Moduli space of BPS domain walls

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$\mathcal{N}=2$ supersymmetric QED with several flavors admits multiple, static BPS domain wall solutions. We determine the explicit two-kink metric and examine the dynamics of colliding domain walls. The multikink metric has a toric Kähler structure and we reduce the Kähler potential to quadrature. In the second part of this paper, we consider semilocal vortices on $\mathbf{R}\times\mathbf{S}^1$. We argue that, in the presence of a suitable Wilson line, the vortices separate into domain wall constituents. These play the role of fractional instantons in two-dimensional gauge theories and sigma models.

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I. INTRODUCTION

The concept of the moduli space is one of the most important tools in the study of solitons. Originally introduced by Manton to describe the classical scattering of monopoles [1], it is now appreciated that the topology and geometry of soliton moduli spaces also encodes many of the quantum properties of supersymmetric theories. Examples include the spectrum of solitonic bound states, and nonperturbative contributions to correlation functions. This has played a pivotal role in untangling the web of dualities in field and string theories.

While the moduli spaces of instantons, monopoles, and vortices have all been studied in detail, less attention has been paid to the moduli space of domain walls. Indeed, in most theories there is a force between widely separated domain walls [2,3],[1] and any attempt to describe the dynamics using a moduli space approximation requires the introduction of a potential [5]. Nevertheless, there do exist models where the force between domain walls vanishes, resulting in a moduli space of static multikink solutions with arbitrary separation. These include a class of generalized Wess-Zumino models [6,7], $\mathcal{N}=1^\star$ theories [8], and massive sigma models [4].

In this paper we shall consider domain walls in $\mathcal{N}=2$ supersymmetric QED with a Fayet-Iliopoulos (FI) parameter and N flavors of electrons.[2] If each electron has a different mass there are N isolated vacua, implying the existence of Bogomol'nyi-Prasad-Sommerfield (BPS) domain walls. As will be reviewed below, a generic domain wall decomposes into several "fundamental" domain walls, each of which carries an independent position and phase collective coordinate. The moduli space of solitons is thus a toric Kähler manifold [4].

In fact, this theory has a much richer spectrum of solitons than one might guess through a naive homotopy group argument. As well as domain walls, there also exist superconducting BPS strings, which carry a global current [10]. Moreover, at least in the strong coupling limit, these strings can end on the domain wall where they are charged under a localized U(1) gauge field [11]. (The gauge field is dual to the phase collective coordinate.) In other words, this field theory provides a simple model of D-brane physics.

The paper is organized as follows. In the next section we introduce the Abelian-Higgs model of interest. We review the first order domain wall equations, as well as the connection to the sigma-model kinks of [4]. We then proceed to consider the metric on the moduli space of solutions and derive the Kähler potential in integral form [Eq. (15)]. In the case of two kinks with identical masses, this integral can be performed [the resulting explicit metric is given in Eq. (17)] and some of the properties and implications of this metric are analyzed. In the second part of this paper, we change track and discuss semilocal vortices on $\mathbf{R}\times\mathbf{S}^1$ with a Wilson line for the flavor symmetry group. Motivated by the analogy with instantons and monopoles [12], we demonstrate that the vortices decompose into multiple kink solutions. This provides a mechanism for calculating fractional instanton effects in strongly coupled two-dimensional sigma models.

A note on the quantum theory: In this paper we concentrate on domain walls in a $d=3+1$ dimensional Abelian gauge theory. However, due to the existence of a Landau pole, the theory is not well defined at the quantum level. The same is true of the theory lifted to $d=4+1$ dimensions, which may be of interest in the context of brane-world scenarios.[3] Since we restrict ourselves to classical aspects of domain walls, no harm is done. However, to find the same domain walls as quantum objects, we must consider the dimensional reduction of the theory to $d=2+1$ or $d=1+1$. Alternatively they appear as instantons in gauge quantum mechanics. In each of these cases, the metric on the moduli space remains the same.

Finally, throughout the paper we stress the many similarities that exist between the domain walls and monopoles, as well as between semilocal vortices and Yang-Mills instantons. These similarities add to the growing evidence that there exists a quantitative correspondence relating these solitons [13,14].

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[1]This remains true even when the domain walls are mutually supersymmetric [4].

[2]This is a slight generalization of the model considered in [4], reducing to it in the strong coupling limit [9].

[3]One cannot lift the theory with a mass gap to dimensions greater than $4+1$ while preserving supersymmetry.

