Coherent state induced star product on $\mathbb{R}^3_\lambda$ and the fuzzy sphere

A. B. Hammou
The Abdus Salam International Center for Theoretical Physics, Strada Costiera 11, Trieste, Italy

M. Lagraa
The Abdus Salam International Center for Theoretical Physics, Strada Costiera 11, Trieste, Italy and Laboratoire de Physique Théorique, Université d'Oran Es-Sénia, 31100, Algeria

M. M. Sheikh-Jabbari
Department of Physics, Stanford University, Stanford, California 94305-4060
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Using the Hopf fibration and starting from a four-dimensional noncommutative Moyal plane $\mathbb{R}^2_\theta\times\mathbb{R}^2_\theta$ we obtain a star product for the noncommutative (fuzzy) $\mathbb{R}^3_\lambda$ defined by $[x^i,x^j]=i\lambda\epsilon_{ijk}x^k$. Furthermore, we show that there is a projection function which allows us to reduce the functions on $\mathbb{R}^3_\lambda$ to that of the fuzzy sphere, and hence we introduce a new star product on the fuzzy sphere. We will then briefly discuss how using our method one can extract information about the field theory on the fuzzy sphere and $\mathbb{R}^3_\lambda$ from the corresponding field theories on $\mathbb{R}^2_\theta\times\mathbb{R}^2_\theta$ space.

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I. INTRODUCTION AND PRELIMINARIES

In the past two years, motivated by string theory [1], the theories on the noncommutative Moyal plane have been extensively studied (for a review see [2]). The Moyal plane can be defined through the functions of operator valued coordinates $\hat{X}^i$ satisfying

$$[\hat{X}^i,\hat{X}^j]=i\Theta^{ij},\quad i=1,\ldots,d, \tag{1.1}$$

where Θ^{ij} is a constant antisymmetric tensor. We will denote such spaces by $\mathbb{R}^d_\Theta$. Let us restrict ourselves to the noncommutative spaces (not space-times) and take $d=3$. In this case, there always exists a rotation which reduces the nonzero components of Θ_{ij} to two,[1] e.g., $\Theta_{12}=-\Theta_{21}=\theta$ and therefore $\mathbb{R}^3_\Theta\simeq\mathbb{R}^2_\theta\times\mathbb{R}$. As a first generalization of the Moyal plane one may consider Θ_{ij} to be linearly X dependent: i.e.,

$$[\hat{X}^i,\hat{X}^j]=i\lambda\epsilon^{ijk}\hat{X}^k,\quad i=1,2,3. \tag{1.2}$$

We will denote this space by $\mathbb{R}^3_\lambda$. It has been shown that these spaces can also arise within string theory [3–6]. Equation (1.2) resembles the su(2) algebra whose generators are $\hat{X}^i/\lambda$. In fact, in general $\hat{X}^i$'s are reducible representations of su(2). The irreducible representations of that algebra given by $(2J+1)\times(2J+1)$ Hermitian matrices will reduce $\mathbb{R}^3_\lambda$ to what is called a fuzzy sphere $S^2_{\lambda,J}$ [7–12].[2] In other words the fuzzy sphere $S^2_{\lambda,J}$ is determined by the algebra (1.2) subjected to

$$\sum_{i=1}^{3}\hat{X}_i^2=\lambda^2J(J+1); \tag{1.3}$$

i.e., the radius of the fuzzy sphere is given by $\lambda\sqrt{J(J+1)}$. Therefore the full $\mathbb{R}^3_\lambda$ can be obtained when we consider the set of fuzzy spheres with all possible radii: $\mathbb{R}^3_\lambda=\Sigma_{J=0}^{\infty}S^2_{\lambda,J}$. We should warn the reader that $\mathbb{R}^3_\lambda$ in the $\lambda\to0$ limit will not reduce to $\mathbb{R}^3$ [8]. This will be seen more explicitly in Sec. II (and in particular Sec. II B). It has been shown that the fuzzy sphere in certain limits can be reduced to the commutative sphere S^2 and also the Moyal plane $\mathbb{R}^2_\theta$ [14].

In order to formulate physics on noncommutative spaces, one should be able to pass to the language of fields (functions) instead of operators, where the algebra of operators is translated to the algebra of functions on a proper space, though with a product different from the usual product of functions. This product is usually called star product. In fact there exists a one-to-one correspondence (called Weyl correspondence) between the operators and the functions [15,16]. Given an operator $\mathcal{O}_f$,

$$\mathcal{O}_f=\int dk\,e^{ik\cdot\hat{X}}\tilde{f}(k), \tag{1.4}$$

the corresponding function is[3]

$$f(x)=\int dk\,e^{ik\cdot x}\tilde{f}(k). \tag{1.5}$$

Then, the algebra of $\hat{X}$ will induce a star product on functions,

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[1] We would like to comment that this is not always possible for the compact noncommutative three manifolds such as noncommutative three torus.

[2] Recently there has been a complete review over the field [13].

[3] We only consider the functions which admit the Fourier expansion.

$$\mathcal{O}_f \cdot \mathcal{O}_g = \int dk dp \, e^{ik\cdot \hat{X}} e^{ip\cdot \hat{X}} \tilde{f}(k)\tilde{g}(p)$$
$$= \int dk dp \, e^{i(k+p)\cdot \hat{X} - (i/2)k_i p_j [\hat{X}^i,\hat{X}^j] + \cdots} \times \tilde{f}(k)\tilde{g}(p). \tag{1.6}$$

For the case of the Moyal plane, the above Hausdorff expansion terminates and then we obtain the so-called Moyal star product,

$$\mathcal{O}_f \cdot \mathcal{O}_g \leftrightarrow f \star g = e^{i\Theta^{ij}(\partial/\partial x_i)(\partial/\partial y_j)} f(x) g(y)|_{x=y}. \tag{1.7}$$

It is easy to check that $x^i \star x^j = x^i x^j + (i/2)\Theta^{ij}$ and hence $\{x^i, x^j\} = x^i \star x^j - x^j \star x^i = i\Theta^{ij}$.

In running the above Weyl-Moyal machinery we have implicitly *assumed* that $e^{ik\cdot\hat{X}} \leftrightarrow e^{ik\cdot x}$ or equivalently $e^{ik\cdot x} = (e\star)^{ik\cdot x} = 1 + ik\cdot x - \frac{1}{2!}(k\cdot x)\star(k\cdot x) + \cdots$. However, to obtain the algebra (1.1) we do not necessarily need to impose this condition (which is in fact a special way of ordering, the Weyl ordering). More explicitly one can define infinitely many star products all resulting in the same algebra. For example, in Eq. (1.7) if we add a general symmetric matrix to Θ^{ij} we will find the same algebra as before. This is equivalent to taking

$$x^i \star x^j = x^i x^j + \frac{i}{2}\Theta^{ij} + A^{ij}, \tag{1.8}$$

where A^{ij}'s are constants and $A^{ij} = A^{ji}$. In fact, the above generalized star products correspond to different ways of ordering in the operator language. For the $\mathbb{R}^2_\theta$ case if we choose $A^{ij} = (L/2)\delta^{ij}$ (this is always possible with a proper rotation), we will obtain a new star product in which

$$(e\star)^{ik\cdot x} = e^{ik\cdot x} e^{-Lk^2/4}. \tag{1.9}$$

More precisely, different star products of Eq. (1.9) resulting from different orderings can be related by introducing the proper weight functions into the "Tr" over the algebra (which in the Moyal case is simply integral over the whole space). Physically this means that instead of the usual simple waves we are expanding our fields in terms of wave packets of the width $\sqrt{L/2}$. It is easy to show that the above star product will lead exactly to the same field theory results as the Moyal star product. More explicitly, different noncommutative versions of a given field theory (corresponding to different star products resulting from different A^{ij}'s) are all related by a field redefinition.[4]

Another natural way of ordering arises if instead of Fourier expansion we use the Laurent expansion of the functions and the corresponding harmonic oscillator basis [17]. Let us consider $\mathbb{R}^2_\theta$ and define

$$z = \frac{x^1 + ix^2}{\sqrt{2\theta}}, \tag{1.10}$$

then $[z, \bar{z}]_\star = 1$. Any function $f(x^1, x^2)$ can be expanded as

$$f(z,\bar{z}) = \sum_{n,m} f_{mn} \bar{z}^m z^n. \tag{1.11}$$

Now replacing z and $\bar{z}$ by harmonic oscillator creation and annihilation operators a and $a^\dagger$ we will obtain the corresponding operator which is "normal ordered." We will show in Sec. II that this normal ordering yields the following star product:

$$z \star \bar{z} = z\bar{z} + 1, \quad \bar{z} \star z = z\bar{z}, \tag{1.12}$$

which exactly corresponds to Eq. (1.8) with $A^{ij} = (\theta/2)\delta^{ij}$.

