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A wide family of singularity-free cosmological models

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In this paper a family of nonsingular cylindrical perfect fluid cosmologies is derived. The equation of state corresponds to a stiff fluid. The family depends on two independent functions under very simple conditions. A sufficient condition for geodesic completeness is provided.

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I. INTRODUCTION

After the discovery of the first regular perfect fluid cosmological model by Senovilla [1], one of the major questions that that spacetime posed was to determine how generic this lack of singularities was. For instance, in the Ruiz-Senovilla [2] barotropic family with the equation of state $p=\gamma\mu, 0<\gamma<1$, the regular models were those corresponding to a radiation fluid, $\gamma=1/3$. These solutions are the separatrix between those models with a singular pressure and energy density, $\gamma>1/3$, and those that just have a singular Weyl curvature, $\gamma<1/3$ [3]. Therefore regular models would be a zero-measure set in this family.

Our aim in this paper is to determine an infinite family of regular cosmological models that need not be so restrictive as the Ruiz-Senovilla family and therefore may indicate that regular models cannot be neglected in the set of solutions of Einstein equations.

We focus on stiff perfect fluids, since they are simple enough to allow almost complete integration of Einstein equations and thereby constitute an excellent arena for checking hypotheses. We shall show that under very simple restrictions regular solutions appear.

The second section of this paper is devoted to deriving solutions of the stiff fluid Einstein equations in a convenient manner for our purposes. In the third section geodesic completeness of the solutions is imposed and the restrictions derived from this assumption are expressed as a sufficient condition.

II. STIFF FLUID COSMOLOGIES

We restrict our attention to spacetimes endowed with an Abelian orthogonally transitive group of isometries G_2, acting on timelike surfaces, since this is the framework where regular cosmological models have so far appeared. We further require that the Killing fields be mutually orthogonal. Adapting the coordinates to these fields we write them as $\{\partial_z,\partial_\phi\}$. Under these assumptions we can write the metric for the spacetime in a convenient coordinate chart [4] $\{t,r,z,\phi\}$,

$$ds^2=e^{2K}(-dt^2+dr^2)+e^{-2U}dz^2+\rho^2e^{2U}d\phi^2, \qquad (1)$$

which has been chosen as isotropic on the nonignorable coordinates t,r. This is *a priori* no restriction and may always be achieved since every two-metric admits an isotropic parametrization. The metric is written in terms of three functions K, U, and ρ that depend only on t and r.

We may interpret the isometry group as cylindrical symmetry in the spacetime provided we have a regular axis where the norm of the angular Killing field vanishes. We shall come back to this issue later on. The range of the coordinates will then be the usual one for cylindrical symmetry:

$$-\infty<t,\ \ z<\infty,\ \ 0<r<\infty,\ \ 0<\phi<2\pi. \qquad (2)$$

The matter content of the spacetime is a perfect fluid of energy density μ, pressure p, and four-velocity u. The energy-momentum tensor is then

$$T^{\mu\nu}=\mu u^\mu u^\nu+p(g^{\mu\nu}+u^\mu u^\nu),$$

$$0\leqslant\mu,\nu\leqslant 3,\quad u^\mu u_\mu=-1. \qquad (3)$$

For a stiff fluid, $\mu=p$. We write down the Einstein equations $R_{\mu\nu}-Rg_{\mu\nu}/2=T_{\mu\nu}$ in a comoving system of coordinates for the perfect fluid, that is, $u=e^{-K}\partial_t$. After some simplifications the equations read

$$U_{tt}-U_{rr}+\frac{1}{\rho}(U_t\rho_t-U_r\rho_r)=0, \qquad (4a)$$

$$\rho_{tt}-\rho_{rr}=0, \qquad (4b)$$

$$K_t\rho_r+K_r\rho_t=\rho_{tr}+U_t\rho_r+U_r\rho_t+2\rho U_tU_r, \qquad (4c)$$

$$K_t\rho_t+K_r\rho_r=\frac{\rho_{tt}+\rho_{rr}}{2}+U_t\rho_t+U_r\rho_r+\rho(U_t^2+U_r^2)+p\rho e^{2K}, \qquad (4d)$$

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$$K_{rr}-K_{tt}+\frac{U_r\rho_r-U_t\rho_t}{\rho}+U_r^2-U_t^2=pe^{2K}, \tag{4e}$$

$$K_r+\frac{p_r}{2p}=0, \tag{4f}$$

$$K_t+\frac{\rho_t}{\rho}+\frac{p_t}{2p}=0, \tag{4g}$$

where the last pair of equations are just the Euler and continuity equations for the perfect fluid.

