On the spectrum of $(1+1)$-dimensional QCD with $SU(N_c)$ currents

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Extending previous work, we calculate the fermionic spectrum of two-dimensional QCD (QCD$_2$) in the formulation with $SU(N_c)$ currents. Together with the results in the bosonic sector this allows us to address the as yet unresolved task of finding the single-particle states of this theory as a function of the ratio of the numbers of flavors and colors, $\lambda = N_f/N_c$, anew. We construct the Hamiltonian matrix in the DLCQ formulation as an algebraic function of the harmonic resolution K and the continuous parameter λ in the Veneziano limit. We find that the fermion momentum is a function of λ in the discrete approach. A universality, existing only in two dimensions, dictates that the well-known 't Hooft and large N_f spectra be reproduced in the limits $\lambda \rightarrow 0$ and ∞, which we confirm. We identify their single-particle content which is surprisingly the same as in the bosonic sectors. All multiparticle states are classified in terms of their constituents. These findings allow for an identification of the lowest single particles of the adjoint theory. While we do not succeed in interpreting this spectrum completely, evidence is presented for the conjecture that adjoint QCD$_2$ has a bosonic and an independent fermionic Regge trajectory of single-particle states.

I. INTRODUCTION

Two-dimensional QCD will remain an interesting model for strong interaction physics until a first principles calculation of the low-lying spectrum of four-dimensional QCD is available. The theory with one flavor of fundamental fermions coupled to non-Abelian gauge fields was solved by 't Hooft in his seminal paper [1] in the limit of a large number of colors N_c. It is the prime example for the solution of a confining gauge theory and exhibits one Regge trajectory of noninteracting mesons, while not possessing dynamical gluonic degrees of freedom. The theory can also be solved when the number of fundamental (flavored) fermions N_f is large. This is the Abelian limit of the theory, and it comprises a single meson with mass $g^2 N_f/\pi$ [2]. So far it has, however, proven impossible to solve the theory with fermions in the adjoint representation. This is unfortunate, because adjoint fermions simulate the transverse gluons of realistic four-dimensional QCD. The latter has, of course, $N_f = 3$ fermions in the fundamental representation of $SU(N_c = 3)$, but it has been established for several theories [1,3,15] that the large N_c limit is often a good approximation. The difficulties with solving adjoint two-dimensional QCD (QCD$_2$) can be traced to the fact that parton pair production is not suppressed by factors $1/N_c$, contrary to the 't Hooft model. One therefore expects a rich spectrum of multiple Regge trajectories. Adjoint QCD$_2$ has been discussed in the literature for almost a decade [5–18]. Many interesting facets of this theory have been revealed, e.g. a confining/screening transition with a linearly decreasing string tension at vanishing fermion mass [10,19], an exponential rise of the density of states with the bound state mass which is reminiscent of string theory [6], and the fact that the theory becomes supersymmetric at a special value of the fermion mass [7]. Still, frustratingly little about the (single-particle) solutions of this theory is known.

Using the framework of discretized light-cone quantization (DLCQ) [20], the numerical eigenvalue spectrum of adjoint QCD has been obtained by Dalley and Klebanov [5].

The results have been improved in Refs. [6,13]. The asymptotic spectrum of the theory has been calculated by Kutasov in the continuum [7]. There are mainly two reasons which prohibit the extraction of the single-particle solutions from these results. First, especially the numerical results are obscured by the fact that the standard formulation in terms of fermionic operators contains many multiparticle states. Secondly, in large N_c calculations one is used to identifying single-particle states with single-trace states since the work of 't Hooft [1]. It was recently established that this is not necessarily correct if one deals with fields in the adjoint representation [13,15,16]. This might have consequences for results derived with this assumption [6,9,4,11]. It seems therefore that not so much the lack of results but their interpretation is the main obstacle for solving the theory. To improve the situation in both respects we will adopt a strategy which is special to massless theories in two dimensions. In the massless case, the light-cone Hamiltonian can be written as a pure current-current interaction, and its Hilbert space splits up into sectors of different representations of the current algebra. The formulation of the theory in terms of $SU(N_c)$ currents which form a Kac-Moody algebra is therefore a preferred choice which we will use in the present work. In this formulation, to be described in Sec. II, many of the multiparticle states will be absent, because only two of the current blocks give rise to single-particle states [9]. The bosonic states lie in the so-called current block of the identity which was considered in Ref. [16]. The adjoint block gives rise to the fermionic bound states, to be calculated here, which we need for the interpretation of the full spectrum, because the bosonic spectrum contains multiparticle states with fermionic constituents [13]. We will use the framework of DLCQ to realize the dynamical operators on a finite-dimensional Fock basis. It turns out that the momentum operator plays a special role in the fermionic sector, and we will describe this and other peculiarities in Sec. III. In Sec. IV we will construct the fermionic light-cone Hamiltonian in terms of the discrete momentum modes of the cur-

rents. The Hamiltonian is an algebraic function of the cutoff in current number and, most importantly, of the ratio $\lambda = N_f/N_c$.

This explicit λ dependence of the Hamiltonian and the eigenvalue spectrum allows us to exploit a universality existing only in two dimensions, as part of our strategy to elucidate the spectrum of adjoint QCD_2. The universality established in Ref. [9] assures that the massive spectrum and interactions of two-dimensional gauge fields coupled to massless matter are largely independent of the representation of the matter fields, given they have the same chiral anomaly. All information on the matter representation beyond its Kac-Moody level is encoded in the massless sector of the theory. There is, however, no strict factorization between massive and massless sectors, although the Hamiltonian is determined by the states from the massive sector only. In particular, massive states will have well-defined discrete symmetry quantum numbers only when accompanied by massless states [9]. It is clear that the universality can hold in two dimensions only. In four dimensions massive and massless modes are known to be strongly interacting. So far, this universality has been understood in light-front quantization only. The universality specifically predicts that the massive spectrum of the Yang-Mills theory coupled to one adjoint $SU(N_c)$ fermion is the same as the spectrum of the theory coupled to $N_f=N_c$ flavors of fundamental fermions. If this is true, we should obtain the 't Hooft spectrum in the limit of vanishing λ and a single meson in the large N_f limit in our numerical calculations. This exercise is performed in Secs. V A and V B with the predicted result. This confirms that the universality can be applied to the present case and provides a strong test of the numerics. In both limits the multiparticle states decouple and we succeed in describing the spectra in terms of their single-particle content, thus classifying all multiparticle states by their constituents. This helps us to understand the adjoint spectrum in Sec. V C. While we are able to identify the low-lying single particle states and to construct some of the multiparticle states, a complete solution of the theory remains elusive. Motivated by empirical findings and an analysis of the spectrum at intermediate values of λ in Sec. V D, we are led to the *conjecture* that there are two Regge trajectories in the adjoint theory: a bosonic and a fermionic one. We discuss the speculative character of these results, tests and possible improvements in the concluding Sec. VI. In the Appendix we display a calculation of the first corrections in λ to the mass to the 't Hooft mesons. The agreement with the results of a recent perturbative analysis [8] provides further support for the usefulness of the present formulation of massless QCD_2 in terms of current operators.

II. QCD IN TWO DIMENSIONS

The aim of the present work is to compute the massive spectrum of $SU(N_c)$ Yang-Mills gauge fields coupled to massless fermions in some representation r in two dimensions. The Veneziano limit, where both N_f and N_c are large, is understood throughout. A universality, existing only for massless two-dimensional gauge theories [9], predicts that

the massive spectrum of the theory is the same, whether one adjoint $SU(N_c)$ Majorana fermion or $N_f=N_c$ flavors of fundamental Dirac fermions are coupled to the gauge fields. This means that we can formulate the theory in terms of adjoint fields, while interpreting the results in terms of fundamentals. This gives us a continuous parameter at hand, namely $\lambda=N_f/N_c$, which allows us to couple the Yang-Mills fields to matter in different representations by simply altering its value, while still keeping N_f and N_c large. This in turn gives deeper insight into the theory, since the spectra in the limits $\lambda\to0$ ('t Hooft model) and $\lambda\to\infty$ (large N_f model) are well understood, whereas the single-particle content of the adjoint theory remains largely unknown. Consequently, the main focus is on the case $\lambda=1$, while we will try to infer as much information as possible from the 't Hooft and large N_f models by analyzing them in the formulation with current operators.

