Compact AdS space, brane geometry, and the AdS/CFT correspondence

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The AdS/CFT correspondence can be realized in spaces that are globally different but share the same asymptotic behavior. Two known cases are a compact AdS space and the space generated by a large number of coincident branes. We discuss the physical consistency, in the sense of the Cauchy problem, of these two formulations. We show that the role of the boundary in the compact AdS space is equivalent to that of the flat asymptotic region in the brane space. We also show, by introducing a second coordinate chart for the pure AdS space, that a point at its spatial infinity corresponds to a horizon in the brane system.

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I. INTRODUCTION

According to the Maldacena conjecture [1] the large N limit of $SU(N)$ superconformal field theories in n dimensions can be described by supergravity on anti–de Sitter (AdS) space-time in $n+1$ dimensions. This is known as the AdS/conformal field theory (CFT) correspondence. By supergravity one means the tree level approximation of string or M theory defined on $\mathrm{AdS}_{n+1}\times \mathrm{M}_d$, where M_d is some d-dimensional compactification space. In this correspondence (see also Refs. [2–5]) the AdS space shows up both as a near horizon geometry of a set of coincident D3-branes or as a solution of ten dimensional supergravity (a Dirichlet p-brane or Dp-brane is a $p+1$ dimensional hyperplane where strings are allowed to end [6,7]).

Precise prescriptions for the realization of the AdS/CFT correspondence were presented in [8,9] by considering Poincaré patches of AdS space. The Poincaré coordinate system allows a simple definition for the flat boundary where the conformal field theory is defined. However there are some differences in the spaces considered in these references that we will discuss in this article. Gubser, Klebanov and Polyakov [8] started with a space generated by a large number N of coincident D3-branes. This space can be approximated by an AdS near the branes and a flat space far from them. On the other side, Witten [9] has considered an AdS space in Poincaré coordinates but compactified by the inclusion of the boundary. These two formulations lead to equivalent results in the sense that conformal boundary correlation functions are the same (see also [10,11]).

The approximation of the D3-brane metric as an AdS space near the branes is valid as long as the axial AdS coordinate is smaller than a parameter that increases with N. In the Maldacena conjecture the large N limit is considered. So one might think that in this case the D3-brane space becomes a pure AdS (without the boundary). However a consistent quantization is not possible in an AdS space without boundary because of the absence of a well defined Cauchy problem [12,13]. In the formulation of [8] the AdS space is always complemented by the flat space asymptotic region far from the branes. We will see that this guarantees a well posed Cauchy problem. In the work of [9] there is no asymptotic flat region so it is necessary to introduce a compactification of the AdS space for physical consistency, as we will discuss in Sec. III.

Recently we have investigated the quantization of scalar fields in the AdS bulk in terms of Poincaré coordinates [14,15]. The compactification in this coordinate system requires the introduction of a point at infinity which can only be properly accommodated in a second coordinate chart. The two coordinate charts must match at some finite value of the axial coordinates implying a discretization of the field spectrum. Then it is possible to find a one to one mapping between bulk and boundary quantum states, at least for scalar fields [16]. One can then ask: Does this extra point at infinity have any physical role or is it just a mathematical tool for a consistent quantization? We answer this question in Sec. II by constructing explicitly a second coordinate chart complementing the original Poincaré one. We will see that the point at infinity represents, in the pure AdS space, the horizon that is found in the D3-brane metric. Curiously the complete compactification of AdS space in Poincaré coordinates introduces a new horizon not present in the brane system. We also find an interpretation for this horizon.

II. AdS SPACE AND COMPACTIFICATION

We will start considering a pure AdS space of $n+1$ dimensions. This space can be represented as the hyperboloid ($\Lambda=\mathrm{const}$)

$$X_0^2+X_{n+1}^2-\sum_{i=1}^{n} X_i^2=\Lambda^2 \tag{1}$$

in a flat $n+2$ dimensional space with measure

$$ds^2=-dX_0^2-dX_{n+1}^2+\sum_{i=1}^{n} dX_i^2 . \tag{2}$$

The so called global coordinates ρ,τ,Ω_i for AdS_{n+1} can be defined by [2,3]

