Finite temperature induced fermion number in the nonlinear σ model in 2+1 dimensions

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We compute the finite temperature induced fermion number for fermions coupled to a static nonlinear sigma model background in 2+1 dimensions, in the derivative expansion limit. While the zero temperature induced fermion number is well known to be topological (it is the winding number of the background), at finite temperature there is a temperature dependent correction that is nontopological—this finite T correction is sensitive to the detailed shape of the background. At low temperature we resum the derivative expansion to all orders, and we consider explicit forms of the background as a $\mathcal{CP}^1$ instanton or as a baby Skyrmion.

I. INTRODUCTION

The phenomenon of induced fermion number arises due to the interaction of fermions with nontrivial topological backgrounds (e.g., solitons, vortices, monopoles, Skyrmions), and has many applications ranging from polymer physics to particle physics [1–8]. At zero temperature, the induced fermion number is a topological quantity (modulo spectral flow effects), and is related to the spectral asymmetry of the relevant Dirac operator, which counts the difference between the number of positive and negative energy states in the fermion spectrum. Mathematical results, such as index theorems and Levinson's theorem, imply that the fractional part of the zero temperature induced fermion number is topological in the sense that it is determined by the asymptotic properties of the background fields [9–14]. This topological character of the induced fermion number is a key feature of its application in certain model field theories [15–18]. At finite temperature, the situation is very different—the induced fermion number is generically nontopological, and is not a sharp observable [19–21]. Several explicit examples of kinks [22–24] and sigma models [19,20] in 1+1 dimensions, and monopoles [19,26–28] in 3+1 dimensions, have been analyzed in detail. In this paper we compute the finite temperature induced fermion number for fermions coupled to a static nonlinear sigma model background in 2+1 dimensions.

At zero temperature, the induced fermion number for fermions coupled to a static nonlinear sigma model background in 2+1 dimensions has been studied extensively [29–32]. The $T=0$ induced fermion number is equal to the winding number of the sigma model background field, and this result may be interpreted in terms of a topological current density. This system has applications both in condensed matter physics [31,34], where the sigma model provides a phenomenological model for a two dimensional Heisenberg ferromagnet, and in particle and nuclear physics, where the sigma model can be used as a "baby Skyrmion" model

[29,30,33,35] that mimics many properties of the (3+1)-dimensional Skyrme model for baryons [16]. One motivation for this paper is to compute the induced fermion number at nozero temperature and to understand in detail the origin of the nontopological T dependent contributions, as similar effects will occur in the (3+1)-dimensional Skyrme case.

In Sec. II we define what is meant by finite temperature induced fermion number, and indicate how it can be computed. In Sec. III we use the derivative expansion to compute the finite temperature induced fermion number for a nonlinear σ-model background in 2+1 dimensions. At low temperature we resum the derivative expansion to all orders, and we consider explicit forms of the background as a $\mathcal{CP}^1$ instanton or as a baby Skyrmion. We conclude in Sec. IV with some comments concerning the relation of our results to the well known $T=0$ results, and concerning the possible extension to (3+1)-dimensional Skyrme models.

II. FINITE TEMPERATURE INDUCED FERMION NUMBER

The induced fermion number is an expectation value of the second quantized fermion operator $N=\frac{1}{2}\int dx[\Psi^\dagger,\Psi]$. For a given static classical background field configuration, the fermion field operator Ψ can be expanded in a complete set of eigenstates of the Dirac Hamiltonian H. Expectation values of N can then be computed. At zero temperature, the fermion number is a vacuum expectation value $\langle N\rangle_0 \equiv\langle 0|N|0\rangle$, and is related to the spectral asymmetry of the Dirac Hamiltonian [6]

$$\langle N\rangle_0 = -\frac{1}{2}(\text{spectral asymmetry})$$

$$= -\frac{1}{2}\int_{-\infty}^{\infty} dE\, \sigma(E)\text{sgn}(E). \quad (1)$$

Here $\sigma(E)$ is the spectral function of the Dirac Hamiltonian H:

$$\sigma(E) = \frac{1}{\pi} \text{Im Tr}\left(\frac{1}{H-E-i\epsilon}\right). \tag{2}$$

At nonzero temperature, T, the induced fermion number is a *thermal* expectation value:

$$\langle N \rangle_T = \frac{\text{Tr}(e^{-\beta H} N)}{\text{Tr}(e^{-\beta H})}$$

$$= -\frac{1}{2}\int_{-\infty}^{\infty} dE \ \sigma(E)\tanh\left(\frac{\beta E}{2}\right) \tag{3}$$

where $\beta \equiv 1/T$. Notice that this finite temperature expression (3) reduces smoothly to the zero temperature expression (1) as $\beta \to \infty$. In fact, the nonzero temperature provides a physically meaningful and mathematically elegant regularization of the spectral asymmetry.

The spectrum of the fermions is independent of the temperature in our approximation of a static classical background field. All information about the fermion spectrum, in the presence of the background, is encoded in the spectral function $\sigma(E)$ defined in Eq. (2). Thus, knowledge about the spectral function $\sigma(E)$ is the key to evaluating Eq. (1) or Eq. (3). Actually, to compute the induced fermion number (either at $T=0$, or $T>0$) one only needs the *odd* part of the spectral function, as is clear from Eqs. (1) and (3). Thus, the calculational problem is to find the odd part of $\sigma(E)$.

