

$$h_{46}(d)\equiv-\frac{d^4-13d^2-24d+12}{6d(d-1)},\quad h_{47}(d)\equiv\frac{d^4-2d^3-d^2-4d+2}{d(d-1)}.$$

APPENDIX E: CLOSED-FORM FORMULAS FOR THE FIXED POINT LOCATION

In the following we derive the approximate formula for the position of the non-Gaussian fixed point discussed in Sec. V C. Here we restrict our considerations to the case $d>2$.

In a first approximation we set $\lambda_k=\lambda_*=0$, $\beta_k=\beta_*=0$ and determine g_* from the condition $\eta_{N*}=2-d$ alone. Since $\beta_*=0$, we may solve this equation for g_* in closed form which leads to

$$g_*=\frac{2-d}{B_1(0,0,\lambda_*;d)-(d-2)B_2(0,0,\lambda_*;d)}. \tag{E1}$$

As $\lambda_*=0$, it boils down to

$$g_*=(4\pi)^{d/2-1}\{h_{43}(d)\Phi^1_{d/2-1}(0)+h_{46}(d)\tilde{\Phi}^1_{d/2-1}(0)+h_{44}(d)\Phi^2_{d/2}(0)+h_{47}(d)\tilde{\Phi}^2_{d/2}(0)\}^{-1}. \tag{E2}$$

Here the $h_i(d)$ are again d-dependent coefficients which are defined in Appendix D 2. It is remarkable that the solution (E2) coincides precisely with the corresponding approximate solution (H2) of Ref. [2] with $\alpha=1$, obtained in the framework of the Einstein-Hilbert truncation.

Employing the exponential shape function (3.11) with $s=1$, and setting $d=4$, for instance, Eq. (E2) yields $g_*\approx0.590$. Here we used that, for this shape function, $\Phi^1_1(0)$

$=\pi^2/6$, $\Phi_2^2(0)=1$, $\tilde{\Phi}_1^1(0)=1$, and $\tilde{\Phi}_2^2(0)=1/2$; see Appendix F.

In order to improve upon this approximation scheme, we determine $(\lambda_*, g_*, \beta_*)$ from a set of Taylor-expanded $\boldsymbol{\beta}$ functions. Using Eqs. (F1)–(F4) we expand the $\boldsymbol{\beta}$ functions (4.23), (4.24) and (4.25) about $\lambda_k=g_k=\beta_k=0$ and obtain

$$\boldsymbol{\beta}_\lambda(\lambda_k, g_k, \beta_k; d) = -2\lambda_k + \nu_d d g_k + \mathcal{O}(\mathrm{g}^2),$$

$$\boldsymbol{\beta}_g(\lambda_k, g_k; \alpha, d) = (d-2) g_k - (d-2)\omega_d g_k^2 + \mathcal{O}(\mathrm{g}^3),$$

$$\boldsymbol{\beta}_\beta(\lambda_k, g_k, \beta_k; d) = \gamma_d + (4-d)\beta_k + \mathcal{O}(\mathrm{g}^2). \tag{E3}$$

Here γ_d, ν_d, and ω_d are defined as in Eqs. (5.8), (5.12) and (5.18), respectively, and $\mathcal{O}(\mathrm{g}^n)$ stands for terms of nth and higher orders in the couplings $\mathrm{g}_1(k)=\lambda_k$, $\mathrm{g}_2(k)=g_k$ and $\mathrm{g}_3(k)=\beta_k$. Now g_* is obtained as the nontrivial solution to $\boldsymbol{\beta}_g=0$, which reads

$$g_* = \omega_d^{-1} = (4\pi)^{d/2-1}\{h_{43}(d)\Phi_{d/2-1}^1(0) + h_{44}(d)\Phi_{d/2}^2(0)\}^{-1}. \tag{E4}$$

Inserting Eq. (E4) into $\boldsymbol{\beta}_\lambda=0$ leads to

$$\lambda_* = \frac{\nu_d d}{2\omega_d} = \frac{d(d-3)}{2}\Phi_{d/2}^1(0)\{h_{43}(d)\Phi_{d/2-1}^1(0) + h_{44}(d)\Phi_{d/2}^2(0)\}^{-1}. \tag{E5}$$

Quite remarkably, also these results agree completely with those of Ref. [2] which follow from the pure Einstein-Hilbert truncation. [See Eqs. (H6) and (H7) of this reference.]

