Time transfer and frequency shift to the order $1/c^4$ in the field of an axisymmetric rotating body

Bernard Linet*
Laboratoire de Mathématiques et Physique Théorique, CNRS/UMR 6083, Université François Rabelais, F-37200 Tours, France

Pierre Teyssandier†
Département Systèmes de Référence Temps et Espace, CNRS/UMR 8630, Observatoire de Paris, 61 avenue de l'Observatoire, F-75014 Paris, France

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Within the weak-field, post-Newtonian approximation of the metric theories of gravity, we determine the one-way time transfer up to the order $1/c^4$, the unperturbed term being of order $1/c$, and the frequency shift up to the order $1/c^4$. We adapt the method of the world function developed by Synge to the Nordvedt-Will parametrized post-Newtonian (PPN) formalism. We get an integral expression for the world function up to the order $1/c^3$ and we apply this result to the field of an isolated, axisymmetric rotating body. We give a new procedure enabling us to calculate the influence of the mass and spin multipole moments of the body on the time transfer and the frequency shift up to the order $1/c^4$. We obtain explicit formulas for the contributions of the mass, of the quadrupole moment and of the intrinsic angular momentum. In the case where the only PPN parameters different from zero are β and γ, we deduce from these results the complete expression of the frequency shift up to the order $1/c^4$. We briefly discuss the influence of the quadrupole moment and of the rotation of the Earth on the frequency shifts in the ESA's Atomic Clock Ensemble in Space mission.

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I. INTRODUCTION

Owing to recent progress in absolute frequency measurements of some optical transitions with a femtosecond laser, it seems possible to achieve in the near future atomic clocks having a time-keeping accuracy of the order of 10^{-18} in fractional frequency [1]. Since a reduced gravity would significantly increase clock performances, it is envisaged to install such clocks on board artificial satellites and to compare them with terrestrial clocks by exchange of electromagnetic signals. Already, a spatial experiment such as the ESA's Atomic Clock Ensemble in Space (ACES) mission [2,3] is planned for 2006, the purpose being to obtain an accuracy of order 10^{-16} in fractional frequency.

At a level of uncertainty about 10^{-18}, a fully relativistic treatment of time or frequency transfers must be performed up to the order $1/c^4$ [23]. As far as we know, the corresponding calculations have not been carried out. For the time transfer, the main relativistic correction of order $1/c^3$ is the well-known Shapiro time delay [4]. Other corrections due to the quadrupole moment and to the intrinsic angular momentum have been studied by several authors [5]. Gravitational corrections of order $1/c^2$ in the frequency transfers were theoretically determined and experimentally checked a long time ago [6]. These corrections are now commonplace in the Global Positioning System. The relativistic theory of the frequency transfers have been recently extended up to the terms of order $1/c^3$ [7], justifying the results previously given in [8] without any detail. However, it must be pointed out that, in [7], some terms of order $1/c^3$ due to the quadrupole moment J_2 of the Earth are bounded without any explicit calculation. Furthermore, the time or frequency transfers have been calculated within the limited framework of general relativity, which prevents us from discussing new tests of gravitational theories.

The present paper is a first step towards a general theory of the time and frequency transfers, including all the terms of order $1/c^4$, within the post-Newtonian (PN) approximation of any metric theory of gravity. Using the Nordtvedt-Will parametrized post-Newtonian (PPN) formalism [9], we bring the complete determination of these effects in the field of an isolated, axisymmetric rotating body, the gravitational field being assumed stationary. Of course, modelling a mission in the vicinity of the Earth at a level of accuracy about 10^{-18} will require us to add the effects due to the tidal gravitational field induced by the Sun and the Moon.

We assume that the photons ensuring the transfers follow null geodesics. The problems that we have to tackle come down to the following ones, relative to a couple of points $x_A=(ct_A,\boldsymbol{x}_A)$ and $x_B=(ct_B,\boldsymbol{x}_B)$ connected by a null geodesic: (i) to calculate the (coordinate) time transfer t_B-t_A as a function of $(\boldsymbol{x}_A,\boldsymbol{x}_B)$; (ii) to determine the vectors tangent to the null geodesic at x_A and x_B. Solving this second problem is indeed indispensable to calculate the frequency shift between x_A and x_B.

The method generally employed to study the questions related to the propagation of light in a gravitational field is based on the solution of the null geodesic equations (see, e.g., [5,10–13] for investigations in the linearized, weak-field limit of general relativity). However, the theory of the world function developed by Synge [14] presents the great advantage to spare the trouble of integrating the geodesic equations. Once the world function is determined, it is possible to solve straightforwardly the two above-mentioned problems. This method is particularly elegant for the stationary, axisymmetric field and we apply it in the present paper.

*Electronic address: linet@celfi.phys.univ-tours.fr
†Electronic address: Pierre.Teyssandier@obspm.fr

We find a new procedure enabling us to determine the influence of the mass and spin multipole moments of the body. Explicit calculations are given for the contributions of the mass, of the quadrupole moment and of the intrinsic angular momentum of the rotating body.

The paper is organized as follows. In Sec. II the relevant properties of the world function $\Omega(x_A, x_B)$ are recalled and the general expression of this function in the post-Newtonian limit of any metric theory is given. The corresponding expression of the time transfer $t_B - t_A$ is derived up to the order $1/c^4$. In Sec. III we determine the expression of $\Omega(x_A, x_B)$ and of $t_B - t_A$ within the ten-parameter PN formalism of Nordtvedt and Will. Then, in Sec. IV we focus our attention on the case of an isolated, axisymmetric rotating body. We show that it is possible to determine the contributions of the mass and spin multipole moments by straightforward differentiations of a single function. Retaining only the terms due to the mass M, to the quadrupole moment J_2 and to the intrinsic angular momentum $\boldsymbol{S}$ of the rotating body, we obtain explicit expressions for the time transfer up to the order $1/c^4$ and for the tangent vectors at x_A and x_B up to the order $1/c^3$. In Sec. V the frequency shift is developed up to the order $1/c^4$ in the case where β and γ are the only nonvanishing PPN parameters. We find detailed expressions for the contributions of J_2 and $\boldsymbol{S}$ and we discuss the possible influence of these terms in the ACES mission. We give our conclusions in Sec. VI.

In this paper G is the Newtonian gravitational constant and c is the speed of light in a vacuum. The Lorentzian metric of space-time is denoted by g. The signature adopted for g is $(+---)$. We suppose that the space-time is covered by one global coordinate system $(x^\mu) = (x^0, \boldsymbol{x})$, where $x^0 = ct$, t being a time coordinate, and $\boldsymbol{x} = (x^i)$, the x^i being quasi-Cartesian coordinates. We assume that the curves of equations $x^i = \text{const}$ are timelike, which means that $g_{00} > 0$ anywhere. We employ the vector notation $\boldsymbol{a}$ in order to denote either $(a^1, a^2, a^3) = (a^i)$ or $(a_1, a_2, a_3) = (a_i)$. Considering two such quantities $\boldsymbol{a}$ and $\boldsymbol{b}$ with, for instance, $\boldsymbol{a} = (a^i)$, we use $\boldsymbol{a}\cdot\boldsymbol{b}$ to denote $a^i b^i$ if $\boldsymbol{b} = (b^i)$ or $a^i b_i$ if $\boldsymbol{b} = (b_i)$ (the Einstein convention on the repeated indices is used). The quantity $|\boldsymbol{a}|$ stands for the ordinary Euclidean norm of $\boldsymbol{a}$.

II. THE WORLD FUNCTION AND ITS POST-NEWTONIAN LIMIT

A. Definition and fundamental properties

Consider two points x_A and x_B in a space-time with a given metric $g_{\mu\nu}$ and assume that x_A and x_B are connected by a unique geodesic path Γ. Throughout this paper, λ denotes the unique affine parameter along Γ which fulfills the boundary conditions $\lambda_A = 0$ and $\lambda_B = 1$. The so-called world function of space-time [14] is the two-point function $\Omega(x_A, x_B)$ defined by

$$\Omega(x_A, x_B) = \frac{1}{2}\int_0^1 g_{\mu\nu}(x^\alpha(\lambda))\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}d\lambda, \tag{1}$$

the integral being taken along Γ. It is easily seen that $\Omega(x_A, x_B) = \varepsilon[s_{AB}]^2/2$, where s_{AB} is the geodesic distance between x_A and x_B and $\varepsilon = 1, 0, -1$ for timelike, null and spacelike geodesics, respectively. It results from definition (1) that the world function $\Omega(x_A, x_B)$ is unchanged if we perform any admissible coordinate transformation.

The utility of the world function for our purpose comes from the following properties [14].

(i) Two points x_A and x_B are linked by a light ray if and only if the condition

$$\Omega(x_A, x_B) = 0 \tag{2}$$

is fulfilled. Thus, $\Omega(x_A, x) = 0$ is the equation of the light cone $\mathcal{C}(x_A)$ at x_A. This fundamental property shows that if $\Omega(x_A, x_B)$ is known, it is possible to determine the travel time $t_B - t_A$ of a photon connecting two points x_A and x_B as a function of t_A, $\boldsymbol{x}_A$ and $\boldsymbol{x}_B$. It must be pointed out, however, that solving the equation $\Omega(ct_A, \boldsymbol{x}_A, ct_B, \boldsymbol{x}_B) = 0$ for t_B yields two distinct solutions t_B^+ and t_B^- since the timelike curve $x^i = x_B^i$ cuts the light cone $\mathcal{C}(x_A)$ at two points x_B^+ and x_B^-, x_B^+ being in the future of x_B^-. In the present paper we always regard x_A as the point of emission of the photon and x_B as the point of reception, and we are concerned only with the determination of $t_B^+ - t_A$ as a function of t_A, $\boldsymbol{x}_A$ and $\boldsymbol{x}_B$. We put

$$t_B^+ - t_A = \mathcal{T}(t_A, \boldsymbol{x}_A, \boldsymbol{x}_B), \tag{3}$$

and we call $\mathcal{T}(t_A, \boldsymbol{x}_A, \boldsymbol{x}_B)$ the (coordinate) time transfer function. Of course, it is also possible to introduce another time transfer function giving $t_B^+ - t_A$ as a function of the instant of reception t_B^+ and of $\boldsymbol{x}_A$, $\boldsymbol{x}_B$, but we do not use it here.

(ii) The vectors $(dx^\alpha/d\lambda)_A$ and $(dx^\alpha/d\lambda)_B$ tangent to the geodesic Γ, respectively, at x_A and x_B are given by

$$\left(g_{\alpha\beta}\frac{dx^\beta}{d\lambda}\right)_A = -\frac{\partial\Omega}{\partial x_A^\alpha}, \quad \left(g_{\alpha\beta}\frac{dx^\beta}{d\lambda}\right)_B = \frac{\partial\Omega}{\partial x_B^\alpha}. \tag{4}$$

As a consequence, if $\Omega(x_A, x_B)$ is explicitly known, the determination of these vectors does not require the integration of the differential equations of the geodesic. Let us note that it can be proved that the tangent vectors (4) are null when (2) holds.