II. GAUGE THEORY DOMAIN WALLS

Our starting point is $d=3+1$, $\mathcal{N}=2$ supersymmetric U(1) gauge theory coupled to N hypermultiplets. The bosonic part of the Lagrangian is given by

$$\mathcal{L}=\frac{1}{4e^2}F^2+\frac{1}{2e^2}|\partial\phi|^2+\sum_{i=1}^{N}(|\mathcal{D}q_i|^2+|\mathcal{D}\tilde{q}_i|^2)-\sum_{i=1}^{N}|\phi-m_i|^2(|q_i|^2+|\tilde{q}_i|^2)-\frac{e^2}{2}\left(\sum_{i=1}^{N}|q_i|^2-|\tilde{q}_i|^2-\zeta\right)^2-\frac{e^2}{2}\left|\sum_{i=1}^{N}\tilde{q}_iq_i\right|^2 \quad (1)$$

Each scalar field q_i has charge $+1$ under the gauge group, while the $\tilde{q}_i$ have charge -1. Each pair is assigned a complex mass m_i which may always be chosen to satisfy $\Sigma_i m_i=0$. The complex scalar field ϕ lives in the vector multiplet and is neutral under the gauge group. Finally, we require that ζ, the real FI parameter appearing in the D term, is nonzero. This ensures the theory lies in its Higgs phase. Without loss of generality[4] we set $\zeta>0$.

For vanishing masses the theory enjoys an SU(N) flavor symmetry and a moduli space of vacua given by $T^\star\mathbf{CP}^{N-1}$. In this paper we will be interested in the case of nonzero, distinct masses: $m_i\neq m_j$ for $i\neq j$. This breaks the flavor symmetry to the maximal torus $\mathrm{U}(1)^{N-1}$ and lifts all but N isolated vacua, lying on the zero section of $T^\star\mathbf{CP}^{N-1}$,

$$\text{Vacuum } i: \quad \phi=m_i, \quad |q_j|^2=\zeta\delta_{ij}, \quad |\tilde{q}_j|^2=0.$$

For generic masses m_i, there exist BPS domain wall solutions interpolating between any given pair of vacua. However, in order to find a moduli space of domain walls we need to restrict the mass parameters to be real:[5] $\mathrm{Im}(m_i)=0$. This immediately leads to the important corollary that there is a natural ordering to the vacua. We choose the ordering $m_{i+1}<m_i$ for all i.

Since certain fields will not appear in the domain wall solutions discussed below, we set them to zero at this stage,

$$\mathrm{Im}(\phi)=\tilde{q}_i=F=0. \quad (2)$$

Their sole role was to complete the supersymmetry multiplets, and to cancel a potential gauge anomaly. (In fact the field strength will be resurrected below when we come to discuss dynamics.) In particular, from now on the field ϕ will always be assumed to be purely real. We choose the domain wall to lie in the (x^2-x^3) plane, so that the only nonzero space-time field variations are in the $x\equiv x^1$ direction. We write $\partial\equiv\partial_1$. The BPS equations, first derived in [16], can be determined by simply completing the square in the Hamiltonian,

$$\mathcal{H}=\frac{1}{2e^2}(\partial\phi)^2+\sum_{i=1}^{N}|\mathcal{D}q_i|^2+\sum_{i=1}^{N}(\phi-m_i)^2|q_i|^2+\frac{e^2}{2}\left(\sum_{i=1}^{N}|q_i|^2-\zeta\right)^2$$
$$=\frac{1}{2e^2}\left[\partial\phi\mp e^2\left(\sum_{i=1}^{N}|q_i|^2-\zeta\right)\right]^2+\sum_{i=1}^{N}|\mathcal{D}q_i\mp(\phi-m_i)q_i|^2\pm T.$$

For the kink interpolating between the ith vacuum at $x\to-\infty$ and the jth vacuum at $x\to+\infty$, the topological charge T is given by

$$T=\left[\sum_{i=1}^{N}(\phi-m_i)|q_i|^2-\phi\zeta\right]_{-\infty}^{+\infty}=\zeta(m_i-m_j), \quad (3)$$

where we have chosen $j>i$ which requires use of the upper signs in the Hamiltonian. The Bogomol'nyi equations are therefore given by

$$\partial\phi=e^2\left(\sum_{i=1}^{N}|q_i|^2-\zeta\right), \quad (4)$$

$$\mathcal{D}q_i=(\phi-m_i)q_i. \quad (5)$$

It is simple to show that for $j<k<i$, these Bogomol'nyi equations require $q_k\equiv0$. For this reason, we now restrict attention to the maximal domain wall interpolating between the 1st and Nth vacua, which has tension $T=\zeta(m_1-m_N)$. Any other domain wall may be embedded maximally in a theory with fewer flavors.

The second Bogomol'nyi equation (5) is easily solved,

$$q_i=\sqrt{\zeta}\exp\left[\psi-m_i(x-x_0)-\sum_{a=1}^{N-2}\alpha_i^a r_a\right], \quad (6)$$

where α is a fixed, rank $(N-2)$ real matrix satisfying $\Sigma_i\alpha_i^a=\Sigma_i m_i\alpha_i^a=0$, and the complex function ψ is determined by

$$\partial\psi=\phi+iA,$$

with $A\equiv A_1$ the gauge potential. By Eq. (2), the imaginary part of ψ is pure gauge. It may be set to zero when considering static solutions, but will play an important role when we turn to the dynamics of domain walls. Most important in the solution (6) are the putative collective coordinates. These are the center of mass x_0 and the parameters r_a, $a=1,\dots,N-2$ which are related to the separation of neighboring domain walls. Each is complex, with real and imaginary parts,

[4]A possible complex FI parameter which would appear in the F term in Eq. (1) has been set to zero using the $\mathrm{SU}(2)_R$ R symmetry of the action.