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$$X_0=\Lambda \sec\rho \cos\tau$$
$$X_i=\Lambda \tan\rho\, \Omega_i \quad \left(\sum_{i=1}^{n} \Omega_i^2=1\right); \tag{3}$$
$$X_{n+1}=\Lambda \sec\rho \sin\tau,$$

with ranges $0\leqslant\rho<\pi/2$ and $0\leqslant\tau<2\pi$. The line element has the form

$$ds^2=\frac{\Lambda^2}{\cos^2(\rho)}[-d\tau^2+d\rho^2+\sin^2(\rho)d\Omega^2]. \tag{4}$$

In order to identify τ as a usual time coordinate it is necessary to unwrap it. This can be done by taking copies of the original AdS space that together represent the AdS covering space [12]. For simplicity we will continue to call this covering space AdS as is usual in the literature.

A consistent quantum field theory in AdS space requires the addition of a boundary at spatial infinity: $\rho=\pi/2$ in global coordinates. This compactification of the space makes it possible to impose appropriate conditions and find a well defined Cauchy problem. (Otherwise massless particles could go to or come from spatial infinity in finite times.) This result was established in [12,13].

On the other hand, AdS space can be represented by Poincaré coordinates z, $\vec{x}$, t that are more useful for the study of the AdS/CFT correspondence. These coordinates are defined by

$$X_0=\frac{1}{2z}(z^2+\Lambda^2+\vec{x}^2-t^2)$$
$$X_j=\frac{\Lambda x^j}{z} \quad (j=1,\ldots,n-1);$$
$$X_n=-\frac{1}{2z}(z^2-\Lambda^2+\vec{x}^2-t^2)$$
$$X_{n+1}=\frac{\Lambda t}{z}, \tag{5}$$

where $\vec{x}$ has $n-1$ components and $0\leqslant z<\infty$. In this case the AdS_{n+1} measure with Lorentzian signature reads

$$ds^2=\frac{\Lambda^2}{(z)^2}[dz^2+(d\vec{x})^2-dt^2]. \tag{6}$$

It is important to see how the compactification discussed in global coordinates can be realized in this system. The AdS boundary ($\rho=\pi/2$ in global coordinates) corresponds to the region $z=0$ described by usual Minkowski coordinates $\vec{x}$, t plus a "point" at infinity ($z\rightarrow\infty$). The point at infinity cannot be accommodated in the original Poincaré chart [14,15] so we have to introduce a second coordinate system to represent it properly.

It is convenient to introduce first an auxiliary variable that will connect the two charts. Let us define the auxiliary variable u as the argument of a monotonic function $f(u)$ such that

$$z=\frac{1}{f(u)}. \tag{7}$$

This way the point $z\rightarrow\infty$ is mapped at the zero of $f(u)$. The simplest choice for this function is a linear one as

$$f(u)=c_o+c_1 u \tag{8}$$

with c_0,c_1 constants. The relation (7) is not defined for $u=-c_0/c_1$ that corresponds to the point at infinity. Also the variables z and u are not related at the point $z=0$. As the zero value of z would be reached for infinite u we can take relation (7) to be valid in the interval $\delta\leqslant z<\infty$ for some small positive δ. This implies a finite range for u. For convenience we choose $c_0=1/\delta$, $c_1=-1$ so that $0\leqslant u\leqslant 1/\delta$ and

$$z=\frac{1}{1/\delta-u}. \tag{9}$$

Then we can define the second coordinate chart $(z',\vec{x},t)$ with

$$z'=\frac{1}{u}. \tag{10}$$

Now the point $z\rightarrow\infty$ is represented in the second chart at $z'=\delta$. The coordinates z and z' of the two charts are related by

$$\frac{1}{z'}=\frac{1}{\delta}-\frac{1}{z} \tag{11}$$

with range $\delta\leqslant z'<\infty$.

The metric of the second coordinate system involves a Poincaré-like factor

$$ds^2=\frac{\Lambda^2}{z'^2}\left(\frac{\delta^2}{(z'-\delta)^6}dz'^2+\frac{(z'-\delta)^2}{\delta^2}[(d\vec{x})^2-dt^2]\right). \tag{12}$$

Now the compact AdS space is described by the coordinate charts corresponding to Eqs. (6) and (12). For example, for an AdS_5 we can calculate the Ricci scalar curvature for the two charts finding

$$R=-20\,\frac{1}{\Lambda^2} \tag{13}$$

for both, as expected since they describe parts of the same AdS space.

