

**Finite spin-foam-based theory of three- and four-dimensional quantum gravity**

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Starting from Ooguri's construction for  $BF$  theory in three (and four) dimensions, we show how to construct a well defined theory with an infinite number of degrees of freedom. The spin network states that are kept invariant by the evolution operators of the theory are exact solutions of the Hamiltonian constraint of quantum gravity proposed by Thiemann. The resulting theory is the first example of a well defined, finite, consistent, spin-foam-based theory in a situation with an infinite number of degrees of freedom. Since it solves the quantum constraints of general relativity it is also a candidate for a theory of quantum gravity. It is likely, however, that the solutions constructed correspond to a spurious sector of solutions of the constraints. The richness of the resulting theory makes it an interesting example to be analyzed by forthcoming techniques that construct the semiclassical limit of spin network quantum gravity.

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Attempts to construct a well defined and consistent theory of quantum gravity have recently received a significant boost through the introduction by Ashtekar and Lewandowski [1] of mathematical tools for performing well defined calculations in the context of theories of connections modulo gauge transformations in infinite dimensional situations. The introduction of these mathematical tools has actually had impact in two different avenues of quantization of general relativity: the canonical and the covariant (path integral) approach.

In the canonical approach, Thiemann [2] was able to construct a finite, well defined, anomaly-free representation of the quantum Hamiltonian constraint. In a separate development, similar techniques were used to define an equally consistent operator on the space of Vassiliev knot invariants [3]. Thiemann's Hamiltonian operates on a space of diffeomorphism invariant spin networks. The algebra of two Hamiltonians with different lapses is therefore an Abelian one, and it is faithfully implemented quantum mechanically. Controversy however remains about if this is "the right" implementation of a Hamiltonian constraint. For instance, it was noticed that a similar implementation in a space of nondiffeomorphism invariant states also yielded an Abelian algebra [4]. The constraint also appears to contain a rather large number of spurious solutions. For instance, applying the Thiemann construction in  $2+1$  dimensions [2], one encounters many quantum states in addition to the usual solutions of the Witten quantization. In this case one can remove the undesired states by the choice of inner product, and this construction works rather naturally in  $2+1$  dimensions. In  $3+1$  dimensions, an example of potentially spurious solution is to consider states  $\langle \psi |$  with support on spin networks with regular (nonextraordinary) vertices. Since the action of Thiemann's Hamiltonian on a bra state is to remove an extraordinary line (a line ending in two vertices that are planar and with two of the three incoming lines collinear), a state  $\langle \psi |$  that does not contain extraordinary lines is automatically annihilated by the constraint. These states are quite problematic since it is difficult to imagine how a semiclassical theory could be built on them that did not approximate an arbitrary

metric, including metrics that do not satisfy the Einstein equations. Getting rid of undesired quantum states is tantamount to "imposing the Einstein equations," and therefore is expected to be a difficult task in  $3+1$  dimensions. It is therefore not entirely surprising that it was possible to do it in  $2+1$  dimensions. These concerns are in our view enough to motivate an active program of searching for alternatives to Thiemann's quantization, although as should be evident from the above discussion, *do not imply* that there is something definitely "wrong" about the construction up to now. It might be that in the end Thiemann's quantization *does* lead to the correct theory of quantum gravity, albeit via an elaborate choice of inner product.

The aforementioned mathematical techniques have also had an impact in the construction of path integrals for general relativity, an approach that has come to be known as "spin foams" (see [5] for a recent review and references). Initial interest in this approach arose quite independently of gravity, in the study of topological field theories. In the spin foam approach to topological field theories one expands the partition function of the theory in terms of the basis of gauge invariants constructed with spin networks and performs the integral over connections of the path integral. To perform this integral one goes to the dual lattice, and is left with an expression that is a function of the valences associated to the faces of the dual lattice. One can understand the resulting path integral as a time evolution. If one slices the "spin foam," the intersections of the faces of the dual lattice with a plane produce lines associated with a spin inherited from that of the face of the dual lattice, that is, one reconstructs a "spatial" spin network. When one expands the action, one chooses a discretization of the expression. To recover the continuum theory one therefore has to either refine the discretization indefinitely or perhaps perform a sum over all possible discretizations in the hope that the sum will be dominated by the finer discretizations. In general these procedures produce difficulties. Refining the lattice is problematic to implement in practice with irregular lattices [6], and for a nonrenormalizable theory is very likely to lead to di-