[5]This is entirely analogous to the situation with monopoles in higher rank gauge groups, in which a moduli space only exists if the vacuum expectation value is real [15].

$$x_0 = X_0 + i\theta_0, \quad r_a = R_a + i\theta_a, \tag{7}$$

and will provide $N-1$ complex coordinates on the domain wall moduli space. When m_i and α_i^a are rational, the corresponding θ is periodic. In contrast, when m_i and α_i^a are irrational, $\theta \in \mathbf{R}$. Note that there exists some ambiguity in fixing the matrix α_a^i which is related to the possiblity of performing coordinate redefinitions on the moduli space. This ambiguity may be naturally removed by insisting that, asymptotically, the parameters R coincide with the relative separations of far-separated domain walls. We shall do this explicitly for the two-kink metric but in general it remains an open problem.

However, we must not be too hasty in concluding that multidomain wall solutions exist, since we have still to satisfy the first Bogomol'nyi equation (4) which now reads

$$\frac{1}{\zeta e^2}\partial^2 \,\mathrm{Re}(\psi) = \sum_{i=1}^{N} \exp[2\,\mathrm{Re}(\psi) - 2m_i(x-X) - 2\alpha_i^a R_a] - 1. \tag{8}$$

Note that we have left the sum over $a=1,\ldots,N-2$ implicit in this equation, and shall continue to do so for the remainder of the paper. This nonlinear, somewhat unpleasant, differential equation, which defines $\mathrm{Re}(\psi)$ as a function of the real variables $(x-X)$ and R_a, is further complicated by the boundary conditions,

$$\mathrm{Re}(\psi) \to \begin{cases} m_1(x-X) + \alpha_1^a R_a, & x \to -\infty, \\ m_N(x-X) + \alpha_N^a R_a, & x \to +\infty. \end{cases}$$

I do not know if solutions exist for all values of the dimensionful parameter ζe^2. However, it is possible to write down a formal solution as a perturbative series in the dimensionless parameter e^{-2}. The strong coupling expansion takes the form

$$\mathrm{Re}(\psi) = \sum_{p=0}^{\infty} \frac{1}{e^{2p}}\psi_p. \tag{9}$$

Then, in the strict strong coupling limit $e^2 \to \infty$, the solution is

$$\exp(2\psi_0) = \left(\sum_{i=1}^{N} \exp[-2m_i(x-X) - 2\alpha_i^a R_a] \right)^{-1}, \tag{10}$$

which indeed has the correct boundary conditions. This may be understood as the long-wavelength approximation to the true solution to Eq. (8). The remaining ψ_p for $p>1$ are determined in an iterative fashion by the equation

$$\partial^2 \sum_{p=0}^{\infty} \frac{1}{e^{2(p+1)}}\psi_p = \zeta \sum_{n=1}^{\infty} \frac{1}{n!}\left(\sum_{p=1}^{\infty} \frac{2}{e^{2p}}\psi_p \right)^n,$$

which ensures that $\psi_p \to 0$ as $x \to \pm\infty$, so the boundary conditions are preserved. Thus, there exist solutions to Eq. (8) enjoying the full compliment of $N-1$ complex collective coordinates, at least in a neighborhood of $e^{-2}=0$. The size of this neighborhood is determined by the radius of convergence of the sum (9), given by the limit $|\psi_p/\psi_{p+1}|$ as $p \to \infty$. It would be interesting to determine whether the solutions exhibit a phase transition as the coupling constant is varied, or whether the radius of convergence is infinity or (more disappointing) zero.

The strong coupling limit $e^2 \to \infty$, which played an important role in determining the existence of the solution, is familiar from linear sigma models [9]. From the expression for the scalar potential (1), it is clear that this limit restricts us to the Higgs branch $\mathcal{V} = T^\star \mathbf{CP}^{N-1}$. The presence of mass terms for the hypermultiplets induces a potential on the Higgs branch which, by supersymmetry requirements, is proportional to the length of a triholomorphic Killing vector on $\mathcal{V}$ [17]. This Killing vector is determined by the global flavor symmetry preserved by the masses m_i. Domain walls in such massive sigma models have been extensively studied in the literature and, in particular, the existence of multikink solutions was demonstrated in [4] using both Morse theory as well as more direct techniques. It was further shown in [4] that the coordinates R_a do indeed parametrize the separations between "fundamental kinks," each of which interpolates from the i vacuum to the $(i+1)$ vacuum. The maximal kink considered here is comprised of $(N-1)$ fundamental kinks. A similar decomposition of kinks has also been found in $\mathrm{SU}(N) \times \mathbf{Z}_2$ gauge models [18].

