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Cosmic censorship, area theorem, and self-energy of particles

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The (zeroth-order) energy of a particle in the background of a black hole is given by Carter's integrals. However, exact calculations of a particle's *self-energy* (first-order corrections) are still beyond our present reach in many situations. In this paper we use Hawking's area theorem in order to derive bounds on the self-energy of a particle in the vicinity of a black hole. Furthermore, we show that self-energy corrections *must* be taken into account in order to guarantee the validity of Penrose's cosmic censorship conjecture.

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I. INTRODUCTION

Spacetime singularities that arise in gravitational collapse are always hidden inside of black holes. This is the essence of the (weak) cosmic censorship conjecture, put forward by Penrose thirty years ago [1]. The conjecture, which is widely believed to be true, has become one of the cornerstones of general relativity. Moreover, it is being envisaged as a basic principle of nature. However, despite the flurry of activity over the years, the validity of this conjecture is still an open question (see, e.g., [2–10], and references therein).

The destruction of a black-hole event horizon is ruled out by this principle because it would expose the inner singularities to distant observers. Moreover, the horizon area of a black hole, A, is associated with an entropy $S_{BH}=A/4\hbar$ (we use gravitational units in which $G=c=1$). Thus, without any obvious physical mechanism to compensate for the loss of the black hole's enormous entropy, the destruction of the black-hole event horizon would violate the generalized second law (GSL) of thermodynamics [11]. For these two reasons, any process which seems, at first sight, to remove the black-hole horizon is expected to be unphysical. For the advocates of the cosmic censorship principle the task remains to find out how such candidate processes eventually fail to remove the horizon.

As is well known, the Kerr-Newman metric with $M^2-Q^2-a^2<0$ (where M,Q, and a are the mass, charge, and specific angular momentum, respectively) does not contain an event horizon, and it therefore describes a naked singularity. Thus, one may try to ''over spin'' (or ''over charge'') a black hole by dropping into it a rotating (or a charged) particle. Such gedanken experiments allow one to test the consistency of the conjecture. It turns out that the *test* particle approximation actually allows a black hole to ''jump over'' extremality in these type of gedanken experiments. We show that in order to guarantee the integrity of the black-hole event horizon one *must* take into account the *self-energy* of the particle (first-order interactions between the black hole and the object).

Furthermore, a well-established theorem in the physics of black holes is Hawking's area theorem [12], according to which the black-hole surface area should increase (or remain unchanged) in such gedanken experiments. We show that it is possible to use the area theorem in order to derive bounds on the self-energy of particles in the black hole spacetime. It should be further noted that in recent years there is a growing interest in the calculation of the self-interaction of a particle in the spacetime of a black hole (see, e.g., [13], and references therein). The flurry of activity in this area of research is motivated by the prospects of detection of gravitational waves in the future by gravitational wave detectors such as the Laser Interferometer Space Antenna (LISA) [14].

II. SELF-ENERGY OF A PARTICLE WITH ANGULAR MOMENTUM

We consider a particle which is lowered towards an extremal Kerr black hole. To *zeroth* order in particle-hole interaction the energy (energy-at-infinity) $\mathcal{E}^{(0)}$ of the object in the black-hole spacetime is given by Carter's [15] integrals (constants of motions). As shown by Christodoulou [16] (see also [17]), $\mathcal{E}^{(0)}(r=r_+)=\Omega^{(0)}J$ at the point of capture, where $\Omega^{(0)}=a/(r_+^2+a^2)$ is the angular velocity of the black hole, J is the conserved angular momentum of the particle, and $r_+=M$ is the location of the black-hole horizon [18].

One should also consider *first*-order interactions between the black hole and the particle's angular momentum. As the particle spirals into the black hole it interacts with the black hole, so the horizon generators start to rotate, such that at the point of assimilation the black-hole angular velocity Ω has changed from $\Omega^{(0)}$ to $\Omega^{(0)}+\Omega_c^{(1)}$. The corresponding first-order energy correction is $\mathcal{E}_{self}^{(1)}=\Omega_c^{(1)}J$. On dimensional analysis one expects $\Omega_c^{(1)}$ to be of the order of $O(J/M^3)$. In fact, Will [19] has performed a perturbation analysis for the problem of a *ring* of particles rotating around a slowly spinning (neutral) black hole, and found $\Omega_c^{(1)}=J/4M^3$. As would be expected from a perturbative approach, $\Omega_c^{(1)}$ is proportional to J. To our best knowledge, no exact calculation of $\Omega_c^{(1)}$ has been performed for generic (Kerr-Newman) black holes, nor for the case of a *single* particle (in which case the system loses the axial symmetry which characterized it in the case of a ring of matter). We therefore write $\mathcal{E}_{self}^{(1)}=\omega J^2$, and obtain

$$\mathcal{E}=\mathcal{E}^{(0)}+\mathcal{E}_{self}^{(1)}=\frac{J}{2M}+\omega J^2, \tag{1}$$

for the particle's energy at the point of capture.