Consider now a stationary space-time. In this case, we use exclusively coordinates (x^μ) such that the metric does not depend on x^0. Then, the world function is a function of $x_B^0 - x_A^0$, $\boldsymbol{x}_A$ and $\boldsymbol{x}_B$, and Eq. (3) reduces to a relation of the form

$$t_B^+ - t_A = \mathcal{T}(\boldsymbol{x}_A, \boldsymbol{x}_B). \tag{5}$$

The time transfer function $\mathcal{T}(\boldsymbol{x}_A, \boldsymbol{x}_B)$ plays a central role in the present paper because a comparaison between Eqs. (2) and (5) immediately shows that the vectors $(l^\mu)_A$ and $(l^\mu)_B$ defined by their covariant components

$$(l_0)_A = 1, \quad (l_i)_A = c\frac{\partial}{\partial x_A^i}\mathcal{T}(\boldsymbol{x}_A, \boldsymbol{x}_B), \tag{6}$$

$$(l_0)_B = 1, \quad (l_i)_B = -c\frac{\partial}{\partial x_B^i}\mathcal{T}(\boldsymbol{x}_A, \boldsymbol{x}_B), \tag{7}$$

are tangent to the ray at x_A and x_B, respectively. It must be pointed out that these tangent vectors correspond to an affine parameter such that $l_0=1$ along the ray (note that such a parameter does not coincide with λ). Generally, extracting the time transfer formula (5) from Eq. (2), and then using Eqs. (6) and (7), will be more straightforward than deriving the vectors tangent at x_A and x_B from Eq. (4) and then imposing the constraint (2). We shall use Eqs. (6) and (7) in Sec. IV.

To conclude, let us emphasize that the method of the world function works as long as $\Omega(x_A,x_B)$ is a well-defined, single-valued function of x_A and x_B. This condition is satisfied in any region of space-time in which any points x_A and x_B are connected by one and only one geodesic, a feature which excludes the existence of conjugate points. This requirement is certainly fulfilled in experiments performed in the solar system and more generally for observations of stars belonging to our Galaxy.

B. General expression of the world function in the post-Newtonian limit

To begin, let us assume that the metric may be written as

$$g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu} \tag{8}$$

throughout space-time, with $\eta_{\mu\nu}=\mathrm{diag}(1,-1,-1,-1)$. Let $\Gamma_{(0)}$ be the straight line defined by the parametric equations $x^\alpha=x^\alpha_{(0)}(\lambda)$, with

$$x^\alpha_{(0)}(\lambda)=(x^\alpha_B-x^\alpha_A)\lambda+x^\alpha_A, \quad 0\leqslant\lambda\leqslant 1. \tag{9}$$

With this definition, the parametric equations of the geodesic Γ connecting x_A and x_B may be written in the form

$$x^\alpha(\lambda)=x^\alpha_{(0)}(\lambda)+X^\alpha(\lambda), \quad 0\leqslant\lambda\leqslant 1, \tag{10}$$

where the quantities $X^\alpha(\lambda)$ satisfy the boundary conditions

$$X^\alpha(0)=0, \quad X^\alpha(1)=0. \tag{11}$$

Inserting Eq. (8) and $dx^\mu(\lambda)/d\lambda=x^\mu_B-x^\mu_A+dX^\mu(\lambda)/d\lambda$ in Eq. (1), then developing and noting that

$$\int_0^1 \eta_{\mu\nu}(x^\mu_B-x^\mu_A)\frac{dX^\nu}{d\lambda}d\lambda=0 \tag{12}$$

by virtue of Eq. (11), we find the rigorous formula

$$\begin{aligned}\Omega(x_A,x_B)=&\,\Omega^{(0)}(x_A,x_B)\\&+\frac{1}{2}(x^\mu_B-x^\mu_A)(x^\nu_B-x^\nu_A)\int_0^1 h_{\mu\nu}(x^\alpha(\lambda))d\lambda\\&+\frac{1}{2}\int_0^1\Bigg[g_{\mu\nu}(x^\alpha(\lambda))\frac{dX^\mu}{d\lambda}\frac{dX^\nu}{d\lambda}\\&+2(x^\mu_B-x^\mu_A)h_{\mu\nu}(x^\alpha(\lambda))\frac{dX^\nu}{d\lambda}\Bigg]d\lambda,\end{aligned} \tag{13}$$

where the integrals are taken over Γ and $\Omega^{(0)}(x_A,x_B)$ is the world function in Minkowski space-time

$$\Omega^{(0)}(x_A,x_B)=\frac{1}{2}\eta_{\mu\nu}(x^\mu_B-x^\mu_A)(x^\nu_B-x^\nu_A). \tag{14}$$

Henceforth, we shall only consider weak gravitational fields generated by self-gravitating extended bodies within the slow-motion, post-Newtonian approximation. So, we assume that the potentials $h_{\mu\nu}$ may be expanded as follows:

$$\begin{aligned}h_{00}&=\frac{1}{c^2}h^{(2)}_{00}+\frac{1}{c^4}h^{(4)}_{00}+O(6),\\ h_{0i}&=\frac{1}{c^3}h^{(3)}_{0i}+O(5), \quad h_{ij}=\frac{1}{c^2}h^{(2)}_{ij}+O(4).\end{aligned} \tag{15}$$

From these expansions and from the Euler-Lagrange equations satisfied by any geodesic Γ, namely,

$$\frac{d}{d\lambda}\left(g_{\alpha\beta}\frac{dx^\beta}{d\lambda}\right)=\frac{1}{2}\partial_\alpha h_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}, \tag{16}$$

it results that $X^\mu(\lambda)=O(2)$ and that $dx^\mu/d\lambda=x^\mu_B-x^\mu_A+O(2)$. As a consequence, $h_{\mu\nu}(x^\alpha(\lambda))=h_{\mu\nu}(x^\alpha_{(0)}(\lambda))+O(4)$ and the third and fourth terms in the right-hand side (rhs) of Eq. (13) are of order $1/c^4$. These features result in an expression for $\Omega(x_A,x_B)$ as follows:

$$\Omega(x_A,x_B)=\Omega^{(0)}(x_A,x_B)+\Omega^{(PN)}(x_A,x_B)+O(4), \tag{17}$$

where

$$\begin{aligned}\Omega^{(PN)}(x_A,x_B)=&\,\frac{1}{2c^2}(x^0_B-x^0_A)^2\int_0^1 h^{(2)}_{00}(x^\alpha_{(0)}(\lambda))d\lambda\\&+\frac{1}{2c^2}(x^i_B-x^i_A)(x^j_B-x^j_A)\\&\times\int_0^1 h^{(2)}_{ij}(x^\alpha_{(0)}(\lambda))d\lambda+\frac{1}{c^3}(x^0_B-x^0_A)\\&\times(x^i_B-x^i_A)\int_0^1 h^{(3)}_{0i}(x^\alpha_{(0)}(\lambda))d\lambda,\end{aligned} \tag{18}$$

the integrals being taken over the line $\Gamma_{(0)}$ defined by Eq. (9).

The formulas (17) and (18) yield the general expression of the world function up to the order $1/c^3$ within the framework of the 1 PN approximation. We shall see in Sec. III C that this approximation is sufficient to determine the time transfer function $\mathcal{T}(t_A,\mathbf{x}_A,\mathbf{x}_B)$ up to the order $1/c^4$. It is worthy of note that the method used above would as well lead to the expression of the world function in the linearized weak-field limit previously found by Synge [14].

We shall put henceforth $\boldsymbol{R}_{AB}=\boldsymbol{x}_B-\boldsymbol{x}_A$ and $R_{AB}=|\boldsymbol{R}_{AB}|$. Defining the quantities $N^{\mu}=(x_B^{\mu}-x_A^{\mu})/R_{AB}$, Eqs. (14) and (18) might be easily rewritten with these notations.

C. Time transfer at the order $1/c^4$

Suppose that x_B is the point of reception of a photon emitted at x_A. Taking Eq. (17) into account, Eq. (2) may be written in the form

$$\Omega^{(0)}(x_A,x_B)+\Omega^{(PN)}(x_A,x_B)=O(4),$$

which implies the relation

$$t_B^{+}-t_A=\frac{1}{c}R_{AB}-\frac{\Omega^{(PN)}(ct_A,\boldsymbol{x}_A,ct_B^{+},\boldsymbol{x}_B)}{cR_{AB}}+O(4). \tag{19}$$

Using iteratively this relation, we find for the time transfer function

$$\mathcal{T}(t_A,\boldsymbol{x}_A,\boldsymbol{x}_B)=\frac{1}{c}R_{AB}-\frac{\Omega^{(PN)}(ct_A,\boldsymbol{x}_A,ct_A+R_{AB},\boldsymbol{x}_B)}{cR_{AB}}+O(5). \tag{20}$$

This formula shows that the time transfer $\mathcal{T}(t_A,\boldsymbol{x}_A,\boldsymbol{x}_B)$ can be explicitly calculated up to the order $1/c^4$ when $\Omega^{(PN)}(x_A,x_B)$ is known. This fundamental result will be exploited in the following sections.

The quantity $\Omega^{(PN)}(ct_A,\boldsymbol{x}_A,ct_A+R_{AB},\boldsymbol{x}_B)$ in Eq. (20) may be written in an integral form using Eq. (18), in which R_{AB} and $R_{AB}\lambda+ct_A$ are substituted for $x_B^0-x_A^0$ and for $x_{(0)}^0(\lambda)$, respectively. Hence

$$\mathcal{T}(t_A,\boldsymbol{x}_A,\boldsymbol{x}_B)=\frac{1}{c}R_{AB}\left\{1-\frac{1}{2c^2}\int_0^1\left[h_{00}^{(2)}(z^{\alpha}(\lambda))+h_{ij}^{(2)}(z^{\alpha}(\lambda))N^iN^j+\frac{2}{c}h_{0i}^{(3)}(z^{\alpha}(\lambda))N^i\right]d\lambda\right\}+O(5), \tag{21}$$

the integrals being taken over the line defined by the parametric equations $x^{\alpha}=z^{\alpha}(\lambda)$, where

$$z^0(\lambda)=R_{AB}\lambda+ct_A,\quad z^i(\lambda)=R_{AB}N^i\lambda+x_A^i,\quad 0\leqslant\lambda\leqslant 1. \tag{22}$$

It must be noted that the line defined by Eq. (22) is the null geodesic of a Minkowski metric from x_A, the direction cosines of which are $N^i=(x_B^i-x_A^i)/R_{AB}$.

III. WORLD FUNCTION AND TIME TRANSFER WITHIN THE NORDTVEDT-WILL PPN FORMALISM

A. Metric in the 1 PN approximation

In this section we use the Nordtvedt-Will post-Newtonian formalism involving ten parameters $\beta,\ \gamma,\ \xi,\ \alpha_1,\ \ldots,\zeta_4$ [9]. We introduce slightly modified notations in order to be closed of the formalism recently proposed by Klioner and Soffel [16] as an extension of the post-Newtonian framework elaborated by Damour, Soffel and Xu [17] for general relativity. In particular, we denote by $\boldsymbol{v}_r$ the velocity of the center of mass O relative to the universe rest frame [24].