In order to study field theories on the $\mathbb{R}^3_\lambda$ we need to build the corresponding star product. Along the above arguments, depending on the ordering we use for the operators we will find various star products on $\mathbb{R}^3_\lambda$ (and similarly on $S^2_{\lambda,J}$). If we take the Weyl ordering [i.e., imposing the condition $(e\star)^{ik\cdot x} = e^{ik\cdot x}$] we will end up with the following star product:

$$x^i \star x^j = x^i x^j + \frac{i\lambda}{2}\epsilon^{ijk} x_k. \tag{1.13}$$

However, this star product is not so convenient for doing field theory on $\mathbb{R}^3_\lambda$ (it is suitable for perturbative expansions in powers of λ). In this work using the normal ordering of operators on the Moyal plane, we obtain a new star product on $\mathbb{R}^3_\lambda$. To obtain the star product we start with a four dimensional Moyal plane, $\mathbb{R}^2_\theta \times \mathbb{R}^2_\theta$, parametrized by z_1, z_2 and choose the star product induced by the normal ordering. Recalling the Hopf fibration for $\mathbb{R}^3_\lambda$ [13], we show that the algebra of operators on $\mathbb{R}^3_\lambda$ (for $\lambda = \theta$) is equivalent to a subalgebra of the functions on $\mathbb{R}^2_\theta \times \mathbb{R}^2_\theta$ which are invariant under $z_1 \to e^{i\alpha} z_1$ and $z_2 \to e^{i\alpha} z_2$, or equivalently $\mathbb{R}^3_\lambda \simeq (\mathbb{R}^2_\theta \times \mathbb{R}^2_\theta)/S^1$. In this way one can read off the form of the star product in $\mathbb{R}^3_\lambda$ induced by the star product on the four-dimensional Moyal plane. In other words there exists a dictionary which allows us to translate $\mathbb{R}^3_\lambda$, the algebra of functions, and hence the field theory on that, into that of $\mathbb{R}^2_\theta \times \mathbb{R}^2_\theta$. As we discussed, the representations of the $\mathbb{R}^3_\lambda$ can be understood as a sum of irreducible representations on fuzzy spheres with different radii. We show that there is a projection operator, P_J, which projects the functions on $\mathbb{R}^3_\lambda$ on the $S^2_{\lambda,J}$. Hence we can extend our dictionary to translate the field theories on the fuzzy sphere in four-dimensional field theories on the Moyal plane.

The paper is organized as follows. In Sec. II, we first review the harmonic oscillator basis and coherent states and then use this basis to extract a new star product on the Moyal plane. We use this star product to read off the induced star product on the $\mathbb{R}^3_\lambda$. We also show how the operators (and functions) and, in particular, the derivative operators on $\mathbb{R}^3_\lambda$

[4]We are grateful to L. Susskind for a discussion on this point.

are related to the four-dimensional operators (and functions). In Sec. III, we introduce the projection operator P_J which enables us to single out an irreducible $(2J+1)\times(2J+1)$ dimensional representation out of the algebra of functions on the $\mathbb{R}^3_\lambda$. We have moved some other useful identities involving P_J to the Appendixes. In Sec. IV, we discuss how by using our dictionary, the field theories on $\mathbb{R}^3_\lambda$ and the fuzzy sphere can be studied through field theories on the four-dimensional Moyal plane. The last section contains our conclusions and discussions.

II. STAR PRODUCT ON $\mathbb{R}^3_\lambda$

In this section, first we will review and generalize the star product deduced from coherent states [8,11,18] and then we will construct the star product on fuzzy three-dimensional vector space $\mathbb{R}^3_\lambda$ and its projection on the fuzzy spheres with given radius.

The sphere can be interpreted as the Hopf fibration,

$$S^3=\{\vec{z}\in\mathbb{C}^2;\ \ \bar{z}z=\rho^2\}\to S^2=\{x=(x^1,x^2,x^3)\in\mathbb{R}^3\}, \tag{2.1}$$

with

$$x^i=\tfrac{1}{2}\bar{z}_\alpha\sigma^i_{\alpha\beta}z_\beta, \quad i=1,2,3, \tag{2.2}$$

where the bar denotes complex conjugation and σ^i's are Pauli matrices. In this approach fields are functions of complex variables $z_\alpha,\bar{z}_\alpha$, $\alpha=1,2$. The relation $\bar{z}z=\rho^2$ leads to $\Sigma_i(x^i)^2=x^{02}$, with

$$x^0=\tfrac{1}{2}\bar{z}_\alpha z_\alpha. \tag{2.3}$$

Since x^i's are invariant under the transformation

$$z\to e^{i\alpha}z, \quad \bar{z}\to e^{-i\alpha}\bar{z}.$$

The above Hopf fibration can be viewed as coordinates on $S^2=CP^1\equiv S^3/U(1)$.

To get a noncommutative version of the above Hopf fibration it is enough to make the coordinates z_α and $\bar{z}_\alpha$ be noncommutative: $[z_\alpha,\bar{z}_\beta]_\star=\delta_{\alpha\beta}$ [with the star product defined in Eq. (1.12)]. Then the corresponding operator language is obtained by replacing coordinates z_α and $\bar{z}_\alpha$ with the creation and annihilation operators of a two-dimensional harmonic oscillator a_α and $a^\dagger_\alpha$,

$$[a_\alpha,a^\dagger_\beta]=\delta_{\alpha\beta}.$$

We note that the coordinates z_α are scaled so that they are dimensionless [as in Eq. (1.10)] and hence θ is scaled to one. However, θ can always be reintroduced on a dimensional analysis.

Given

$$\hat{X}^i=\tfrac{1}{2}a^\dagger_\alpha\sigma^i_{\alpha\beta}a_\beta, \tag{2.4}$$

it is straightforward to show that

$$[\hat{X}^i,\hat{X}^j]=i\epsilon^{ijk}\hat{X}^k$$

(if we reintroduce θ, the above will become $[\hat{X}^i,\hat{X}^j]=i\theta\epsilon^{ijk}\hat{X}^k$). This is the key observation which relates $\mathbb{R}^2_\theta\times\mathbb{R}^2_\theta$ to the $\mathbb{R}^3_\lambda$ (with $\lambda=\theta$). In this section using the above realization we obtain an explicit form of the star product on $\mathbb{R}^3_\lambda$.

A. Coherent states

Let $|n_1,n_2\rangle$ represent the energy eigenstates of the two-dimensional harmonic oscillators whose creation and annihilation operators $a^\dagger_\alpha$ and a_α ($\alpha=1,2$) satisfy the above commutation relations. To any vector $\vec{z}\in\mathbb{C}^2$ one can assign a coherent state,

$$|z_1,z_2\rangle\equiv|\vec{z}\rangle=e^{-(\bar{z}z/2)}e^{z_\alpha a^\dagger_\alpha}|0,0\rangle, \tag{2.5}$$

where $\bar{z}z=\bar{z}_\alpha z_\alpha$. The coherent states $|\vec{z}\rangle$ are normalized $\langle\vec{z}|\vec{z}\rangle=1$, form an (overcomplete) basis for the Hilbert space $\mathcal{H}$, and are eigenstates of the annihilation operators $a_\alpha|\vec{z}\rangle=z_\alpha|\vec{z}\rangle$. They are not orthonormal but satisfy

$$\langle\vec{\eta}|\vec{z}\rangle=e^{-(\bar{\eta}\eta/2)-(\bar{z}z/2)+\bar{\eta}z}. \tag{2.6}$$

The completeness relation reads

$$\int d\mu(\bar{z},z)|\vec{z}\rangle\langle\vec{z}|=1, \tag{2.7}$$

where $d\mu(\bar{z},z)=(1/\pi^2)d\bar{z}_1dz_1d\bar{z}_2dz_2$ is the measure on the two-dimensional complex plane $\mathbb{C}^2$.