The assimilation of the particle results with a change $\Delta M=\mathcal{E}$ in the black-hole mass, and a change $\Delta(Ma)=J$ in its angular momentum. The condition for the black hole to preserve its integrity after the assimilation of the particle ($a_{new}\leqslant M_{new}$) is

$$\frac{Ma+J}{M+\mathcal{E}}\leqslant M+\mathcal{E}, \tag{2}$$

or equivalently [substituting $\mathcal{E}$ from Eq. (1)],

$$0\leqslant J^2\left(4M\omega+\frac{1}{2M^2}\right), \tag{3}$$

which is automatically satisfied. We therefore conclude that the black-hole horizon cannot be removed by the assimilation of the particle—cosmic censorship is upheld.

We next consider the case of a particle which is lowered towards a *near* extremal Kerr black hole. The condition for the black hole to preserve its integrity, Eq. (2), yields

$$0\leqslant\left(\frac{J}{M}-\varepsilon\right)^2+J^2\left(4M\omega-\frac{1}{2M^2}\right), \tag{4}$$

where $r_{\pm}\equiv M\pm\varepsilon$. Perhaps somewhat surprisingly, the situation is more involved than in the extremal case: the test particle approximation ($\omega=0$) actually allows a *near* extremal Kerr black hole to ''jump over'' extremality by capturing a particle with angular momentum. One must refer to the self-energy of the particle (first-order interactions between the black hole and the object's angular momentum) in order to insure the validity of the cosmic censorship conjecture. In fact, we may reverse the line of reasoning: with the plausible assumption of cosmic censorship, it is possible to infer a lower bound on the self-energy of the particle, $\mathcal{E}^{(1)}_{self}\geqslant J^2/8M^3$.

We next generalize our results to the Kerr-Newman case. The energy of the particle at the point of capture is now given by

$$\mathcal{E}=\mathcal{E}^{(0)}+\mathcal{E}^{(1)}_{self}=\frac{aJ}{r_+^2+a^2}+\omega J^2. \tag{5}$$

The black-hole condition $M^2-a^2-Q^2\geqslant 0$ (after the assimilation of the particle) now reads

$$0\leqslant(M+\mathcal{E})^2-\left(\frac{Ma+J}{M+\mathcal{E}}\right)^2-Q^2, \tag{6}$$

which implies

$$0\leqslant\left(\frac{2a}{M^2+a^2}J-\varepsilon\right)^2+J^2\left(2\omega\frac{M^2+a^2}{M}-\frac{1}{M^2+a^2}\right). \tag{7}$$

Thus, one may derive a necessary condition for the validity of the cosmic censorship conjecture (a lower bound on the self-energy $\mathcal{E}^{(1)}_{self}$),

$$\mathcal{E}^{(1)}_{self}\geqslant\frac{M}{2(M^2+a^2)^2}J^2. \tag{8}$$

Furthermore, if the resulting configuration (after the assimilation of the particle) is a black hole, then according to Hawking's *area theorem* [12] there should be a growth (or no change) in the area of the black hole. The surface area A of a Kerr-Newman black hole is given by $A=4\pi(2Mr_+-Q^2)$, where $r_+=M+(M^2-a^2-Q^2)^{1/2}$ is the location of the black-hole outer horizon. Substituting $M\rightarrow M+\mathcal{E}$ and $Ma\rightarrow Ma+J$, one may use the area theorem ($A_{old}\leqslant A_{new}$) to derive a lower bound on the particle self-energy,

$$\mathcal{E}^{(1)}_{self}\geqslant\frac{r_+^2}{2M\alpha^2}J^2, \tag{9}$$

where $\alpha=A/4\pi$. This bound is valid for any Kerr-Newman black hole (not necessarily a near extremal one).

We note that the bound Eq. (8) derived from the cosmic censorship conjecture is stronger than the one derived from the area theorem, Eq. (9). (There is an equality in the extremal limit, where $r_+\rightarrow M$.) Thus, the analysis is self-consistent—provided cosmic censorship is respected, there is a growth in the black-hole surface area.