Although our method is not confined to any particular assumption on the matter, we suppose here that each source of the field is described by the energy-momentum tensor of a perfect fluid

$$T^{\mu\nu}=\rho c^2\left[1+\frac{1}{c^2}\left(\Pi+\frac{p}{\rho}\right)\right]u^{\mu}u^{\nu}-pg^{\mu\nu},$$

where ρ is the rest mass density, Π is the specific energy density (ratio of internal energy density to rest mass density), p is the pressure and u^{μ} is the unit 4-velocity of the fluid. In this section and in the following one, $\boldsymbol{v}$ is the coordinate velocity $d\boldsymbol{x}/dt$ of an element of the fluid. We introduce the conserved mass density ρ^* given by

$$\rho^*=\rho\sqrt{-g}u^0=\rho\left[1+\frac{1}{c^2}\left(\frac{1}{2}v^2+3\gamma U\right)+O(4)\right], \tag{23}$$

where $g=\det(g_{\mu\nu})$ and U is the Newtonian-like potential

$$U(x^0,\boldsymbol{x})=G\int\frac{\rho^*(x^0,\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}d^3\boldsymbol{x}'. \tag{24}$$

In order to obtain a more simple form than the usual one for the potentials h_{0i}, we suppose that the chosen (x^{μ}) are related to a standard post-Newtonian gauge $(\overline{x^{\mu}})$ by the transformation

$$x^0=\overline{x^0}+\frac{1}{c^3}[(1+2\xi+\alpha_2-\zeta_1)\partial_t\chi-2\alpha_2\boldsymbol{v}_r\cdot\boldsymbol{\nabla}\chi],$$

$$x^i=\overline{x^i}, \tag{25}$$

where χ is the superpotential defined by

$$\chi(x^0,\boldsymbol{x})=\frac{1}{2}G\int\rho^*(x^0,\boldsymbol{x}')|\boldsymbol{x}-\boldsymbol{x}'|d^3\boldsymbol{x}'. \tag{26}$$

Moreover, we define $\hat{\rho}$ by

$$\hat{\rho}=\rho^*\bigg[1+\frac{1}{2}(2\gamma+1-2\xi+\alpha_3+\zeta_1)\frac{v^2}{c^2} +(1-2\beta+\xi+\zeta_2)\frac{U}{c^2}+(1+\zeta_3)\frac{\Pi}{c^2} +(3\gamma-2\xi+3\zeta_4)\frac{p}{\rho^* c^2}-\frac{1}{2}(\alpha_1-\alpha_3)\frac{v_r^2}{c^2} -\frac{1}{2}(\alpha_1-2\alpha_3)\frac{\boldsymbol{v}_r\cdot\boldsymbol{v}}{c^2}+O(4)\bigg]. \tag{27}$$

Then, the post-Newtonian potentials read

$$h_{00}=-\frac{2}{c^2}w+\frac{2\beta}{c^4}w^2+\frac{2\xi}{c^4}\phi_W+\frac{1}{c^4}(\zeta_1-2\xi)\phi_v -\frac{2\alpha_2}{c^4}v_r^i v_r^j\partial_{ij}\chi+O(6), \tag{28}$$

$$\boldsymbol{h}\equiv\{h_{0i}\}=\frac{2}{c^3}\left[\left(\gamma+1+\frac{1}{4}\alpha_1\right)\boldsymbol{w}+\frac{1}{4}\alpha_1 w\boldsymbol{v}_r\right]+O(5), \tag{29}$$

$$h_{ij}=-\frac{2\gamma}{c^2}w\,\delta_{ij}+O(4), \tag{30}$$

where

$$w(x^0,\boldsymbol{x})=G\int\frac{\hat{\rho}(x^0,\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}d^3\boldsymbol{x}' +\frac{1}{c^2}[(1+2\xi+\alpha_2-\zeta_1)\partial_{tt}\chi-2\alpha_2\boldsymbol{v}_r\cdot\boldsymbol{\nabla}(\partial_t\chi)], \tag{31}$$

$$\phi_W(x^0,\boldsymbol{x})=G^2\int\frac{\rho^*(x^0,\boldsymbol{x}')\rho^*(x^0,\boldsymbol{x}'')(\boldsymbol{x}-\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|^3} \times\left(\frac{\boldsymbol{x}'-\boldsymbol{x}''}{|\boldsymbol{x}-\boldsymbol{x}''|}-\frac{\boldsymbol{x}-\boldsymbol{x}''}{|\boldsymbol{x}'-\boldsymbol{x}''|}\right)d^3\boldsymbol{x}'\,d^3\boldsymbol{x}'', \tag{32}$$

$$\phi_v(x^0,\boldsymbol{x})=G\int\frac{\rho^*(x^0,\boldsymbol{x}')[\boldsymbol{v}(x^0,\boldsymbol{x}')\cdot(\boldsymbol{x}-\boldsymbol{x}')]^2}{|\boldsymbol{x}-\boldsymbol{x}'|^3}d^3\boldsymbol{x}', \tag{33}$$

$$\boldsymbol{w}(x^0,\boldsymbol{x})=G\int\frac{\rho^*(x^0,\boldsymbol{x}')\boldsymbol{v}(x^0,\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}d^3\boldsymbol{x}'. \tag{34}$$

B. Determination of the world function and of the time transfer

For the post-Newtonian metric given by Eqs. (28)–(34), it follows from Eq. (18) that $\Omega(x_A,x_B)$ may be written up to the order $1/c^3$ in the form given by Eq. (17) with

$$\Omega^{(PN)}(x_A,x_B)=\Omega_w^{(PN)}(x_A,x_B)+\Omega_{\boldsymbol{w}}^{(PN)}(x_A,x_B) +\Omega_{\boldsymbol{v}_r}^{(PN)}(x_A,x_B), \tag{35}$$

where

$$\Omega_w^{(PN)}(x_A,x_B)=-\frac{1}{c^2}[(x_B^0-x_A^0)^2+\gamma R_{AB}^2] \times\int_0^1 w(x_{(0)}^\alpha(\lambda))d\lambda, \tag{36}$$

$$\Omega_{\boldsymbol{w}}^{(PN)}(x_A,x_B)=\frac{2}{c^3}\left(\gamma+1+\frac{1}{4}\alpha_1\right)(x_B^0-x_A^0) \times\boldsymbol{R}_{AB}\cdot\int_0^1\boldsymbol{w}(x_{(0)}^\alpha(\lambda))d\lambda, \tag{37}$$

$$\Omega_{\boldsymbol{v}_r}^{(PN)}(x_A,x_B)=\frac{1}{2c^3}\alpha_1(x_B^0-x_A^0)(\boldsymbol{R}_{AB}\cdot\boldsymbol{v}_r) \times\int_0^1 w(x_{(0)}^\alpha(\lambda))d\lambda, \tag{38}$$

the integrals being calculated along the line defined by Eq. (9).

The corresponding time transfer function is easily obtained by using Eq. (20) or Eq. (21). We get

$$\mathcal{T}(t_A,\boldsymbol{x}_A,\boldsymbol{x}_B)=\frac{1}{c}R_{AB}+\frac{1}{c^3}(\gamma+1)R_{AB}\int_0^1 w(z^\alpha(\lambda))d\lambda -\frac{2}{c^4}\boldsymbol{R}_{AB}\cdot\left[\left(\gamma+1+\frac{1}{4}\alpha_1\right)\int_0^1\boldsymbol{w}(z^\alpha(\lambda))d\lambda +\frac{1}{4}\alpha_1\boldsymbol{v}_r\int_0^1 w(z^\alpha(\lambda))d\lambda\right]+O(5), \tag{39}$$

the integrals being evaluated along the curve defined by Eq. (22).

Let us emphasize that, since $w=U+O(2)$, w may be replaced by the Newtonian-like potential U in expressions (36)–(39).

C. Case of stationary sources

In what follows, we suppose that the gravitational field is generated by a unique stationary source. Then, $\partial_t\chi=0$ and the potentials w and $\boldsymbol{w}$ do not depend on time. In this case, the integration involved in Eqs. (36)–(38) can be performed by a method due to Buchdahl [15]. Introducing the auxiliary variables $\boldsymbol{y}_A=\boldsymbol{x}_A-\boldsymbol{x}'$ and $\boldsymbol{y}_B=\boldsymbol{x}_B-\boldsymbol{x}'$, and replacing in Eq. (9) the parameter λ by $u=\lambda-1/2$, a straightforward calculation yields

$$\int_0^1 w(\boldsymbol{x}_{(0)}(\lambda))d\lambda = G\int \hat{\rho}(\boldsymbol{x}')F(\boldsymbol{x}',\boldsymbol{x}_A,\boldsymbol{x}_B)d^3\boldsymbol{x}', \tag{40}$$

$$\int_0^1 \boldsymbol{w}(\boldsymbol{x}_{(0)}(\lambda))d\lambda = G\int \rho^*(\boldsymbol{x}')\boldsymbol{v}(\boldsymbol{x}')F(\boldsymbol{x}',\boldsymbol{x}_A,\boldsymbol{x}_B)d^3\boldsymbol{x}', \tag{41}$$

where the kernel function $F(\boldsymbol{x}',\boldsymbol{x}_A,\boldsymbol{x}_B)$ has the expression

$$F(\boldsymbol{x}',\boldsymbol{x}_A,\boldsymbol{x}_B)=\int_{-1/2}^{1/2}\frac{du}{\left|(\boldsymbol{y}_B-\boldsymbol{y}_A)u+\frac{1}{2}(\boldsymbol{y}_B+\boldsymbol{y}_A)\right|}.$$

Noting that $\boldsymbol{y}_B-\boldsymbol{y}_A=\boldsymbol{R}_{AB}$, which implies that $|\boldsymbol{y}_B-\boldsymbol{y}_A| = R_{AB}$, we find

$$F(\boldsymbol{x},\boldsymbol{x}_A,\boldsymbol{x}_B)=\frac{1}{R_{AB}}\ln\left(\frac{|\boldsymbol{x}-\boldsymbol{x}_A|+|\boldsymbol{x}-\boldsymbol{x}_B|+R_{AB}}{|\boldsymbol{x}-\boldsymbol{x}_A|+|\boldsymbol{x}-\boldsymbol{x}_B|-R_{AB}}\right). \tag{42}$$

Inserting Eqs. (40), (41) and (42) in Eqs. (36)-(38) and in Eq. (39) will enable one to obtain quite elegant expressions for $\Omega^{(PN)}(x_A,x_B)$ and for $\mathcal{T}(\boldsymbol{x}_A,\boldsymbol{x}_B)$, respectively.

IV. ISOLATED, AXISYMMETRIC ROTATING BODY

Henceforth, we suppose that the light is propagating in the gravitational field of an isolated, axisymmetric rotating body. The gravitational field is assumed to be stationary. The main purpose of this section is to determine the influence of the mass and spin multipole moments of the rotating body on the coordinate time transfer and on the direction of light rays. From these results, it will be possible to obtain a relativistic modelling of the one-way time transfers and frequency shifts up to the order $1/c^4$ in a geocentric nonrotating frame.

Since we treat the case of a body located very far from the other bodies of the universe, the global coordinate system (x^μ) used until now can be considered as a local (i.e. geocentric) one. So, in agreement with the UAI/UGG Resolution B1 (2000) [18], we shall henceforth denote by W and $\boldsymbol{W}$, the quantities w and $\boldsymbol{w}$, respectively-defined by Eqs. (31) and (34), and we shall denote by $G_{\mu\nu}$ the components of the metric. However, we shall continue here with using lower case letters for the geocentric coordinates in order to avoid too heavy notations.

The center of mass O of the rotating body being taken as the origin of the quasi-Cartesian coordinates $(\boldsymbol{x})$, we choose the axis of symmetry as the x^3 axis. We assume that the body is slowly rotating about Ox^3 with a constant angular velocity $\boldsymbol{\omega}$, so that

$$\boldsymbol{v}(\boldsymbol{x})=\boldsymbol{\omega}\times\boldsymbol{x}. \tag{43}$$

In what follows, we put $r=|\boldsymbol{x}|$, $r_A=|\boldsymbol{x}_A|$ and $r_B=|\boldsymbol{x}_A|$. We call θ the angle between $\boldsymbol{x}$ and Ox^3. We consider only the case where all points of the segment joining $\boldsymbol{x}_A$ and $\boldsymbol{x}_B$ are outside the body. We denote by r_e the radius of the smallest sphere centered on O and containing the body (for celestial bodies, r_e is the equatorial radius). In this section, we assume the convergence of the multipole expansions formally derived below at any point outside the body, even if $r<r_e$.