To any operator $\hat{f}$ belonging to the algebra $\hat{\mathcal{A}}_4$ generated by the creation and annihilation operators, we can associate a function $f(\vec{z},\vec{\bar{z}})$ belonging to the algebra of functions on $\mathbb{R}^2_\theta\times\mathbb{R}^2_\theta$ denoted by $\mathcal{A}_4$ and generated by z_α and $\bar{z}_\alpha$ as

$$\langle\vec{z}|\hat{f}|\vec{z}\rangle=f(\vec{z},\vec{\bar{z}}). \tag{2.8}$$

Then the product of operators corresponds to an associative star product of the corresponding functions as

$$(f\star g)(\vec{z},\vec{\bar{z}})=\langle\vec{z}|\hat{f}\hat{g}|\vec{z}\rangle=\int d\mu(\bar{\eta},\eta)\langle\vec{z}|\hat{f}|\vec{\eta}\rangle\langle\vec{\eta}|\hat{g}|\vec{z}\rangle. \tag{2.9}$$

To get the explicit form of the star product (2.9), following [18] we introduce the translation operators,

$$e^{-z_\alpha(\partial/\partial\eta_\alpha)+\eta_\alpha(\partial/\partial z_\alpha)}f(\vec{z},\vec{\bar{z}})=\frac{\langle\vec{z}|\hat{f}|\vec{\eta}\rangle}{\langle\vec{z}|\vec{\eta}\rangle}$$
$$=:e^{(\eta_\alpha-z_\alpha)(\partial/\partial z_\alpha)}:f(\vec{z},\vec{\bar{z}}), \tag{2.10}$$

where : : means that the derivatives are ordered to the right in each term in the Taylor expansion of the exponential. Substituting Eq. (2.10) into Eq. (2.9) and performing the integration we obtain

$$(f\star g)(\vec{z},\vec{\bar{z}})=f(\vec{z},\vec{\bar{z}})\exp\frac{\overleftarrow{\partial}}{\partial z_\alpha}\frac{\vec{\partial}}{\partial \bar{z}_\alpha}g(\vec{z},\vec{\bar{z}}). \tag{2.11}$$

This is a new star product which is resulting from the normal ordering in the operator language. We note that this star product is different from the Moyal star product [18].

Besides the coherent states $|z_\alpha\rangle$ we have the usual two-dimensional harmonic oscillator basis $|n_1,n_2\rangle$,

$$|n_1,n_2\rangle=\frac{(a_1^\dagger)^{n_1}}{\sqrt{(n_1)!}}\frac{(a_2^\dagger)^{n_2}}{\sqrt{(n_2)!}}|0,0\rangle.$$

However, it turns out that for our purpose (reduction of the four-dimensional algebra to that of $\mathbb{R}^3_\lambda$) it is more convenient to use the Schwinger basis,

$$|j,m\rangle=\frac{(a_1^\dagger)^{j+m}}{\sqrt{(j+m)!}}\frac{(a_2^\dagger)^{j-m}}{\sqrt{(j-m)!}}|0,0\rangle, \tag{2.12}$$

where $j=0,\frac{1}{2},1,\ldots,\infty$, and m runs by integer steps over the range $-j\leqslant m\leqslant j$. The coherent state can be expanded in the $|j,m\rangle$ basis,

$$|\vec{z}\rangle=\sum_{j=0}^{\infty}\frac{e^{-(\bar{z}z/2)}}{\sqrt{(2j)!}}\sum_{m=-j}^{m=j}z_1^{j+m}z_2^{j-m}\sqrt{C_{j+m}^{2j}}|j,m\rangle, \tag{2.13}$$

with

$$C_{j+m}^{2j}=\frac{(2j)!}{(j+m)!(j-m)!}.$$

Now we consider the subalgebra $\hat{\mathcal{A}}_3\subset\hat{\mathcal{A}}_4$ generated by $\hat{X}^i$, whose corresponding subalgebra of functions $\mathcal{A}_3\subset\mathcal{A}_4$ is generated by $x^i=\frac{1}{2}\bar{z}_\alpha\sigma^i_{\alpha\beta}z_\beta$. Noting that $\hat{X}^0$ $[\hat{X}^0(\hat{X}^0+1)=\Sigma_i(\hat{X}^i)^2]$ commutes with all $\hat{X}^i$'s (i.e., $[\hat{X}^0,\hat{f}(\hat{X}^i)]=0$ for any function $\hat{f}$), one can define the $\hat{\mathcal{A}}_3$ algebra as the subalgebra of $\hat{\mathcal{A}}_4$ whose elements are commuting with $\hat{X}^0$, i.e.,

$$\forall\hat{f}(a^\dagger,a)\in\hat{\mathcal{A}}_4,\quad[\hat{X}^0,\hat{f}]=0\Rightarrow\hat{f}\in\hat{\mathcal{A}}_3. \tag{2.14}$$

At the level of functions the commutator with $\hat{X}^0$ corresponds to the derivative operator $\mathcal{L}_0$,

$$i[\hat{X}^0,\hat{f}]\leftrightarrow\mathcal{L}_0f\equiv\frac{i}{2}(\bar{z}_\alpha\bar{\partial}_\alpha-z_\alpha\partial_\alpha)f(z,\bar{z}). \tag{2.15}$$

Therefore, the elements of the algebra $\mathcal{A}_3$ are functions of z_α and $\bar{z}_\alpha$ subjected to $\mathcal{L}_0f(z,\bar{z})=0$. We would like to stress that the operator $\mathcal{L}_0$ is in fact a derivative operator with respect to the star product (2.11),

$$\mathcal{L}_0(f\star g)=(\mathcal{L}_0f)\star g+f\star(\mathcal{L}_0g), \tag{2.16}$$

and hence the subalgebra $\mathcal{A}_3$ is closed under the star product (2.11), i.e., $f,g\in\mathcal{A}_3$ then $f\star g\in\mathcal{A}_3$. Using this property and the fact that all the elements of $\mathcal{A}_3$ can be represented as functions of x^i and x^0, one can rewrite the star product of Eq. (2.11) in terms of x^i's and their derivatives. To start with, we recall that

$$\frac{\partial}{\partial z_\alpha}=\frac{1}{2}\bar{z}_\beta\sigma^i_{\beta\alpha}\frac{\partial}{\partial x^i},$$

$$\frac{\partial}{\partial\bar{z}_\alpha}=\frac{1}{2}\frac{\partial}{\partial x^i}\sigma^i_{\alpha\beta}z_\beta. \tag{2.17}$$