Since every regular cosmological model in the literature has a ρ with spacelike gradient, we shall impose as an ansatz that grad ρ be orthogonal to the velocity of the fluid, u. Under this assumption, ρ is a function of r only. But then Eq. (4b) requires that ρ be a linear function of r. After rescaling the coordinates we can take $\rho=r$ and the whole system of equations becomes rather simple:

$$U_{tt}-U_{rr}-\frac{U_r}{r}=0, \tag{5a}$$

$$K_t=U_t+2rU_tU_r, \tag{5b}$$

$$K_r=U_r+r(U_t^2+U_r^2)+pre^{2K}, \tag{5c}$$

$$K_{rr}-K_{tt}+\frac{U_r}{r}+U_r^2-U_t^2=pe^{2K}, \tag{5d}$$

$$K_r+\frac{p_r}{2p}=0, \tag{5e}$$

$$K_t+\frac{p_t}{2p}=0. \tag{5f}$$

The energy-momentum conservation equations can be integrated,

$$p=\alpha e^{-2K}, \tag{6}$$

with $\alpha=\text{const}>0$, and Eq. (5d) is a consequence of the others. We are left then with a two-dimensional reduced wave equation in polar coordinates without source term [Eq. (5a)] and a quadrature for K,

$$K_t=U_t+2rU_tU_r, \tag{7a}$$

$$K_r=U_r+r(U_t^2+U_r^2)+\alpha r, \tag{7b}$$

which can be integrated after providing a solution to the wave equation; namely, the integrability condition for the quadrature (7) is the wave equation, so the whole problem reduces to solving it.

Following [5], for instance, the solution to the Cauchy problem for the wave equation in the plane can be constructed from the 3D solution by ignoring the third variable,

$$U(x,y,t)=\frac{1}{2\pi}\int_0^{2\pi}d\phi\int_0^t dRR\,\frac{g(x+R\cos\phi,y+R\sin\phi)}{\sqrt{t^2-R^2}}+\frac{1}{2\pi}\frac{\partial}{\partial t}\int_0^{2\pi}d\phi\int_0^t dRR\,\frac{f(x+R\cos\phi,y+R\sin\phi)}{\sqrt{t^2-R^2}}, \tag{8}$$

for initial data $U(x,y,0)=f(x,y),U_t(x,y,0)=g(x,y)$. Note that the 2D wave equation does not satisfy Huygens' principle and therefore the domain of dependence is a circle, not just a circumference.

Our problem is even easier since we do not have dependence on the polar angle. Therefore we just have to impose circular symmetry on the initial data $U(r,0)=f(r),U_t(r,0)=g(r)$. The time coordinate may be removed from the integration limits by an appropriate scaling, $R=t\tau$,

$$U(r,t)=\frac{1}{2\pi}\int_0^{2\pi}d\phi\int_0^1 d\tau\frac{\tau}{\sqrt{1-\tau^2}}\times\left\{tg(v)+f(v)+tf'(v)\frac{t\tau^2+r\tau\cos\phi}{v}\right\}, \tag{9}$$

where $v=\sqrt{r^2+t^2\tau^2+2rt\tau\cos\phi}$, choosing the origin of the polar angle at the angle for (x,y). This expression is valid for all values of t.

For instance, the nonsingular spacetime in [6] is generated by

$$f(x)=\frac{\beta}{2}x^2,\quad g(x)=0,\quad \beta>0.$$

It is clear now that we are just integrating the functions f,g on a finite interval of r and that the integrals are well defined provided g,f,f' are continuous. The singularity at $\tau=1$ is harmless under such conditions.

The solution does not share the class of differentiability of the initial data because of the derivative term. We need at least $f\in C^3([0,\infty))$, $g\in C^2([0,\infty))$ in order to have $U,K\in C^2([0,\infty))$ and a well-defined Riemann tensor.