A. Multipole developments of W and $\boldsymbol{W}$

According to Eqs. (31), (34) and (43), the gravitational potentials W and $\boldsymbol{W}$ obey the equations

$$\nabla^2 W=-4\pi G\hat{\rho}, \quad \nabla^2\boldsymbol{W}=-4\pi G\rho^*\boldsymbol{\omega}\times\boldsymbol{x}. \tag{44}$$

It follows from Eq. (44) that the potential W is a harmonic function outside the rotating body. As a consequence, W may be expanded in a multipole series of the form

$$W(\boldsymbol{x})=\frac{GM}{r}\left[1-\sum_{n=2}^{\infty}J_n\left(\frac{r_e}{r}\right)^n P_n(\cos\theta)\right]. \tag{45}$$

In this development, the P_n are the Legendre polynomials and the quantities M, $J_2,\dots,J_n,\dots$ correspond to the generalized Blanchet-Damour mass multipole moments in general relativity [19].

In fact, taking into account the identity

$$\frac{\partial^n}{\partial z^n}\left(\frac{1}{r}\right)=\frac{(-1)^n n!}{r^{1+n}}P_n(z/r), \quad z=x^3,$$

it will be much more convenient for the computation of integral (40) to use the following expansion in a series of derivatives of $1/r$:

$$W(\boldsymbol{x})=GM\left[\frac{1}{r}-\sum_{n=2}^{\infty}\frac{(-1)^n}{n!}J_n r_e^n\frac{\partial^n}{\partial z^n}\left(\frac{1}{r}\right)\right]. \tag{46}$$

According to Eq. (46), the mass density $\hat{\rho}$ can be developed in the multipole series

$$\hat{\rho}(\boldsymbol{x})=M\left[\delta^{(3)}(\boldsymbol{x})-\sum_{n=2}^{\infty}\frac{(-1)^n}{n!}J_n r_e^n\frac{\partial^n}{\partial z^n}\delta^{(3)}(\boldsymbol{x})\right], \tag{47}$$

$\delta^{(3)}(\boldsymbol{x})$ being the Dirac distribution supported by the origin O.

Now, substituting Eq. (43) into Eq. (34) yields for the vector potential $\boldsymbol{W}$

$$\boldsymbol{W}(\boldsymbol{x})=G\int\frac{\rho^*(\boldsymbol{x}')\boldsymbol{\omega}\times\boldsymbol{x}'}{|\boldsymbol{x}-\boldsymbol{x}'|}d^3\boldsymbol{x}'. \tag{48}$$

It is possible to show that this vector may be written as

$$\boldsymbol{W}=-\frac{1}{2}\boldsymbol{\omega}\times\nabla\mathcal{V}, \tag{49}$$

where $\mathcal{V}$ is an axisymmetric function satisfying the Laplace equation $\nabla^2\mathcal{V}=0$ outside the body. Consequently, we can expand $\mathcal{V}$ in a series of the form

$$\mathcal{V}(\boldsymbol{x})=\frac{GI}{r}\left[1-\sum_{n=1}^{\infty}K_n\left(\frac{r_e}{r}\right)^n P_n(\cos\theta)\right], \tag{50}$$

where I and the K_n are constants. Inserting Eq. (50) into Eq. (49) and using the identity

$$(n+1)P_n(z/r)+(z/r)P_n'(z/r)=P_{n+1}'(z/r),$$

we find for $\boldsymbol{W}$ an expansion as follows:

$$\boldsymbol{W}(\boldsymbol{x})=\frac{GI\boldsymbol{\omega}\times\boldsymbol{x}}{2r^3}\left[1-\sum_{n=1}^{\infty}K_n\left(\frac{r_e}{r}\right)^n P_{n+1}'(\cos\theta)\right], \tag{51}$$

which coincides with a result previously obtained by one of us [20]. The coincidence shows that I is the moment of inertia of the body about the z axis. Thus, the quantity $\boldsymbol{S}=I\boldsymbol{\omega}$ is the intrinsic angular momentum of the rotating body. The coefficients K_n are completely determined by the density distribution ρ^* and by the shape of the body [20,21]. Expansion (51) may also be written as

$$\boldsymbol{W}(\boldsymbol{x})=-\frac{1}{2}G\boldsymbol{S}\times\boldsymbol{\nabla}\left[\frac{1}{r}-\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}K_n r_e^n\frac{\partial^n}{\partial z^n}\left(\frac{1}{r}\right)\right]. \tag{52}$$

Consequently, the density of mass current can be developed in the multipole series

$$\rho^*(\boldsymbol{x})(\boldsymbol{\omega}\times\boldsymbol{x}) =-\frac{1}{2}\boldsymbol{S}\times\boldsymbol{\nabla}\left[\delta^{(3)}(\boldsymbol{x})-\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}K_n r_e^n\frac{\partial^n}{\partial z^n}\delta^{(3)}(\boldsymbol{x})\right], \tag{53}$$

a property which will be exploited in the following section.

B. Multipole structure of the world function

The function $\Omega^{(PN)}(x_A,x_B)$ is determined by Eqs. (35)–(38) where w and $\boldsymbol{w}$ are, respectively, replaced by W and $\boldsymbol{W}$. The integrals involved in the rhs of Eqs. (35)–(38) are given by Eqs. (40) and (41). Substituting Eq. (47) into Eq. (40) and using the properties of the Dirac distribution, we obtain

$$\int_0^1 W(\boldsymbol{x}_{(0)}(\lambda))d\lambda=GM\left[1-\sum_{n=2}^{\infty}\frac{1}{n!}J_n r_e^n\frac{\partial^n}{\partial z^n}\right]\times F(\boldsymbol{x},\boldsymbol{x}_A,\boldsymbol{x}_B)|_{\boldsymbol{x}=0}. \tag{54}$$

Similarly, substituting Eq. (53) into Eq. (41), we get

$$\int_0^1 \boldsymbol{W}(\boldsymbol{x}_{(0)}(\lambda))d\lambda=-\frac{1}{2}G\boldsymbol{S}\times\boldsymbol{\nabla}\left[1-\sum_{n=1}^{\infty}\frac{1}{n!}K_n r_e^n\frac{\partial^n}{\partial z^n}\right]\times F(\boldsymbol{x},\boldsymbol{x}_A,\boldsymbol{x}_B)|_{\boldsymbol{x}=0}. \tag{55}$$

These formulas show that the multipole expansion of $\Omega^{(PN)}(x_A,x_B)$ can be thoroughly calculated by straightforward differentiations of the kernel function $F(\boldsymbol{x},\boldsymbol{x}_A,\boldsymbol{x}_B)$ given by Eq. (42). They constitute the essential result of the present paper, from which it would be possible to deduce the multipole expansions giving the time transfer and the frequency shift between $\boldsymbol{x}_A$ and $\boldsymbol{x}_B$ up to the order $1/c^4$.

In order to obtain explicit formulas, we shall only retain the contributions due to M, J_2 and $\boldsymbol{S}$ in the expansion yielding $\Omega_W^{(PN)}$ and $\Omega_{\boldsymbol{W}}^{(PN)}$. Then, denoting the unit vector along the z axis by $\boldsymbol{k}$ and noting that $\boldsymbol{S}=S\boldsymbol{k}$, we get for $\Omega_W^{(PN)}(x_A,x_B)$

$$\Omega_W^{(PN)}(x_A,x_B)=-\frac{GM}{c^2}\frac{(x_B^0-x_A^0)^2+\gamma R_{AB}^2}{R_{AB}}\ln\left(\frac{r_A+r_B+R_{AB}}{r_A+r_B-R_{AB}}\right)+\frac{2GM}{c^2}J_2 r_e^2\frac{(x_B^0-x_A^0)^2+\gamma R_{AB}^2}{[(r_A+r_B)^2-R_{AB}^2]^2}(r_A+r_B)\left(\frac{\boldsymbol{k}\cdot\boldsymbol{x}_A}{r_A}+\frac{\boldsymbol{k}\cdot\boldsymbol{x}_B}{r_B}\right)^2 -\frac{GM}{c^2}J_2 r_e^2\frac{(x_B^0-x_A^0)^2+\gamma R_{AB}^2}{(r_A+r_B)^2-R_{AB}^2}\left[\frac{(\boldsymbol{k}\times\boldsymbol{x}_A)^2}{r_A^3}+\frac{(\boldsymbol{k}\times\boldsymbol{x}_B)^2}{r_B^3}\right]+\cdots \tag{56}$$

and for $\Omega_{\boldsymbol{W}}^{(PN)}(x_A,x_B)$

$$\Omega_{\boldsymbol{W}}^{(PN)}(x_A,x_B)=\left(\gamma+1+\frac{1}{4}\alpha_1\right)\frac{2GS}{c^3}(x_B^0-x_A^0)\times\frac{r_A+r_B}{r_Ar_B}\frac{\boldsymbol{k}\cdot(\boldsymbol{x}_A\times\boldsymbol{x}_B)}{(r_A+r_B)^2-R_{AB}^2}+\cdots. \tag{57}$$

Finally, owing to the limit $|\alpha_1|<0.02$ furnished in [9], we shall henceforth neglect all the multipole contributions in $\Omega_{\boldsymbol{v}_r}^{(PN)}(x_A,x_B)$. Thus, we get

$$\Omega_{\boldsymbol{v}_r}^{(PN)}(x_A,x_B)=\alpha_1\frac{GM}{2c^3}(x_B^0-x_A^0)\frac{\boldsymbol{R}_{AB}\cdot\boldsymbol{v}_r}{R_{AB}}\times\ln\left(\frac{r_A+r_B+R_{AB}}{r_A+r_B-R_{AB}}\right)+\cdots. \tag{58}$$

In this section and in the following one, the symbol $+\cdots$ stands for the contributions of higher multipole moments which are neglected. For the sake of brevity, when $+\cdots$ is used, we systematically omit to mention the symbol $O(n)$ which stands for the neglected post-Newtonian terms.