Note that the above expressions are only true when derivatives are acting on the functions in $\mathcal{A}_3$. Now using the relation

$$\bar{z}_\beta\sigma^i_{\beta\alpha}\sigma^j_{\alpha\rho}z_\rho=2(\delta^{ij}x^0+i\epsilon^{ijk}x^k),$$

we obtain the desired star product in terms of the coordinates x^i,

$$(f\star g)(\vec{x})=\exp\left[\frac{1}{2}(\delta^{ij}x^0+i\epsilon^{ijk}x^k)\frac{\partial}{\partial u^i}\frac{\partial}{\partial v^j}\right]f(\vec{u})g(\vec{v})\Big|_{u=v=x}, \tag{2.18}$$

for any functions $f,g\in\mathcal{A}_3$. Note that the exponential in the expression of the star product (2.18) should be understood by its Taylor expansion. We would like to stress that our new star product (2.18), similar to that of Eq. (2.11), is associative. From Eq. (2.18), it follows that

$$x^i\star x^j=x^ix^j+\tfrac{1}{2}(\delta^{ij}x^0+i\epsilon^{ijk}x^k) \tag{2.19}$$

$$x^0\star x^i=x^0x^i+\tfrac{1}{2}x^i \tag{2.20}$$

$$x^0\star x^0=x^0(x^0+\tfrac{1}{2}), \tag{2.21}$$

where we have used the fact that $x^0=\sqrt{x^ix^i}$ and hence $\partial x^0/\partial x^i=x^i/x^0$. To show the relations (2.20) and (2.21) one can use the following equivalent definition:

$$(f\star g)(x^i,x^0)=f\left(x^i+\frac{1}{2}(\delta^{ij}x^0+i\epsilon^{ijk}x^k)\times\frac{\partial}{\partial y^j},x^0+\frac{1}{2}x^i\frac{\partial}{\partial y^i}\right)g(y^i,y^0)\Big|_{y=x}b.$$

The above is valid for any function which admits Taylor expansion. Using Eq. (2.19) it is easy to check that

$$[x^i,x^j]_\star=i\epsilon^{ijk}x^k. \tag{2.22}$$

Furthermore,

$$[x^0,x^i]_\star=0,$$

$$\delta_{ij}x^i\star x^j=\vec{x}\star\vec{x}=x^0\star(x^0+1).$$

Equation (2.22) shows that the algebra $\mathcal{A}_3$ equipped with the star product (2.18) can be viewed as an algebra of functions on the $\mathbb{R}^3_\lambda$ endowed with a Euclidean metric δ_{ij}. It is easy to see that the star product (2.18) is invariant under the classical SO(3) group. This SO(3) symmetry of $\mathbb{R}^3_\lambda$ is the residual symmetry of the Poisson structure in $\mathbb{R}^2_\theta\times\mathbb{R}^2_\theta$ which is $\mathrm{U}sp(1)\times\mathrm{U}sp(1)$ moded out by the U(1) factor. We would like to note that, as it is clear from our construction, there are two ways for computing the star product of any functions of x^i's: to use definition (2.18) or, to consider the function as functions of z_α and $\overline{z_\alpha}$ and use Eq. (2.11), and of course the result would be the same.

It is worth noting that if instead of the four-dimensional star product of Eq. (2.11) one uses the usual Moyal star product, the reduction to a three-dimensional star product which is expressible only in terms of x^i's, unlike Eq. (2.18), will not have a simple form.

B. The measure

To formulate field theory on $\mathbb{R}^3_\lambda$ we need to find the corresponding measure which should depend only on x^i's and should be related to the four-dimensional measure on $\mathbb{R}^2_\theta\times\mathbb{R}^2_\theta$ given by

$$d\mu(z,\bar{z})=\frac{1}{\pi^2}d\bar{z}_1dz_1d\bar{z}_2dz_2. \tag{2.23}$$

Let us write z_α in a more convenient basis,

$$z_1=R\cos\theta_3e^{i\theta_1},\quad z_2=R\sin\theta_3e^{i\theta_2},$$

with $0\leqslant\theta_3\leqslant\pi/2$ and $0\leqslant\theta_\alpha\leqslant\pi$. In this coordinate system the measure takes the form

$$d\mu(z,\bar{z})=\frac{1}{(2\pi)^2}R^3dR\sin 2\theta_3d2\theta_3d\theta_1d\theta_2.$$

One can easily check that x^i and x^0 (and hence any function of them) depend only on R, $\theta=2\theta_3$, and $\phi=\theta_2-\theta_1$. Therefore, without any loss of generality we can write

$$\int d\mu(z,\bar{z})f(x^i,x^0)=\frac{1}{2\pi}\int_0^\infty R^3dR\times\int_0^\pi\sin\theta d\theta\int_0^{2\pi}d\phi f(x^i,x^0), \tag{2.24}$$

where we have performed the integration over $\theta_2+\theta_1$ which gives a factor of 2π. Remembering that $R^2=\bar{z}z=2x^0$, x^i's in this basis will be of the form

$$x^1=x^0\sin\theta\cos\phi,\quad x^2=x^0\sin\theta\sin\phi,\quad x^3=x^0\cos\theta,$$

with $\Sigma(x^i)^2=(x^0)^2$, which is the spherical coordinate basis. Finally the measure can be expressed as

$$\int d\mu(z,\bar{z})f(x^i,x^0)=\frac{1}{\pi}\int_0^\infty x^0dx^0\int_0^\pi\sin\theta d\theta\times\int_0^{2\pi}d\phi f(x^i,x^0)=\int\frac{d^3x}{\pi x^0}f(x^i,x^0). \tag{2.25}$$

That is, the measure on $\mathbb{R}^3_\lambda$ differs from the usual d^3x in a factor of $1/x^0$. This, in particular, implies that the radial part of the line element for $\mathbb{R}^3_\lambda$, which is the set of all possible fuzzy spheres with different radii, is different from the usual $\mathbb{R}^3$. However, the measure on the fuzzy sphere which is the angular part of the measure of $\mathbb{R}^3_\lambda$ remains the same as usual.

III. REDUCTION TO THE FUZZY SPHERE

The fuzzy sphere, $S^2_{\lambda,J}$, following the discussions in the introduction, is defined as $(2J+1)\times(2J+1)$ irreducible representation of the su(2) algebra of $\mathbb{R}^3_\lambda$. In this section, introducing a proper projection operator, $\hat{P}_J$, we show how the star product of functions on $\mathbb{R}^3_\lambda$ can also be used as the star product on the fuzzy sphere $S^2_{\lambda,J}$.

A. The projection operator $\hat{P}_J$

Let $\hat{\mathcal{A}}_J$ denote the algebra of operators on the fuzzy sphere of radius J,$S^2_{\lambda,J}$ [defined through Eqs. (1.2),(1.3)]. An arbitrary element of $\hat{\mathcal{A}}_J,\hat{f}_J$, can be expanded in the Schwinger basis for fixed J,

$$\hat{f}_J=\sum_{m,m'=-J}^{J}f^J_{m,m'}|J,m\rangle\langle J,m'|, \tag{3.1}$$

on the other hand the operator $\hat{f}\in\hat{\mathcal{A}}_3$ can be written as[5]

$$\hat{f}=\sum_{j=0}^\infty\hat{f}_j. \tag{3.2}$$

Therefore to reduce the algebra $\hat{\mathcal{A}}_3$ to that of $S^2_{\lambda,J},\hat{\mathcal{A}}_J$, it is enough to project it as

$$\hat{f}_J=\hat{P}^\dagger_J\hat{f}\hat{P}_J, \tag{3.3}$$

where

[5]Note that an arbitrary element of $\hat{\mathcal{A}}_4$ can be expanded as $\hat{f}=\Sigma_{j,j',m,m'}f^{jj'}_{m,m'}|j,m\rangle\langle j',m'|$, while elements of $\hat{\mathcal{A}}_3$ are of the form $\hat{f}=\Sigma_{j,m,m'}f^j_{m,m'}|j,m\rangle\langle j,m'|$. The latter follows directly from the definition of the $\hat{\mathcal{A}}_3$ [Eq. (2.14)].

$$\hat{P}_J=\sum_{m=-J}^{J}|J,m\rangle\langle J,m|. \tag{3.4}$$

It is easy to check that $\hat{P}_J$ is a (rank $2J+1$) projection operator, i.e.,

$$\hat{P}_J^\dagger=\hat{P}_J,\quad \hat{P}_J\hat{P}_K=\delta_{JK}\hat{P}_K,\quad \sum_{J=0}^{\infty}\hat{P}_J=1. \tag{3.5}$$