Moreover, the massive sigma models also admit a wide range of BPS solutions including intersecting domain walls [19], string lumps [20], strings ending on domain walls [11], and dyonic domain walls [21]. It seems likely that each of these has a generalization to the finite coupling gauge theory considered here. This is certainly true of strings, as shown in [10], and it is a simple exercise to generalize the BPS equations of all these solitons to the gauge theory context.

III. THE MODULI SPACE OF DOMAIN WALLS

We turn now to the dynamics of domain walls. We work in the moduli space approximation; in other words, we consider solutions to the static BPS equations. The complex collective coordinates are then promoted to fields on the domain wall world volume, $x_0(\xi)$ and $r_a(\xi)$. For small fluctuations, the low-energy dynamics of the soliton are described by a $d=2+1$ dimensional sigma model with target space toplogically of the form [4]

$$\mathcal{M}_{N-1} = \mathbf{R} \times \frac{\mathbf{R} \times \tilde{\mathcal{M}}_{N-1}}{\mathcal{G}},$$

where the first two $\mathbf{R}$ factors parametrize the center of mass and overall phase of the soliton, respectively. Newton's third law ensures each is endowed with a flat metric. As with monopoles, motion along the second $\mathbf{R}$ factor recovers the dyonic domain walls discussed in [21]. All interesting dynamics are encoded in the metric on $\tilde{\mathcal{M}}_{N-1}$, the relative kink moduli space, which has complex dimension $(N-2)$. This inherits both a complex structure from Eq. (7) as well as $(N-2)$ holomorphic U(1) isometries from the Abelian flavor symmetries, and is thus a toric Kähler manifold. The quotient by the discrete group $\mathcal{G}$ acts only on the toric fibers. For generic domain wall masses, $\mathcal{G}=\mathbf{Z}$. For certain, rational choices of masses the second $\mathbf{R}$ factor collapses to $\mathbf{S}^1$, and $\mathcal{G}$ is a finite group. This is identical to monopoles in higher rank gauge groups. Unlike monoples however, the symmetries of the problem allow the metric to contain constant cross terms between the center of mass and relative motion factors—more on these later.

To find the metric, we first study fluctuations around the domain wall background. We concentrate on variations with respect to time, but the final answer may be trivially extended to have $d=2+1$ Lorentz symmetry on the domain wall world volume. The linearized Bogomol'nyi equations are

$$\partial\dot{\phi}=e^2\sum_{i=1}^{N}(\dot{q}_iq_i^\dagger+q_i\dot{q}_i^\dagger), \tag{11}$$

$$\mathcal{D}\dot{q}_i-i\dot{A}q_i=(\phi-m_i)\dot{q}_i+\dot{\phi}q_i, \tag{12}$$

which are augmented by Gauss' law, determining the electric field $E=F_{01}$,

$$\partial E=ie^2\sum_{i=1}^{N}(q_i\mathcal{D}_0q_i^\dagger-q_i^\dagger\mathcal{D}_0q_i). \tag{13}$$

This equation requires $E\neq 0$. However, it may be shown that neither $\mathrm{Im}(\phi)$ nor $\tilde{q}_i$ have zero modes, and so we continue to ignore them as per ansatz (2).[6] We choose to work in $A_0=0$ gauge, in which case the three equations above combine to give

$$\partial^2\left(\frac{\dot{q}_i}{q_i}\right)=\partial(\dot{\phi}+i\dot{A})=2e^2\sum_{j=1}^{N}\dot{q}_jq_j^\dagger,$$

which is valid for all i such that $q_i\neq 0$. The metric on the moduli space is, as usual, inherited from the kinetic terms in the action. After employing the above time-evolution equations, this can be brought into the form,

$$\begin{aligned}\mathcal{L}&=\int dx\,\frac{1}{2e^2}|\dot{\phi}+i\dot{A}|^2+\sum_{i=1}^{N}\dot{q}_i\dot{q}_i^\dagger\\&=\int dx\sum_{j=1}^{N}\dot{q}_j\dot{q}_j^\dagger-\frac{\dot{q}_i}{q_i}\dot{q}_j^\dagger q_j\quad\forall i\ (q_i\neq 0)\\&=\int dx\sum_j|q_j|^2(\dot{\psi}+m_j\dot{x}_o-\alpha_j^a\dot{r}^a)\\&\quad\times(m_j\dot{x}_0^\dagger-\alpha_i^a\dot{r}_a^\dagger),\end{aligned}$$

where, in the last equality, we have used the explicit expression for q_j given in Eq. (6). To make further progress, we must find an expression for $\dot{\psi}$ in terms of the collective coordinates x_0 and r_a. This involves solving Eq. (8) which is currently beyond our reach apart from in the strong coupling $e^2\to\infty$ limit. From now on, we therefore restrict ourselves to this regime of parameter space so that, using Eq. (10), Gauss' law becomes

$$\dot{\psi}_0=\frac{\sum_{i=1}^{N}(m_i\dot{x}_0+\alpha_i^a\dot{r}_a)\exp[-2m_i(x-X)-2\alpha_i^aR_a]}{\sum_{j=1}^{N}\exp[-2m_j(x-X)-2\alpha_j^aR_a]}.$$