Now we use $\boldsymbol{\beta}_\beta$ in order to determine β_*. However, since the term linear in β_k vanishes for $d=4$, the expanded $\boldsymbol{\beta}_\beta$ of Eq. (E3) is not sufficient in this case. Therefore we consider also those terms of second order in the couplings which are linear in β_k. For these terms we find

$$\left.\frac{\partial^2 \boldsymbol{\beta}_\beta}{\partial\lambda_k \partial\beta_k}\right|_{\lambda_k=g_k=\beta_k=0} = 0,$$

$$\alpha_d \equiv \left.\frac{\partial^2 \boldsymbol{\beta}_\beta}{\partial g_k \partial\beta_k}\right|_{\lambda_k=g_k=\beta_k=0} = -(4\pi)^{1-d/2}\{2h_{39}(d) + 2h_{45}(d)\Phi_{d/2}^2(0) - 3\delta_{d,4}\}. \tag{E6}$$

Taking the nonvanishing term of Eq. (E6) into account, and inserting g_* of Eq. (E4) into $\boldsymbol{\beta}_\beta=0$ then leads to

$$\beta_* = \frac{\gamma_d}{d-4-\alpha_d\omega_d^{-1}} = \frac{(4\pi)^{-d/2}\{h_{31}(d)\Phi_{d/2-2}^1(0) + h_{32}(d)\Phi_{d/2-1}^2(0) + h_{33}(d)\Phi_{d/2}^3(0)\}}{d-4+\{2h_{39}(d)+2h_{45}(d)\Phi_{d/2}^2(0)-3\delta_{d,4}\}\{h_{43}(d)\Phi_{d/2-1}^1(0)+h_{44}(d)\Phi_{d/2}^2(0)\}^{-1}}. \tag{E7}$$

Employing the shape function (3.11) with $s=1$ we obtain from Eqs. (E4), (E5) and (E7) in $d=4$ dimensions

$$\lambda_* = \zeta(3)\left(\frac{13\pi^2}{144} + \frac{79}{24}\right)^{-1} \approx 0.287,$$

$$g_* = \left(\frac{13\pi}{144} + \frac{79}{24\pi}\right)^{-1} \approx 0.751, \tag{E8}$$

$$\beta_* = \frac{419(13\pi^2+474)}{(4\pi)^2 906768} \approx 0.0018.$$

Here we used the expressions for the threshold functions derived in Appendix F. The numbers in Eq. (E8) should be compared to the exact result (5.33).

APPENDIX F: PROPERTIES OF THE THRESHOLD FUNCTIONS

In this appendix we summarize various important properties of the threshold functions $\Psi_{n;m}^{p;q}$, $\tilde{\Psi}_{n;m;l}^{p;q}$, Φ_n^p and $\tilde{\Phi}_n^p$ which are defined by Eqs. (A29), (A30) and (A31).

Expanding the generalized threshold functions $\Psi_{n;m}^{p;q}$, $\tilde{\Psi}_{n;m;l}^{p;q}$ about vanishing couplings yields

$$\Psi^{p;q}_{n;m}(g_k\beta_k,-2\lambda_k;d)=\left(-2\frac{d-1}{d-2}\right)^q\Phi^{p+q-m}_n(0)+2(p+q)\left(-2\frac{d-1}{d-2}\right)^q\Phi^{p+q-m+1}_n(0)\lambda_k+2(p+q)(p+q+1)$$
$$\times\left(-2\frac{d-1}{d-2}\right)^q\Phi^{p+q-m+2}_n(0)\lambda_k^2-32\pi q\left(-2\frac{d-1}{d-2}\right)^{q+1}\Phi^{p+q-m-1}_n(0)g_k\beta_k+\mathcal{O}(\mathrm{g}^3), \tag{F1}$$

$$\tilde\Psi^{p;q}_{n;m;0}(g_k\beta_k,-2\lambda_k;d)=\left(-2\frac{d-1}{d-2}\right)^q\tilde\Phi^{p+q-m}_n(0)+2(p+q)\left(-2\frac{d-1}{d-2}\right)^q\tilde\Phi^{p+q-m+1}_n(0)\lambda_k+2(p+q)(p+q+1)$$
$$\times\left(-2\frac{d-1}{d-2}\right)^q\tilde\Phi^{p+q-m+2}_n(0)\lambda_k^2-32\pi q\left(-2\frac{d-1}{d-2}\right)^{q+1}\tilde\Phi^{p+q-m-1}_n(0)g_k\beta_k+\mathcal{O}(\mathrm{g}^3), \tag{F2}$$