In order to do so we have to derive the momentum and energy operators in terms of currents rather than with fermionic operators. We consider the adjoint theory, but shall distinguish N_f and N_c throughout the derivation. As we saw, one can formally interpret the results at different N_f and N_c as distinct theories. The Lagrangian in light-cone coordinates $x^\pm=(x^0\pm x^1)/\sqrt{2}$, where x^+ plays the role of a time, reads

$$\mathcal{L}=\mathrm{Tr}\left[-\frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu}+i\bar{\Psi}\gamma_\mu D^\mu\Psi\right] \quad (1)$$

where $\Psi=2^{-1/4}\binom{\psi}{\chi}$, with ψ and χ being $N_c\times N_f$ matrices. The field strength is $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu+i[A_\mu,A_\nu]$, and the covariant derivative is defined as $D_\mu=\partial_\mu+i[A_\mu,.]$. We work in the light-cone gauge, $A^+=0$, which is consistent if we omit the fermionic zero modes. The massive spectrum is not affected by this omission [21]. We use the convenient Dirac basis $\gamma^0=\sigma_1$, $\gamma^1=-i\sigma_2$. The Lagrangian then becomes

$$\mathcal{L}=\mathrm{Tr}\left[\frac{1}{2g^2}(\partial_-A^-)^2+i\psi^\dagger\partial_+\psi+i\chi^\dagger\partial_-\chi-A^-J\right], \quad (2)$$

with the current

$$J_b^a(x^-)=2:\psi_c^{\dagger a}(x^-)\psi_b^c(x^-):. \quad (3)$$

The use of both upper and lower indices is adopted as a reminder that in general the indices are from different index sets. We can integrate out the nondynamical component A^- of the gauge field and obtain

$$\mathcal{L}=\mathrm{Tr}\left[i\psi^\dagger\partial_+\psi+i\chi^\dagger\partial_-\chi-\frac{g^2}{2}J\frac{1}{\partial_-^2}J\right]. \quad (4)$$

It is obvious that the left-moving fields χ decouple, because their equations of motion do not involve a time derivative, i.e., are constraint equations. Noting the simple expression of the interaction in terms of the currents, it is natural to formulate the theory with $SU(N_c)$ currents as basic degrees of freedom. For reasons of clarity, we will not use the terminology of bosonization or conformal field theory. We shall

rather stick to the definition of the currents as a bilinear product of fermions, Eq. (3), and derive everything based on this definition, which is perfectly possible.

The key issue is to obtain the mass eigenvalues M_n by solving the eigenvalue problem

$$\mathcal{M}^2|\varphi\rangle \equiv 2P^+P^-|\varphi\rangle = M_n^2|\varphi\rangle, \qquad (5)$$

where we act with the light-cone momentum and energy operators, P^+ and P^-, on a state $|\varphi\rangle$. The operators $P^\pm \equiv T^{+\pm}$ can be found by constructing the energy-stress tensor $T^{\mu\nu}$ in the canonical way, and one obtains

$$P^+ = T^{++} = \frac{\pi}{N_c+N_f} \int_{-\infty}^{\infty} dx^- : \mathrm{Tr}[J(x^-)J(x^-)]: \qquad (6)$$

$$P^- = T^{+-} = -\frac{g^2}{2} \int_{-\infty}^{\infty} dx^- : \mathrm{Tr}\left[J(x^-) \frac{1}{\partial_-^2} J(x^-) \right]: . \qquad (7)$$

The Sugawara form of the momentum operator P^+, Eq. (6), might seem somewhat unfamiliar, but an explicit analysis of this construction in terms of the fermionic mode operators yields indeed the above result.[1] To solve the eigenvalue problem, Eq. (5), we have to diagonalize the mass squared operator, which is equivalent to diagonalizing the Hamiltonian P^-, since P^+ is already diagonal. The latter is not as obvious as usual, and we elaborate on this in Sec. III.

We use the standard mode expansion of the fermionic fields

$$\psi_k^j(x^-) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} dp\, e^{-ipx^-} b_k^j(p), \qquad (8)$$

and the mode expansion of currents becomes

$$J_k^j(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx^- e^{+ipx^-} J_k^j(x^-)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dq : b_l^j(q) b_k^l(p-q): . \qquad (9)$$

The canonical anticommutation relation for the fermionic operators

$$\{b_k^j(p), b_m^l(p')\} = \delta(p+p')\delta_m^j \delta_k^l, \qquad (10)$$

determines the commutator of current with fermionic modes

$$[J_k^j(p), b_m^l(p')] = \delta_k^l b_m^j(p+p') - \delta_m^j b_k^l(p+p'). \qquad (11)$$

Recall that in the adjoint theory $b_k^j(-n) = b_j^{\dagger k}(n)$. Due to the occurrence of Schwinger terms the current-current commu-

tator is harder to derive. It is, however, well known that the modes of the currents are subject to the Kac-Moody algebra[2]

$$[J_k^j(p), J_m^l(p')] = pN_f \delta_m^j \delta_k^l \delta(p+p') + \delta_k^l J_m^j(p+p')$$
$$- \delta_m^j J_k^l(k+k'). \qquad (12)$$

The vacuum is defined by

$$J_k^j(p)|0\rangle = 0 \quad \text{and} \quad b_k^j(p)|0\rangle = 0, \quad \forall p \geq 0. \qquad (13)$$

Following the usual DLCQ program [20], we put the system in a box of length $2L$ and impose antiperiodic boundary conditions on the fermionic fields $\psi(x^- - L) = -\psi(x^- + L)$. The currents are by construction subject to periodic boundary conditions. The momentum modes are now discrete, and, as always in light-cone quantization, the longitudinal momenta are non-negative. The smallest momentum $k_{min} = P^+/2K$ is determined by the harmonic resolution $K \equiv P^+L/\pi$, which controls the coarseness of the momentum-space discretization. The continuum limit is obtained by sending K to infinity. In practice one solves the eigenvalue problem, Eq. (5), for growing values of K and extrapolates the spectrum to the continuum by e.g. fitting the eigenvalues to a polynomial in $1/K$. The expansion of the fermion fields, Eq. (8), becomes

$$\psi_k^j(x^-) = \frac{1}{2\sqrt{L}} \sum_{n=\pm 1/2,\pm 3/2,...} B_k^j(n) e^{-i\pi n x^-/L}, \qquad (14)$$

with the discrete field operators $B_k^j(n) \equiv (\pi/L)^{1/2} b_k^j(n\pi/L)$. The current mode operators $J(n)$ are defined by the discrete version of Eq. (9). The momentum operators read

$$P^+ = \left(\frac{\pi}{L}\right) \frac{1}{N_c+N_f} \mathrm{Tr}\left[\frac{1}{2}J(0)J(0) + \sum_{n=1}^{\infty} J(-n)J(n) \right], \qquad (15)$$

$$P^- = \frac{\tilde{g}^2}{2\pi} \sum_{n=1}^{\infty} \frac{1}{n^2} \mathrm{Tr}[J(-n)J(n)], \qquad (16)$$

and become finite-dimensional matrices on the Hilbert space constructed by acting with the current operators of momentum K or smaller on the vacuum defined by Eq. (13). For convenience we introduced the scaled coupling $\tilde{g}^2 \equiv g^2 L/\pi$. We emphasize the appearance of the zero mode contribution in the discrete formulation, as should be clear from $P^+ = \lim_{\epsilon\to 0}[1/(N_c+N_f)]\mathrm{Tr}[(\epsilon/2)J(0)J(0) + \int_\epsilon^\infty dp J(-p)J(p)]$. In the Veneziano limit the operators are realized on a Hilbert space of discrete $SU(N_c)$ singlet Fock states. The fermionic states look like

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[1] The remainder of the product of four fermion operators is the contraction term. Its integral becomes the momentum factor in the usual definition $P^+ = \int_0^\infty dp\, p\, \mathrm{Tr}[b(-p)b(p)]$.