$$\psi \left( \begin{array}{ccc} j_1 & & j_5 \\ & \beta & \\ j_2 & & j_4 \end{array} \right) = \sum_{j_6} (-1)^{\sum_{i=1}^6 j_i} \sqrt{2j_6 + 1} \sqrt{2j_3 + 1} \left\{ \begin{array}{ccc} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{array} \right\} \psi \left( \begin{array}{ccc} j_1 & & j_5 \\ & \beta & \\ j_2 & & j_4 \end{array} \right). \tag{3}$$

where  $\Lambda$  is a cutoff that is needed for the Ooguri action to be finite. The moves allow to untangle any spin network into a trivial one, generating  $6j$  symbols and other coefficients in the process. These moves are well known, they are called recoupling identities and just state that a spin network state is based on a flat connection. For that reason all networks can be reduced to a trivial one (the trivial network depends on the topology of the manifold, for instance on a sphere it is a point, on a torus it is a ‘‘Theta-net’’).

The first move is interesting, since its action appears as the ‘‘inverse’’ of the action of Thiemann’s Hamiltonian constraint for general relativity. In Thiemann’s construction the action of a Hamiltonian constraint is to add a line at a trivalent intersection. The first move allows to remove such a line. But in fact, the result is stronger. It was shown that if one uses the first move to ‘‘undo’’ the action of Thiemann’s Hamiltonian constraint, the end result vanishes [10]. That is, a quantum state whose definition incorporates the first move automatically satisfies the Hamiltonian constraint of quantum gravity (in fact it also solves the generalization of the constraint proposed in [11] as well). We will use this fact to construct the theory.

The theory we propose is defined in the following way: its wave functions are defined by diffeomorphism invariant spin network states that satisfy the *first two moves* of the three listed above. These moves are inverse of each other so the theory is consistent. It is well defined. But the lack of the third move prevents us from ‘‘undoing’’ a nontrivial spin network into the trivial one. The theory therefore has infinitely many inequivalent states, which hints to the fact that in its connection representation version the wave functions are not concentrated on flat connections anymore and the theory has an infinite number of degrees of freedom. Yet, due to the

discussion of the previous paragraph, *its wave functions still solve the Hamiltonian constraint of quantum gravity as proposed by Thiemann*. This is the main result of this paper. We have just constructed, simply by removing the last move, a theory that is finite, well defined and whose states are in the kernel of Thiemann’s Hamiltonian constraint. The theory is well defined in the sense that the evolution operators that arise from the above moves are finite and well defined operators and they satisfy the condition of being projectors,  $\sum_{s'} P(s, s') P(s', s'') = P(s, s'')$ , that as we discussed above was required of evolution operators of a totally constrained theory. The sum over the intermediate spin networks  $s'$  is appropriately restricted (otherwise there is potential for a divergence). In *BF* theory, as Ooguri [9] first discussed, the sum is only over colorations of a given triangulation. Choosing different triangulations just correspond to different representations of the same Hilbert space. In the case of our theory one needs a more subtle structure. The sum is over all colorations and over all inequivalent ‘‘skeletons’’ of spin networks. A skeleton is defined as the minimal spin network one obtains when all triangles are removed. Since the projectors are nonvanishing only if the initial and final spin network share the same skeletonization (the moves keep the skeleton invariant), then the left hand side is indeed finite even if one sums over all skeletons. Different spin networks with the same skeleton correspond, as in Ooguri’s case, to different representations of the Hilbert space.