A more physical interpretation of the finite T fermion number $\langle N \rangle_T$ is provided by noting [20,21] that $\langle N \rangle_T$ in Eq. (3) separates naturally into a $T=0$ piece and a finite T correction

$$\langle N \rangle_T = \langle N \rangle_0 + \int_{-\infty}^{\infty} dE \ \sigma(E)\text{sgn}(E)n(|E|), \tag{4}$$

where

$$n(E) = \frac{1}{e^{\beta E}+1} \tag{5}$$

is the Fermi-Dirac distribution function, and we have used the simple fact that $\tanh(\beta E/2) = 1 - 2n(E)$.

The first term, $\langle N \rangle_0$, in Eq. (4) is known to be topological in the sense that it is invariant under small local deformations of the background which do not change the asymptotic boundary values of the background. Physically, this topological term corresponds to the vacuum polarization response of the system to the presence of the background. The second term in Eq. (4), which is the finite T correction term, is generically nontopological in the sense that it is sensitive to local deformations in the background. Physically, this term corresponds to the plasma response of the system to the presence of the background. It incorporates effects beyond the vacuum response, and so is more sensitive to the entire single-particle fermion spectrum. The caveat "generically" has been included here because for certain very special backgrounds the finite T correction is itself topological. This happens when the Dirac Hamiltonian has a quantum mechanical

supersymmetry relating the positive and negative parts of the spectrum [20,21]. Several explicit examples (in $1+1$ and $3+1$ dimensions) of this distinction between topological and nontopological contributions have been studied in detail already [19–21]. An example in $2+1$ dimensions where the finite T correction is topological is the case of a background flux string [25–27]. In this paper we study a $(2+1)$-dimensional example in which the finite T correction is nontopological.

III. THE NONLINEAR SIGMA MODEL IN 2+1 DIMENSIONS

Consider fermions coupled to a static σ-model background in $2+1$ dimensions:

$$\mathcal{L} = \bar{\psi}(i\gamma^{\mu}\partial_{\mu} - m\hat{n}\cdot\vec{\tau})\psi. \tag{6}$$

The fermions are in the defining representation of SU(2), and the static σ-model background is represented by a 3-vector (in internal space) $\hat{n}(\vec{x})$, which is constrained to take values on S^2:

$$\hat{n}^2(\vec{x}) = 1. \tag{7}$$

In the Lagrangian (6), the τ^a are Pauli matrices, and m is a mass parameter that sets an important energy scale of the single-particle fermion spectrum. Later in the paper we will consider specific profiles for $\hat{n}(\vec{x})$. At $T=0$ the quantization of fermions in this background is well known, and leads to an induced fermion number [29–31]

$$\langle N \rangle_0 = \frac{1}{8\pi}\int d^2x \ \epsilon^{abc}\epsilon^{ij}\hat{n}^a\partial_i\hat{n}^b\partial_j\hat{n}^c. \tag{8}$$

This induced fermion number, $\langle N \rangle_0$, is just the integer winding number of the map $\hat{n}:\mathfrak{R}^2 \to S^2$. With the boundary condition that $\hat{n}(\vec{x})$ has an angle independent limit at spatial infinity, we can compactify the spatial $\mathfrak{R}^2$ to S^2, in which case $\langle N \rangle_0$ gives the integer winding number for maps $\hat{n}:S^2 \to S^2$. Thus, $\langle N \rangle_0$ is topological.

Invoking Lorentz invariance, one interprets the result (8) as the spatial integral, $\int d^2x J^0$, of the zeroth component J^0 of a "topological" current density

$$J^{\mu} = \frac{1}{8\pi}\epsilon^{\mu\nu\lambda}\epsilon^{abc}\hat{n}^a\partial_{\nu}\hat{n}^b\partial_{\lambda}\hat{n}^c. \tag{9}$$

This current is manifestly conserved ($\partial_{\mu}J^{\mu}=0$) without use of the equations of motion, and hence is called "topological." In fact, J^{μ} may (and generally does) have higher derivative corrections beyond the term shown in Eq. (9). However, these are all total derivatives and hence do not contribute to an integrated quantity such as the zero temperature fermion number $\langle N \rangle_0$ in Eq. (8) [10].

At nonzero T, several aspects of this familiar story change. First, as we show explicitly below for this $(2+1)$-dimensional σ-model case, there are higher derivative corrections to J^0 that are not total derivatives, and these

contribute to the finite T correction to the $T=0$ induced fermion number (8). Second, at finite T one can no longer invoke Lorentz invariance (the thermal bath determines a preferred frame), and so one cannot directly infer a topological current density as was done in Eq. (9).

A convenient approach to this calculation is to express Eq. (3) as a contour integral (using the $i\epsilon$ prescription in the spectral function) in terms of the Dirac resolvent

$$R(z) \equiv \mathrm{Tr}\left(\frac{1}{H-z}\right). \tag{10}$$

Thus we have

$$\langle N \rangle_T = -\frac{1}{2} \int_{\mathcal{C}} \frac{dz}{2\pi i} \mathrm{Tr}\left(\frac{1}{H-z}\right) \tanh\left(\frac{\beta z}{2}\right) \tag{11}$$

where $\mathcal{C}$ is the contour $(-\infty+i\epsilon, +\infty+i\epsilon)$ and $(+\infty-i\epsilon, -\infty-i\epsilon)$ in the complex energy plane. For a Hermitian Hamiltonian H, the poles and branch cuts of the resolvent lie on the real axis, so one has two choices for evaluating the contour integral in Eq. (11). First, one can deform the contour $\mathcal{C}$ around the poles and cuts of the resolvent $R(z)$. This approach leads to an integral representation for $\langle N \rangle_T$. Alternatively, one could deform the contour $\mathcal{C}$ around the Matsubara poles, $z_n = (2n+1)i\pi T$, of the $\tanh(\beta z/2)$ function, which lie on the imaginary axis. This leads to a summation representation for $\langle N \rangle_T$. These two representations are, of course, equivalent. The integral representation is just the familiar Sommerfeld-Watson transform of the summation representation.