Inserting this into the Lagrangian, and performing some of the more simple integrals, we find the metric on the $\mathcal{G}$-fold cover of $\mathcal{M}_{N-1}$ is given by

$$\mathcal{L}=\tfrac{1}{2}T|\dot{x}_0|^2+\zeta(\alpha_N^a-\alpha_1^a)\dot{x}_o\dot{r}_a^\dagger+\text{H.c.}+\frac{\partial^2\mathcal{K}}{\partial R_a\partial R_b}\dot{r}_a\dot{r}_b^\dagger. \tag{14}$$

The cross term between the center of mass x_0 and the relative separations r_a reflect the fact that the generic domain wall solution breaks parity, so that the metric is not invariant under $x_0\to -x_0$.[7] In relativistic terminology, the metric is "stationary," as opposed to "static," in the x_0 coordinate. The interesting dynamics are contained within the Kähler potential $\mathcal{K}(R_a)$ which is given by the integral

$$\mathcal{K}=\frac{\zeta}{4}\int dx\,\log\left[\sum_{i=1}^{N}\exp(-2m_ix-2\alpha_i^aR_a)\right]. \tag{15}$$

Note that although this integral is divergent, all such terms are at most linear in R_a, and so do not contribute to the metric. The toric Kähler structure of the metric, which is required by the global and supersymmetries of the theory, is manifest in these coordinates. In particular, the Kähler structure ensures that the bosonic Lagrangian (14) enjoys a supersymmetric extension. Since the domain walls are half BPS, their three-dimensional world volume dynamics preserve $\mathcal{N}=2$ supersymmetry (or four-supercharges). The relevant sigma model was found long ago [22] and includes a four-Fermi term coupling Grassmannian zero modes to the Riemann tensor associated with the metric (15).

As mentioned in the introduction, the function $\mathcal{K}$ contains information about the classical scattering of domain walls, as well as quantum effects in lower dimensional theories. For example, we may dimensionally reduce the theory (1) to $d=1+1$, with $\mathcal{N}=(4,4)$ supersymmetry. The spectrum of solitons is then determined by normalizable, harmonic forms on $\mathcal{M}_k$. The holomorphic subset of these forms survive as states in the $\mathcal{N}=(2,2)$ theory (in which the $\tilde{q}_i$ and their su-

[6]This is not true of the corresponding fermions and the superpartners of $\tilde{q}_i$ do yield fermionic zero modes.

[7]Such terms are absent for higher codimension solitons (vortices, monopoles, instantons) by rotational symmetry. I thank Adam Ritz for a discussion on this issue.

perpartners are removed). Alternatively, if we reduce further to $d=0+1$ quantum mechanics, the integral of the Euler form over $\mathcal{M}_k$ yields the k-instanton contribution to the four-Fermi correlation function.

Let us now restrict attention to the simplest case: two kinks with equal tension $T=\frac{1}{2}M\zeta$. This occurs for the $N=3$ model with the parameters

$$m_i=(\tfrac{1}{2}M,0,-\tfrac{1}{2}M), \quad \alpha_i=\tfrac{1}{6}(\tfrac{1}{2}M,-M,\tfrac{1}{2}M). \tag{16}$$

With this choice, the system is symmetric under $x_0\to -x_0$, ensuring that the $dx_0 dr^{\dagger}$ cross terms in the metric vanish. The moduli space takes the form

$$\mathcal{M}_2=\mathbf{R}\times\frac{\mathbf{S}^1\times\tilde{\mathcal{M}}_2}{\mathbf{Z}_2}.$$

The periodicities of the two angular variables can be found by careful examination of the solutions (6). The $\mathbf{S}^1$ factor is parametrized by $\theta_0\in[0,4\pi/M)$. The relative moduli space is parametrized by the collective coordinates $R\in\mathbf{R}$ and $\theta\in[0,8\pi/M)$. The $\mathbf{Z}_2$ symmetry acts only on these periodic variables,

$$\mathbf{Z}_2:\ \theta_0\to\theta_0+2\pi/M, \quad \theta\to\theta+4\pi/M.$$

Most importantly, with the choice of parameters (16) the Kähler potential given in Eq. (15) simplifies tremendously; in fact, to the point where Mathematica is happy to perform the integral with minimal complaint. We find the relative moduli space metric to be

$$ds^2=\tfrac{1}{16}M\zeta F(R)(dR^2+d\theta^2), \tag{17}$$

where all interactions are encoded in the function F,

$$\begin{aligned} F(R)&=e^{MR/2}\int dy\,\frac{e^{-y}+e^{y}}{(e^{-y}+e^{y}+e^{MR/2})^2}\\ &=\frac{2e^{MR/2}}{(e^{MR}-4)^{3/2}}\Bigg[e^{MR/2}(e^{MR}-4)^{1/2}\\ &\quad+4\log\left(\frac{2+e^{MR/2}-(e^{MR}-4)^{1/2}}{2+e^{MR/2}+(e^{MR}-4)^{1/2}}\right)\Bigg]. \end{aligned}$$