[2] We use the opportunity to correct Ref. [16], where the $\mathcal{T}$-symmetric [cf. Eq. (25)] version of the algebra was used, with no consequences for the results in the bosonic sector.

$$\left| \mathbf{b} + \frac{1}{2}; n_1, \ldots, n_b \right\rangle = (N_c N_f)^{-b/2 - 1/4}$$

$$\times \mathrm{Tr}\left[J(-n_1) J(-n_2) \cdots J(-n_b) \right.$$

$$\left. \times B\left(-\frac{1}{2} \right) \right] |0\rangle, \tag{17}$$

whereas in the bosonic sectors we find the singlets

$$\left| \mathbf{b}; n_1, \ldots, n_b \right\rangle = (N_c N_f)^{-b/2}$$

$$\times \mathrm{Tr}[J(-n_1) J(-n_2) \cdots J(-n_b)] |0\rangle. \tag{18}$$

The additional fermion operator $B_k^i(-1/2)$ in the fermionic states is the source of some rather odd differences between the two sectors, as we shall see in the next section.

III. SPECIALTIES OF THE FERMIONIC SECTOR

The numerical solution to the eigenvalue problem, Eq. (5), in the bosonic sector was presented in Ref. [16]. The calculations in the fermionic sector are not quite as straightforward, and we shall point out the major differences. The first peculiarity in the fermionic sector resides in the action of the momentum operator P^+ on a fermionic state. It turns out that the eigenvalue of P^+ on a fermionic state depends on the ratio $\lambda = N_f / N_c$. To convince ourselves that this is true, we consider the simplest case in discrete formulation

$$P^+ \mathrm{Tr}\left[J(-n) B\left(-\frac{1}{2} \right) \right] |0\rangle$$

$$= [P^+, J_j^i(-n)] B_i^j\left(-\frac{1}{2} \right) |0\rangle + J_j^i(-n)$$

$$\times \left[P^+, B_i^j\left(-\frac{1}{2} \right) \right] |0\rangle + \left[P^+, J_j^i(-n) \right], B_j^i\left(-\frac{1}{2} \right) \right] |0\rangle$$

$$= \frac{\pi}{L}\left(n + \frac{1}{1+\lambda} \right) \mathrm{Tr}\left[J(-n) B\left(-\frac{1}{2} \right) \right] |0\rangle. \tag{19}$$

In other words, only in the adjoint theory the fermion has the familiar momentum. In the 't Hooft limit, a fermion with half-integer momentum contributes the same momentum as a current, whereas in the large N_f limit it has a vanishing contribution. This is a consequence of the discrete formulation. In particular, it is $\mathcal{M}^2 B_j^i(-1/2)|0\rangle = 2 P^+ P^- B_j^i(-1/2)|0\rangle = 0$, although $P^+ B_j^i(-1/2)|0\rangle = (1+\lambda)^{-1} B_j^i(-1/2)|0\rangle$.

Another issue to be addressed here is the size of the Fock space. In the bosonic sector, the singlet states are of the form of Eq. (18), i.e., they are single-trace states of a certain number of currents. This form allows for cyclic permutations of the currents under the trace. The cyclic permutations are related nontrivially due to the Kac-Moody structure of the currents, yet all cyclic permutations of a given state have to be eliminated from the Fock basis. The key difference in the fermionic sector is the absence of these cyclic permutations; the fermion defines the ordering of the state. This is quite

natural, since it basically acts like an adjoint vacuum, as we shall see. This renders the Fock basis much larger than in the bosonic case. The number of states grows like $2^{K-3/2}$, see Table I, with the harmonic resolution K being a half-integer in the fermionic sector due to the momentum of the additional fermion. The different sizes of the Fock bases will help us in interpreting the resulting fermionic spectra, because the spectra in the 't Hooft and large N_f limits of the theory have the same single-particle content as in the bosonic sectors.

We briefly comment on the fact that we can calculate two eigenvalues of the mass squared operator $\mathcal{M}^2 = 2 P^+ P^-$ analytically. Remarkably, we obtain the same functional form $M_{1,2}^2(K)$ as in the bosonic sector. This is somewhat surprising, since in the fermionic sector we lose the uniqueness of the two states with the largest number of currents. At harmonic resolution $K = b + 1/2$ the states

$$\left| b + \frac{1}{2} \right\rangle = \mathrm{Tr}\left[\{J(-1)\}^b B\left(-\frac{1}{2} \right) \right] |0\rangle \tag{20}$$

$$\left| b - \frac{1}{2} \right\rangle = \mathrm{Tr}\left[\{J(-1)\}^{b-2} J(-2) B\left(-\frac{1}{2} \right) \right] |0\rangle, \tag{21}$$

have b and $b-1$ currents, respectively, plus a fermion with momentum 1/2. However, Eq. (21) represents a class of $b-1$ states, rather than a unique state as in the bosonic sector. Explicit calculations show that the image of one of these states under the mass squared operator has no overlap with any of the other states, and its eigenvalue can be trivially extracted in the fashion of Ref. [16]. Hence, two eigenvalues can be evaluated *a priori*. The eigenvalues of the mass squared operator associated with the states, Eqs. (20) and (21), are

$$M_1^2\left(K = b + \frac{1}{2} \right) = \frac{g^2 N_c}{\pi} (1+\lambda)\left(b + \frac{1}{1+\lambda} \right)^2, \tag{22}$$

$$M_2^2\left(K = b + \frac{1}{2} \right) = \frac{g^2 N_c}{\pi} (1+\lambda)\left(b + \frac{1}{1+\lambda} \right)$$

$$\times \left(b - \frac{3}{2} + \frac{1}{1+\lambda} \right). \tag{23}$$

We note that in the 't Hooft limit, all states of the form Eq. (21) have the same eigenvalue.

IV. THE HAMILTONIAN

We construct the light-cone Hamiltonian in the framework of DLCQ. Once the commutation relations, Eqs. (10)–(12), are specified, and the Fock basis is chosen, this is a straightforward generalization of previous work [16], and we can be brief here. In the fermionic sector we get additional contributions to the light-cone Hamiltonian by commuting through zero modes of the currents and acting with them on the extra fermion. Note that annihilation operators may be created by commuting current operators. To streamline the calculations it is useful to distinguish the creation, annihilation and zero-mode part of a commutator

TABLE I. Number of basis states as a function of the number b of currents in a state. The relation of b to the harmonic resolution K is $K=b$ in the bosonic, and $K=b+\frac{1}{2}$ in the fermionic sector.

b	1	2	3	4	5	6	7	8	9	10	11	12	13	14
fermions	1	2	4	8	16	32	64	128	256	512	1024	2048	4096	8192
bosons		1	2	4	6	12	18	34	58	106	186	350	630	1180

$$[A,B] \equiv \lceil A,B \rceil + \lfloor A,B \rfloor + \lfloor A,B \rfloor_0,$$

much in the fashion of Ref. [16]. The resulting Hamiltonian is slightly simpler than in the bosonic case. The number of its terms of leading power in N_c grows exactly quadratic with the number b of currents in a state. In the large N_c limit the action of P^- on a state with b currents, Eq. (17), is then