Thus, in order to evaluate the finite temperature induced fermion number, $\langle N \rangle_T$, one needs the odd part of the spectral function $\sigma(E)$; or, equivalently, the even part of the resolvent $R(z)$. No exact expression is known for $R_{\mathrm{even}}(z)$ for the interaction described by Eq. (6). Therefore, one needs an approximate technique. Here we use the derivative expansion [36], in which one assumes that the natural length scale λ associated with the background field, $\hat{n}(\vec{x})$, is large compared to the Compton wavelength $1/m$ of the fermion field: $\lambda m \gg 1$. That is, we assume that

$$|\vec{\nabla}\hat{n}(\vec{x})| \ll m. \tag{12}$$

By symmetry, this means that we assume that all derivatives of $\hat{n}$ are small compared to the fermion mass scale m. Since the background is static, these derivatives are all spatial derivatives. The derivative expansion of the resolvent (and hence of the induced fermion number) can then be formalized as follows. The Dirac Hamiltonian corresponding to the Lagrangian (6) is

$$H = -i\vec{\alpha}\cdot\vec{\nabla} + m\,\hat{n}\cdot\vec{\tau}\,\gamma^0. \tag{13}$$

Here $(\vec{\alpha} = \gamma^0\vec{\gamma})$ and for definiteness we choose gamma matrices as follows: $\gamma^0 = \sigma^3$, $\vec{\gamma} = -i\vec{\sigma}$. The even part (in z) of the resolvent (10) is

$$\left[\mathrm{Tr}\left(\frac{1}{H-z}\right)\right]_{\mathrm{even}} = \mathrm{Tr}\left(H\frac{1}{H^2-z^2}\right). \tag{14}$$

This can be systematically expanded in terms of derivatives of the background field as follows. First, write the square of the Hamiltonian (13) as

$$H^2 = -\vec{\nabla}^2 + m^2 + im\gamma^j\partial_j\hat{n}\cdot\vec{\tau}$$

$$\equiv H_0^2 + V \tag{15}$$

where $H_0^2 = -\vec{\nabla}^2 + m^2$ is the square of the free Hamiltonian, and the "interaction" is $V = im\gamma^j\partial_j\hat{n}\cdot\vec{\tau}$. Next, since V involves a factor of m and a derivative of $\hat{n}$, by Eq. (12) it is small relative to the natural scale, m^2, of H_0^2. Therefore, the even part of the resolvent in Eq. (14) can be systematically expanded in terms of V:

$$\left[\mathrm{Tr}\left(\frac{1}{H-z}\right)\right]_{\mathrm{even}} = \mathrm{Tr}(\Delta V\Delta V\Delta I) - \mathrm{Tr}(\Delta V\Delta V\Delta V\Delta K)$$

$$+ \mathrm{Tr}(\Delta V\Delta V\Delta V\Delta V\Delta I) - \cdots \tag{16}$$

where we have defined the kinetic part of H as $K = -i\vec{\alpha}\cdot\vec{\nabla}$, the interaction part of H as $I = m\hat{n}\cdot\vec{\tau}\gamma^0$, and the free propagator as

$$\Delta \equiv \frac{1}{-\vec{\nabla}^2 + m^2 - z^2}. \tag{17}$$

In obtaining Eq. (16) we have used the simple facts that $\mathrm{Tr}(\tau^a) = 0$ and $\mathrm{Tr}(\gamma^{i1}\ldots\gamma^{i2n+1}\gamma^0) = 0$. The trace in Eq. (16) involves a matrix trace over the Dirac matrices, $\gamma^\mu = (\sigma^3, -i\vec{\sigma})$, and the isospin matrices, $\vec{\tau}$, as well as a functional trace over the propagators Δ. These traces can be done in momentum space or in position space. In position space, as we go to higher derivative orders we make use of the following operator identity:

$$\frac{1}{-\vec{\nabla}^2 + a^2}V(\vec{x}) = V(\vec{x})\frac{1}{-\vec{\nabla}^2 + a^2}$$

$$+ 2(\vec{\nabla}V(\vec{x}))\cdot\vec{\nabla}\frac{1}{(-\vec{\nabla}^2+a^2)^2}$$

$$+ (\vec{\nabla}^2 V(\vec{x}))\frac{1}{(-\vec{\nabla}^2+a^2)^2}$$

$$+ 4(\partial_i\partial_j V(\vec{x}))\partial_i\partial_j\frac{1}{(-\vec{\nabla}^2+a^2)^3}$$

$$+ 4[\vec{\nabla}(\vec{\nabla}^2 V(\vec{x}))]\cdot\vec{\nabla}\frac{1}{(-\vec{\nabla}^2+a^2)^3}$$

$$+ \cdots \tag{18}$$

where the parentheses around the derivatives on the right-

hand side (RHS) of Eq. (18) indicate that the gradient operator acts on $V(\vec{x})$ only. At any given order of the derivative expansion we collect together the required number of derivatives of $\hat{n}(\vec{x})$, as given by repeated application of the above identity, noting that $V(\vec{x})$ itself involves one derivative of $\hat{n}(\vec{x})$. This, however, becomes rather clumsy for higher order terms in the derivative expansion and in practice it is easier to do the computation in momentum space. The leading term in the derivative expansion involves two derivatives, which come directly from the two factors of V in the first term of Eq. (16):

$$\left[\mathrm{Tr}\left(\frac{1}{H-z}\right)\right]^{(2)}_{\mathrm{even}} = \mathrm{Tr}(\Delta V \Delta V \Delta I)^{(2)}$$

$$= \frac{-m^3}{2\pi(m^2-z^2)^2}$$

$$\times \int d^2x \, \epsilon^{ij}\epsilon^{abc}\hat{n}^a\partial_i\hat{n}^b\partial_j\hat{n}^c. \qquad (19)$$