Further, with this second chart we find a horizon (infinite singularity in the spatial part of ds^2) at $z'=\delta$. This was not apparent in the original Poincaré chart. We are going to see in the next section that this horizon corresponds to the one found in the D3-brane system. Some other aspects of Poincaré coordinate description of AdS space have been studied in [17].

III. BRANES AND AdS SPACE

The brane system is one of the physical settings for the AdS/CFT correspondence. Let us now study the ten dimensional geometry generated by N coincident D3-branes and its relation to the compactified AdS space. The metric can be written as [6,8]

$$ds^2=\left(1+\frac{\Lambda^4}{r^4}\right)^{-1/2}(-dt^2+d\vec{x}^2)+\left(1+\frac{\Lambda^4}{r^4}\right)^{1/2}(dr^2+r^2d\Omega_5^2) \quad (14)$$

where we are using the same symbol Λ for a constant that now satisfies $\Lambda^4=N/2\pi^2T_3$ where T_3 is the tension of a single D3-brane. The metric (14) has a horizon at $r=0$ with zero perpendicular area (apart from the S_5 term).

It is interesting to look at the space corresponding to Eq. (14) in two limiting cases where it assumes simpler asymptotic forms: large and small r compared to Λ. Considering first the region $r\gg\Lambda$ (far from the horizon) the space is asymptotically a ten dimensional Minkowski space:

$$(ds^2)_{far}=-dt^2+d\vec{x}^2+dr^2+r^2d\Omega_5^2 . \quad (15)$$

Now looking at the near horizon region $r\ll\Lambda$ we can approximate the metric (14) as

$$(ds^2)_{near}=\frac{r^2}{\Lambda^2}(-dt^2+d\vec{x}^2)+\frac{\Lambda^2}{r^2}dr^2+\Lambda^2d\Omega_5^2 . \quad (16)$$

Changing the axial coordinate according to $z=\Lambda^2/r$, as in Refs. [1,8], the metric that will describe the brane system as long as $r/\Lambda\ll1$ takes the form

$$ds^2=\frac{\Lambda^2}{z^2}[dz^2+(d\vec{x})^2-dt^2]+\Lambda^2d\Omega_5^2 \quad (17)$$

corresponding to $\mathrm{AdS}_5\times S_5$. This corresponds to the Poincaré chart (6) apart from the S_5 factor. Note however that the horizon $r=0$ which corresponds to the limit $z\to\infty$ is not included in this chart as a consequence of the lack of a relation between z and r at $r=0$. It is interesting to note that from the point of view of a pure AdS space, one has to include this point as a requirement for a consistent quantization. Considering the brane space this point is already present, corresponding to the brane location. The inclusion of this point in the AdS space is possible by introducing one more coordinate chart as discussed in the previous section. Explicitly, the point $r=0$ corresponds to $z'=\delta$. So, indeed the horizon found in the second chart at $z'=\delta$ corresponds to the brane horizon.

Let us now examine the large r region of the D3-branes space. A massless particle moving in the $r\to\infty$ direction will arrive at an asymptotically Minkowski space as in Eq. (15). Then it would spend an infinite time to reach spatial infinity. So, the Cauchy problem is well posed for the D3-branes space and it is geodesically complete. This is the physical setting of Gubser, Klebanov and Polyakov [8].

Further it is interesting to consider the limit $\Lambda\to\infty$ as suggested by the Maldacena conjecture. The larger we take Λ the larger is the range of r for which the AdS approximation (17) for the brane metric (14) holds. So one could naively disregard the asymptotic flat space region in this limit. Then one would find an AdS space without the boundary, where particles could enter or leave the space in finite times. This would lead to the absence of a well defined Cauchy problem.

If one chooses to disregard the flat space region, boundary conditions should be imposed at $r\to\infty$ in order to recover physical consistency. That means, in the limit $\Lambda\to\infty$ we should not represent the branes space by just a pure AdS space but rather by a compactified AdS including the hypersurface at $z=0$ besides the point z at infinity. This is Witten's [9] physical setting for the AdS/CFT correspondence.