$$
\begin{aligned}
P^-\left|\mathbf{b}+\frac{1}{2};n_1,\ldots,n_b\right\rangle = {} & \frac{\tilde{g}^2 N_c}{2\pi}\sum_{i=1}^{b}\left(\sum_{m=1}^{n_i-1}\frac{1}{(m-n_i)^2}-\sum_{m=1}^{n_i}\frac{1}{m^2}\right)\left|\mathbf{b}+\mathbf{1}+\frac{1}{2};n_1,n_2,\ldots,n_i-m,m,\ldots,n_b,\frac{1}{2}\right\rangle \\
& + \frac{\tilde{g}^2 N_c}{2\pi}\sum_{i=1}^{b}\left(\frac{\lambda}{n_i}+\sum_{m=1}^{n_i-1}\frac{1}{m^2}\right)\left|\mathbf{b}+\frac{1}{2};n_1,n_2,\ldots,n_b,\frac{1}{2}\right\rangle \\
& + \frac{\tilde{g}^2 N_c}{2\pi}\sum_{i=1}^{b-1}\left(\sum_{m=0}^{n_i-1}\frac{1}{(m+n_i)^2}-\sum_{m=1}^{n_i-1}\frac{1}{m^2}\right)\left|\mathbf{b}+\frac{1}{2};n_1,n_2,\ldots,n_i+m,n_{i+1}-m,\ldots n_b,\frac{1}{2}\right\rangle \\
& + \lambda\frac{\tilde{g}^2 N_c}{2\pi}\sum_{j=1}^{b-1}\sum_{i=1}^{b-j}\left[\frac{1}{(\Sigma_{q=i}^{j+i}n_q)^2}-\frac{1}{(\Sigma_{q=i+1}^{j+i}n_q)^2}\right]n_{i+j} \\
& \times\left|\mathbf{b}-\mathbf{j}+\frac{1}{2};n_1,n_2,\ldots,n_{i-1},\sum_{q=i}^{j+i}n_q,n_{j+i+1},\ldots,n_b,\frac{1}{2}\right\rangle \\
& + \frac{\tilde{g}^2 N_c}{2\pi}\sum_{j=1}^{b-2}\sum_{i=1}^{b-j-1}\sum_{m=0}^{n_{i+j+1}-1}\left(\frac{1}{(m+\Sigma_{q=i}^{i+j}n_q)^2}-\frac{1}{(m+\Sigma_{q=i+1}^{i+j}n_q)^2}\right) \\
& \times\left|\mathbf{b}-\mathbf{j}+\frac{1}{2};n_1,n_2,\ldots,\sum_{q=i}^{i+j}n_q+m,n_{i+j+1}-m,n_{i+j+2},\ldots,n_b,\frac{1}{2}\right\rangle \\
& + \frac{\tilde{g}^2 N_c}{2\pi}\sum_{i=1}^{b-1}\left[\frac{1}{(\Sigma_{q=b-i}^{b}n_q)^2}-\frac{1}{(\Sigma_{q=b-i+1}^{b}n_q)^2}\right]\left|\mathbf{b}-\mathbf{i}+\frac{1}{2};n_1,n_2,\ldots,n_{b-i-1},\sum_{q=b-i}^{b}n_q,\frac{1}{2}\right\rangle \\
& + \frac{\tilde{g}^2 N_c}{2\pi}\left(\frac{1}{n_1^2}+\frac{1}{n_b^2}\right)\left|\mathbf{b}+\frac{1}{2};n_1,n_2,\ldots,n_b,\frac{1}{2}\right\rangle. \quad (24)
\end{aligned}
$$

As in the bosonic case [16], the terms in the Hamiltonian have a different N_c and N_f behavior. Only the terms in lines two and four of Eq. (24) contain N_f. These terms will be absent in the 't Hooft limit and will be dominant in the large N_f limit.

V. NUMERICAL RESULTS

We solve the eigenvalue problem, Eq. (5), numerically to obtain the mass spectrum of the theory as a function of the harmonic resolution K and the ratio of the number of colors and flavors, λ. Remember that the Veneziano limit is always understood. In the sequel, we will use both parameters to extract information from the spectra. We recover the large N_f

limit when $\lambda \to \infty$, the 't Hooft model in the limit $\lambda \to 0$, and the adjoint model, of chief interest, at $\lambda = 1$. It turns out that the complexity of the spectra grows in this order, and we will start their interpretation with the rather simple large N_f limit. Since we are using a discrete formulation, we may aim to understand *all* states in the spectra.

Some words on the numerical algorithm seem in order. The number of states grows exponentially with the harmonic resolution, cf. Table I, and much faster than in the bosonic sector. However, the situation is still better than in the formulation of the theory with fermionic operators [13]; we have at $K=25/2$ a roughly four times smaller basis. To further reduce the computational effort, we could use the Z_2 symmetry of the Hamiltonian which is invariant under the
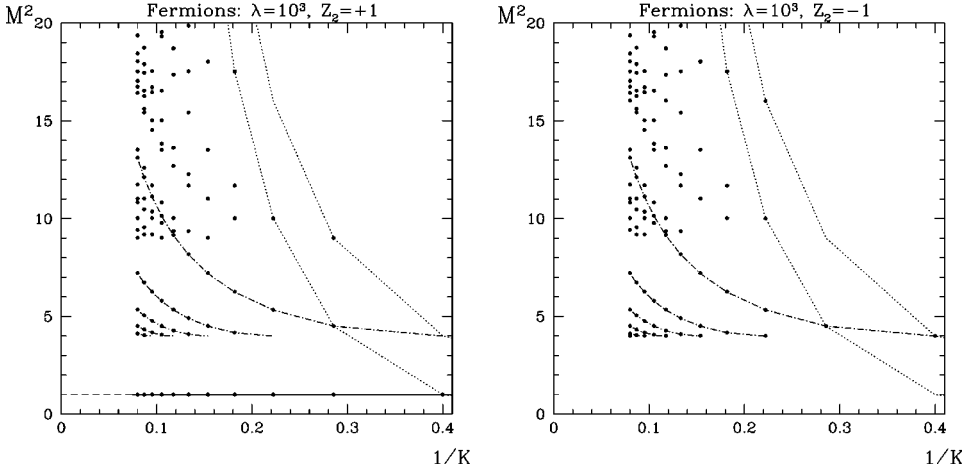
FIG. 1. Fermionic spectrum in the large N_f limit. Left: (a) Z_2 even sector. Right: (b) Z_2 odd sector. Solid (dash-dotted) lines connect associated single(multi)-particle eigenvalues at different K. Dotted lines connect analytically calculable eigenvalues. Dashed lines are extrapolations to the continuum limit. Note that masses are in units $g^2 N_f / \pi$.

transformation

$$\mathcal{T} J_{ij}(n) = -J_{ij}(n). \tag{25}$$

It is straightforward to convince oneself that the action of this operator on a state with b currents is

$$\mathcal{T} \left| \mathbf{b} + \frac{1}{2}; n_1, n_2, \ldots, n_b \right\rangle$$

$$= \sum_{i=0}^{2^{b-1}} (-)^{p_i+1} \left| \mathbf{p}_i + \mathbf{1/2}; \tau_i(n_b, n_{b-1}, \ldots, n_1), \frac{1}{2} \right\rangle, \tag{26}$$

where the τ_i consist of p_i partial sums of the b momenta, in the sense that i runs over all possibilities to place $0, 1, \ldots, b-1$ commas between the momenta while summing those momenta which are not separated by a comma, e.g.

$$\mathcal{T} \left| 7/2; n_1, n_2, n_3, \frac{1}{2} \right\rangle$$

$$= \left| 7/2; n_3, n_2, n_1, \frac{1}{2} \right\rangle - \left| 5/2; n_3, n_2 + n_1, \frac{1}{2} \right\rangle$$

$$- \left| 5/2; n_3 + n_2, n_1, \frac{1}{2} \right\rangle + \left| 3/2; n_3 + n_2 + n_1, \frac{1}{2} \right\rangle. \tag{27}$$

Since we do not work in an orthogonal basis, it is not very helpful to block diagonalize the Hamiltonian with respect to this symmetry. We will rather determine the Z_2 parity of an eigenstate *a posteriori* by calculating the expectation value of the operator $\mathcal{T}$ in this state. The main benefit of the *a priori* symmetrization, namely the reduction of the numerical effort to diagonalize a smaller matrix, is of course lost this way. However, the separation of the Z_2 odd and even eigenfunctions is very useful when interpreting the results, because it reduces the density of eigenvalues to roughly one half.