Note that, despite appearances to the contrary, the function F is real and positive definite. The apparent singularity at $e^{MR}=4$ is quite illusory: F is smooth at that point with value 4/3. As mentioned in the introduction, this function contains information about the spectrum of domain walls in $d=1+1$ dimensional gauge theories, as well as instanton effects in gauge quantum mechanics. Here we merely extract some simple physics concerning the classical dynamics. First, let us consider the limit of far-separated domain walls. As $R\to\infty$, the metric becomes

$$\begin{aligned} ds^2\to&\tfrac{1}{8}M\zeta[1-(2MR-4)e^{-MR}+\mathcal{O}(e^{-2MR})]\\ &\times(dR^2+d\theta^2). \end{aligned}$$

The constant term, with the factor of $M/8$ shows that we have correctly identified R as the large-distance separation between the kinks [this can be traced back to the choice of normalization of α in Eq. (16)]. The leading order correction to free motion is exponentially suppressed as expected for any soliton in a theory with a mass gap. The long-range velocity dependent force is

$$\ddot{R}=-e^{-MR}(M^2R-3M)(\dot{R}^2-\dot{\theta}^2).$$

For suitably large $\dot{\theta}$, the two kinks repel. If, instead, the kinks have constant relative phase, the overall minus sign implies that the velocity dependent force is attractive at large separation. In fact, numerical studies show that this attractive force persists for all values of R. Thus kinks moving initially apart will continue on such a trajectory, slowing but not halting. In contrast, kinks moving towards each other will increase their speed. Assuming the velocities remain small so that the moduli space approximation is valid, we can determine their fate by examining the metric as they approach. In the limit $R\to-\infty$, the function $F\to\frac{1}{2}\pi\exp(MR/2)$. After changing to coordinates

$$L=\left(\frac{\pi\zeta}{2M}\right)^{1/2}\exp(MR/4)\in\mathbf{R}^+,$$

then, as $L\to 0$, the metric becomes

$$ds^2\to dL^2+\tfrac{1}{16}M^2L^2d\theta^2,$$

which, for the specific range $\theta\in[0,8\pi/M)$, is nonsingular. Thus the two-kink moduli space is smooth, the collision is elastic, and the domain walls rebound with their phases exchanged, $\theta\to\theta+4\pi/M$.

IV. FRACTIONAL VORTICES

In this section, we discuss the relationship between the gauge theory domain walls considered above, and BPS semilocal vortices [23,24] (for a review, see [25]). The latter solitons are vortices in an Abelian gauge theory with a multicomponent Higgs field. This is precisely the model of Eq. (1) if we set the mass terms m_i to zero. The vortices have $\tilde{q}_i=\phi=0$, but a nonzero magnetic field, say $B=F_{13}$ for a vortex string extended in the x_2 direction. The BPS equations are

$$B=e^2\left(\sum_{i=1}^{N}|q_i|^2-\zeta\right),$$

$$\mathcal{D}_1q_i=\mathcal{D}_3q_i\,.$$

Suppose we dimensionally reduce on the x_3 direction, so that $\partial_3\equiv 0$ on all dynamical fields, and we further rename $A_3\equiv\mathrm{Re}(\phi)$. Then the vortex equations are precisely the domain wall equations (4) and (5) if, upon dimension reduction, we impose a Wilson line for the SU(N) flavor symmetry, introducing the masses m_i.

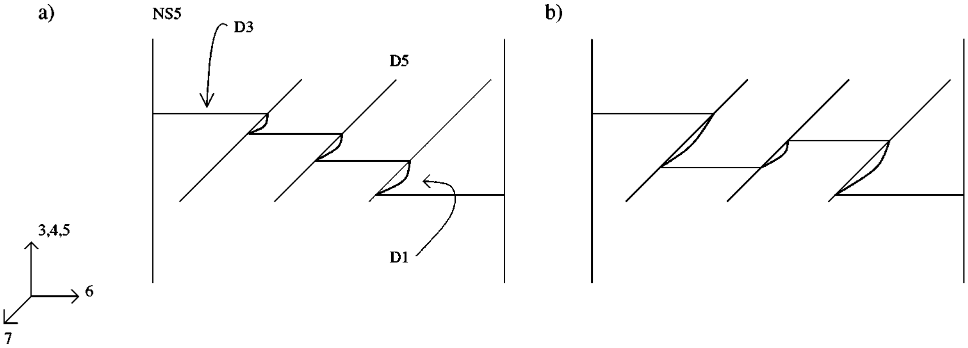

FIG. 1. Semilocal vortices from IIB branes. (a) has $\tilde{q}_i=0$, and finite mass BPS states exist. In (b), $\tilde{q}_i\neq 0$ and BPS states do not exist. The middle D-string has opposite orientation to the others, and breaks supersymmetry.