C. Time transfer function up to the order $1/c^4$

Let us substitute R_{AB} for $x_B^0 - x_A^0$ into Eqs. (56)–(58) and insert the corresponding expression of $\Omega^{(PN)}$ into Eq. (20). We get an expression for the time transfer function as follows:

$$\mathcal{T}(\boldsymbol{x}_A,\boldsymbol{x}_B)=\frac{1}{c}R_{AB}+\mathcal{T}_M(\boldsymbol{x}_A,\boldsymbol{x}_B)+\mathcal{T}_{J_2}(\boldsymbol{x}_A,\boldsymbol{x}_B)+\mathcal{T}_{\boldsymbol{S}}(\boldsymbol{x}_A,\boldsymbol{x}_B)+\mathcal{T}_{\boldsymbol{v}_r}(\boldsymbol{x}_A,\boldsymbol{x}_B)+\cdots, \tag{59}$$

where

$$\mathcal{T}_M(\boldsymbol{x}_A,\boldsymbol{x}_B)=(\gamma+1)\frac{GM}{c^3}\ln\left(\frac{r_A+r_B+R_{AB}}{r_A+r_B-R_{AB}}\right), \tag{60}$$

$$\mathcal{T}_{J_2}(\boldsymbol{x}_A,\boldsymbol{x}_B)=-(\gamma+1)\frac{GM}{c^3}\frac{J_2r_e^2R_{AB}}{(r_A+r_B)^2-R_{AB}^2}\times\left[\frac{2(r_A+r_B)}{(r_A+r_B)^2-R_{AB}^2}\left(\frac{\boldsymbol{k}\cdot\boldsymbol{x}_A}{r_A}+\frac{\boldsymbol{k}\cdot\boldsymbol{x}_B}{r_B}\right)^2-\frac{(\boldsymbol{k}\times\boldsymbol{x}_A)^2}{r_A^3}-\frac{(\boldsymbol{k}\times\boldsymbol{x}_B)^2}{r_B^3}\right], \tag{61}$$

$$\mathcal{T}_{\boldsymbol{S}}(\boldsymbol{x}_A,\boldsymbol{x}_B)=-\left(\gamma+1+\frac{1}{4}\alpha_1\right)\frac{2GS}{c^4}\frac{r_A+r_B}{r_Ar_B}\times\frac{\boldsymbol{k}\cdot(\boldsymbol{x}_A\times\boldsymbol{x}_B)}{(r_A+r_B)^2-R_{AB}^2}, \tag{62}$$

$$\mathcal{T}_{\boldsymbol{v}_r}(\boldsymbol{x}_A,\boldsymbol{x}_B)=-\alpha_1\frac{GM}{2c^4}\frac{\boldsymbol{R}_{AB}\cdot\boldsymbol{v}_r}{R_{AB}}\ln\left(\frac{r_A+r_B+R_{AB}}{r_A+r_B-R_{AB}}\right). \tag{63}$$

The time transfer is thus explicitly determined up to the order $1/c^4$. The term of order $1/c^3$ given by Eq. (60) is the well-known Shapiro time delay. Equations (61) and (62) extend results previously found for $\gamma=1$ and $\alpha_1=0$ [5]. However, our derivation is more straightforward and yields formulas which are more convenient to calculate the frequency shifts. As a final remark, it is worthy of note that $\mathcal{T}_M$ and $\mathcal{T}_{J_2}$ are symmetric in $(\boldsymbol{x}_A,\boldsymbol{x}_B)$, while $\mathcal{T}_{\boldsymbol{S}}$ and $\mathcal{T}_{\boldsymbol{v}_r}$ are antisymmetric in $(\boldsymbol{x}_A,\boldsymbol{x}_B)$.

D. Directions of light rays at x_A and x_B up to the order $1/c^3$

In order to determine the vectors tangent to the ray path at x_A and x_B, we use Eqs. (6) and (7) where $\mathcal{T}$ is replaced by the expression given by Eq. (59). For the sake of brevity, we put henceforth $\boldsymbol{l}_A=\{(l_i)_A\}$ and $\boldsymbol{l}_B=\{(l_i)_B\}$. We find

$$\boldsymbol{l}_A(\boldsymbol{x}_A,\boldsymbol{x}_B)=-\boldsymbol{N}_{AB}+\boldsymbol{l}_M(\boldsymbol{x}_A,\boldsymbol{x}_B)+\boldsymbol{l}_{J_2}(\boldsymbol{x}_A,\boldsymbol{x}_B)+\boldsymbol{l}_{\boldsymbol{S}}(\boldsymbol{x}_A,\boldsymbol{x}_B)+\boldsymbol{l}_{\boldsymbol{v}_r}(\boldsymbol{x}_A,\boldsymbol{x}_B)+\cdots, \tag{64}$$

$$\boldsymbol{l}_B(\boldsymbol{x}_A,\boldsymbol{x}_B)=-\boldsymbol{N}_{AB}-\boldsymbol{l}_M(\boldsymbol{x}_B,\boldsymbol{x}_A)-\boldsymbol{l}_{J_2}(\boldsymbol{x}_B,\boldsymbol{x}_A)+\boldsymbol{l}_{\boldsymbol{S}}(\boldsymbol{x}_B,\boldsymbol{x}_A)+\boldsymbol{l}_{\boldsymbol{v}_r}(\boldsymbol{x}_B,\boldsymbol{x}_A)+\cdots, \tag{65}$$

where $\boldsymbol{l}_M$, $\boldsymbol{l}_{J_2}$, $\boldsymbol{l}_{\boldsymbol{S}}$ and $\boldsymbol{l}_{\boldsymbol{v}_r}$ stand for the contributions of $\mathcal{T}_M$, $\mathcal{T}_{J_2}$, $\mathcal{T}_{\boldsymbol{S}}$ and $\mathcal{T}_{\boldsymbol{v}_r}$, respectively. Putting

$$\boldsymbol{n}_A=\frac{\boldsymbol{x}_A}{r_A},\quad \boldsymbol{n}_B=\frac{\boldsymbol{x}_B}{r_B},\quad \boldsymbol{N}_{AB}=\frac{\boldsymbol{x}_B-\boldsymbol{x}_A}{R_{AB}},$$

we get from Eq. (60)

$$\boldsymbol{l}_M(\boldsymbol{x}_A,\boldsymbol{x}_B)=-(\gamma+1)\frac{2GM}{c^2}\frac{(r_A+r_B)\boldsymbol{N}_{AB}+R_{AB}\boldsymbol{n}_A}{(r_A+r_B)^2-R_{AB}^2}. \tag{66}$$

From Eq. (61), we get

$$\begin{aligned}\boldsymbol{l}_{J_2}(\boldsymbol{x}_A,\boldsymbol{x}_B)=&(\gamma+1)\frac{GMJ_2r_e^2}{c^2}\frac{r_A+r_B}{[(r_A+r_B)^2-R_{AB}^2]^2}\Bigg\{\boldsymbol{N}_{AB}\Bigg[2(\boldsymbol{k}\cdot\boldsymbol{n}_A+\boldsymbol{k}\cdot\boldsymbol{n}_B)^2\frac{(r_A+r_B)^2+3R_{AB}^2}{(r_A+r_B)^2-R_{AB}^2}-\left(\frac{1-(\boldsymbol{k}\cdot\boldsymbol{n}_A)^2}{r_A}\right.\\&\left.+\frac{1-(\boldsymbol{k}\cdot\boldsymbol{n}_B)^2}{r_B}\right)\frac{(r_A+r_B)^2+R_{AB}^2}{r_A+r_B}\Bigg]+2\boldsymbol{n}_A\frac{R_{AB}}{r_A+r_B}\Bigg[(\boldsymbol{k}\cdot\boldsymbol{n}_A+\boldsymbol{k}\cdot\boldsymbol{n}_B)^2\frac{3(r_A+r_B)^2+R_{AB}^2}{(r_A+r_B)^2-R_{AB}^2}-\frac{1}{2}[1-3(\boldsymbol{k}\cdot\boldsymbol{n}_A)^2]\\&\times\frac{(3r_A+r_B)(r_A+r_B)-R_{AB}^2}{r_A^2}+(r_A+r_B)\left(\frac{2(\boldsymbol{k}\cdot\boldsymbol{n}_A)(\boldsymbol{k}\cdot\boldsymbol{n}_B)}{r_A}-\frac{1-(\boldsymbol{k}\cdot\boldsymbol{n}_B)^2}{r_B}\right)\Bigg]\\&-4\boldsymbol{k}\frac{R_{AB}}{r_A}\left[(\boldsymbol{k}\cdot\boldsymbol{n}_A)\frac{(3r_A+r_B)(r_A+r_B)-R_{AB}^2}{2r_A(r_A+r_B)}+(\boldsymbol{k}\cdot\boldsymbol{n}_B)\right]\Bigg\}.\end{aligned} \tag{67}$$

From Eqs. (62) and (63), we derive the other contributions which are of order $1/c^3$

$$\boldsymbol{l}_{\boldsymbol{S}}(\boldsymbol{x}_A,\boldsymbol{x}_B)=\left(\gamma+1+\frac{1}{4}\alpha_1\right)\frac{2GS}{c^3}\frac{r_A+r_B}{r_A[(r_A+r_B)^2-R_{AB}^2]}\times\left\{\boldsymbol{k}\times\boldsymbol{n}_B+\frac{2r_Ar_B\boldsymbol{k}\cdot(\boldsymbol{n}_A\times\boldsymbol{n}_B)}{(r_A+r_B)^2-R_{AB}^2}\left[\frac{(3r_A+r_B)(r_A+r_B)-R_{AB}^2}{2r_A(r_A+r_B)}\boldsymbol{n}_A+\boldsymbol{n}_B\right]\right\}, \tag{68}$$

$$\boldsymbol{l}_{\boldsymbol{v}_r}(\boldsymbol{x}_A,\boldsymbol{x}_B)=\alpha_1\frac{GM}{c^3}\left[\frac{\boldsymbol{v}_r-(\boldsymbol{v}_r\cdot\boldsymbol{N}_{AB})\boldsymbol{N}_{AB}}{2R_{AB}}\ln\left(\frac{r_A+r_B+R_{AB}}{r_A+r_B-R_{AB}}\right)+(\boldsymbol{v}_r\cdot\boldsymbol{N}_{AB})\frac{(r_A+r_B)\boldsymbol{N}_{AB}+R_{AB}\boldsymbol{n}_A}{(r_A+r_B)^2-R_{AB}^2}\right]. \tag{69}$$

We note that the mass and the quadrupole moment yield contributions of order $1/c^2$, while the intrinsic angular momentum and the velocity relative to the universe rest frame yield contributions of order $1/c^3$.

E. Sagnac terms in the time transfer function

In an experiment like ACES, recording the time of emission t_A will be more practical than recording the time of reception t_B^+. So, it will be very convenient to form the expression of the time transfer $\mathcal{T}(\boldsymbol{x}_A,\boldsymbol{x}_B)$ from $\boldsymbol{x}_A(t_A)$ to $\boldsymbol{x}_B(t_B^+)$ in terms of the position of the receiver B at the time of emission t_A. For any quantity $Q_B(t)$ defined along the world line of the station B, let us put $\widetilde{Q}_B=Q_B(t_A)$. Thus we may write $\widetilde{\boldsymbol{x}}_B$, $\widetilde{r}_B$, $\widetilde{\boldsymbol{v}}_B$, $\widetilde{v}_B=|\widetilde{\boldsymbol{v}}_B|$, etc.