Surprisingly there is no need to impose cylindrical symmetry on the solution, since we have a regular axis at $r=0$ provided that f,g are regular there, and we have already required it. According to [4], the axis is regular if

$$\lim_{r\to 0}\frac{\langle \text{grad}\ \Delta,\text{grad}\ \Delta\rangle}{4\Delta}=e^{2(U-K)}|_{r=0}=1,$$

$$\Delta=\langle\partial_\phi,\partial_\phi\rangle=r^2e^{2U}; \tag{10}$$

therefore we need that $K(0,t)=U(0,t)$ for every value of t. But at the axis Eqs. (7a),(7b) which determine K are rather simple,

$$K_r=U_r,\quad K_t=U_t, \tag{11}$$

and therefore $K(0,t)=U(0,t)+\mathrm{const}$ and the condition of regularity at the axis is fulfilled by either taking the constant of integration equal to zero or conveniently rescaling the angular coordinate.

Note that this requirement of regularity excludes a time-like gradient of ρ in the vicinity of the axis.

III. GEODESIC COMPLETENESS

The metric that is obtained after integrating the system (7a),(7b) has regular components in the whole spacetime, but this does not suffice in order to have a nonsingular spacetime. We shall consider that a spacetime is regular [7] if it is causally geodesically complete, that is, if every causal geodesic may be extended to all values of its affine parametrization.

This means analyzing the geodesic equations for diagonal cylindrically symmetric spacetimes. This was done in [8]. Those results can be summarized as follows.

Theorem. A cylindrically symmetric diagonal metric in the form (1) with C^2 metric functions f,g,ρ is future causally geodesically complete provided that along causal geodesics the following conditions apply.

(1) For large values of t and increasing r:

(a) $K_r+K_t\geqslant 0$, and either $K_r\geqslant 0$ or $|K_r|\lesssim K_r+K_t$.

(b) $(K+U)_r+(K+U)_t\geqslant 0$, and either $(K+U)_r\geqslant 0$ or $|(K+U)_r|\lesssim(K+U)_r+(K+U)_t$.

(c) $(K-U-\ln\rho)_r+(K-U-\ln\rho)_t\geqslant 0$, and either $(K-U-\ln\rho)_r\geqslant 0$ or $|(K-U-\ln\rho)_r|\lesssim(K-U-\ln\rho)_r+(K-U-\ln\rho)_t$.

(2) For large values of t, constant b exist such that

$$\left.\begin{matrix} K(t,r)-U(t,r) \\ 2K(t,r) \\ K(t,r)+U(t,r)+\ln\rho(t,r)\end{matrix}\right\}\geqslant -\ln|t|+b.$$

A similar result can be stated for past-pointing geodesics by just reversing the sign of the time derivatives in condition 1.

Note that we have omitted the condition for nonradial geodesics with decreasing r in [8], since according to Eq. (97) in that reference, nonradial causal geodesics would reach $r=0$ for finite t, that is, before t becomes singular, contradicting the fact that geodesics should be singular there. Therefore the axis cannot be reached by geodesics with nonzero angular momentum.

All we have to do now in order to have a geodesically complete model is to check whether conditions 1 and 2 are satisfied.

We first show that the conditions on the derivatives are always satisfied for a stiff fluid model.

(1) According to Eqs. (7a),(7b) we have

$$K_t+K_r=U_t+U_r+r(U_t+U_r)^2+\alpha r.$$

Several possibilities are open: When U_t+U_r is positive, K_t+K_r is positive. If U_t+U_r is negative and $|U_t+U_r|\geqslant 1$, the quadratic term is larger and K_t+K_r is again positive for large values of r. Finally, if U_t+U_r is negative and $|U_t+U_r|\leqslant 1$, it is the pressure term αr which overcomes the negative term for large r.

The same sort of reasoning is valid to conclude that the radial derivative

$$K_r=U_r+r(U_t^2+U_r^2)+\alpha r$$

is positive for large values of r.

(b) We can apply the same argument to $K+U$,

$$(K+U)_t+(K+U)_r=2(U_t+U_r)+r(U_t+U_r)^2+\alpha r,$$

in order to show that these derivatives and $(K+U)_r$ are positive for large values of the radial coordinate.

(c) Finally, the third condition is always satisfied,

$$\begin{aligned}(K-U-\ln\rho)_t&+(K-U-\ln\rho)_r\\ &=r(U_t+U_r)^2-\frac{1}{r}+\alpha r\geqslant 0,\end{aligned}$$

$$(K-U-\ln\rho)_r=r(U_t^2+U_r^2)-\frac{1}{r}+\alpha r\geqslant 0,$$

for increasing r.