$\hat{P}_J$ can also be studied as an operator in $\hat{\mathcal{A}}_4$. Noting the definition of the $|j,m\rangle$ basis [Eq. (2.12)], it is straightforward to check that

$$a_\alpha\hat{P}_J=\hat{P}_{J-1/2}a_\alpha,\quad a_\alpha^\dagger\hat{P}_J=\hat{P}_{J+1/2}a_\alpha^\dagger, \tag{3.6}$$

and therefore

$$[a_\beta^\dagger a_\alpha,\hat{P}_J]=0. \tag{3.7}$$

From Eq. (3.6) we learn that the projected creation and annihilation operators are zero

$$a_{J\alpha}^\dagger=\hat{P}_Ja_\alpha^\dagger\hat{P}_J=0,\qquad a_{J\alpha}=\hat{P}_Ja_\alpha\hat{P}_J=0,$$

and Eq. (3.7) results in

$$[\hat{X}^i,\hat{P}_J]=0, \tag{3.8}$$

or equivalently $\hat{P}_J$ is only a function of $\hat{X}^0$, and hence any function in $\hat{\mathcal{A}}_3$ commutes with $\hat{P}_J$. (This can be used as another equivalent definition for $\hat{\mathcal{A}}_3$.) Then,

$$\hat{f}\in\hat{\mathcal{A}}_3,\quad \hat{f}_J=\hat{P}_J^\dagger\hat{f}\hat{P}_J=\hat{f}\hat{P}_J, \tag{3.9}$$

and hence

$$\hat{\mathcal{A}}_J=\hat{P}_J\hat{\mathcal{A}}_3=\hat{\mathcal{A}}_3\hat{P}_J=\hat{P}_J\hat{\mathcal{A}}_4\hat{P}_J. \tag{3.10}$$

The operator $\hat{P}_J$ can also be expanded in terms of the coherent states. First we recall Eq. (2.13),

$$|\vec{z}\rangle=\sum_{j=0}^{j=\infty}|\vec{z}\rangle_j,$$

$$|\vec{z}\rangle_j=\frac{e^{-(\overline{z}z/2)}}{\sqrt{(2j)!}}\sum_{m=-j}^{m=j}z_1^{j+m}z_2^{j-m}\sqrt{C_{j+m}^{2j}}|j,m\rangle. \tag{3.11}$$

We note that $|\vec{z}\rangle_j$ are orthogonal to each other (${}_j\langle\vec{z}|\vec{z}\rangle_k=0$ for any $j\neq k$) but not normalized to one. With the definition of $|\vec{z}\rangle_j$'s and $\hat{P}_j$ we have

$$\hat{P}_j|\vec{z}\rangle=|\vec{z}\rangle_j, \tag{3.12}$$

and therefore

$$\hat{P}_j=\int d\mu(\overline{z}z)|\vec{z}\rangle_{j\,j}\langle\vec{z}|. \tag{3.13}$$

The above operator relations can be written in terms of functions and the corresponding star product (2.11) or (2.18). However, we are interested in the explicit form of the function corresponding to $\hat{P}_J$. To obtain that one can use the coherent states,

$$\langle\vec{z}|\hat{P}_J|\vec{z}\rangle=\frac{1}{(2J)!}e^{-\overline{z}z}(\overline{z}z)^{2J}=\frac{1}{(2J)!}e^{-2x^0}(2x^0)^{2J}=P_J(x^0). \tag{3.14}$$

Then Eq. (3.5) will read as

$$P_J(x^0)\star P_K(x^0)=\delta_{JK}P_K(x^0), \tag{3.15}$$

with the star product given in Eq. (2.18). To show this last equation we should expand $P_J(x^0)$ in powers of x^0 and then use the following identity:

$$(x^0)^{2l}\star P_J(x^0)=(x^0)^{2l}P_{J-l}, \tag{3.16}$$

with $P_{J-l}=0$ for $J-l<0$. The proof is shown in Appendix A.

B. More on the projection operator P_J

So far we have shown how using projection operator P_J, the algebra of functions on $\mathbb{R}^2_\theta\times\mathbb{R}^2_\theta$ and on $\mathbb{R}^3_\lambda$ ($\mathcal{A}_4$ and $\mathcal{A}_3$, respectively), can be reduced to that of the fuzzy sphere, $\mathcal{A}_J$ defined by Eq. (3.10). In this subsection we would like to elaborate more on the projection operator P_J and its properties.

Let us define f_j as

$$f_j(z,\overline{z})={}_j\langle\vec{z}|\hat{f}|\vec{z}\rangle_j=\langle\vec{z}|\hat{P}_j\hat{f}\hat{P}_j|\vec{z}\rangle=P_j\star f\star P_j, \tag{3.17}$$

for any $\hat{f}\in\hat{\mathcal{A}}_4$. It is clear that by definition f_j is a function in $\mathcal{A}_j$. If we start with the operators in $\hat{\mathcal{A}}_3$ instead (i.e., $\hat{f}\in\hat{\mathcal{A}}_3$) then

$$f_j(x^i,x^0)=\langle\vec{z}|\hat{f}\hat{P}_j|\vec{z}\rangle=f\star P_j=P_j\star f, \tag{3.18}$$

and $f(x^i,x^0)=\Sigma_{j=0}^{\infty}f_j(x^i,x^0)$. With the above definition we have

$$(f\star g)_j=f_j\star g_j,$$

and

$$f_j\star g_{j'}=0,\qquad j\neq j'.$$

By a simple analysis one can show that the x-dependence of $f_j(x^i,x^0)$ is of the form,

$$f_j(x^i,x^0)=\tilde{f}_j(\tilde{x}^i)P_j(x^0), \tag{3.19}$$

where $\tilde{x}^i=x^i/x^0$ is the angular part of x^i's. In other words,

$$\tilde{f}_j(\tilde{x}^i)=(f\star P_j)/P_j. \tag{3.20}$$

Here we will show some more identities involving P_j (some more are gathered in Appendix A) which turns out to be useful in working out the field theory manipulations on the fuzzy sphere. The generators of the algebra $\mathcal{A}_J$ are

$$x_J^i = x^i \star P_J(x^0) = x^i P_{J-1/2}(x^0) = J\tilde{x}^i P_J(x^0). \tag{3.21}$$

x^0 projected on $S^2_{\lambda,J}$, as we expect gives the radius J, i.e.,

$$x_J^0 = x^0 \star P_J(x^0) = J P_J(x^0). \tag{3.22}$$

To evaluate the above star products either Eq. (2.11) or Eq. (2.18) may be used. In the same way one can show that

$$[x_J^i, x_J^j]_\star = [x^i, x^j]_\star \star P_J(x^0) = i\epsilon^{ijk} x^k \star P_J(x^0) = i\epsilon^{ijk} x_J^k,$$

$$[x_J^0, x_J^i]_\star = 0,$$

$$\vec{x}_J \star \vec{x}_J = \delta_{ij} x_J^i \star x_J^j = J(J+1) P_j(x^0). \tag{3.23}$$

Hence, x_J^i can be viewed as coordinates of a sphere of a given radius J embedded into the fuzzy space $\mathbb{R}^3_\lambda$. Each sphere is described by the algebra of functions $\mathcal{A}_J \subset \mathcal{A}_3$ generated by x_J^i. The spheres of different radii $0 \leqslant J < \infty$ fill the whole $\mathbb{R}^3_\lambda$ or in terms of algebras $\oplus_{J=0}^{\infty} \mathcal{A}_J = \mathcal{A}_3$. Finally we would like to present an important identity, the proof of which is shown in Appendix A,

$$f(\tilde{x}^i) \star x^0 = f(\tilde{x}^i) x^0, \tag{3.24}$$

where $\tilde{x}^i = x^i / x^0$, and therefore any function of the angular coordinates $\tilde{x}^i$ commutes with any function which only has a radial dependence. This is in fact what one intuitively expects as the radial coordinate x^0 labels different representations of the su(2) algebra.