It is interesting to note that if we consider the whole space to be of the AdS form [Eq. (17)] there is a horizon with infinite area at $z=0$. This is not present in the D3-branes model and it is a consequence of closing the AdS space as required for physical consistency once the asymptotic flat space region has been removed. This emphasizes the differences between the spaces considered in equivalent formulations of AdS/CFT correspondence.

Note also that the boundary of the space defined by the metric (17), apart from the point at $z\to\infty$, corresponds to $z=0$ which naively has $8+1$ dimensions. But as z approaches zero the term $\Lambda^2d\Omega_5^2$ becomes irrelevant with respect to the AdS part. So we can think of the $z=0$ hypersurface as just $3+1$ dimensional. Then naturally the CFT lives in 4 dimensions, although the brane model is defined in 10 dimensions.

IV. CONCLUSIONS

The AdS/CFT correspondence can be realized in different spaces. One of them is the space generated by N coincident D3-branes and another is a compact AdS space. We have discussed the physical consistency of these two formulations from the point of view of the Cauchy problem. The brane space is consistent thanks to the existence of a flat asymptotic region far from the branes and of a horizon on their location.

For the compact AdS space, consistency comes from the inclusion of its boundary: a hypersurface at $z=0$ plus a point at $z\to\infty$. As this point at infinity is not properly represented in the Poincaré patch we introduced a second coordinate chart. In this chart that point is found at $z'=\delta$ which corresponds to a horizon. This horizon was not apparent in the Poincaré patch although it is present in the brane system. This provides a nice physical interpretation for the inclusion of the point at infinity in the AdS case. We have also found a horizon at $z=0$ that is not present in the brane system. The two horizons together are responsible for the physical consistency of the pure AdS case. In fact one can think that the AdS boundary corresponds just to a single horizon which confines particles inside the space. This can be seen by looking at global coordinates where this boundary is represented simply as the hypersurface $\rho = \pi/2$. In this regard it is important to mention that the connectedness of the AdS boundary was proved in [18].

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[1] J. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998).
[2] O. Aharony, S.S. Gubser, J. Maldacena, H. Ooguri, and Y. Oz, Phys. Rep. **323**, 183 (2000).
[3] J.L. Petersen, Int. J. Mod. Phys. A **14**, 3597 (1999).
[4] I.R. Klebanov, ''TASI Lectures: Introduction to the AdS/CFT Correspondence,'' hep-th/0009139.
[5] D. Bigatti and L. Susskind, ''TASI Lectures on the Holographic principle,'' hep-th/0002044.
[6] G. Horowitz and A. Strominger, Nucl. Phys. **B360**, 197 (1991).
[7] J. Polchinski, ''TASI Lectures on D-branes,'' hep-th/9611050.
[8] S.S. Gubser , I.R. Klebanov, and A.M. Polyakov, Phys. Lett. B **428**, 105 (1998).
[9] E. Witten, Adv. Theor. Math. Phys. **2**, 253 (1998).
[10] W. Mueck and K.S. Viswanathan, Phys. Rev. D **58**, 041901 (1998).
[11] D.Z. Freedman, S.D. Mathur, A. Matusis, and L. Rastelli, Nucl. Phys. **B546**, 96 (1999).
[12] S.J. Avis, C.J. Isham, and D. Storey, Phys. Rev. D **18**, 3565 (1978).
[13] P. Breitenlohner and D.Z. Freedman, Phys. Lett. **115B**, 197 (1982); Ann. Phys. (N.Y.) **144**, 249 (1982).
[14] H. Boschi-Filho and N.R.F. Braga, Phys. Lett. B **505**, 263 (2001).
[15] H. Boschi-Filho and N.R.F. Braga, Nucl. Phys. **B608**, 319 (2001).
[16] H. Boschi-Filho and N.R.F. Braga, Phys. Lett. B **525**, 164 (2002).
[17] B. McInnes, Nucl. Phys. **B602**, 132 (2001).
[18] E. Witten and S.-T. Yau, Adv. Theor. Math. Phys. **3**, 1635 (1999).