$$\tilde\Psi^{p;q}_{n;m;1}(g_k\beta_k,-2\lambda_k;d)=\left(-2\frac{d-1}{d-2}\right)^q(\tilde\Phi^{p+q-m-1}_n(0)+n\tilde\Phi^{p+q-m}_{n+1}(0))+2(p+q)\left(-2\frac{d-1}{d-2}\right)^q$$
$$\times(\tilde\Phi^{p+q-m}_n(0)+n\tilde\Phi^{p+q-m+1}_{n+1}(0))\lambda_k+2(p+q)(p+q+1)\left(-2\frac{d-1}{d-2}\right)^q$$
$$\times(\tilde\Phi^{p+q-m+1}_n(0)+n\tilde\Phi^{p+q-m+2}_{n+1}(0))\lambda_k^2-32\pi q\left(-2\frac{d-1}{d-2}\right)^{q+1}(\tilde\Phi^{p+q-m-2}_n(0)$$
$$+n\tilde\Phi^{p+q-m-1}_{n+1}(0))g_k\beta_k+\mathcal{O}(\mathrm{g}^3). \tag{F3}$$

Here $\mathcal{O}(\mathrm{g}^3)$ stands for terms of third and higher orders in the couplings $\mathrm{g}_1(k)=\lambda_k$, $\mathrm{g}_2(k)=g_k$ and $\mathrm{g}_3(k)=\beta_k$. Quite remarkably, every fixed order of these expansions depends only on the "conventional" threshold functions Φ^p_n and $\tilde\Phi^p_n$ at vanishing arguments.

By using Eq. (A31) the corresponding expansions of Φ^p_n and $\tilde\Phi^p_n$ about vanishing argument can be read off directly from Eqs. (F1) and (F2). They are given by

$$\begin{aligned}\Phi^p_n(-2\lambda_k)&=\Phi^p_n(0)+2p\Phi^{p+1}_n(0)\lambda_k+2p(p+1)\Phi^{p+2}_n(0)\lambda_k^2+\mathcal{O}(\lambda_k^3)\\ \tilde\Phi^p_n(-2\lambda_k)&=\tilde\Phi^p_n(0)+2p\tilde\Phi^{p+1}_n(0)\lambda_k+2p(p+1)\tilde\Phi^{p+2}_n(0)\lambda_k^2+\mathcal{O}(\lambda_k^3).\end{aligned} \tag{F4}$$

For $n=0$ the threshold functions are universal in the sense that they do not depend on $R^{(0)}(y)$. In fact, setting $n=0$ in $\Psi^{p;q}_{n;m}$, $\tilde\Psi^{p;q}_{n;m;l}$, Φ^p_n and $\tilde\Phi^p_n$ leads to

$$\Psi^{p;q}_{0;m}(v,w;d)=\tilde\Psi^{p;q}_{0;m;l}(v,w;d)=(1+w)^{-p}\left(32\pi v-\frac{d-2}{2(d-1)}(1+w)\right)^{-q} \tag{F5}$$

and

$$\Phi^p_0(w)=\tilde\Phi^p_0(w)=(1+w)^{-p}. \tag{F6}$$

There exists a second class of universal values of certain threshold functions. Using the boundary conditions for $R^{(0)}(y)$ one may easily verify that, for vanishing argument and for $n+1=p\geqslant 1$, Φ^p_n assumes the universal value

$$\Phi^p_{p-1}(0)=\frac{1}{\Gamma(p)}. \tag{F7}$$

Let us now be more specific and opt for the family of exponential cutoffs (3.11). In this case the integral that defines Φ^p_n can be carried out analytically for the vanishing argument. Using the integral representation of the polylogarithm [49],

$$\mathrm{Li}_n(x)=\frac{1}{\Gamma(n)}\int_0^\infty dz\frac{xz^{n-1}}{e^z-x}, \tag{F8}$$

one obtains for $p=0,\ldots,4$

$$\Phi^0_n(0)=n(n+1)s^{-n}\zeta(n+1) \tag{F9}$$

$$\Phi^1_n(0)=ns^{-n}\{\zeta(n+1)-\mathrm{Li}_{n+1}(1-s)\} \tag{F10}$$

$$\Phi^2_n(0)=\begin{cases} s^{2-n}(1-s)^{-1}\mathrm{Li}_{n-1}(1-s), & s\neq 1,\\ 1, & s=1\end{cases} \tag{F11}$$