IV. FIELD THEORY ON $\mathbb{R}^3_\lambda$ AND THE FUZZY SPHERE

In previous sections we have given the necessary mathematical tools to construct $\mathbb{R}^3_\lambda$ and $S^2_{\lambda,J}$ algebra of the operators, the algebra of functions, and the star product on them, in terms of the four-dimensional Moyal plane $\mathbb{R}^2_\theta \times \mathbb{R}^2_\theta$. In addition, introducing the projection operator, P_J, we discussed how the fuzzy sphere algebra is resulting from that of $\mathbb{R}^3_\lambda$. In this section as an application of our mathematical construction we show how the action of field theories on $\mathbb{R}^3_\lambda$ and the fuzzy sphere are induced from the corresponding actions on the Moyal plane. Hence we can deduce field theoretical information on $\mathbb{R}^3_\lambda$ and $S^2_{\lambda,J}$ from the more familiar and simpler case of the Moyal plane. However for reducing the actions, besides what we have already introduced one should know what are the derivative operators on $\mathbb{R}^3_\lambda$ in terms of the derivative operators on $\mathbb{R}^2_\theta \times \mathbb{R}^2_\theta$. These derivatives are needed for writing the kinetic terms of actions. In this section we show that the derivative along the radial coordinate of $\mathbb{R}^3_\lambda$, as we expect, is a discrete one.

A. Derivative operators

The derivative operators are generally operators which satisfy the Leibniz rule with respect to star product,

$$\mathcal{D}(f \star g) = \mathcal{D}f \star g + f \star \mathcal{D}g.$$

In the Moyal plane, where the noncommutativity parameter is a constant, the usual ∂_α and $\overline{\partial_\alpha}$ are proper derivative operators. However, it is easy to see that the usual $\partial_i = \partial / \partial x^i$ are not good derivatives with respect to the star product of Eq. (2.18). It is clear that, in the operator language, any operator which acts as a commutator will satisfy the Leibniz rule. (In the Moyal plane, i.e., $[a_\alpha, \hat{\phi}] \leftrightarrow \overline{\partial_\alpha}\phi$ and $[a_\alpha^\dagger, \hat{\phi}] \leftrightarrow -\partial_\alpha \phi$.)

In Sec. III, we showed that $\mathcal{L}_0$ [given by Eq. (2.15)] is in fact a derivative operator with respect to the S^1 direction which is moded out for reducing $\mathbb{R}^2_\theta \times \mathbb{R}^2_\theta$ to $\mathbb{R}^3_\lambda$. From the su(2) algebra of $\mathbb{R}^3_\lambda$, which is the algebra of angular momenta, we learn that $i[\hat{X}^i, .]$ gives the proper derivatives, but not all of them; yet the radial derivative is not specified. In terms of functions,

$$i[\hat{X}^i, \hat{\phi}] \leftrightarrow \mathcal{L}_i \phi \equiv \langle z | i[\hat{X}^i, \hat{\phi}] | z \rangle$$
$$= \frac{i}{2} \sigma^i_{\alpha\beta} (\overline{z_\alpha}\, \overline{\partial_\beta} - z_\beta \partial_\alpha) \phi(z, \overline{z}). \tag{4.1}$$

One can explicitly show that $\mathcal{L}_i$ do satisfy the Leibniz rule with respect to star products of Eqs. (2.11) and (2.18). Using Eq. (2.17) one can rewrite $\mathcal{L}_i$ in terms of x^i and ∂_i, and of course the result is the usual angular momentum operator, i.e., $\mathcal{L}_i = \epsilon_{ijk} x^j \partial_k$.

As we discussed the radial coordinate x^0 can only take discrete values of $j = 0, \frac{1}{2}, 1, \ldots$. Therefore, the derivative in this direction is expected to be a discrete difference and hence it is not necessarily fulfilling the Leibniz rule. On the other hand we note that $\partial / \partial x^0$, despite being a derivative, is not a Hermitian operator. One can check that the operator

$$\Delta = \tfrac{1}{2} (\overline{z_\alpha}\, \overline{\partial_\alpha} + z_\alpha \partial_\alpha), \tag{4.2}$$

which corresponds to $x^0 (\partial / \partial x^0)$, is the Hermitian operator which appears in the kinetic terms of the actions. To show that Δ is in fact acting like a discrete derivative it is enough to note that

$$\Delta P_j = 2x^0 (P_{j-1/2} - P_j) = [2jP_j - (2j+1) P_{j+1/2}].$$

Hence, for any arbitrary function of x,

$$\phi(x^i, x^0) = \sum_{j=0} \phi_j(x) = \sum_j \tilde{\phi}(\tilde{x}^i) P_j(x^0),$$

we have

$$P_J\star\Delta\phi(x)=P_J\star\sum \tilde\phi(\tilde x^i)\star\Delta P_j(x^0)$$

$$=\sum \tilde\phi(\tilde x^i)\star P_J\star\Delta P_j(x^0)$$

$$=2J(\tilde\phi_J[\tilde x)-\tilde\phi_{J-1/2}(\tilde x)]P_J, \qquad (4.3)$$

which clearly shows that Δ gives the expected discrete derivative.

We have introduced four Hermitian operators which are the proper derivatives expressed in terms of x^i, x^0 and can be used instead of ∂_α and $\overline{\partial_\alpha}$. We also note that

$$[\mathcal{L}_0,\mathcal{L}_i]=0, \quad [\mathcal{L}_0,\Delta]=0, \quad [\mathcal{L}_i,\Delta]=0. \qquad (4.4)$$

Equation (4.4) confirms our previous arguments on the reduction of the $\mathcal{A}_4$ algebra to $\mathcal{A}_3$ and $\mathcal{A}_J$.

We close this part by commenting that for the formulation of field theories one should also include the time direction which is commutative with the space directions. The time derivative is therefore the same as usual.

B. Field theories on $\mathbb{R}^3_\lambda\times\mathbb{R}$

Along our previous discussions, given a field theory on $\mathbb{R}^2_\theta\times\mathbb{R}^2_\theta\times\mathbb{R}$ ($\mathbb{R}$ stands for the time direction) the corresponding field theory on $\mathbb{R}^3_\lambda\times\mathbb{R}$ is obtained by restricting the fields to be only x-dependent (or elements of $\mathcal{A}_3$ algebra). As an explicit example let us consider the scalar theory on $\mathbb{R}^2_\theta\times\mathbb{R}^2_\theta\times\mathbb{R}$,

$$S=\int dt d\mu(\bar z,z)[\partial_t\phi\star\partial_t\phi-\partial_\alpha\phi\star\overline{\partial_\alpha}\phi+V_\star(\phi)], \qquad (4.5)$$

where the star products are that of Eq. (2.11) and $V_\star$ is the potential term in which all the products between ϕ's are carried out with the star product of Eq. (2.11). We should remind the reader that with the star product (2.11), unlike the Moyal star product,

$$\int d\mu(\bar z,z) f\star g\neq\int d\mu(\bar z,z) fg; \qquad (4.6)$$

i.e., we cannot remove the star product in the quadratic terms of the action. However, still we have the cyclicity of the star product inside the integral,

$$\int d\mu(\bar z,z) f_1\star f_2\star\cdots\star f_n$$

$$=\int d\mu(\bar z,z) f_n\star f_1\star\cdots\star f_{n-1}. \qquad (4.7)$$

The same is also true for the star product of Eq. (2.18),

$$\int \frac{d^3x}{x^0} f_1(x)\star f_2(x)\star\cdots\star f_n(x)$$

$$=\int\frac{d^3x}{x^0} f_n(x)\star f_1(x)\star\cdots\star f_{n-1}(x). \qquad (4.8)$$

Restricting ϕ to be only x-dependent, the potential term is replaced with the same functional form of ϕ, with the star product of Eq. (2.18). The kinetic term is more involved, because $\phi\in\mathcal{A}_3$ ($\mathcal{L}_0\phi=0$), but we cannot conclude that $\partial_\alpha\phi,\overline{\partial_\alpha}\phi\in\mathcal{A}_3$. However, with a little algebra one can show that

$$\phi\star\Box_4\phi=\tfrac{1}{2}\phi\star(x^0\partial_i\partial_i\phi). \qquad (4.9)$$

Recalling the discussion of Sec. II B, one can write the action (4.5) in terms of $\mathbb{R}^3_\lambda\times\mathbb{R}$ parameters,

$$S=\int dt\frac{d^3x}{\pi x^0}\left\{\partial_t\phi\star\partial_t\phi+\frac{1}{2}\phi\star(x^0\partial_i\partial_i\phi)+V_\star(\phi)\right\}. \qquad (4.10)$$

We would like to comment that, since ∂_i are not derivative operators with respect to the star product (2.18), the spatial part of the kinetic term should be handled with special care. On the other hand if one tries to use the ''proper derivative'' (i.e., $\Delta,\mathcal{L}_i$), as it is shown in Appendix B, the form of kinetic terms of the action are not as simple as inserting the star product into the commutative expressions.