To compute the induced fermion number to this (leading) order of the derivative expansion, we evaluate the z integral in Eq. (11) as a sum over Matsubara modes. This leads to the following contribution to the induced fermion number:

$$\langle N \rangle^{(2)}_T = \left(\sum_{n=-\infty}^{\infty} \frac{m^3 T}{2\pi[m^2+((2n+1)\pi T)^2]^2}\right)$$

$$\times \int d^2x \, \epsilon^{ij}\epsilon^{abc}\hat{n}^a\partial_i\hat{n}^b\partial_j\hat{n}^c$$

$$= \left(\sum_{n=-\infty}^{\infty} \frac{4m^3 T}{[m^2+((2n+1)\pi T)^2]^2}\right)\langle N \rangle_0. \qquad (20)$$

The prefactor in Eq. (20) is temperature dependent, but in the $T \to 0$ limit it reduces smoothly to one:

$$\sum_{n=-\infty}^{\infty} \frac{4m^3 T}{[m^2+((2n+1)\pi T)^2]^2} = 1 - \left(\frac{2m}{T}\right)e^{-m/T} + \cdots. \qquad (21)$$

Here we have used the general low temperature expansion:

$$T\sum_{n=-\infty}^{\infty} \frac{1}{[((2n+1)\pi T)^2+m^2]^p}$$

$$\sim \frac{m^{1-2p}}{2\sqrt{\pi}}\frac{\Gamma\left(p-\frac{1}{2}\right)}{\Gamma(p)} - \frac{(2mT)^{1-p}}{m\Gamma(p)}e^{-m/T} + \cdots. \qquad (22)$$

Thus, in the $T \to 0$ limit, the leading derivative expansion term (20) produces the entire zero temperature answer, $\langle N \rangle_0$. And at nonzero temperature, this order of the derivative ex-

pansion gives the result (20), which is topological because it is simply the $T=0$ result multiplied by a smooth function of temperature.

To see the nontopological contribution to $\langle N \rangle_T$ we need to go to the next order in the derivative expansion, which involves four derivatives of $\hat{n}(\vec{x})$. Recalling that $V = im\gamma^j\partial_j\hat{n}(\vec{x})\cdot\vec{\tau}$ already includes one derivative of $\hat{n}(\vec{x})$, we see that at fourth order we can get contributions from the first three terms in Eq. (16):

$$\left[\mathrm{Tr}\left(\frac{1}{H-z}\right)\right]^{(4)}_{\mathrm{even}} = \mathrm{Tr}(\Delta V \Delta V \Delta I)^{(4)}$$

$$- \mathrm{Tr}(\Delta V \Delta V \Delta V \Delta K)^{(4)}$$

$$+ \mathrm{Tr}(\Delta V \Delta V \Delta V \Delta V \Delta I)^{(4)} - \cdots. \qquad (23)$$

After a straightforward expansion one finds

$$\left[\mathrm{Tr}\left(\frac{1}{H-z}\right)\right]^{(4)}_{\mathrm{even}} = -\frac{m^3}{12\pi}\frac{m^2+5z^2}{(m^2-z^2)^4}$$

$$\times \int d^2x (\epsilon^{ij}\epsilon^{abc}\hat{n}^a\partial_i\hat{n}^b\partial_j\hat{n}^c)(\partial_k\hat{n}^d\partial_k\hat{n}^d)$$

$$- \frac{m^3}{12\pi(m^2-z^2)^3}$$

$$\times \int d^2x [2\vec{\nabla}^2(\epsilon^{ij}\epsilon^{abc}\hat{n}^a\partial_i\hat{n}^b\partial_j\hat{n}^c)$$

$$+ 3\partial_i(\epsilon^{ij}\epsilon^{abc}(\vec{\nabla}^2\hat{n}^a)\partial_j\hat{n}^b\hat{n}^c)]. \qquad (24)$$

There are several important comments to be made about this fourth order contribution. First, the second term contains total spatial derivatives, which vanish after doing the $\vec{x}$ integral. Second, the first term is nontopological and cannot be expressed as a total derivative. This term gives a nonzero contribution to the spatial integral, and furthermore this contribution [unlike the winding number integral in Eq. (19)] depends explicitly on the length scale of the background field $\hat{n}(\vec{x})$. Third, to see how this is compatible with the fact that the zero T fermion number is topological, we observe that after doing the z integral in Eq. (11), we obtain

$$\langle N \rangle^{(4)}_T = \left[\frac{m^3 T}{12\pi}\sum_{n=-\infty}^{\infty}\frac{m^2-5((2n+1)\pi T)^2}{[m^2+((2n+1)\pi T)^2]^4}\right]$$

$$\times \int d^2x (\epsilon^{ij}\epsilon^{abc}\hat{n}^a\partial_i\hat{n}^b\partial_j\hat{n}^c)(\partial_k\hat{n}^d\partial_k\hat{n}^d). \qquad (25)$$