The construction works in three and four dimensions and is not confined to trivalent intersections. If one wishes to consider intersections of higher valence (which is especially of interest in four dimensions) one needs to consider additional recoupling moves. For four valent intersections the move to consider is [13]

$$\Psi \left( \begin{array}{ccc} & j_1 & \\ & | & \\ j_6 & \beta & j_5 \\ & | & \\ j_2 & & j_4 \\ & | & \\ & j_7 & \\ & | & \\ & j_3 & \end{array} \right) = \prod_{i=5, k=2}^{10,5} \sqrt{2j_i + 1} \sqrt{2I_k + 1} \{6j\} \{15j\} \Psi \left( \begin{array}{ccc} j_1 & & j_4 \\ & I_1 & \\ j_2 & & j_3 \end{array} \right) \tag{4}$$

and its inverse. These moves contain the ones listed before as particular cases by setting valences to zero. One would need two additional moves in order to have complete recoupling corresponding to a flat connection if one has four valent intersections.

We need to elaborate a bit on the precise nature of the space of states that we are proposing. In spite of similarities with gravity, *BF* theory has an evolution operator that acts on a space of *combinatorial* spin networks. The evolution operator ‘‘propagates in space’’ in the sense that its action

creates new vertices. These aspects seems to imply that there should be little connection between these evolution operators and Thiemann's Hamiltonian constraint. The latter operates on regular (not combinatorial), diffeomorphism invariant, spin networks. Its action is only concentrated at vertices and only produces new "extraordinary" vertices that are not "seen" if one further operates with the constraint. Yet, it is remarkable that a state that is left invariant by our evolution operator (acting on the space of diffeomorphism invariant spin networks) manages to be annihilated by the Hamiltonian constraint of Thiemann. It can be seen as if the condition implied by our theory is "stronger" on the states than the one implied by the vanishing of Thiemann's Hamiltonian. To give an analogy (it has only a partial meaning as we will soon discuss), consider the Hamiltonian constraint of classical general relativity (indices omitted)  $EEF=0$ . One can consider a theory whose constraint is  $F=0$  and therefore all its solutions will also be annihilated by  $EEF=0$ . The analogy here would be  $BF$  theory. It has been known for some time [10] that the solutions of  $BF$  theory (chromatic evaluations) are trivially annihilated by Thiemann's Hamiltonian. The theory we are proposing today would be roughly of the same kind as the one defined by a "Hamiltonian constraint"  $EF=0$ . All its solutions would still be included in Thiemann's theory, but it has a richer solution space than that of the theory defined by  $F=0$ . In reality this analogy is too naive. The realization of the Hamiltonian constraint of our theory in terms of classical variables is unknown, but given the way we constructed it, is very likely (as we mentioned before) to be highly nonlocal (it is "simple in knot space," which suggests a very complex nature in connection space).

We have therefore constructed a well defined theory, with an infinite number of degrees of freedom, which manages to solve in the sense discussed above, the Hamiltonian constraint of (Euclidean) quantum gravity as proposed by Thiemann. The theory exists in either three or four dimensions. In

three dimensions it is of little physical interest, since there gravity is finite dimensional and therefore the theory we constructed is clearly unphysical. The four dimensional theory we constructed, since its states manage to solve Thiemann's Hamiltonian constraint, is in principle a candidate for a quantum theory of (Euclidean) gravity in four dimensions. It should be evident from the way we constructed the theory (it was not derived from an action, it is only formulated in terms of moves by removing one of the moves of  $BF$  theory) that it is unlikely that it will be connected with the correct physics of four dimensional quantum gravity. Nevertheless, we believe it is a valuable example in that it embodies many features considered desirable in a theory of quantum gravity. It is consistent in the sense that the evolution operators are projectors. There are explicit solutions to the Wheeler-DeWitt equation. The evolution operators do not suffer the "locality" [12] issues that apparently arise in Thiemann's formulation. We expect that when further progress is achieved in the analysis of a semiclassical limit for spin-network based theories [14], our theory could be analyzed and ruled out as not containing the correct classical physics of general relativity. Both the theory in three and the one in four space-time dimensions will however be quite nontrivial and rich examples to be analyzed, which go beyond the flatness of  $BF$  theory and nevertheless are finite and well defined. The fact that the states solve the Hamiltonian constraint of quantum gravity and include "propagation in space" sheds further light on the Hamiltonian proposed by Thiemann and may even imply that the theory constructed has more physical relevance than the one we can establish today.

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