A. The large N_f limit

In the large N_f limit we find the expected meson [2] in the fermionic Z_2 even sector at mass

$$M_M^2(K) \equiv \frac{g^2 N_f}{\pi}, \tag{28}$$

cf. Fig. 1. All other states are multiparticle states built from this state, and are in this sense trivial. It is, nevertheless, important that we understand *all* states in the spectrum to see what we can learn in order to decode the adjoint spectrum. Let us first focus on the fermionic sectors. It is easy to write down a formula for the mass of a multiparticle state consisting of noninteracting partons in DLCQ. If we take into account the finding of Sec. III that the momentum of a fermion depends on λ, it reads

$$M_{p_1, p_2, \ldots, p_b}^2(n_1, n_2, \ldots, n_{b-1}; K) = \left(K - \frac{1}{2} + \frac{1}{1+\lambda} \right)$$

$$\times \left(\frac{M_{p_b}^2 \left(K - \Sigma_{i=1}^{b-1} n_i + (b-1) \left[\frac{1}{2} - (1-\lambda)^{-1} \right] \right)}{K - \Sigma_{i=1}^{b-1} n_i + (b-2) \left[\frac{1}{2} - (1-\lambda)^{-1} \right]} + \sum_{i=1}^{b-1} \frac{M_{p_i}^2(n_i)}{n_i - \frac{1}{2} + (1+\lambda)^{-1}} \right). \tag{29}$$

The masses of the multiparticle states grow like the momentum cutoff K, i.e., diverge in the continuum limit. We found it therefore, contrary to previous work [13,16], appropriate to connect the multiparticle states reflecting this fact in Fig. 1. Incidentally, this makes the labeling of states easier. A multiparticle state with b constituents is characterized by $b-1$ momenta
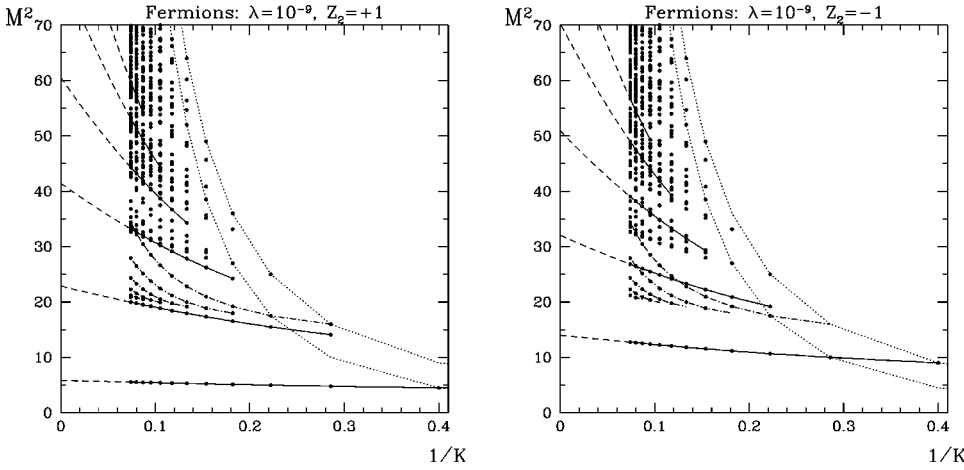
FIG. 2. Fermionic spectra of the 't Hooft limit in the Z_2 even (a) and odd (b) sectors. Solid (dash-dotted) lines connect associated single(multi)-particle eigenvalues at different K. Dotted lines connect analytically calculable eigenvalues. Dashed lines are extrapolations to the continuum limit. Masses are in units $g^2 N_c / \pi$.

n_i and b numbers p_i specifying its single-particle constituents. Equation (29) is slightly more general than needed here and holds also in the 't Hooft limit. In the large N_f limit there is only one single-particle state and it has a constant mass, Eq. (28). If we denote it by the operator $A_M^\dagger(n)$, the Fock basis in the fermionic sector looks like

$$|1\rangle_F = A_M^\dagger\left(K - \frac{1}{2}\right) B^\dagger\left(\frac{1}{2}\right)|0\rangle \tag{30}$$

$$|2;n\rangle_F = A_M^\dagger(n) A_M^\dagger\left(K - \frac{1}{2} - n\right) B^\dagger\left(\frac{1}{2}\right)|0\rangle \tag{31}$$

$$|3;n,m\rangle_F = A_M^\dagger(n) A_M^\dagger(m) A_M^\dagger\left(K - \frac{1}{2} - n - m\right)$$

$$\times B^\dagger\left(\frac{1}{2}\right)|0\rangle, \text{ etc.} \tag{32}$$

Fock basis states with b mesons will thus be constructed by assigning meson momenta as all partitions of $K - 1/2$ into b integers, i.e. exactly like the states, Eq. (17), except that now we are operating with meson rather than current operators. The role of the fermion in the states will be discussed in Sec. V C. Here it serves as a convenient tool for book-keeping. The assignment of quantum numbers π_T of the Z_2 symmetry, Eq. (25), is clear: the single particle state Eq. (30) is a two-parton state and therefore according to Eq. (26) Z_2 even, $\pi_T = +1$. It comprises a boson and a fermion with unique momentum partition and we find it indeed only in the spectrum of the fermionic Z_2 even sector. The states with two mesons, Eq. (31), are actually three parton states in the fermionic sector. Therefore the states where the mesons have the same momentum transform to minus themselves under the Z_2 symmetry, and are present only in the Z_2 odd sector. All other momentum partitions should be present in both Z_2 sectors. This is exactly what we see in the spectra. The generalization to the b parton states is obvious, and reproduces exactly the spectra in Fig. 1.

An analogous construction can be used in the bosonic sectors. The spectra are depicted in Fig. 5(a) of Ref. [16]. Again we build up the Fock basis from solutions of the dy-

namical eigenvalue problem, Eq. (5). But now the meson operators will act on the vacuum itself, rather than on the fermion and the vacuum

$$|1;n\rangle_B = A_M^\dagger(n) A_M^\dagger(K - n)|0\rangle \tag{33}$$

$$|2;n,m\rangle_B = A_M^\dagger(n) A_M^\dagger(m) A_M^\dagger(K - n - m)|0\rangle, \text{ etc.} \tag{34}$$

Consequently, we have to discard all states with momentum partitions equivalent under cyclic permutations. This is, of course, nothing other than constructing the current basis in the bosonic sectors, as we did in Ref. [16]. A two-meson state is a two-parton state in the bosonic sector. Hence, the Z_2 quantum numbers are opposite as in the fermionic sector. In particular, since there are no cyclic permutations of momenta of the currents, the two-meson states are absent altogether in the bosonic Z_2 odd sector, as observed. The generalization to b partons is again obvious, and we thus completely constructed the spectrum of the large N_f limit of the theory.

B. The 't Hooft limit

In the 't Hooft limit we find exactly the same eigenvalues as in previous work [16], Eq. (29), namely

$$M^2 = 5.88, 13.44, 23.04, 32.27, 41.68, 51.24, 60.93,$$

$$70.76, 80.97, 90.90, \tag{35}$$

cf. Fig. 2, which is in very good agreement with 't Hooft's original solution [1]. These continuum results are obtained by fitting the eigenvalue trajectories $M_i^2(K)$ of the single-particle states to polynomials of second order in $1/K$ and subsequently extrapolating to $K \to \infty$. If we compare the spectrum, Fig. 2, to the bosonic sector, Fig. 2 of Ref. [16], we find much more multiparticle states and, of course, a shift of half a unit in momentum. As in the large N_f limit, the multiparticle states decouple, but now we have several single-particle states. Namely, the spectrum at resolution K contains $K - 1/2$ 't Hooft mesons. The ith meson makes its first appearance at resolution $K = i + 1/2$ and has $\pi_T = (-1)^{i+1}$. Note that the single-particle masses are functions of K. If we would set up an orthonormal current basis for this
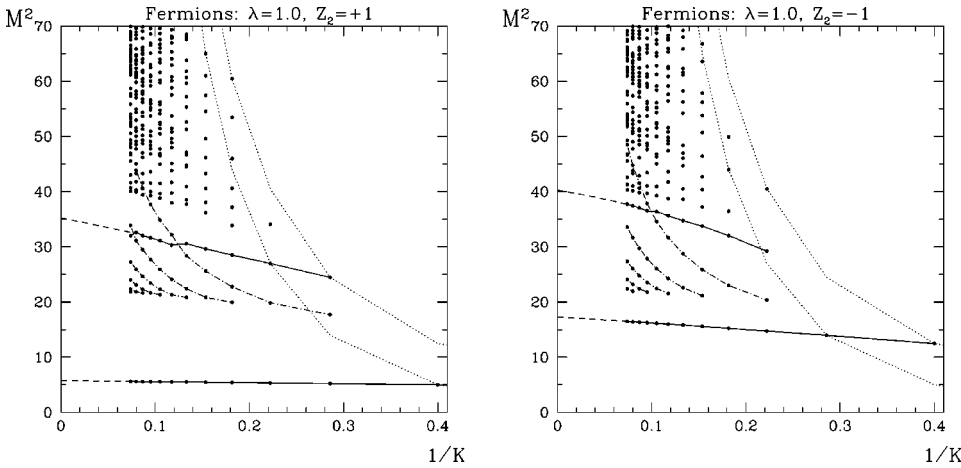
FIG. 3. Fermionic spectra of the theory with adjoint fermions in the Z_2 even (a) and odd (b) sectors. Solid (dash-dotted) lines connect conjectured single(multi)-particle eigenvalues at different K. Dotted lines connect analytically calculable eigenvalues. Dashed lines are extrapolations to the continuum limit. Masses are in units g^2N_c/π.

problem to factorize the Hamiltonian into its single- and multiparticle blocks, the task would be to diagonalize a $(K-1/2)\times(K-1/2)$ matrix to find the masses of these mesons. Instead we are diagonalizing a $2^{K-3/2}$ dimensional matrix. The first procedure is, however, due to the tedious evaluation of the scalar product of Kac-Moody states more expensive than actually diagonalizing the much larger matrix [16].