C. Field theories on the fuzzy sphere

In this section we would like to discuss how using our formalism one can study field theory on the fuzzy sphere $S^2_{\lambda,J}$. As we have previously discussed in Sec. III B, any function f on $\mathbb{R}^3_\lambda$ can be written as

$$f=\sum_j f_j(x^i,x^0)=\sum_j \tilde f_j(\tilde x^i)P_j(x^0),$$

where $\tilde x^i=x^i/x^0$ are the angular part of x^i's, and $\tilde f_j(\tilde x^i)$ is the corresponding function on $S^2_{\lambda,J}$ given by Eq. (3.20). It is easy to show that

$$Tr\hat P_J=\sum_{j,m}\langle j,m|\hat P_J|j,m\rangle=\sum_m\langle J,m|J,m\rangle$$

$$=\int d\mu(z,\bar z)P_J(z,\bar z)$$

$$=\int\frac{d^3x}{\pi x^0}P_J(x^0)=2J+1,$$

which is giving the dimension of an irreducible representation of spin J. Using the above observation one can, therefore, deduce the field theory on the fuzzy sphere starting from the one on $\mathbb{R}^3_\lambda$ by simply performing the integration over x^0 and dividing the result by the factor $2J+1$ for a sphere of radius J. Then as a result we have

$$\int \frac{d\Omega}{4\pi}\tilde{f}_{1J}\star\tilde{f}_{2J}\star\cdots\star\tilde{f}_{nJ}=\frac{1}{(2J+1)}\times\int\frac{d^3x}{\pi x^0}(f_1\star f_2\star\cdots\star f_n)\star P_J, \tag{4.11}$$

where the star product of the functions defined on $S^2_{\lambda,J}$ is defined in terms of the star product (2.18) of the corresponding functions on $\mathbb{R}^3_\lambda$ through Eq. (3.20). Now the action for a scalar field, for example, on the fuzzy sphere of a given radius J will be expressed in $\mathbb{R}^3_\lambda$ as

$$S_J=\frac{1}{(2J+1)}\int dt\frac{d^3x}{\pi x^0}\times[\partial_t\phi\star\partial_t\phi-\mathcal{L}_i\phi\star\mathcal{L}_i\phi+V_\star(\phi)]\star P_J, \tag{4.12}$$

and as we have already explained, performing the integration over the x^0 part will result in

$$S_J=\int dt\frac{d\Omega}{4\pi}[\partial_t\tilde{\phi}_J\star\partial_t\tilde{\phi}_J-\mathcal{L}_i\tilde{\phi}_J\star\mathcal{L}_i\tilde{\phi}_J+V_\star(\tilde{\phi}_J)]. \tag{4.13}$$

Noting the fact that any $(2J+1)\times(2J+1)$ Hermitian matrix can be expanded in terms of spherical harmonic, Y_{lm} $l\leqslant 2J$, instead of working with matrices, spherical harmonics are usually used for field theory manipulations on the fuzzy sphere. So, to complete our dictionary for the fuzzy sphere, we would like to show how the Y_{lm}'s can be expressed in terms of x coordinates. This will automatically lead to the proper star product between the Y_{lm}'s. The classical spherical harmonics are a set of orthonormal functions obeying

$$\mathcal{L}_i^2Y_{l,m}=l(l+1)Y_{l,m},$$
$$\mathcal{L}_3Y_{l,m}=mY_{l,m}. \tag{4.14}$$

Following [8] we introduce the highest weight functions in $\mathcal{A}_3$ algebra,[6]

$$\psi_{l,l}(x)=\sum_j c_{j,l}x_+^l\star P_j, \tag{4.15}$$

where $x_+=x^1+ix^2$ and c_l are normalization factors. Then, by virtue of Eq. (3.16), for each term (of specific value of j) in the sum $l=0,1,\ldots,2j$ and in addition $\mathcal{L}_+\psi_{l,l}=0$ for all $(l\leqslant 2j)$. In order to reduce on the $S^2_{\lambda,J}$, one should multiply Eq. (4.15) by P_J,

$$\psi^J_{l,l}(x)=c_{J,l}x_+^l\star P_J,\quad l\leqslant 2J. \tag{4.16}$$

Acting on $\psi^J_{l,l}$ with the operator $\mathcal{L}_-$ will lead to the functions (4.14),[7]

$$Y^J_{l,m}=N_{l,m}(\mathcal{L}_-)^{l-m}\psi^J_{l,l}, \tag{4.17}$$

where m runs by integer steps over the range $-l\leqslant m\leqslant l$. Any function in $S^2_{\lambda,J}$ can be expanded in terms of $Y^J_{l,m}$ as

$$\Phi_J(x)=\sum_{(l,m)}a_{l,m}Y^J_{l,m}, \tag{4.18}$$

where the sum over (l,m) means $l=0,\ldots,2J$, $-l\leqslant m\leqslant l$, and $a_{l,m}$ are complex coefficients obeying

$$a_{l,-m}=(-)^m a^*_{l,m}, \tag{4.19}$$

which guarantees the reality condition $\Phi^*(x)=\Phi(x)$. However, we note that in Eq. (4.16) still there is x^0 dependence, which can be removed by integration over x^0 or simply by dividing by P_J,

$$\tilde{Y}^J_{l,m}=Y^J_{l,m}/P_J, \tag{4.20}$$

and therefore, any function $\tilde{\Phi}$ on $S^2_{\lambda,J}$ can be expanded as in Eq. (4.18)

$$\tilde{\Phi}_J(x)=\sum_{(l,m)}a_{l,m}\tilde{Y}^J_{l,m}, \tag{4.21}$$

where now $\tilde{Y}^J_{l,m}$ depend only on the angular part of x as it should be.

V. DISCUSSION

In this paper we have constructed a dictionary which translates different descriptions of $\mathbb{R}^3_\lambda$ (the set of fuzzy spheres with all possible radii) or the fuzzy spheres into each other. The fuzzy sphere and $\mathbb{R}^3_\lambda$ may be studied through the operators which are functions of the coordinates $\hat{X}^i$, the generators of the su(2) algebra. On the other hand there is always a corresponding algebra of functions with the proper star product which is equivalent to the operator algebra. The latter is the appropriate language for performing field theories. Starting from the four-dimensional Moyal plane and reducing that on a circle, we have constructed $\mathbb{R}^3_\lambda$ and from there we read off the star product induced from the Moyal plane. Then we discussed how the algebra of functions on the fuzzy sphere can be realized as different sectors of the algebra of functions on $\mathbb{R}^3_\lambda$ (and of course with the same star product). If we reintroduce θ in our expressions, we would obtain a factor of θ in the exponential factor in Eq. (2.18). Then, it can be checked directly from the definition of our star product (2.18) that in the λ (or θ) $\rightarrow 0$ limit we will find the usual product of functions.

[6]Note that, unlike the commutative case, any function on $\mathbb{R}^3_\lambda$ can be expanded through the spherical harmonics.