Now, let us introduce the instantaneous coordinate distance $\boldsymbol{D}_{AB}=\widetilde{\boldsymbol{x}}_B-\boldsymbol{x}_A$ and its norm D_{AB}. Since we want to know $t_B^+-t_A$ up to the order $1/c^4$, we can use the Taylor expansion of $\boldsymbol{R}_{AB}$

$$\boldsymbol{R}_{AB}=\boldsymbol{D}_{AB}+(t_B^+-t_A)\widetilde{\boldsymbol{v}}_B+\frac{1}{2}(t_B^+-t_A)^2\widetilde{\boldsymbol{a}}_B+\frac{1}{6}(t_B^+-t_A)^3\widetilde{\boldsymbol{b}}_B+\cdots,$$

where $\boldsymbol{a}_B$ is the acceleration of B and $\boldsymbol{b}_B=d\boldsymbol{a}_B/dt$. Using iteratively this expansion together with Eq. (59), we get

$$\begin{aligned}\mathcal{T}(\boldsymbol{x}_A,\boldsymbol{x}_B)=&\ \mathcal{T}(\boldsymbol{x}_A,\widetilde{\boldsymbol{x}}_B)+\frac{1}{c^2}\boldsymbol{D}_{AB}\cdot\widetilde{\boldsymbol{v}}_B\\&+\frac{1}{2c^3}D_{AB}\left[\frac{(\boldsymbol{D}_{AB}\cdot\widetilde{\boldsymbol{v}}_B)^2}{D_{AB}^2}+\widetilde{v}_B^2+\boldsymbol{D}_{AB}\cdot\widetilde{\boldsymbol{a}}_B\right]\\&+\frac{1}{c^4}\Bigg[(\boldsymbol{D}_{AB}\cdot\widetilde{\boldsymbol{v}}_B)(\widetilde{v}_B^2+\boldsymbol{D}_{AB}\cdot\widetilde{\boldsymbol{a}}_B)\\&+\frac{1}{2}D_{AB}^2\left(\widetilde{\boldsymbol{v}}_B\cdot\widetilde{\boldsymbol{a}}_B+\frac{1}{3}\boldsymbol{D}_{AB}\cdot\widetilde{\boldsymbol{b}}_B\right)\Bigg]\\&+\frac{1}{c}\frac{\boldsymbol{D}_{AB}}{D_{AB}}\cdot\widetilde{\boldsymbol{v}}_B[\mathcal{T}_M(\boldsymbol{x}_A,\widetilde{\boldsymbol{x}}_B)+\mathcal{T}_{J_2}(\boldsymbol{x}_A,\widetilde{\boldsymbol{x}}_B)]\\&+\frac{1}{c^2}D_{AB}\widetilde{\boldsymbol{v}}_B\cdot[\boldsymbol{l}_M(\widetilde{\boldsymbol{x}}_B,\boldsymbol{x}_A)+\boldsymbol{l}_{J_2}(\widetilde{\boldsymbol{x}}_B,\boldsymbol{x}_A)]\\&+\cdots,\end{aligned} \tag{70}$$

where $\mathcal{T}(\boldsymbol{x}_A,\widetilde{\boldsymbol{x}}_B)$ is obtained by substituting $\widetilde{\boldsymbol{x}}_B$, $\widetilde{r}_B$ and $\boldsymbol{D}_{AB}$, respectively for $\boldsymbol{x}_B$, r_B and $\boldsymbol{R}_{AB}$ into the time transfer function defined by Eqs. (59)–(63). This expression extends the previous formula [7] to the next order $1/c^4$. The second, the third and the fourth terms in Eq. (70) represent pure Sagnac terms of order $1/c^2$, $1/c^3$ and $1/c^4$, respectively. The fifth and the sixth terms are contributions of the gravitational field mixed with the coordinate velocity of the receiving station. Since these last two terms are of order $1/c^4$, they may be calculated for the arguments $(\boldsymbol{x}_B,\boldsymbol{x}_A)$ (note the order of the arguments in $\boldsymbol{l}_M$ and $\boldsymbol{l}_{J_2}$).

V. FREQUENCY SHIFT IN THE FIELD OF AN AXISYMMETRIC ROTATING BODY

A. General formulas

Consider a clock $\mathcal{O}_A$ on A and a clock $\mathcal{O}_B$ on B delivering, respectively, the proper frequencies f_A and f_B and suppose that $\mathcal{O}_A$ is sending photons to $\mathcal{O}_B$. The one-way frequency transfer from $\mathcal{O}_A$ and $\mathcal{O}_B$ is characterized by the ratio f_A/f_B which may be written as $f_A/f_B=(f_A/\nu_A)(\nu_A/\nu_B)(\nu_B/f_B)$ where ν_A is the proper frequency of the photon as measured on A at the instant of emission and ν_B is the proper frequency of the same photon as measured on B at the instant of receipt. The ratios f_A/ν_A and f_B/ν_B are obtained by local measurements performed on A and B, respectively [7]. So, in the present paper, we are concerned only with the theoretical determination of ν_A/ν_B. This ratio is given by the well-known relation

$$\frac{\nu_A}{\nu_B}=\frac{u_A^\mu(l_\mu)_A}{u_B^\mu(l_\mu)_B} \tag{71}$$

where $u_A^\mu=(dx^\mu/ds)_A$ and $u_B^\mu=(dx^\mu/ds)_B$ are, respectively, the unit 4-velocity of the clock $\mathcal{O}_A$ and of the clock $\mathcal{O}_B$, and $(l_\mu)_A$ and $(l_\mu)_B$ are the null tangent vectors at the point of emission x_A and at the point of reception x_B, respectively.

Let us denote by $\boldsymbol{v}_A=(d\boldsymbol{x}/dt)_A$ and $\boldsymbol{v}_B=(d\boldsymbol{x}/dt)_B$ the coordinate velocities of the clocks on A and B, respectively. Since the gravitational field is assumed to be stationary, the formula (71) giving the frequency shift between x_A and x_B may be written as

$$\frac{\nu_A}{\nu_B}=\frac{u_A^0}{u_B^0}\times\frac{q_A}{q_B},\quad q_A=1+\frac{1}{c}\boldsymbol{l}_A\cdot\boldsymbol{v}_A,\quad q_B=1+\frac{1}{c}\boldsymbol{l}_B\cdot\boldsymbol{v}_B, \tag{72}$$

where $\boldsymbol{l}_A$ and $\boldsymbol{l}_B$ are the quantities, respectively, defined by Eqs. (6) and (7).

It is possible to calculate the ratio q_A/q_B up to the order $1/c^4$ from our results in Sec. IV since $\boldsymbol{l}_A$ and $\boldsymbol{l}_B$ are given up to the order $1/c^3$, respectively, by Eqs. (64) and (65). Denoting by $\boldsymbol{l}^{(n)}/c^n$ the $O(n)$ terms in $\boldsymbol{l}$, q_A/q_B may be expanded as

$$\frac{q_A}{q_B}=1-\frac{1}{c}\frac{\boldsymbol{N}_{AB}\cdot(\boldsymbol{v}_A-\boldsymbol{v}_B)}{1-\boldsymbol{N}_{AB}\cdot\dfrac{\boldsymbol{v}_B}{c}}+\frac{1}{c^3}[\boldsymbol{l}_A^{(2)}\cdot\boldsymbol{v}_A-\boldsymbol{l}_B^{(2)}\cdot\boldsymbol{v}_B] +\frac{1}{c^4}[\boldsymbol{l}_A^{(3)}\cdot\boldsymbol{v}_A-\boldsymbol{l}_B^{(3)}\cdot\boldsymbol{v}_B]+\frac{1}{c^4}\boldsymbol{N}_{AB}\cdot[(\boldsymbol{l}_B^{(2)}\cdot\boldsymbol{v}_B)(\boldsymbol{v}_A -2\boldsymbol{v}_B)+(\boldsymbol{l}_A^{(2)}\cdot\boldsymbol{v}_A)\boldsymbol{v}_B]+O(5). \tag{73}$$

In order to be consistent with this expansion, we have to perform the calculation of u_A^0/u_B^0 at the same level of approximation. For a clock delivering a proper time τ, $1/u^0$ is the ratio of the proper time $d\tau$ to the coordinate time dt. To reach the suitable accuracy, it is therefore necessary to take into account the terms of order $1/c^4$ in g_{00}. For the sake of simplicity, we shall henceforth confine ourselves to the fully conservative metric theories of gravity without preferred location effects, in which all the PPN parameters vanish except β and γ. Since the gravitational field is assumed to be stationary, the chosen coordinate system is then a standard post-Newtonian gauge and the metric reduces to its usual form

$$G_{00}=1-\frac{2}{c^2}W+\frac{2\beta}{c^4}W^2+O(6),$$

$$\{G_{0i}\}=\frac{2(\gamma+1)}{c^3}\boldsymbol{W}+O(5),$$

$$G_{ij}=-\left(1+\frac{2\gamma}{c^2}W\right)\delta_{ij}+O(4), \tag{74}$$

where W given by Eq. (31) reduces to

$$W(\boldsymbol{x})=U(\boldsymbol{x})+\frac{G}{c^2}\int\frac{\rho^*(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}\times\left[\left(\gamma+\frac{1}{2}\right)v^2+(1-2\beta)U+\Pi+3\gamma\frac{p}{\rho^*}\right]d^3\boldsymbol{x}', \tag{75}$$

and $\boldsymbol{W}$ is given by Eq. (48). As a consequence, for a clock moving with the coordinate velocity $\boldsymbol{v}$, the quantity $1/u^0$ is given by the formula

$$\frac{1}{u^0}\equiv\frac{d\tau}{dt}=1-\frac{1}{c^2}\left(W+\frac{1}{2}v^2\right)+\frac{1}{c^4}\left[\left(\beta-\frac{1}{2}\right)W^2-\left(\gamma+\frac{1}{2}\right)Wv^2-\frac{1}{8}v^4+2(\gamma+1)\boldsymbol{W}\cdot\boldsymbol{v}\right]+O(6), \tag{76}$$

from which it is easily deduced that

$$\frac{u_A^0}{u_B^0}=1+\frac{1}{c^2}\left(W_A-W_B+\frac{1}{2}v_A^2-\frac{1}{2}v_B^2\right)+\frac{1}{c^4}\left\{(\gamma+1)(W_Av_A^2-W_Bv_B^2)+\frac{1}{2}(W_A-W_B)[W_A-W_B+v_A^2-v_B^2+2(1-\beta)(W_A+W_B)]+\frac{3}{8}v_A^4-\frac{1}{4}v_A^2v_B^2-\frac{1}{8}v_B^4-2(\gamma+1)(\boldsymbol{W}_A\cdot\boldsymbol{v}_A-\boldsymbol{W}_B\cdot\boldsymbol{v}_B)\right\}+O(6). \tag{77}$$

It follows from Eqs. (73) and (77) that the frequency shift $\delta\nu/\nu$ is given by

$$\frac{\delta\nu}{\nu}\equiv\frac{\nu_A}{\nu_B}-1=\left(\frac{\delta\nu}{\nu}\right)_c+\left(\frac{\delta\nu}{\nu}\right)_g, \tag{78}$$

where $(\delta\nu/\nu)_c$ is the special-relativistic Doppler effect

$$\left(\frac{\delta\nu}{\nu}\right)_c=-\frac{1}{c}\boldsymbol{N}_{AB}\cdot(\boldsymbol{v}_A-\boldsymbol{v}_B)+\frac{1}{c^2}\left[\frac{1}{2}v_A^2-\frac{1}{2}v_B^2-[\boldsymbol{N}_{AB}\cdot(\boldsymbol{v}_A-\boldsymbol{v}_B)](\boldsymbol{N}_{AB}\cdot\boldsymbol{v}_B)\right]-\frac{1}{c^3}\left[[\boldsymbol{N}_{AB}\cdot(\boldsymbol{v}_A-\boldsymbol{v}_B)]\left(\frac{1}{2}v_A^2-\frac{1}{2}v_B^2+(\boldsymbol{N}_{AB}\cdot\boldsymbol{v}_B)^2\right)\right]+\frac{1}{c^4}\left[\frac{3}{8}v_A^4-\frac{1}{4}v_A^2v_B^2-\frac{1}{8}v_B^4-[\boldsymbol{N}_{AB}\cdot(\boldsymbol{v}_A-\boldsymbol{v}_B)](\boldsymbol{N}_{AB}\cdot\boldsymbol{v}_B)\left(\frac{1}{2}v_A^2-\frac{1}{2}v_B^2+(\boldsymbol{N}_{AB}\cdot\boldsymbol{v}_B)^2\right)\right]+O(5) \tag{79}$$

and $(\delta\nu/\nu)_g$ contains all the contributions of the gravitational field, eventually mixed with kinetic terms

$$\left(\frac{\delta\nu}{\nu}\right)_g=\frac{1}{c^2}(W_A-W_B)-\frac{1}{c^3}\{(W_A-W_B)[\boldsymbol{N}_{AB}\cdot(\boldsymbol{v}_A-\boldsymbol{v}_B)]-\boldsymbol{l}_A^{(2)}\cdot\boldsymbol{v}_A+\boldsymbol{l}_B^{(2)}\cdot\boldsymbol{v}_B\}+\frac{1}{c^4}\Bigg((\gamma+1)(W_Av_A^2-W_Bv_B^2)+\frac{1}{2}(W_A-W_B)$$
$$\times\{W_A-W_B+2(1-\beta)(W_A+W_B)+v_A^2-v_B^2-2[\boldsymbol{N}_{AB}\cdot(\boldsymbol{v}_A-\boldsymbol{v}_B)](\boldsymbol{N}_{AB}\cdot\boldsymbol{v}_B)\}+\boldsymbol{N}_{AB}\cdot[(\boldsymbol{l}_B^{(2)}\cdot\boldsymbol{v}_B)(\boldsymbol{v}_A-2\boldsymbol{v}_B)$$
$$+(\boldsymbol{l}_A^{(2)}\cdot\boldsymbol{v}_A)\boldsymbol{v}_B]+[\boldsymbol{l}_A^{(3)}-2(\gamma+1)\boldsymbol{W}_A]\cdot\boldsymbol{v}_A-[\boldsymbol{l}_B^{(3)}-2(\gamma+1)\boldsymbol{W}_B]\cdot\boldsymbol{v}_B\Bigg)+O(5). \quad (80)$$