The prefactor is T dependent, as was the prefactor in the 2-derivative contribution in Eq. (20). However, using Eq. (22), as $T \to 0$, the prefactor of the fourth order term (25) behaves as

$$\frac{m^3 T}{12\pi} \sum_{n=-\infty}^{\infty} \frac{m^2 - 5((2n+1)\pi T)^2}{[m^2 + ((2n+1)\pi T)^2]^4}$$

$$\sim \frac{m^3}{12\pi}\left[\left(\frac{15}{16m^5} - \frac{1}{8m^2 T^3}e^{-m/T} + \cdots\right)\right.$$

$$\left. - \left(\frac{15}{16m^5} - \frac{5}{8m^3 T^2}e^{-m/T} + \cdots\right)\right]$$

$$\sim -\left(\frac{m}{96\pi T^3}\right)e^{-m/T}\left(1 - 5\frac{T}{m} + \cdots\right) \qquad (26)$$

so that it vanishes (exponentially) as $T \to 0$. Thus, at $T=0$ the nontopological contribution to the fermion number vanishes, at this order in the derivative expansion. Note that in Eq. (21) the leading constant term 1 survives the $T \to 0$ limit, while in Eq. (26) the leading constant terms cancel, leaving a function that vanishes as $T \to 0$.

Already at this next-to-leading order of the derivative expansion, we have established in Eq. (25) that the induced fermion number acquires a nontopological temperature dependent contribution at finite T. We now consider higher orders in the derivative expansion. The next term beyond Eq. (24) has 6 derivatives and receives contributions from each of the first five terms in the expansion (16). While it is systematic to write down all these 6-derivative terms, it becomes a lengthy expression due to the many ways of contracting the spacetime and SU(2) indices. This proliferation of terms rapidly becomes worse as we go to higher and higher orders in the derivative expansion. To proceed, we consider instead the low temperature limit. In this case we find that we can evaluate the leading contribution to all orders of the derivative expansion, and resum this leading low temperature correction to the zero temperature answer (8). This follows the procedure used in [19] for a $(1+1)$-dimensional sigma model, and is based on simple dimensional analysis. From the general expansion (16) we see that terms either involve (a) an even number of V's and one interaction term I, or (b) an odd number of V's and one kinetic term K. Consider a term of form (a) with ν vertex insertions of V. Since V has already a derivative of $\hat{n}(\vec{x})$, the total number of derivatives, d, of $\hat{n}(\vec{x})$ in this term satisfies $2 \leq \nu \leq d$. This term also has $\nu+1$ propagators $\triangle$. Thus, by dimensional reasoning, this term must behave as

$$\text{Tr}((\Delta V)^\nu \Delta I)^{(d)} \sim \frac{m^{\nu+1}}{(m^2 - z^2)^{(\nu+d)/2}} \mathcal{A}_d^{\nu+1}[\hat{n}(\vec{x})] \quad (27)$$

where $\mathcal{A}_d^{\nu+1}[\hat{n}(\vec{x})]$ is some functional of $\hat{n}(\vec{x})$ with d derivatives and $(\nu+1)$ factors of $\hat{n}(\vec{x})$. On the other hand, a term of the form (b) behaves as

$$\text{Tr}((\Delta V)^\nu \Delta K)^{(d)} \sim \frac{m^\nu}{(m^2 - z^2)^{(\nu+d-1)/2}} \mathcal{A}_d^\nu[\hat{n}(\vec{x})]. \quad (28)$$

We now consider the T dependence of the prefactors of these types of contributions. First, for $d \geq 4$ the constant terms in the low T ($T \ll m$) expansion must all cancel at any given order d of the derivative expansion, since the final answer must be just the 2-derivative topological term (8) at $T=0$. This is a very stringent test of the derivative expansion at finite temperature. We have explicitly checked that this is indeed satisfied for $d=4$ and $d=6$. The leading corrections all have the same exponential factor, with a prefactor coefficient that depends on the power of the propagator. From Eq. (22) we see that the dominant prefactor at low temperature occurs when p (the power to which the propagator is raised) is as large as possible. This means that for the terms in Eqs. (27) and (28) we must take $\nu=d$. This means that each derivative just comes from one of the insertions of V. This fact dramatically simplifies the derivative expansion (in this low T limit), because the propagator factors can simply be moved out of the trace without generating further derivatives (since these would be subleading). We also learn that the terms of the form (27) dominate over those of the form (28). These simplifications permit us to go to any order of the derivative expansion.

To compute the actual form of the derivtive expansion contribution in this low temperature limit, we have to calculate $\mathcal{A}_d^{\nu+1}[\hat{n}(\vec{x})]$ in Eq. (27), which involves doing the (matrix) traces over the Dirac and isospin matrices:

$$\text{tr}[IV^d] = \text{tr}[m\,\sigma^3 \otimes (\hat{n}(\vec{x}) \cdot \vec{\tau})$$

$$\times (m^2 \partial_i \hat{n}^a \partial_j \hat{n}^b \sigma^i \sigma^j \otimes \tau^a \tau^b)^{d/2}] \qquad (29)$$

where $(\sigma^3, -i\vec{\sigma})$ and $\vec{\tau}$ are the Pauli matrices in the Dirac and isospin spaces respectively. Using the fact that i and j can only take values 1 or 2, and that $\sigma^i \sigma^j = \delta^{ij} + i\,\epsilon^{ijk}\sigma^k$, the above trace becomes

$$m^{d+1} \text{tr}[\sigma^3 \otimes (\hat{n}(\vec{x}) \cdot \vec{\tau})(\partial_i \hat{n}^a \partial_i \hat{n}^a$$

$$- \epsilon^{ij}\epsilon^{abc}\partial_i \hat{n}^a \partial_j \hat{n}^b \sigma^3 \otimes \tau^c)^{d/2}]. \qquad (30)$$