Past-pointing geodesics are treated analogously without any additional problem, since reversing the sign of time derivatives does not alter the positivity of the quadratic terms. Consequently only condition 2 yields a restriction.

(2) As we shall see, this condition amounts to studying U at the axis for large values of the time coordinate.

(a) This condition is trivial since $(K-U)|_{r=0}=0$,

$$\begin{aligned}(K-U)(t,r)&=\int_0^r dr'(K-U)_r(t,r')\\ &=\int_0^r dr'\{r'(U_t^2(t,r')+U_r^2(t,r'))+\alpha r\}\\ &>0.\end{aligned}$$

(b) The previous reasoning for ruling out singularities for decreasing radius leaves us with two possibilities: increasing radius and constant radius, $r=0$. Since, according to Eq. (11),

$$K(r,t)=U(0,t)+\int_0^r dr' K_r(r',t),$$

and we have already checked that K_r is positive for large t and increasing r, we just have to study the term $U|_{r=0}$, that is, we have reduced the problem to analyzing the behavior of K at the axis for large values of t.

(c) Similarly,

$$\begin{aligned}(K+U)(r,t)+\ln r&=2U(0,t)+\ln r\\ &\quad+\int_0^r dr'(K+U)(r',t),\end{aligned}$$

and we have already checked the positivity of K_r+U_r as in the previous condition. The logarithmic term does not mean a problem for increasing radius. Again we are left with controlling the behavior of the U term.

Summarizing our results so far, in order to have a causally geodesically complete spacetime we just have to require that $U|_{r=0}$ does not decrease faster than a negative logarithm for large values of the absolute value of the time coordinate. The condition on the solution of the Cauchy problem for the wave equation at the axis becomes a bit simpler,

$$\begin{aligned}K(0,t)&=U(0,t)\\&=\int_0^1 d\tau\frac{\tau}{\sqrt{1-\tau^2}}\{tg(|t|\tau)+f(|t|\tau)+|t|\tau f'(|t|\tau)\}\\&\geqslant-\frac{1}{2}\ln|t|+b,\end{aligned}\tag{12}$$

since the dependence on the polar angle is lost.

This bound can be attained, for example, by the initial data,

$$f(r)=\frac{|a|-1}{2}\ln r,\quad g(r)=\frac{a}{\pi}\frac{\ln r}{r},$$

since for this choice of functions the solution to the Cauchy problem for the wave equation is

$$\begin{aligned}U(0,t)&=\frac{1}{2}(|a|+a\operatorname{sgn}t-1)\ln|t|+\frac{1}{2}(|a|-a\operatorname{sgn}t-1)\ln 2\\&=\begin{cases}\frac{1}{2}(2|a|-1)\ln|t|-\ln 2, & \operatorname{sgn}at>0,\\ \frac{1}{2}(2|a|-1)\ln 2-\frac{1}{2}\ln|t|, & \operatorname{sgn}at<0.\end{cases}\end{aligned}$$

The behavior of the terms in Eq. (12) is rather different. The term U_f, dependent on f, the initial value of U, is even in the time coordinate, as is to be expected when the initial time derivative of U is zero. On the contrary, the term U_g, dependent on g, is odd in t.

This means that if U_f satisfies Eq. (12) for positive time, it is automatically satisfied for negative time. On the contrary, if U_g satisfies Eq. (12) for positive time, it is only satisfied for negative time if it is also satisfied by U_{-g} for positive time. Therefore three different possibilities are open depending on the value of

$$\lim_{r\to\infty}\frac{f(r)+rf'(r)}{rg(r)}.\tag{13}$$

If Eq. (13) is infinite, we need $U_f(t)>-\frac{1}{2}\ln|t|+b$ for large values of t in order to have geodesic completeness. This means that $f(r)+rf'(r)>-\frac{1}{2}\ln r+k$ for large r.

If Eq. (13) is zero, we need $|U_g(t)|<\frac{1}{2}\ln|t|+b$ for large values of t in order to have geodesic completeness. This means that $|g(r)|<(1\pi r)\ln r+k$ for large r.

If Eq. (13) is finite, then U_f and U_g are of the same order for large values of t or of $-t$ and geodesic completeness will depend on the value of the limit.