$$\Phi^3_n(0)=\begin{cases} [2(n-1)(1-s)^2]^{-1}s^{3-n}\{(2-s)\mathrm{Li}_{n-2}(1-s)-s\mathrm{Li}_{n-3}(1-s)\}, & n\neq 1, s\neq 1\\ [2^{n-1}(n-1)]^{-1}(2^{n-1}-1), & n\neq 1, s=1\\ -[2(1-s)^2]^{-1}s^{3-n}\{(2-s)\mathrm{Li}^{(1,0)}_{-1}(1-s)-s\mathrm{Li}^{(1,0)}_{-2}(1-s)\}, & n=1, s\neq 1\\ \ln(2), & n=s=1\end{cases} \tag{F12}$$

$$\Phi^4_n(0)=\begin{cases} [6(n-1)(n-2)(1-s)^3]^{-1}s^{4-n}\{2(s^2-3s+3)\mathrm{Li}_{n-3}(1-s)-3s(2-s)\mathrm{Li}_{n-4}(1-s) & \\ \quad +s^2\mathrm{Li}_{n-5}(1-s)\}, & n\neq 1,2, s\neq 1,\\ [(n-1)(n-2)]^{-1}(1-2^{3-n}+3^{2-n}), & n\neq 1,2, s=1,\\ [6(2n-3)(1-s)^3]^{-1}s^{4-n}\{2(s^2-3s+3)\mathrm{Li}^{(1,0)}_{n-3}(1-s)-3s(2-s)\mathrm{Li}^{(1,0)}_{n-4}(1-s) & \\ \quad +s^2\mathrm{Li}^{(1,0)}_{n-5}(1-s)\}, & n\in\{1,2\}, s\neq 1,\\ \ln(27/16), & n=s=1,\\ \ln(4/3), & n=2, s=1.\end{cases} \tag{F13}$$

Here we defined

$$\mathrm{Li}^{(k,l)}_n(x)\equiv \frac{d^k}{dn^k}\frac{d^l}{dx^l}\mathrm{Li}_n(x) \tag{F14}$$

and used the relations

$$\mathrm{Li}_n(1)=\zeta(n),\quad \mathrm{Li}^{(0,1)}_n(x)=\frac{\mathrm{Li}_{n-1}(x)}{x} \tag{F15}$$

with ζ denoting the Riemann zeta function. For nonvanishing arguments an analytic solution to the integrals defining the threshold functions is not known.

For the exponential cutoff (3.11) with $s=1$ there even exists a very useful relation among $\Phi^p_n(0)$ and $\tilde{\Phi}^p_n(0)$. One may easily verify that

$$\Phi^p_n(0)=\tilde{\Phi}^{p-1}_n(0). \tag{F16}$$

This relation allows us to calculate the $\tilde{\Phi}^p_n(0)$ integrals analytically as well.

APPENDIX G: PROOF OF THE INEQUALITY (6.14)

In this appendix we prove the inequality (6.14). As a first step we consider the function

$$f(a,y)=a\sqrt{a(a-y)}-\sqrt{1-y} \tag{G1}$$

with $a>1$ and $0\leqslant y\leqslant 2/5$. [For $a=1$ this function vanishes identically: $f(1,y)\equiv 0$.] An upper bound for this function may be obtained as follows. For the first two derivatives of f with respect to y we obtain

$$\begin{aligned} f^{(0,1)}(a,y)&\equiv\frac{d}{dy}f(a,y)\\ &=-\frac{a^2}{2\sqrt{a(a-y)}}+\frac{1}{2\sqrt{1-y}},\\ f^{(0,2)}(a,y)&\equiv\frac{d^2}{dy^2}f(a,y)\\ &=-\frac{a^3}{4[a(a-y)]^{3/2}}+\frac{1}{4(1-y)^{3/2}}.\end{aligned} \tag{G2}$$

Solving $f^{(0,1)}(a,y)=0$ for y leads to the *single* solution

$$y=y_0\equiv\frac{a(a+1)}{a^2+a+1}. \tag{G3}$$

Since $f^{(0,2)}(a,y_0)=(a^2+a+1)^{3/2}(a^3-1)/(4a^3)>0$, we have $f(a,y_0)\leqslant f(a,y)$ for all $y\in[0,1]$ and $a>1$. Hence, for $a>1$ fixed but arbitrary, f monotonically decreases in the interval $y\in[0,y_0]$ where $y_0<1$.