Using the binomial expansion for expanding the above power and noting that (i) $\text{Tr}(\sigma^3)^{m+1} = 2$ when m is odd and zero otherwise, and (ii) $(\vec{\tau} \cdot \vec{a})(\vec{\tau} \cdot \vec{b}) = \vec{a} \cdot \vec{b} + i(\vec{a} \times \vec{b}) \cdot \vec{\tau}$, the expression (30) becomes

$$-4m^{d+1}\frac{q(\vec{x})}{|\vec{v}(\vec{x})|}\sum_{m=1,3,5,...}^{d/2}\binom{d/2}{m}(\partial_i \hat{n}^a \partial_i \hat{n}^a)^{(d/2-m)}|\vec{v}(\vec{x})|^m$$

$$(31)$$

where we have defined the following combinations:

$$\vec{v}(\vec{x}) \equiv \epsilon^{ij}\partial_i \hat{n} \times \partial_j \hat{n}, \quad q(\vec{x}) \equiv \epsilon^{ij}\epsilon^{abc}\hat{n}^a \partial_i \hat{n}^b \partial_j \hat{n}^c. \quad (32)$$

Finally, performing the functional trace over the propagators we find

$$\mathrm{Tr}((\triangle V)^\nu \triangle I)^{(d)} \sim - \frac{m^{d+1}}{2\pi d(m^2 - z^2)^d}$$

$$\times \int d^2 x \frac{q(\vec{x})}{|\vec{v}(\vec{x})|} [(\partial_i \hat{n}^a \partial_i \hat{n}^a + |\vec{v}(\vec{x})|)^{d/2}$$

$$- (\partial_i \hat{n}^a \partial_i \hat{n}^a - |\vec{v}(\vec{x})|)^{d/2}]. \tag{33}$$

So, in the low T limit ($T \ll m$), the derivative expansion becomes very compact. The leading low temperature contribution, with d spatial derivatives, to the induced fermion number is then found by summing over the Matsubara poles of the prefactor (33), and then using the low temperature expansion in Eq. (22). This gives the following:

$$\langle N \rangle_T^{(d)} \sim - \frac{mT}{\pi} \frac{1}{d!(2T)^d} e^{-m/T} \int d^2 x \frac{q(\vec{x})}{|\vec{v}(\vec{x})|} [(\partial_i \hat{n}^a \partial_i \hat{n}^a$$

$$+ |\vec{v}(\vec{x})|)^{d/2} - (\partial_i \hat{n}^a \partial_i \hat{n}^a - |\vec{v}(\vec{x})|)^{d/2}]. \tag{34}$$

Note that at any order d of the derivative expansion, the prefactor term in Eq. (33) leads to a temperature dependent prefactor in the induced fermion number that vanishes as $T \to 0$. Furthermore, the leading low T terms in Eq. (34) have a simple form that can be resummed to all orders in the derivative expansion. Combining with the 2-derivative term in Eq. (20), we obtain the following all-orders result for the low temperature correction to the induced fermion number in the (2+1)-dimensional nonlinear σ model:

$$\langle N \rangle_T \sim \langle N \rangle_0 - \frac{mT}{\pi} e^{-m/T} \int d^2 x \frac{q(\vec{x})}{|\vec{v}(\vec{x})|}$$

$$\times \left[\cosh\left(\frac{\sqrt{\partial_i \hat{n}^a \partial_i \hat{n}^a + |\vec{v}(\vec{x})|}}{2T} \right) \right.$$

$$\left. - \cosh\left(\frac{\sqrt{\partial_i \hat{n}^a \partial_i \hat{n}^a - |\vec{v}(\vec{x})|}}{2T} \right) \right]. \tag{35}$$

This is the main result of this paper. It shows explicitly that the topological zero temperature fermion number, given by the winding number $\langle N \rangle_0$ in Eq. (8), acquires a T dependent correction which is nontopological. This temperature dependent correction cannot be expressed in terms of the winding number, and is sensitive to the detailed shape of the background field $\hat{n}(\vec{x})$, not just its asymptotic values. Also note that the $T \to 0$ limit is well defined in the derivative expansion regime where Eq. (12) is satisfied. This resummed expression (35) is analogous to a similar all-orders resummed result found in [19] for the (1+1)-dimensional sigma model background. We find it remarkable that this all-orders resummation can also be done for the (2+1)-dimensional sigma model, as it is a significantly more complicated model.

Thus far, the only thing we have assumed about $\hat{n}(\vec{x})$ is that it takes values in S^2, see Eq. (7), and that it has a "gentle" spatial profile, see Eq. (12). We now consider two specific explicit forms for the background $\hat{n}(\vec{x})$.

First, suppose $\hat{n}(\vec{x})$ has the form of a 2-dimensional $\mathcal{CP}^1$ instanton, in which case the winding number $\langle N \rangle_0$ in Eq. (8) is just the instanton number, as is well known for the zero temperature system [37]. The instanton field $\hat{n}(\vec{x})$ satisfies the first order instanton equation

$$\partial_i \hat{n}^a = \pm \epsilon_{ij} \epsilon^{abc} \hat{n}^b \partial_j \hat{n}^c \tag{36}$$