It must be emphasized that the formulas (76) and (77) are valid within the PPN framework without adding special assumption, provided that β and γ are the only nonvanishing post-Newtonian parameters. On the other hand, Eq. (80) is valid only for stationary gravitational fields. In the case of an axisymmetric rotating body, we shall obtain an approximate expression of the frequency shift by inserting the following developments in Eq. (80), yielded by Eqs. (64)–(69):

$$\boldsymbol{l}_A^{(2)}/c^2=\boldsymbol{l}_M(\boldsymbol{x}_A,\boldsymbol{x}_B)+\boldsymbol{l}_{J_2}(\boldsymbol{x}_A,\boldsymbol{x}_B)+\cdots,$$

$$\boldsymbol{l}_A^{(3)}/c^3=\boldsymbol{l}_{\boldsymbol{S}}(\boldsymbol{x}_A,\boldsymbol{x}_B)+\cdots,$$

$$\boldsymbol{l}_B^{(2)}/c^2=-\boldsymbol{l}_M(\boldsymbol{x}_B,\boldsymbol{x}_A)-\boldsymbol{l}_{J_2}(\boldsymbol{x}_B,\boldsymbol{x}_A)+\cdots,$$

$$\boldsymbol{l}_B^{(3)}/c^3=\boldsymbol{l}_{\boldsymbol{S}}(\boldsymbol{x}_B,\boldsymbol{x}_A)+\cdots,$$

the function $\boldsymbol{l}_{\boldsymbol{S}}$ being now given by Eq. (68) written with $\alpha_1=0$. Let us recall that the symbol $+\cdots$ stands for the contributions of the higher multipole moments which are neglected.

B. Application in the vicinity of the Earth

In order to perform numerical estimates of the frequency shifts in the vicinity of the Earth, we suppose now that A is on board the International Space Station (ISS) orbiting at the altitude $H=400$ km and that B is a terrestrial station. It will be the case for the ACES mission. We use $r_B=6.37\times10^6$ m and $r_A-r_B=400$ km. For the velocity of ISS, we take $v_A=7.7\times10^3$ m/s and for the terrestrial station, we have $v_B\leqslant465$ m/s. The other useful parameters concerning the Earth are as follows: $GM=3.986\times10^{14}$ m^3/s^2, $r_e=6.378\times10^6$ m, $J_2=1.083\times10^{-3}$; for $n\geqslant3$, the multipole moments J_n are in the order of 10^{-6}. With these values, we get $W_B/c^2\approx GM/c^2r_B=6.95\times10^{-10}$ and $W_A/c^2\approx GM/c^2r_A=6.54\times10^{-10}$. From these data, it is easy to deduce the following upper bounds: $|\boldsymbol{N}_{AB}\cdot\boldsymbol{v}_A/c|\leqslant2.6\times10^{-5}$ for the satellite, $|\boldsymbol{N}_{AB}\cdot\boldsymbol{v}_B/c|\leqslant1.6\times10^{-6}$ for the ground station and $|\boldsymbol{N}_{AB}\cdot(\boldsymbol{v}_A-\boldsymbol{v}_B)/c|\leqslant2.76\times10^{-5}$ for the first-order Doppler term.

Our purpose is to obtain correct estimates of the effects in Eq. (80) which are greater than or equal to 10^{-18} for an axisymmetric model of the Earth. At this level of approximation, it is not sufficient to take into account the J_2 terms in $(W_A-W_B)/c^2$. First, the higher-multipole moments J_3, $J_4,\ldots$ yield a contribution of order 10^{-15} in W_A/c^2. Second, owing to the irregularities in the distribution of masses, the expansion of the geopotential in a series of spherical harmonics is probably not convergent at the surface of the Earth. For these reasons, we do not expand $(W_A-W_B)/c^2$ in Eq. (80).

However, for the higher-order terms in Eq. (80), we can apply the explicit formulas obtained in the previous section. Indeed, since the difference between the geoid and the reference ellipsoid is less than 100 m, W_B/c^2 may be written as [22]

$$\frac{1}{c^2}W_B=\frac{GM}{c^2r_B}+\frac{GMr_e^2J_2}{2c^2r_B^3}(1-3\cos^2\theta)+\frac{1}{c^2}\triangle W_B,$$

where the residual term $\triangle W_B/c^2$ is such that $|\triangle W_B/c^2|\leqslant10^{-14}$. At a level of experimental uncertainty about 10^{-18}, this inequality allows us to retain only the contributions due to M, J_2 and $\boldsymbol{S}$ in the terms of orders $1/c^3$ and $1/c^4$. As a consequence, the formula (80) reduces to

$$\left(\frac{\delta\nu}{\nu}\right)_g=\frac{1}{c^2}(W_A-W_B)+\frac{1}{c^3}\left(\frac{\delta\nu}{\nu}\right)_M^{(3)}+\frac{1}{c^3}\left(\frac{\delta\nu}{\nu}\right)_{J_2}^{(3)}+\cdots$$
$$+\frac{1}{c^4}\left(\frac{\delta\nu}{\nu}\right)_M^{(4)}+\frac{1}{c^4}\left(\frac{\delta\nu}{\nu}\right)_{\boldsymbol{S}}^{(4)}+\cdots, \quad (81)$$

where the different terms involved in the rhs are separately made explicit and discussed in what follows.

Using the identity $(r_A+r_B)^2-R_{AB}^2=2r_Ar_B(1+\boldsymbol{n}_A\cdot\boldsymbol{n}_B)$, it may be seen that $(\delta\nu/\nu)_M^{(3)}$ is given by

$$\left(\frac{\delta\nu}{\nu}\right)_M^{(3)}=-\frac{GM(r_A+r_B)}{r_Ar_B}\Bigg[\left(\frac{\gamma+1}{1+\boldsymbol{n}_A\cdot\boldsymbol{n}_B}-\frac{r_A-r_B}{r_A+r_B}\right)\times[\boldsymbol{N}_{AB}\cdot(\boldsymbol{v}_A-\boldsymbol{v}_B)]+(\gamma+1)\times\frac{R_{AB}}{r_A+r_B}\frac{\boldsymbol{n}_A\cdot\boldsymbol{v}_A+\boldsymbol{n}_B\cdot\boldsymbol{v}_B}{1+\boldsymbol{n}_A\cdot\boldsymbol{n}_B}\Bigg]. \tag{82}$$

The contribution of this third-order term is bounded by 5×10^{-14} for $\gamma=1$, in accordance with a previous analysis [7].

C. Influence of the quadrupole moment at the order $1/c^3$

Defining the quantity K_{AB} by

$$K_{AB}=\frac{(r_A-r_B)^2}{r_Ar_B},$$

it is easily deduced from Eqs. (67) and (80) that the term $(\delta\nu/\nu)_{J_2}^{(3)}$ in Eq. (81) is given by

$$\begin{aligned}\left(\frac{\delta\nu}{\nu}\right)_{J_2}^{(3)}&=\frac{GM}{2r_e}J_2[\boldsymbol{N}_{AB}\cdot(\boldsymbol{v}_A-\boldsymbol{v}_B)]\left[\left(\frac{r_e}{r_A}\right)^3[3(\boldsymbol{k}\cdot\boldsymbol{n}_A)^2-1]-\left(\frac{r_e}{r_B}\right)^3[3(\boldsymbol{k}\cdot\boldsymbol{n}_B)^2-1]\right]\\
&+\frac{\gamma+1}{2}\frac{GM\,J_2r_e^2(r_A+r_B)}{r_A^2r_B^2}\frac{1}{(1+\boldsymbol{n}_A\cdot\boldsymbol{n}_B)^2}\Bigg\{[\boldsymbol{N}_{AB}\cdot(\boldsymbol{v}_A-\boldsymbol{v}_B)]\Bigg[(\boldsymbol{k}\cdot\boldsymbol{n}_A+\boldsymbol{k}\cdot\boldsymbol{n}_B)^2\frac{5-3\boldsymbol{n}_A\cdot\boldsymbol{n}_B+2K_{AB}}{1+\boldsymbol{n}_A\cdot\boldsymbol{n}_B}\\
&-\left(1-\frac{r_A(\boldsymbol{k}\cdot\boldsymbol{n}_B)^2+r_B(\boldsymbol{k}\cdot\boldsymbol{n}_A)^2}{r_A+r_B}\right)(3-\boldsymbol{n}_A\cdot\boldsymbol{n}_B+K_{AB})\Bigg]+\frac{R_{AB}}{r_A+r_B}(\boldsymbol{n}_A\cdot\boldsymbol{v}_A+\boldsymbol{n}_B\cdot\boldsymbol{v}_B)(\boldsymbol{k}\cdot\boldsymbol{n}_A\\
&+\boldsymbol{k}\cdot\boldsymbol{n}_B)^2\frac{7-\boldsymbol{n}_A\cdot\boldsymbol{n}_B+2K_{AB}}{1+\boldsymbol{n}_A\cdot\boldsymbol{n}_B}-\frac{R_{AB}}{r_A}(\boldsymbol{n}_A\cdot\boldsymbol{v}_A)[1-3(\boldsymbol{k}\cdot\boldsymbol{n}_A)^2]\frac{r_A+r_B(2+\boldsymbol{n}_A\cdot\boldsymbol{n}_B)}{r_A+r_B}-\frac{R_{AB}}{r_B}(\boldsymbol{n}_B\cdot\boldsymbol{v}_B)[1\\
&-3(\boldsymbol{k}\cdot\boldsymbol{n}_B)^2]\frac{r_A(2+\boldsymbol{n}_A\cdot\boldsymbol{n}_B)+r_B}{r_A+r_B}+R_{AB}\Bigg[2\left(\frac{\boldsymbol{n}_A\cdot\boldsymbol{v}_A}{r_A}+\frac{\boldsymbol{n}_B\cdot\boldsymbol{v}_B}{r_B}\right)(\boldsymbol{k}\cdot\boldsymbol{n}_A)(\boldsymbol{k}\cdot\boldsymbol{n}_B)-(\boldsymbol{n}_A\cdot\boldsymbol{v}_A)\frac{1-(\boldsymbol{k}\cdot\boldsymbol{n}_B)^2}{r_B}\\
&-(\boldsymbol{n}_B\cdot\boldsymbol{v}_B)\frac{1-(\boldsymbol{k}\cdot\boldsymbol{n}_A)^2}{r_A}\Bigg]-2\frac{R_{AB}}{r_A}(\boldsymbol{k}\cdot\boldsymbol{v}_A)\left[\boldsymbol{k}\cdot\boldsymbol{n}_A\frac{r_A+r_B(2+\boldsymbol{n}_A\cdot\boldsymbol{n}_B)}{r_A+r_B}+\boldsymbol{k}\cdot\boldsymbol{n}_B\right]\\
&-2\frac{R_{AB}}{r_B}(\boldsymbol{k}\cdot\boldsymbol{v}_B)\left[\boldsymbol{k}\cdot\boldsymbol{n}_A+\boldsymbol{k}\cdot\boldsymbol{n}_B\frac{r_A(2+\boldsymbol{n}_A\cdot\boldsymbol{n}_B)+r_B}{r_A+r_B}\right]\Bigg\}.\end{aligned} \tag{83}$$