Note that values of the integrand close to $\tau=0$ do not influence the result for large values of t, since we may split the integral into two terms,

$$U(0,t)=\int_0^{x_c/t}+\int_{x_c/t}^1,$$

and the first one is bounded and negligible for large t. Therefore the main contribution to U comes from the second term, which must satisfy the required asymptotic behavior.

IV. EXAMPLES

A simple and wide family of functions that satisfy Eq. (12) can be written in terms of polynomials. Consider

$$f(r)=\sum_{i=0}^{n}a_ir^i,\quad g(r)=\sum_{i=0}^{m}b_ir^i.\tag{14}$$

If n,m are even numbers, U can be analytically integrated in terms of polynomials. For our purposes we just require U at the axis, which can be integrated for a larger set of functions. Since U is linear in f and g, we may analyze the monomials independently. For $f(r)=r^n$, $g(r)=r^m$ we obtain

$$\begin{aligned}U_f(t)&=\frac{n!!}{(n-1)!!}\left(\frac{\pi}{2}\right)^{(1+(-1)^{n+1})/2}|t|^n,\\U_g(t)&=\frac{m!!}{(m+1)!!}\left(\frac{\pi}{2}\right)^{(1+(-1)^{m+1})/2}|t|^m t.\end{aligned}\tag{15}$$

These expressions are valid even for $n,m=-1$ taking $(-1)!!:=1$, although they may not be very practical.

According to Eq. (12) we have two different possibilities for obtaining a singularity-free model.

If f,g are polynomials in r respectively of degree n,m and $n>m+1$, we have a nonsingular model if a_n is positive.

If f,g are polynomials in r respectively of degree $n,n-1$, U_f and U_g at the axis are polynomials of degree n and we have a nonsingular model if U_f dominates over U_g. This happens if the leading term of U_f is greater than the one of U_g, that is,

$$\left(\frac{n!!}{(n-1)!!}\right)^2\left(\frac{2}{\pi}\right)^{(-1)^n}\frac{a_n}{|b_{n-1}|}>1.\tag{16}$$

Using Stirling's formula for approximation of factorials, an easy and safe bound would be

$$\left(n+\frac{1}{2}\right)a_n>|b_{n-1}|.\tag{17}$$

This family of nonsingular cosmological models is large indeed, as can be seen by restricting the range to a finite dimensional space of polynomial functions: if we consider the space of functions U for which $U|_{r=0}$ is a polynomial of degree equal to or lower than n, the subset of singularity-free models comprises an open set, according to Eq. (17).

This result can be generalized, since

$$\int_0^1 d\tau \frac{\tau^{p+1}}{\sqrt{1-\tau^2}} = \frac{\sqrt{\pi}}{2} \frac{\Gamma((p+2)/2)}{\Gamma((p+3)/2)} \tag{18}$$

allows integration for every real value of the exponent p. Therefore, for $f(r)=r^p, g(r)=r^q$, we obtain

$$U_f(t) = \sqrt{\pi} \frac{\Gamma((p+2)/2)}{\Gamma((p+1)/2)} |t|^p,$$

$$U_g(t) = \frac{\sqrt{\pi}}{2} \frac{\Gamma((q+2)/2)}{\Gamma((q+3)/2)} |t|^q t, \tag{19}$$

which allow generalization of the geodesic completeness requirements that were previously derived for polynomial functions to linear combinations of powers of r. Additionally, one has just to care about the class of differentiability of U, which demands that $p \geqslant 2,\ q > 2$.

V. CONCLUSIONS

We have analyzed a wide family of stiff perfect fluid cosmological models with cylindrical symmetry. The issue of causal geodesic completeness has been reduced to just the behavior at the axis of the initial value problem for a sourceless 2D wave equation, which is the only one left after simplifying Einstein equations. A sufficient condition for geodesic completeness is provided, which is very easy to check and to implement. The case of polynomial initial data has been discussed and allows a fairly large set of nonsingular cosmological models. We think that this set is wide enough to preclude considering nonsingular models as isolated points in a space of cosmological models.

The role of pressure in these models is obviously determinant, since stiff perfect fluids are a limit case for energy conditions, corresponding to a sound velocity equal to that of light. On the contrary, dust perfect fluids are always singular according to the Raychaudhuri equation. Intermediate cases remain open for discussion, even though partial results have been obtained [3].

It is interesting to notice that nonseparability of the models in these coordinates is fundamental for geodesic completeness. In [9] separable cosmologies were studied and *none* of them was found to be regular.

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