where the $\pm$ signs refer to the anti-instanton–instanton cases respectively [one can easily check this by substituting Eq. (36) in the expression (8) to find the sign of $\langle N \rangle_0$]. Now, in general we have that $|\vec{v}(\vec{x})|^2 = 2(\partial_i \hat{n}^a \partial_i \hat{n}^a)^2 - 2(\partial_i \hat{n} \cdot \partial_j \hat{n})^2$. But for an instanton background $\hat{n}$ satisfying Eq. (36), it follows that $(\partial_i \hat{n} \cdot \partial_j \hat{n})^2 = \frac{1}{2}(\partial_i \hat{n} \cdot \partial_i \hat{n})^2$. Therefore, $|\vec{v}(\vec{x})| = (\partial_i \hat{n} \cdot \partial_i \hat{n})$, and $q(\vec{x}) = \mp(\partial_i \hat{n} \cdot \partial_i \hat{n})$. Then the low T expansion (35) simplifies somewhat:

$$\langle N \rangle_T \sim \langle N \rangle_0 \pm \frac{mT}{\pi} e^{-m/T} \int d^2 x \left[\cosh\left(\frac{\sqrt{2\partial_i \hat{n} \cdot \partial_i \hat{n}}}{2T} \right) - 1 \right]. \tag{37}$$

To be even more explicit, we choose $\hat{n}$ to be a k-instanton solution, in which case $\langle N \rangle_0 = k$. The explicit $\mathcal{CP}^1$ instantons are most easily expressed in terms of the following stereographically projected fields ω^1 and ω^2:

$$\omega^1 = \frac{2n^1}{(1-n^3)}, \quad \omega^2 = \frac{2n^2}{(1-n^3)}. \tag{38}$$

Defining $\omega = \omega^1 + i\omega^2$, the original sigma model field $\hat{n}(\vec{x})$ is

$$\hat{n}(\vec{x}) = \frac{1}{\frac{|\omega|^2}{4} + 1} \begin{pmatrix} \mathrm{Re}\,\omega \\ \mathrm{Im}\,\omega \\ \frac{|\omega|^2}{4} - 1 \end{pmatrix}. \tag{39}$$

As is well known, the instanton equation (36) then becomes a simple Cauchy-Riemann condition for ω, so that instantons are characterized by a meromorphic function of $z = x^1 + ix^2$. For example,

$$\omega(z) = \left[\frac{(z - z_0)}{\lambda} \right]^k \tag{40}$$

represents k instantons at the location z_0, each with a common length scale λ. For such an instanton,

$$\partial_i \hat{n} \cdot \partial_i \hat{n} = \left(\frac{2k^2}{\lambda^{2k}} \right) |z - z_0|^{2k-2} \frac{1}{\left[1 + \frac{1}{4\lambda^{2k}} |z - z_0|^{2k} \right]^2}. \tag{41}$$

The spatial integrals in Eq. (37) can now be done, and one finds

$$\langle N\rangle_T \sim k \pm \frac{4km}{T}e^{-m/T}\sum_{n=0}^{\infty}\frac{1}{(2n+2)!}\left(\frac{2k}{2^{1/k}\lambda T}\right)^{2n}$$

$$\times \beta\left(n+1+\frac{n}{k},n+1-\frac{n}{k}\right) \qquad (42)$$

where $\beta(m,n)=\Gamma(m)\Gamma(n)/\Gamma(m+n)$ is the Euler beta function. The first term, k, in Eq. (42) is the zero temperature fermion number $\langle N\rangle_0$, which is simply the instanton number k, and which is manifestly independent of the instanton scale λ. On the other hand, the finite T correction in Eq. (42) is manifestly dependent on the instanton scale λ, reflecting its nontopological nature. It is interesting to note that this correction term is a convergent series in $1/\lambda T$, for any instanton number k. For a single instanton ($k=1$), it simplifies further to

$$\langle N\rangle_T \sim 1-\frac{4m}{T}e^{-m/T}\sum_{n=0}^{\infty}\frac{1}{(2n+1)(2n+2)!}\left(\frac{1}{\lambda T}\right)^{2n}$$

$$= 1-(4m\lambda)e^{-m/T}\int_0^{1/\lambda T}\frac{\cosh y-1}{y^2}dy. \qquad (43)$$

The second physically interesting form for the sigma model background field $\hat{n}$ is the "baby Skyrmion" case

$$\hat{n}(r,\phi)=\begin{pmatrix}\cos\phi\sin\theta(r)\\\sin\phi\sin\theta(r)\\\cos\theta(r)\end{pmatrix}. \qquad (44)$$

Here, r and ϕ are the 2-dimensional polar coordinates and $\theta(r)$ is some radial profile function. The "baby Skyrmion" is the 2 dimensional analog of the 3 dimensional Skyrme model for baryons, with a "hedgehog" ansatz for the meson fields [16]. For such an ansatz, the winding number density is

$$q(\vec{x})=\epsilon^{abc}\epsilon^{ij}\hat{n}^a\partial_i\hat{n}^b\partial_j\hat{n}^c=\frac{2}{r}\sin\theta(r)\theta'(r) \qquad (45)$$

so that the zero temperature induced fermion number (8) is

$$\langle N\rangle_0=\frac{1}{8\pi}\left[4\pi\int_0^{\infty}dr\frac{d\theta}{dr}\sin\theta(r)\right]$$

$$= \frac{1}{2}[\cos\theta(0)-\cos\theta(\infty)]. \qquad (46)$$

This shows clearly that $\langle N\rangle_0$ is topological: it only depends on the asymptotic values of the profile function $\theta(r)$, not on its specific spatial profile or scale. The low temperature correction to the fermion number in this case is obtained from Eq. (35) by noting that for the ansatz (44),

$$\partial_i\hat{n}^a\partial_i\hat{n}^a=(\theta'(r))^2+\left(\frac{\sin\theta(r)}{r}\right)^2, \qquad (47)$$