One has $|\boldsymbol{v}_A/c|=2.6\times10^{-5}$, $|\boldsymbol{v}_B/c|\leqslant1.6\times10^{-6}$ and $K_{AB}=3.77\times10^{-3}$. A crude estimate can be obtained by neglecting in Eq. (83) the terms involving the scalar products $\boldsymbol{n}_B\cdot\boldsymbol{v}_B$ and $\boldsymbol{k}\cdot\boldsymbol{v}_B$. Since the orbit of the ISS is almost circular, the scalar product $\boldsymbol{n}_A\cdot\boldsymbol{v}_A$ can also be neglected. On these assumptions, we find, for $\gamma=1$,

$$\left|\frac{1}{c^3}\left(\frac{\delta\nu}{\nu}\right)_{J_2}^{(3)}\right|\leqslant1.3\times10^{-16}. \tag{84}$$

As a consequence, it will perhaps be necessary to take into account the $O(3)$ contributions of J_2 in the ACES mission. This conclusion is to be compared with the order of magnitude given in [7] without a detailed calculation. Of course, a better estimate might be found if the inclination $i=51.6$ deg of the orbit with respect to the terrestrial equatorial plane and the latitude $\pi/2-\theta_B$ of the ground station were taken into account.

D. Frequency shifts of order $1/c^4$

The term $(\delta\nu/\nu)_M^{(4)}$ in Eq. (81) is given by

$$\begin{aligned}\left(\frac{\delta\nu}{\nu}\right)_M^{(4)}&=(\gamma+1)\left(\frac{GM}{r_A}v_A^2-\frac{GM}{r_B}v_B^2\right)-\frac{GM(r_A-r_B)}{2\,r_Ar_B}(v_A^2-v_B^2)+\frac{1}{2}\left(\frac{GM}{r_Ar_B}\right)^2[(r_A-r_B)^2+2(\beta-1)(r_A^2-r_B^2)]\\
&-\frac{GM(r_A+r_B)}{r_Ar_B}\Bigg[\left(\frac{2(\gamma+1)}{1+\boldsymbol{n}_A\cdot\boldsymbol{n}_B}-\frac{r_A-r_B}{r_A+r_B}\right)[\boldsymbol{N}_{AB}\cdot(\boldsymbol{v}_A-\boldsymbol{v}_B)](\boldsymbol{N}_{AB}\cdot\boldsymbol{v}_B)+\frac{\gamma+1}{1+\boldsymbol{n}_A\cdot\boldsymbol{n}_B}\frac{R_{AB}}{r_A+r_B}\{(\boldsymbol{n}_A\cdot\boldsymbol{v}_A)(\boldsymbol{N}_{AB}\cdot\boldsymbol{v}_B)\\
&-[\boldsymbol{N}_{AB}\cdot(\boldsymbol{v}_A-2\boldsymbol{v}_B)](\boldsymbol{n}_B\cdot\boldsymbol{v}_B)\}\Bigg].\end{aligned} \tag{85}$$

The dominant term $(\gamma+1)GMv_A^2/r_A$ in Eq. (85) induces a correction to the frequency shift which amounts to 10^{-18}. So, it will certainly be necessary to take this correction into account in experiments performed in the foreseeable future.

The terms $(\delta\nu/\nu)_{\boldsymbol{S}}^{(4)}$ is the contribution of the intrinsic angular momentum to the frequency shift. Substituting Eqs. (51) and (68) into Eq. (80), it may be seen that

$$\left(\frac{\delta\nu}{\nu}\right)_{\boldsymbol{S}}^{(4)}=(\mathcal{F}_{\boldsymbol{S}})_A-(\mathcal{F}_{\boldsymbol{S}})_B, \tag{86}$$

where

$$(\mathcal{F}_{\boldsymbol{S}})_A=(\gamma+1)\frac{GS}{r_A^2}\left(1+\frac{r_A}{r_B}\right)\boldsymbol{v}_A\cdot\left\{\frac{\boldsymbol{k}\times\boldsymbol{n}_B}{1+\boldsymbol{n}_A\cdot\boldsymbol{n}_B}-\frac{r_B}{r_A+r_B}\boldsymbol{k}\times\boldsymbol{n}_A+\frac{\boldsymbol{k}\cdot(\boldsymbol{n}_A\times\boldsymbol{n}_B)}{(1+\boldsymbol{n}_A\cdot\boldsymbol{n}_B)^2}\left[\frac{r_A+r_B(2+\boldsymbol{n}_A\cdot\boldsymbol{n}_B)}{r_A+r_B}\boldsymbol{n}_A+\boldsymbol{n}_B\right]\right\}, \tag{87}$$

$$(\mathcal{F}_{\boldsymbol{S}})_B=(\gamma+1)\frac{GS}{r_B^2}\left(1+\frac{r_B}{r_A}\right)\boldsymbol{v}_B\cdot\left\{\frac{\boldsymbol{k}\times\boldsymbol{n}_A}{1+\boldsymbol{n}_A\cdot\boldsymbol{n}_B}-\frac{r_A}{r_A+r_B}\boldsymbol{k}\times\boldsymbol{n}_B-\frac{\boldsymbol{k}\cdot(\boldsymbol{n}_A\times\boldsymbol{n}_B)}{(1+\boldsymbol{n}_A\cdot\boldsymbol{n}_B)^2}\left[\boldsymbol{n}_A+\frac{r_A(2+\boldsymbol{n}_A\cdot\boldsymbol{n}_B)+r_B}{r_A+r_B}\boldsymbol{n}_B\right]\right\}. \tag{88}$$

In order to make easier the discussion, it is useful to introduce the angle ψ between $\boldsymbol{x}_A$ and $\boldsymbol{x}_B$ and the angle i_p between the plane of the photon path and the equatorial plane. These angles are defined by

$$\cos\psi=\boldsymbol{n}_A\cdot\boldsymbol{n}_B,\quad 0\leqslant\psi<\pi,$$

$$\boldsymbol{k}\cdot(\boldsymbol{n}_A\times\boldsymbol{n}_B)=\sin\psi\cos i_p,\quad 0\leqslant i_p<\pi.$$

With these definitions, it is easily seen that

$$\frac{\boldsymbol{k}\cdot(\boldsymbol{n}_A\times\boldsymbol{n}_B)}{1+\boldsymbol{n}_A\cdot\boldsymbol{n}_B}=\cos i_p\tan\frac{\psi}{2}.$$

Let us apply our formulas to ISS. Due to the inequality $v_B/v_A\leqslant 6\times10^{-2}$, the term $(\mathcal{F}_{\boldsymbol{S}})_B$ in Eq. (86) may be neglected. From Eq. (87), it is easily deduced that

$$|(\mathcal{F}_{\boldsymbol{S}})_A|\leqslant(\gamma+1)\frac{GS}{r_A^2}\left(1+\frac{r_A}{r_B}\right)\frac{2+3|\tan\psi/2|}{|1+\cos\psi|}v_A.$$

Assuming $0\leqslant\psi\leqslant\pi/2$, we have $(2+3|\tan\psi/2|)/|1+\cos\psi|\leqslant 5$. Inserting this inequality in the previous one and taking for the Earth $GS/c^3r_A^2=3.15\times10^{-16}$, we find

$$\left|\frac{1}{c^4}\left(\frac{\delta\nu}{\nu}\right)_{\boldsymbol{S}}^{(4)}\right|\leqslant(\gamma+1)\times10^{-19}. \tag{89}$$

Thus, we get an upper bound which is slightly greater than the one estimated by retaining only the term $h_{0i}v^i/c$ in Eq. (77). However, our formula confirms that the intrinsic angular momentum of the Earth will not affect the ACES experiment.

VI. CONCLUSION

In this paper, we have shown that the world function $\Omega(x_A,x_B)$ constitutes a powerful tool for determining the (coordinate) time transfer and the frequency shift in a weak gravitational field. Our main results are established within the Nordtvedt-Will PPN formalism. We have found the general expression of $\Omega(x_A,x_B)$ up to the order $1/c^3$. This result yields the expression of the time transfer function $\mathcal{T}(t_A,\boldsymbol{x}_A,\boldsymbol{x}_B)$ at the order $1/c^4$. We point out that γ and α_1 are the only post-Newtonian parameters involved in the expressions of the world function and of the time transfer function.

We have treated in detail the case of an isolated, axisymmetric rotating body, assuming that the gravitational field is stationary and that the body is moving with a constant velocity $\boldsymbol{v}_r$ relative to the universe rest frame. We have given a systematic procedure for calculating the terms due to the multipole moments in the world function $\Omega(x_A,x_B)$ and in the time transfer function $\mathcal{T}(\boldsymbol{x}_A,\boldsymbol{x}_B)$. These terms are obtained by straightforward differentiations of a kernel function. We have explicitly derived the contributions due to the mass M, to the quadrupole moment J_2 and to the intrinsic angular momentum $\boldsymbol{S}$ of the rotating body.

Restricting our attention to the case where only β and γ are different from zero, we have then determined the general expression of the frequency shift up to the order $1/c^4$. We have obtained the contributions of J_2 at the order $1/c^3$. Our method would give as well the quadrupole contribution at the order $1/c^4$ in case of necessity. We have found the complete evaluation of the effect of the intrinsic angular momentum $\boldsymbol{S}$, which is of order $1/c^4$. It is noteworthy that our formulas contain terms which have not been taking into account until now.

Within the limits of our model, the formulas that we have established yield all the gravitational corrections to the frequency shifts up to 10^{-18} in the vicinity of the Earth. We have applied our results to the ACES mission. We have found that the influence of the quadrupole moment at the order $1/c^3$ is in the region of 10^{-16}. For the effect of the intrinsic angular momentum, we have obtained an upper bound which is greater than the currently accepted estimate but which remains three orders of magnitude less than the expected accuracy in an experiment like ACES. Finally, it must be noted that our results could be applied to the two-way time/frequency transfers. In particular, the $O(3)$ contributions of J_2 to the two-way frequency transfers would probably deserve to be carefully calculated.

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[23] Let us prevent possible confusion. When we talk about calculating terms of order $1/c^4$ in the time transfer, we mean terms of absolute order $1/c^4$. As a consequence, these terms are of order $1/c^3$ relative to the leading term, which is a distance divided by c. For the frequency shifts, there is no ambiguity since we are calculating only frequency ratios.

[24] This velocity is noted $\boldsymbol{w}$ in Ref. [9].