We already wrote down the formula for the masses of the multiparticle states, Eq. (29). Also the designation of Z_2 quantum numbers from the partitions of the parton momenta stays the same as in the large N_f limit. It should be noted, however, that due to the difference in effective momenta of the fermion in the states, the first multiparticle 't Hooft state appears at $K=7/2$, as opposed to $K=5/2$ in the large N_f limit. As in the large N_f limit, we reproduce the distribution of the Z_2 even and odd states, and understand the spectra completely in terms of their single-particle content. In particular, there is no need for recurring to properties of the massless sector of the theory, because we can construct all quantum numbers from the information of the massive spectrum. This will change substantially in the adjoint case which we consider next.

C. Adjoint fermions

Solving the eigenvalue problem, Eq. (5) in the adjoint model, $\lambda=1$, we obtain precisely the same eigenvalues as previous works with fermions as basic degrees of freedom, e.g. Ref. [13], with *antiperiodic* boundary conditions for the fermions. This is not surprising because the formulation of the theory with $SU(N_c)$ currents rather than with fermions is in essence a change of basis.

If we look at the eigenvalue trajectories (mass squared as a function of K) in Fig. 3, the structure of the spectrum looks similar to the 't Hooft case. We see immediately four single-particle candidates which qualify by their quasilinear trajectories. In the continuum limit they have the eigenvalues

$$M_{F_1}^2=5.75, \quad M_{F_2}^2=17.29,$$

$$M_{F_3}^2=35.25, \quad M_{F_3}^2=40.24, \quad (36)$$

in units g^2N_c/π; see also Table II. It is, however, questionable if these states are indeed single-particle states. There are a couple of problems which prevent a straightforward interpretation of the adjoint spectrum, which will become clear when we compare the adjoint to the 't Hooft spectrum. In the adjoint spectrum we find kinks in the single-particle trajectories, and also the multiparticle trajectories are distorted. This is clear evidence for an interaction between these states. Since the multiparticle states do not decouple, a mass formula analogous to Eq. (29) cannot be exact. Furthermore, the masses of the single-particle states of the fermionic and bosonic sectors are not degenerate as in the 't Hooft case, but differ significantly. This in turn means that we cannot have a fermionic and bosonic Regge trajectory of single-particle states, if all of them give rise to multiparticle states: there would be simply too many states to account for in a discrete Fock basis.

As a way out of this dilemma, we make the following

TABLE II. Eigenvalues of the lowest four (suspectedly) single-particle states in the adjoint case. The masses are given in units g^2N_c/π. The masses in the 't Hooft case at $K=b+\frac{1}{2}$ are exactly the same as in the bosonic sector at $K=b+1$. The mass of the only single-particle state in the large N_f limit is independent of the cutoff: $M_{N_f}^2(K)\equiv1.0000(g^2N_f/\pi)$.

$2K$	$M_{F_1}^2$	$M_{F_2}^2$	$M_{F_3}^2$	$M_{F_4}^2$
3	4.5000			
5	5.0000	12.5000		
7	5.2227	14.0000	24.5000	
9	5.3456	14.7645	27.0000	29.2451
11	5.4222	15.2575	28.5484	32.0373
13	5.4741	15.5908	29.6419	33.7443
15	5.5111	15.8311	30.5931	34.7280
17	5.5388	16.0113	30.3593	35.6396
19	5.5602	16.1509	31.1301	36.3496
21	5.5771	16.2618	31.6091	36.5054
23	5.5908	16.3518	32.0304	37.0575
25	5.6021	16.4261	32.6060	37.4225
27	5.6115	16.4884	32.0123	37.7243
∞	5.75	17.29	35.25	40.24

conjecture which we will try to support by empirical facts. Namely, we view the sole fermion in the states of smallest discrete momentum $k_{\min}=P^+/2K$ acting on the vacuum as the finite K expression for an "adjoint vacuum," as it appears in the bosonized version of this theory [4]. It is clear that the correct expression should involve a fermionic zero mode, which is absent in the present discrete approach. It is recovered in the continuum limit $K\to\infty$. This conjecture makes the following interpretation of the spectrum plausible. The approximate vacuum will introduce couplings between states that are decoupled in the continuum limit. Most of these states will be multiparticle states, which are necessary to describe the theory correctly at finite harmonic resolution K. Since the approximation occurs only in the fermionic sector, it seems natural that the artifacts associated with this finite K effect also originate in this sector. In other words, we expect *only* the fermionic single-particle states to give rise to multiparticle states. We can check this conjecture by looking at the spectra at small λ. There we expect that the multiparticle states are still very well described by the DLCQ formula for the spectrum of free particles, Eq. (29). On the other hand, the masses of the lowest single-particle boson and fermion are already noticeably different. By constructing two sets of multiparticle states out of two bosons and two fermions, respectively, we convinced ourselves that the eigenvalues are well described by a pair of free fermions, but not of bosons. On the other hand, we expect the coupling between the single- and multiparticle states to vanish as the harmonic resolution K grows. In particular, the deviations from a multiparticle mass formula analogous to Eq. (29) should decrease with a power of K. We checked that the discrepancies vanish indeed like $1/K^2$, providing additional support for the approximate-vacuum conjecture.

Furthermore, analyzing the spectrum as a function of the continuous parameter λ, it seems plausible that the number of single-particle states stays the same as in the 't Hooft case; see also the discussion in the next section. This hypothesis can in principle be tested at $\lambda=1$ by using an approximate multiparticle mass formula to eliminate the multiparticle states. The fact that most single-particle states asymptotically become multiparticle states at large λ cannot affect us here. At finite λ the problem is to show that most states are multiparticle states, although they are not exactly following a mass formula in the fashion of Eq. (29).

Summing up the above findings, the conclusion is the following. Finding that the adjoint bosonic single-particle states do not form multiparticle states and conjecturing that the number of fermionic single-particle states grows linearly with K, we get the same situation concerning the ratio of single- to multiparticle states as in the 't Hooft case. We then conclude that the number of single-particle states is the same as in the 't Hooft model, even in the bosonic sector. Since the masses of the fermionic and bosonic single particles are not degenerate as opposed to the 't Hooft case, we find two adjoint Regge trajectories, a bosonic and a fermionic one. The determination of the functional dependence of the single-particle states on the excitation number requires further investigations which are beyond the scope of the present work.

D. Intermediate cases and eigenfunctions

It is instructive to study the spectrum of eigenvalues as a function of the continuous parameter $\lambda=N_f/N_c$. In Fig. 4 we plotted the spectrum of two-dimensional Yang-Mills theories coupled to massless matter in representations characterized by λ, in the Veneziano limit. We show all sectors of the theory, i.e. the bosonic and fermionic Z_2 even and odd spectra, as a function of λ. A crucial observation is that the 't Hooft mesons develop differently in the fermionic and bosonic sectors as λ grows, and the degeneracy of their masses is lifted. In particular, the mass of the lightest meson in the fermionic sector decreases as λ grows with a slope of $\hat{e}_1^{(f)}{}'(1)=-1.39$; see the Appendix. It reaches the minimum of its parabolic trajectory $M_{F_1}^2(\lambda)$ at $\lambda=1/3$. Asymptotically it rises linearly with λ. On the other hand, the mass of the lightest boson increases monotonously. This scenario for small λ is expected, since the mass of the lightest state in a theory has to decrease in second order perturbation theory. It is evidence for the conjecture that the theory is incomplete if only its bosonic sector is considered [7], because the lowest boson cannot be the lightest state of the full theory. In general, we obtain the first corrections in λ to the 't Hooft meson masses in the bosonic sector in complete agreement with the perturbative calculations by Engelhardt [18], as we will show in more detail in the Appendix.