This situation is reminiscent of the relationship between monopoles and instantons. In this case, dimensional reduction of the self-dual Yang-Mills equations, with a Wilson line for the SU(N) gauge symmetry, results in the BPS monopole equation. In fact, in the monopole case, there is a deeper relationship between instantons and monopoles, discovered by Lee and Yi [12]. These authors consider SU(N) instantons compactified on $\mathbf{R}^3\times\mathbf{S}^1$ with a Wilson line around the $\mathbf{S}^1$, breaking the gauge group to the maximal torus. Such configurations are known as calorons. The dimension of the moduli space of calorons is the same as that of instantons in flat space; for Pontyagrin number k, there are $4kN$ moduli. Lee and Yi show that these collective coordinates may be understood as the position and internal phase of kN magnetic monopoles of N different types. Recall that on flat $\mathbf{R}^3$, an SU(N) gauge theory plays host to only $(N-1)$ different types of monopoles, one for each simple root of the Lie algebra. For the caloron, the extra monopole is associated to the affine root of the Lie algebra, and arises because of the periodic nature of the Higgs field. The simplest way to see this result is using the string setup of D0-D4-branes compactified on a circle.

In the remainder of this paper, we would like to show that a similar phenomenon happens for semilocal vortices. Related observations were also made in [21]. As in the case of instantons and monopoles, the simplest way to see this result is through a brane construction. In fact, we choose to model the $d=2+1$ theory which is simply the dimensional reduction of the model considered up to now,[8] which we subsequently compactify on $\mathbf{R}^{1,1}\times\mathbf{S}^1$. There are (at least) two ways to construct semilocal vortices in string theory, and each has its advantages. The first method uses IIA string theory with a background of D2- and D6-branes, together with an NS-NS B_2 field,

$$D2:\ 012,$$

$$N\times D6:\ 0123456,$$

with antiself dual $B_{\mu\nu}$ for $\mu,\nu=3,4,5,6$. The theory on the D2-brane is the U(1) gauge theory of interest, with $\zeta\sim|B|$. In this setup, the Chern-Simons terms on the D2-brane world volume imply that the semilocal vortex may be thought of as a D0-brane absorbed within the D2-brane. Unfortunately, moduli counting is difficult to see from this perspective, as the D0-brane does not preserve supersymmetry when it is removed from the D2-D6-bound state. Nevertheless, we shall have use for this construction later.

The second construction of semilocal vortices uses the IIB Hanany-Witten setup

$$2\times NS5:\ 012345,$$

$$N\times D5:\ 012789,$$

$$D3:\ 0126,$$

$$k\times D1:\ 07.$$

The full configuration is drawn in Fig. 1. The low-energy theory on the D3-brane is the $d=2+1$, $\mathcal{N}=4$ $U(1)$ gauge theory with N flavors described in Eq. (1). Although this setup employs more branes, it has the advantage that all parameters have geometric origins. For example, the real FI parameter is given by the relative positions of the NS5-branes,

$$x^8|_{NS5}=x^9|_{NS5}=0,\quad \Delta x^7|_{NS5}=\zeta,$$

while the complex masses are related to the relative positions of the D5-branes in the x^4 and x^5 directions. For the purposes of discussing vortices, we set the masses to zero. The vevs of q_i and $\tilde{q}_i$ are determined by the positions of the D3-brane segments suspended between the D5-branes. We denote the $(N+1)$ segments as $D3_\alpha$. For BPS vortices to exist, we require $\tilde{q}_i=0$. In the brane picture, this translates to

$$x^8|_{D3_\alpha}=x^9|_{D3_\alpha}=0,\quad x^7|_{D3_{\alpha+1}}>x^7|_{D3_\alpha}.$$

In this case, k parallel D-strings, lying in the x^7 direction, may connect the first D3-brane segment with the last, while preserving supersymmetry, see Fig. 1(a). This is the semilocal vortex with magnetic charge k. Each of these D-strings splits into N separate segments as illustrated in the picture. Since each of these segments is free to roam the (x^1-x^2) plane, the moduli space of semilocal vortices has real dimension $2kN$. This counting is indeed correct, as shown in [24]. Moreover, this provides the first piece of numerological evidence that, as for instantons and monopoles, the charge k

[8]The $d=3+1$ dimensional theory has a Landau pole. This does not affect the classical discussion of this paper but, due to the omniscience of string theory, makes a brane discussion more subtle.

semilocal vortex may decompose into kN parallel domain walls when placed on a circle.

One may wonder why a single segment of D-string does not qualify as a BPS state of the theory. To see this, note that the flux from such a D-string is deposited on the D3-brane segment, where it spreads out. Since the D3-brane is noncompact only in two spatial dimensions, this state has a logarithmically divergent mass. Only when the D-strings form a closed, oriented path from the first D3-brane to the last can this flux escape onto the NS5-branes, leading to a state of finite energy. Note also that the brane picture suggests that finite mass states also exist when $\tilde{q}_i \neq 0$, see Fig. 1(b). However, since the D-strings are not parallel in this case, the state breaks supersymmetry.