$$(\partial_i\hat{n}\cdot\partial_j\hat{n})^2=(\theta'(r))^4+\left(\frac{\sin\theta(r)}{r}\right)^4$$

in which case,

$$|\vec{v}(r,\phi)|=\frac{2|\theta'(r)\sin\theta(r)|}{r}. \qquad (48)$$

Thus, the low temperature induced fermion number for the "baby Skyrmion" background is

$$\langle N\rangle_T \sim \langle N\rangle_0 - 2mTe^{-m/T}\int_0^{\infty}r\,dr\,\mathrm{sgn}(\theta'(r)\sin\theta(r))$$

$$\times\left[\cosh\left(\frac{|\theta'(r)|+\dfrac{|\sin\theta(r)|}{r}}{2T}\right)\right.$$

$$\left.-\cosh\left(\frac{|\theta'(r)|-\dfrac{|\sin\theta(r)|}{r}}{2T}\right)\right]. \qquad (49)$$

The background field $\theta(r)$ has to satisfy certain boundary conditions. Requiring that, for a localized soliton solution, the chiral field as $r\to\infty$ should equal its vacuum value, i.e., $\hat{n}(r,\phi)$ should be independent of the angle ϕ, fixes $\theta(\infty)=k\pi$, where k is an integer. At the origin, finiteness of the soliton energy requires $\theta(0)=n\pi$, where n is an integer. As an example, consider the following ansatz for the radial profile function $\theta(r)$:

$$\theta(r)=2\arctan\left(\frac{r}{2\lambda}\right). \qquad (50)$$

Then

$$\theta'(r)=\frac{4\lambda}{r^2+4\lambda^2}=\frac{\sin\theta(r)}{r}. \qquad (51)$$

Substituting this in Eq. (49) we obtain

$$\langle N\rangle_T \sim 1-4m\lambda e^{-m/T}\int_0^{1/\lambda T}dy\,\frac{\cosh y-1}{y^2} \qquad (52)$$

which is, incidentally, exactly the finite temperature fermion number (43) for the $\mathcal{CP}^1$ single instanton background.

The most important thing to be learned from the baby Skyrmion result (49) is that while the zero temperature fermion number $\langle N\rangle_0$ only depends on the asymptotic values of the radial profile function $\theta(r)$, the finite T correction is sensitive to the details of the shape of $\theta(r)$. This means, for example, that in a variational calculation where the asymptotic values of $\theta(r)$ are fixed (to keep the Skyrme number fixed), but the shape of $\theta(r)$ is varied, the finite temperature fermion number will vary.

IV. CONCLUSIONS

To conclude, we have computed the finite temperature induced fermion number for fermions in a static nonlinear σ-model background in $2+1$ dimensions. This calculation illustrates the splitting of the induced fermion number into a zero temperature piece, which is topological (here, the winding number of the sigma model background), and a finite temperature correction which is nontopological [20]. The calculation was done in the derivative expansion limit where the spatial derivatives of the background fields are assumed to be small compared to the fermion mass scale parameter m. We found that it is possible to resum the derivative expansion to all orders, in the low temperature limit where $T \ll m$. Such a resummation was done previously [19] in the $(1+1)$-dimensional σ-model case, but we find it remarkable that it can also be done in the more complicated $(2+1)$-dimensional case. These results were then applied to two specific background fields in the σ-model case: a 2-dimensional $\mathcal{CP}^1$ instanton, and a 2-dimensional "baby Skyrmion." In each case, the finite temperature correction can be computed in the low temperature limit. For an instanton background the final expression for the induced fermion number $\langle N \rangle_T$ becomes a simple convergent series in $1/\lambda T$ where λ is the length scale of the instanton, thus explicitly illustrating the nontopological character of the finite temperature corrections in $\langle N \rangle_T$.

Finally, we make several comments on possible extensions of this work. As has been emphasized recently at zero temperature in [31,32], the induced charge (and, more generally, induced current) is only one part of the induced effective action for fermions in a σ-model background. There is, in addition, a Hopf term which is topological, but which does not contribute to the induced current (since this is defined through a functional variation of the effective action with respect to a gauge field). This Hopf term has important implications for the spin and statistics of solitons [31,32,34]. It would be interesting to investigate these issues at finite temperature, in the language of finite T effective actions. Another result from our calculation is the implication it has for

the chiral sigma models in $3+1$ dimensions. There, the zero temperature calculation is very similar to the $(2+1)$-dimensional case. At finite temperature we learn from our calculation that there will be a finite temperature correction and that it will be nontopological—that is, while the zero temperature induced fermion number is the Skyrme number, the temperature dependent correction will be a much more complicated functional of the Skyrme background. This is already clear from the general argument about finite T fermion number in [20], together with the numerical results (see, e.g., [38]) for the sensitive dependence of the fermion spectrum on the scale of the Skyrme background. That is, since the single-particle fermion spectrum is not symmetric and is sensitive to the scale of the background, the finite temperature correction is also sensitive to this scale, and thus must be nontopological. From a calculational standpoint, we learn that while at zero temperature the induced fermion number arises from the leading nontrivial order of the derivative expansion [15,36], at finite temperature it is necessary to go beyond this order. In other words, at finite temperature there are contributions that are not total derivatives, and these contribute to the integrated charge. It is a much more difficult problem to investigate higher orders of the derivative expansion in the $(3+1)$-dimensional Skyrme model. However, the results of this paper for the $(2+1)$-dimensional model suggest that it should be possible to explore higher order of the derivative expansion, at least in the low temperature limit. It would also be of interest in this context to extend the systematic renormalization approach developed in [14,39], which is ideally suited to numerical evaluation of induced charges, to finite temperature.

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