Concerning the global λ dependence of the spectrum, Fig. 4, we emphasize that the eigenvalues are smooth functions of λ, and we seem to find no indication that the adjoint theory is special. We see a lot of level crossings, some of which are obscured by eigenvalue repulsion due to finite harmonic resolution. Some of the reduced eigenvalues $\hat{M}^2 \equiv M^2/(1+\lambda)$ are almost stationary as a function of the parameter λ, amongst them chiefly the suspected single-particle states. Note, however, the somewhat artificial definition of the reduced masses, which was used in order to fit the spectrum for all values of λ into one plot.

It would be very interesting if one could find a criterion for a state to be a single-particle state, or if one could formulate a good observable, e.g., a structure function, that would allow one to distinguish single- from multiparticle states. We display four adjoint eigenfunctions in Fig. 5. Apart from the striking repetitive pattern in the different parton sectors, we see that the multiparticle state in this plot is distinct from the single-particle states. In the extreme cases, $\lambda=0$ and $\lambda\to\infty$, we obtain the following behavior of the wave functions. The 't Hooft eigenfunctions are very similar to the ones in the adjoint case. They can in principle be calculated from the standard formulation of the theory with fermion fields, which is equivalent to a change of basis. At large N_f the wave functions are converging very slowly. They look very much like those in the 't Hooft limit for $K<10$. At that point, most of the amplitudes become suppressed while keeping their shape, and the amplitudes of states with a large number of currents become heavily peaked.

VI. SUMMARY AND DISCUSSION

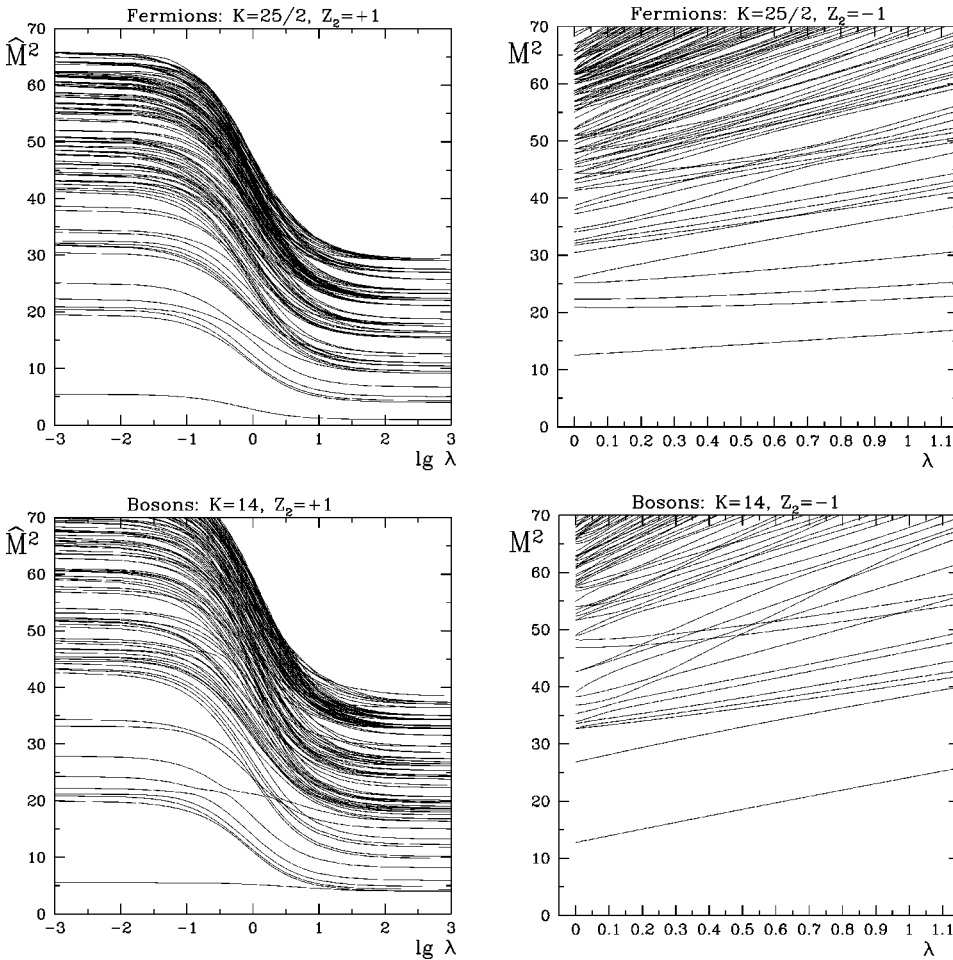In this article we presented the spectrum of two dimensional Yang-Mills theories coupled to massless matter in a

FIG. 4. The spectrum of two-dimensional QCD in all sectors of the theory as a function of λ. Plotted are the lowest 100 eigenvalues in the fermionic sectors (top row) and the bosonic sectors (bottom row). Left column: Z_2 even sectors, reduced eigenvalues $\hat{M}^2 \equiv M^2/(1+\lambda)$ vs lg λ. Right column: Z_2 odd sectors, actual eigenvalues vs λ.
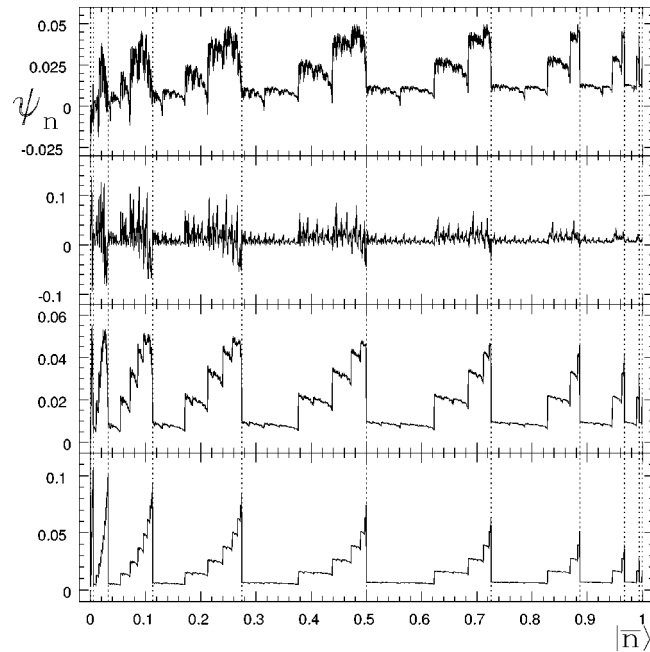

FIG. 5. The wave functions of four adjoint states at $K=25/2$: (a) $M^2=5.6021$, (b) $M^2=16.4261$, (c) $M^2=21.8688$, and (d) $M^2=32.6060$ (from bottom to top). Plotted are the amplitudes ψ_n vs $\bar{n}=n/2^{K-3/2}$. The third state is the only multiparticle state in this plot and its eigenfunction clearly has a different shape. The number of currents in a basis state changes at the dashed lines.