Furthermore, $y_0=y_0(a)$ is a monotonically increasing function of a for all $a\geqslant 1$. Therefore we have that $y_0(a)\geqslant y_0(a=1)=2/3$. As a consequence, f monotonically decreases in the interval $0 \leqslant y \leqslant 2/3$ for any value of $a>1$. Since we restricted our considerations to $y \in [0,2/5]$ we obtain $f(a,y) \leqslant f(a,0)$.

$(\mathcal{R}_k)_{\overline{\phi\sigma}}$ may be obtained from f by replacing

$$a \to \frac{\Lambda_l(4,0)/k^2 + R^{(0)}(\Lambda_l(4,0)/k^2)}{\Lambda_l(4,0)/k^2},$$

$$y \to \frac{R}{3\Lambda_l(4,0)} = \frac{4}{l(l+3)}, \tag{G4}$$

and multiplying the result by $9\beta_k(\Delta_l(4,0))^2/8$. Note that for all $l \geqslant 2$ we have $\Lambda_l(4,0) \geqslant 5R/6$ so that $y \leqslant 2/5$ is indeed satisfied. Moreover, $a>1$ is satisfied as long as $R^{(0)}(\Lambda_l(4,0)/k^2)>0$. If $R^{(0)}(\Lambda_l(4,0)/k^2)=0$, the cutoff in the scalar sector is zero anyway: $(\mathcal{R}_k)_{\overline{\sigma\sigma}} = (\mathcal{R}_k)_{\overline{\phi\sigma}} = (\mathcal{R}_k)_{\overline{\phi\phi}} = 0$. Hence, for positive values of β_k, $f(a,y) \leqslant f(a,0)$ leads to

$$(\mathcal{R}_k)_{\overline{\phi\sigma}} \leqslant \frac{9}{8}\beta_k k^4 R^{(0)}(\Lambda_l(4,0)/k^2)\{2\Lambda_l(4,0)/k^2 + R^{(0)}(\Lambda_l(4,0)/k^2)\}. \tag{G5}$$

Next we insert $(\mathcal{R}_k)_{\overline{\sigma\sigma}}$ and $(\mathcal{R}_k)_{\overline{\phi\phi}}$ of Eq. (6.12) into $(\mathcal{R}_k)_{\overline{\phi\phi}}(\mathcal{R}_k)_{\overline{\sigma\sigma}} - (\mathcal{R}_k)^2_{\overline{\phi\sigma}}$ and then use Eq. (G5). This yields

$$\begin{aligned}(\mathcal{R}_k)_{\overline{\phi\phi}}(\mathcal{R}_k)_{\overline{\sigma\sigma}} - (\mathcal{R}_k)^2_{\overline{\phi\sigma}} &\geqslant \left(\frac{k^4 R^{(0)}(\Lambda_l(4,0)/k^2)}{32\pi g_k}\right)^2 \\ &\times \Big\{ 18\pi g_k \beta_k (2\Lambda_l(4,0)/k^2 \\ &+ R^{(0)}(\Lambda_l(4,0)/k^2)) - \frac{3}{16}\Big\}. \end{aligned} \tag{G6}$$

Obviously the positivity condition $(\mathcal{R}_k)_{\overline{\phi\phi}}(\mathcal{R}_k)_{\overline{\sigma\sigma}} - (\mathcal{R}_k)^2_{\overline{\phi\sigma}} > 0$ now boils down to

$$v(k^2,l,R) + 2\Lambda_l(4,0)/k^2 + R^{(0)}(\Lambda_l(4,0)/k^2) > \frac{1}{96\pi g_k \beta_k}. \tag{G7}$$

Here v is the non-negative function of k, l and R which represents the contributions to $-(\mathcal{R}_k)^2_{\overline{\phi\sigma}}$ neglected on the RHS of Eq. (G6). We see that Eq. (6.14) is a sufficient condition for the inequality (G7) to be valid, i.e., for $(\mathcal{R}_k)_{\overline{\phi\phi}}(\mathcal{R}_k)_{\overline{\sigma\sigma}} - (\mathcal{R}_k)^2_{\overline{\phi\sigma}}$ to be positive. This is what we wanted to prove.

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