We turn now to the question of vortices on $\mathbf{R}\times\mathbf{S}^1$, where the circle is taken to have radius R. To realize this in string theory, we may work with either of the brane constructions above, and make x^2 periodic. Since the IIA setup is the less complicated (and easier to draw) we choose to work in the D2-D6 system. If a Wilson line for the SU(N) flavor symmetry is introduced, upon T duality the D6-branes become D5-branes separated in the x^2 direction. The flat D2-brane becomes a D-string oriented in the x^1 direction. The result is the familiar D1-D5 system. The presence of the B_2 field, which is not affected by T duality, induces a force between the D-string and the D5-branes. This reproduces the N vacuum states, in which the D-string lies within a given D5-brane. What becomes of the vortex? At the position of the vortex, the D2-brane is wrapping an internal cycle and thus, locally, is not extended in the x^2 direction. Upon T duality, we therefore expect a D-string wrapping the dual S^1. The final configuration for $N=2$ is shown in Fig. 2. The D-string interpolates to one of the vacuum states at $x_1\rightarrow\pm\infty$, lying within one of the D5-branes. However, at the location of the vortex, it leaves its asymptotic location to wind k times around the circle ($k=1$ is drawn in the picture). On the way around, it may make a temporary home in any one of the other $(N-1)$ D5-branes it encounters. The existence of such BPS kinky D-string configurations was predicted in [26], and further examined in [16] where it was shown that they indeed correspond to the gauge theory domain walls discussed in this paper.

Let us discuss the $N=2$ field theory in more detail. This theory contains only two vacua, located at $\phi=\pm m$. (Since the mass m arose from a flavor Wilson line, it is periodic so we must have $m<\pi/R$.) In flat space, this would imply the existence of a single kink, in which ϕ monotonically increases from $\phi=-m$ to $\phi=+m$. There is also a corresponding antikink in which ϕ decreases, in the other direction. However, as the brane picture clearly demonstrates, when placed on $\mathbf{R}\times\mathbf{S}^1$, there exists a further *kink* (as opposed to antikink) solution which interpolates from $\phi=+m$ to $\phi=-m$. This preserves the same supersymmetry as the original kink and is possible because ϕ arises from a Wilson line, $\phi=\int_{S^1}A$. Invariance under large gauge transformations means that ϕ has period $2\pi/R$. This allows ϕ to interpolate from $+m>0$ to $-m<0$ with $\phi'>0$ at all times. For the theory on the circle, the mass of the original kink is

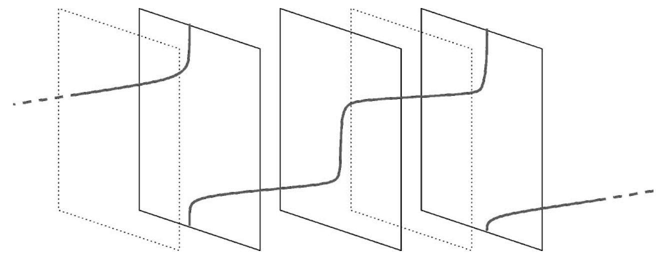

FIG. 2. Kinks as fractional vortices: The $N=2$, $k=1$ model. The infinite array of branes is periodic mod 2. The 4 collective coordinates of the vortex are seen as the positions and phases of two kinks.

$2\tilde{\zeta}m$, where $\tilde{\zeta}=2\pi R\zeta$. In the limit $R\rightarrow 0$, we keep $\tilde{\zeta}$ fixed, to ensure that this kink remains in the spectrum. In contrast, the new kink which interpolates from $+m$ to $-m$, has mass $\tilde{\zeta}(2\pi/R-2m)$. As $R\rightarrow 0$, with $\tilde{\zeta}$ fixed, this kink decouples, and we recover the situation described at the beginning of this paper.

A periodic ϕ allows for the possibility of multiple domain wall solutions for theories with only two, or indeed one, vacuum states. It would be interesting to examine the metric on these moduli spaces in more detail, especially in light of the fact that they give a Kähler deformation of the (semilocal) vortex moduli space.

As stressed above, the preceding discussion is entirely analogous to that of instantons and monopoles given in [12]. In that case, the calorons have found an interesting application in supersymmetric quantum field theories. It has long been known that certain strongly coupled four-dimensional field theories receive nonperturbative contributions that have the characteristics of fractional instantons. The decomposition of instantons into monopoles when compactified on a circle gives a physical manifestation of this phenomenon and, when coupled with holomorphy, may be used to compute fractional instanton effects in four-dimensional gauge theories in the weakly coupled regime [27,28]. Fractional instantons also appear in certain two-dimensional gauge theories and sigma models, the most familiar example being the $\mathcal{N}=(2,2)$, $\mathbf{CP}^{N-1}$ sigma model. As for the four-dimensional case, compactification on a circle gives a new method to compute these effects using controlled, semiclassical techniques.

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