[7]Here we will not try to compute the exact values of the normalization factors which are very important if one wants to do explicit field theory calculation.

Here we have concentrated on completing the mathematical tools. Using our results one can easily study the field theories on a fuzzy sphere through the Moyal field theories. As a result we would like to mention the IR-UV mixing, which is a general feature of noncommutative Moyal field theories. So, using our method, we expect to be able to trace the same phenonemon for the fuzzy sphere field theories. This, in fact, has been explicitly checked by using the spherical harmonics [12,19]. There are several interesting open problems one can address here. Using our formulation we have a straightforward way of introducing fermions on the fuzzy sphere, starting from the fermions on the Moyal plane. Furthermore, we have a simple handle on the vector gauge fields on the fuzzy sphere.

The other interesting question is the solitonic solutions on the fuzzy sphere [20,21]. In our approach, one can easily obtain the solitonic solutions on the fuzzy sphere from the solitonic solutions on the Moyal plane which respect the rotational su(2) symmetry. As an explicit example we would like to note that the fuzzy sphere itself can be thought of as a solitonic solution in the Moyal field theory, as it can be identified with the projector, P_j. We postpone a full study of such solutions to future works.

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APPENDIX A: PROOF OF SOME USEFUL IDENTITIES THAT HAVE BEEN USED IN THE PAPER

The proof for Eq. (3.16) is

$$(x^0)^{2l}\star P_j(x^0)=(x^0)^{2l}P_{j-l}. \tag{A1}$$

To show the above we should perform the star product explicitly, i.e.,

$$(x^0)^{2l}\star P_j(x^0)=\left(x^0+\frac{1}{2}x^i\frac{\partial}{\partial y^i}\right)^{2l}P_j(y^0)\bigg|_{y=x}$$
$$=(x^0)^{2l}\sum_{n=0}^{2l}C_n^{2l}\left(\frac{x^i}{2x^0}\frac{\partial}{\partial y^i}\right)^n P_j(y^0)\bigg|_{y=x}, \tag{A2}$$

with $C_n^{2l}=[(2l)!/n!(2l-n)!]$. Since $\partial_i P_j=(x^i/x^0)\partial_0 P_j$, where $\partial_0=\partial/\partial x^0$ is the derivative with respect to x^0, after straight forward calculations one can show that

$$(x^0)^{2l}\star P_j(x^0)=(x^0)^{2l}\sum_{n=0}^{2l}C_n^{2l}\frac{1}{2^n}\partial_0^n P_j(x^0)$$
$$=(x^0)^{2l}\left(1+\frac{1}{2}\frac{\partial}{\partial x^0}\right)^{2l}P_j. \tag{A3}$$

On the other hand it is easy to check that

$$\left(1+\frac{1}{2}\frac{\partial}{\partial x^0}\right)P_j=P_{j-1/2}. \tag{A4}$$

Therefore,

$$(x^0)^{2l}\star P_j(x^0)=\begin{cases}(x^0)^{2l}P_{j-l}(x^0) & \text{for} \quad j\geqslant l,\\ 0 \quad \text{for} \quad j<l.\end{cases} \tag{A5}$$

Equivalently one can prove Eq. (3.16) using the operator language. From Eq. (3.6) it follows that:

$$:(\hat{x}^0)^{2l}:\hat{P}_j=(\tfrac{1}{2})^{2l}a^\dagger_{\alpha_1}\cdots a^\dagger_{\alpha_{2l}}a_{\alpha_1}\cdots a_{\alpha_{2l}}\hat{P}_j$$
$$=(\tfrac{1}{2})^{2l}a^\dagger_{\alpha_1}\cdots a^\dagger_{\alpha_{2l}}\hat{P}_{j-l}a_{\alpha_1}\cdots a_{\alpha_{2l}}. \tag{A6}$$

Applying the coherent states we will find Eq. (3.16).

Some more identities:

$$x_J^i\star x_J^j=x^i\star x^j\star P_J(x^0)=[x^ix^j+\tfrac{1}{2}(\delta^{ij}x^0+i\epsilon^{ijk}x^k)]\star P_J(x^0)$$
$$=x^ix^j\frac{1}{(2J-2)!}e^{-2x^0}(2x^0)^{2J-2}+\tfrac{1}{2}$$
$$\times(\delta^{ij}x^0+i\epsilon^{ijk}x^k)\frac{1}{(2J-1)!}e^{-2x^0}(2x^0)^{2J-1}$$
$$=[J(J-\tfrac{1}{2})\tilde{x}^i\tilde{x}^j+\tfrac{1}{2}J(\delta^{ij}+i\epsilon^{ijk}\tilde{x}^k)]P_J, \tag{A7}$$

and

$$x_J^0\star x_J^0=x^0x^0\frac{1}{(2J-2)!}e^{-2x^0}(2x^0)^{2J-2}$$
$$+\frac{1}{2}\frac{\delta^{ij}x^0)}{(2J-1)!}e^{-2x^0}(2x^0)^{2J-1}$$
$$=\left[\frac{2J(2J-1)}{4}+\frac{J}{2}\right]\star P_J=J^2P_J, \tag{A8}$$

from which we deduce

$$\vec{x}_J\star\vec{x}_J=\delta_{ij}x_J^i\star x_J^j=(x^0)^2\frac{1}{(2J-2)!}e^{-2x^0}(2x^0)^{2J-2}$$
$$+\frac{3}{2}x^0\frac{1}{(2J-1)!}e^{-2x^0}(2x^0)^{2J-1}$$
$$=x_J^0\star(x_J^0+1)=J(J+1)P_j(x^0). \tag{A9}$$

Another important relation is the star product between x^0 and $\tilde{x}^i$,

$$x^0\star\tilde{x}^i=\left(x^0+\frac{1}{2}x^j\frac{\partial}{\partial x^j}\right)\tilde{x}^i$$

$$=x^0\tilde{x}^i+\frac{1}{2}x^j\left(\frac{\delta^{ij}}{x^0}-\frac{x^ix^j}{(x^0)^3}\right)$$

$$=x^0\tilde{x}^i=\tilde{x}^i\star x^0, \tag{A10}$$

which in turn results in Eq. (3.24). Then using the above relations one can check that

$$(x^i)^{2l}\star P_J(x^0)=(\tilde{x}^i)^{2l}\star(x^0)^{2l}\star P_J(x^0)$$

$$=(\tilde{x}^i)^{2l}\star[(x^0)^{2l}P_{J-l}(x^0)]=(x^i)^{2l}P_{J-l}(x^0). \tag{A11}$$

APPENDIX B: MORE ON KINETIC TERMS

If we write the kinetic terms of the four dimensional theory as

$$\phi\star\Box_4\phi, \tag{B1}$$

where

$$\Box_4=\partial_\alpha\overline{\partial_\alpha}, \tag{B2}$$

then, for ϕ's with $\mathcal{L}_0\phi=0$, in terms of the three-dimensional derivatives we have,

$$-\phi\star(x^0\Box_4\phi)=\phi\star(\mathcal{L}_i\mathcal{L}_i+\Delta\Delta)\phi+\phi\star\Delta\phi. \tag{B3}$$

The last term $\phi\star\Delta\phi$ is there because Δ is not a derivative (or in other words, it is a discrete derivative in the radial direction). On the other hand, we note that

$$x^0\Box_4\phi=\Box_4(x^0\phi)-\phi-\Delta\phi. \tag{B4}$$

Consequently, we obtain

$$-\phi\star\Box_4(x^0\phi)=\phi\star(\mathcal{L}_i\mathcal{L}_i+\Delta\Delta)\phi+\phi\star\phi. \tag{B5}$$

As another way of writing the kinetic term we have

$$\int\phi\star\Delta\phi=\int\left(\frac{1}{2}\partial_\alpha\phi\star\overline{\partial_\alpha}\phi-\phi\star\phi\right)$$

$$=-\frac{1}{2}\int\phi\star[(\Box_4+2)\phi]. \tag{B6}$$

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