representation characterized by the ratio of the numbers of flavor and color λ in the Veneziano limit. We derived the Hamiltonian in the fermionic sector in the framework of DLCQ as an algebraic function of the harmonic resolution K and the ratio λ. Surprisingly, we found the momentum of an adjoint fermion depending on λ in this discrete approach. This is explained by the fact that we have zero modes of the current operators in the theory, while fermionic zero modes are recovered in the continuum limit only. The well-known spectra in the 't Hooft and the large N_f limits were reproduced. Although this is not surprising taking into account the universality established in Ref. [9] which is a specialty of two dimensions, it is nevertheless a strong check on the numerics. We found the bosonic and fermionic spectra to be degenerate in the 't Hooft limit, and the only meson of the large N_f limit in the fermionic sector. The multiparticle states decouple completely in these limits and a construction of the spectra in terms of their single-particle content was achieved. This allowed for a complete classification of all states including their statistics and symmetry properties in both cases. In trying to apply this knowledge to the adjoint case, we were only partly successful. We presented evidence for the *conjecture* that the vacuum is only approximately realized in the DLCQ formulation of the fermionic sector of the theory. This conjecture allowed us to understand the empirical finding that the multiparticle states have only fermionic single-
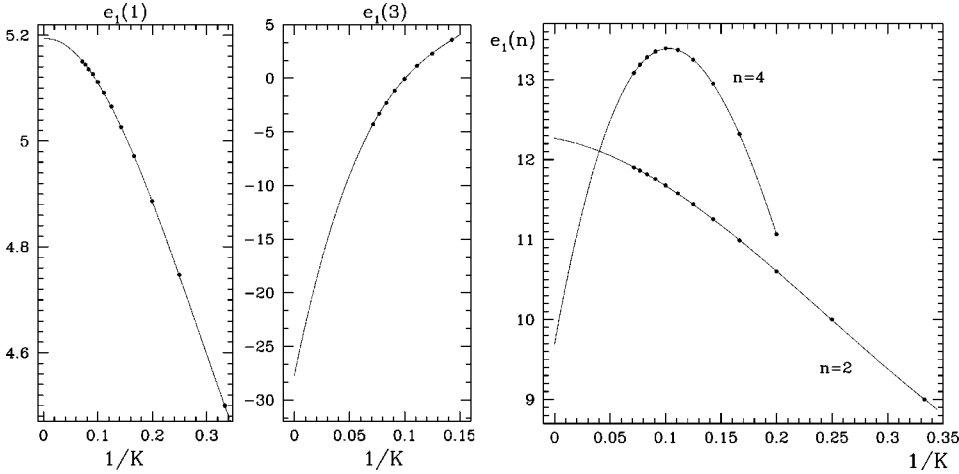
FIG. 6. The coefficients $e_1(n)$ of the first correction in λ to the 't Hooft meson masses $M_n^2(\lambda=0)$ vs $1/K$.

particle constituents. This fact was deduced from an analysis of the spectra at intermediate values of λ. The approximate vacuum induces couplings between single- and multiparticle states which were found to decrease with the harmonic resolution like $1/K^2$, i.e. consistent with the above conjecture. We motivated the *hypothesis* that the number of fermionic single-particle states is the same in the 't Hooft and adjoint cases by pointing out the smooth transition of the spectra into each other by the continuous parameter λ. We then concluded that there has to be a second Regge trajectory of bosonic single-particle states, because at each K the size of the Fock basis and the number of multiparticle states determined by the kinematics of their fermionic constituents allows for exactly $K-1$ additional states. Although we were as of yet unable to give the complete solution of the adjoint theory in terms of its single-particle states, it seems thus that their number grows linearly with the harmonic resolution. Their masses tend to grow more rapid with the excitation number n, maybe like $M^2 \propto n^2$, rather than linear as in the 't Hooft model.

The two Regge trajectory conjecture is not in contradiction with the expectation of a multi-Regge structure at non-vanishing fermion mass m [7], or with the related appearance of a Hagedorn spectrum signaling the confinement/screening transition as m vanishes [12]. When a fermion mass is turned on, the description of the theory in terms of Kac-Moody currents breaks down and the theory turns from screening into a confining phase. It has been pointed out by Kutasov how in the massless theory the two seemingly contradictory facts of having a vanishing string tension together with the absence of a Hagedorn transition can be reconciled [12]. In short, the exponentially rising density of states characteristic of a Hagedorn transition can be explained by a large degeneracy in the massless sector of the theory which is lifted if a fermion mass is turned on. This sector is, of course, completely inaccessible in the present approach.

In summary, we hope to have added some new pieces of information to the adjoint QCD$_2$ puzzle. The major practical goal remains to identify all single-particle states of the theory unambiguously. While we were unable to present a complete solution, we described a practical algorithm to extract the single-particle spectrum. Using the spectral information presented here, one could identify all multiparticle states by the characteristic K dependence of their masses dictated by an approximate multiparticle mass formula. While this seems in principle possible, it is nevertheless beyond the scope of the present work. This exercise can also serve as a quantitative test of the hypothesis that the number of single particles is the same in the 't Hooft and adjoint models. Furthermore, the approximate-vacuum conjecture can be ruled out, if one could show that the deviations from the discrete multiparticle mass formula do not fall off everywhere with the resolution K. Improvements of the results might be possible by attacking the theory from a very different point. The adjoint theory is supersymmetric at a specific value of the fermion mass $m = g\sqrt{N_c}$ [12]. In the light of recent progress in the evaluation of supersymmetric theories [17], this might be an interesting alternative.

APPENDIX: RECOVERING THE CORRECTIONS TO THE 't HOOFT MASSES

In a recent paper, Engelhardt [18] calculated the first corrections in $\lambda = N_f/N_c$ to the masses of the lowest four 't Hooft mesons, namely the slopes $e_1(n)$, $n=1,2,3,4$ in the expansion

$$M_n^2(\lambda) = M_n^2(0) + e_1(n)\lambda + \cdots,$$

where the masses are in units $g^2 N_c/\pi$. Surprisingly, some of these corrections are negative and large. This seems to contradict the results of Ref. [16], Fig. 5(b), and we shall reanalyze them here. The slope of the curves $M_n^2(\lambda)$ is strongly dependent on the harmonic resolution K for small λ. If we plot the slopes as a function of $1/K$ we see a consistent picture arising. We fitted the slopes for the lowest four 't Hooft mesons, Fig. 6, to a polynomial of third order in $1/K$, and obtain in the continuum limit

$$\hat{e}_1(n) = 5.19, 12.27 - 27.7, 9.69, \tag{A1}$$

to be compared to Engelhardt's values

$$e_1(n) = 5.1, 12.0, -30.5, 9.1. \tag{A2}$$

Note that Engelhardt's values are lower bounds (although one expects very small corrections), whereas extrapolations towards the continuum in DLCQ tend to be upper bounds. In this sense, the agreement is fairly good.

We emphasize that for small K one is totally misled as to what the continuum limit might be for the slopes $e_1(3)$ and $e_1(4)$; cf. Fig. 6. For instance, the slope of the third state, $e_1(3)|_{K \to \infty} = -27.7$, is still positive at the fairly large resolution $K = 10$. This shows the importance of Fock states with a large number of currents, and renders two-current approximations questionable. Note that the mass of this state increases linearly for large enough λ, although it starts with a large negative slope; cf. Fig. 5(b) of Ref. [16]. The first correction to the 't Hooft mass is a good approximation only up to $\lambda \simeq 0.01$.

It is important to note that the corresponding (and degenerate) 't Hooft single-particle states in the fermionic sector have different corrections in λ. Following the development of the three lowest 't Hooft mesons in the fermionic sector we obtain the following slopes:

$$\hat{e}_1^{(f)}(n) = -1.36, 1.94, -15.38. \tag{A3}$$

The lowest state, which develops into the lightest adjoint state as λ grows to unity, has a negative correction, as expected from second order perturbation theory for the lowest state of a theory. It is thus clear that the single-particle states in the fermionic and bosonic sectors are distinct entities, although their masses are degenerate in the 't Hooft limit. The fourth lightest state has a rather irregular trajectory $\hat{e}_1(4,K)$ which prevents us from extrapolating to the continuum. This is easily understood when one compares the spectra in the bosonic and the fermionic 't Hooft sectors. In the bosonic sector, all four lowest single-particle states are lighter than the lowest multiparticle states in their sector. In the fermionic sector the fourth single-particle state lies in the two-particle continuum formed by the lightest massive 't Hooft meson.[3] In perturbation theory, such a situation has to be taken care of by constructing states which contain admixtures of degenerate states with higher parton numbers [18]. We see that in the present discrete approach we run into the same difficulties.

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[3] The lightest 't Hooft meson proper with $M^2(n=0)=0$ is absent from the spectrum, since we work formally in the adjoint theory.

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