III. SELF-ENERGY OF A CHARGED PARTICLE

We next consider a charged particle of rest mass μ, charge q, and proper radius R, which is (slowly) descent into a (near extremal) Kerr black hole. The total energy $\mathcal{E}$ of the particle in a black-hole spacetime is made up of two distinct contributions: (1) $\mathcal{E}_0$, the energy associated with the body's mass (redshifted by the gravitational field), and (2) $\mathcal{E}^{(1)}_{self}$, the gravitationally induced self-energy of the charged particle.

The first contribution, $\mathcal{E}_0$, is given by Carter's [15] integrals for a particle moving in a black-hole background,

$$\mathcal{E}_0=\frac{\mu l(r_+-r_-)}{2\alpha}[1+O(l^2/r_+^2)], \tag{10}$$

where $r_{\pm}=M\pm(M^2-a^2)^{1/2}$ are the locations of the black-hole (event and inner) horizons, and l is the proper distance from the horizon. Namely,

$$l=l(r,\theta)=\int_{r_+}^{r}\sqrt{g_{rr}}\,dr, \tag{11}$$

with $g_{rr}=(r^2+a^2\cos^2\theta)/(r-r_+)(r-r_-)$.

The second contribution, $\mathcal{E}^{(1)}_{self}$, reflects the effect of the spacetime *curvature* on the particle's electrostatic *self-interaction*. The physical origin of this force is the distortion of the charge's long-range Coulomb field by the spacetime curvature. This can also be interpreted as being due to the image charge induced inside the (polarized) black hole [20,21]. The self-interaction of a charged particle in the black-hole background results with a repulsive (i.e., directed away from the black hole) self-force. A variety of techniques have been used to demonstrate this effect in black-hole spacetimes [22–30]. In particular, the contribution of this effect to the particle's (self) energy in the Schwarzschild background is $\mathcal{E}^{(1)}_{self}=Mq^2/2r^2$, which implies $\mathcal{E}^{(1)}_{self}=q^2/8M$ in the vicinity of the black hole. However, in the generic case of a spinning Kerr black hole, the self-energy was calculated only for the specific case in which the particle is located along the symmetry axis of the black hole. We therefore write $\mathcal{E}^{(1)}_{self}=\eta q^2$.

The gradual approach to the black hole must stop when the proper distance from the body's center of mass to the black-hole horizon equals R, the body's radius. One therefore finds

$$\mathcal{E}=\frac{\mu R(r_+-r_-)}{2\alpha}+\eta q^2, \tag{12}$$

for the particle's energy at the point of capture (this expression is valid for an arbitrary value of the azimuthal angel θ).

An assimilation of the charged particle results with a change $\Delta M=\mathcal{E}$ in the black-hole mass, and a change $\Delta Q=q$ in its charge. The condition for the black hole to preserve its integrity after the assimilation of the charge is therefore

$$0\leqslant(M+\mathcal{E})^2-\left(\frac{M}{M+\mathcal{E}}\right)^2-q^2, \tag{13}$$

or equivalently,

$$0\leqslant q^2(4M\eta-1)+2\varepsilon\mu R/M. \tag{14}$$

We emphasize that Eq. (14) implies that the test particle approximation (that is, taking $\eta=0$) allows to over charge a black hole.

The Coulomb energy of a charged particle is given by fq^2/R, where f is a numerical factor of order unity which depends on how the charge is distributed inside the body. The Coulomb energy attains its minimum, $q^2/2R$, when the charge is uniformly spread on a thin shell of radius R, which implies $f\geqslant 1/2$ (an homogeneous charged sphere, for instance, has $f=3/5$). Therefore, any charged body which respects the weak (positive) energy condition must be larger than $r_c\equiv q^2/2\mu$. Thus, a necessary condition for the validity of the cosmic censorship conjecture is [see Eq. (14)]

$$0\leqslant q^2(4M\eta-1+\varepsilon/M), \tag{15}$$

which implies a lower bound on the self-energy, $\mathcal{E}^{(1)}_{self}$, of a charged particle,

$$\mathcal{E}^{(1)}_{self}\geqslant\frac{q^2}{4M}\left(1-\frac{\varepsilon}{M}\right). \tag{16}$$

We next apply Hawking's area theorem to the gedanken experiment. If the resulting configuration is a black hole, then the area theorem [12] (namely, $A_{old}\leqslant A_{new}$) imposes a lower bound on the particle self-energy,

TABLE I. Self-energy of a particle in the vicinity of a black hole.

Type of self-energy	Type of a black hole	Lower bound on self-energy	Exact calculation
Rotational	Kerr-Newman	$\frac{r_+^2}{2M\alpha^2}J^2$	?
Electrostatic	Kerr (symmetry axis)	$\frac{M}{2\alpha}q^2$	$\frac{M}{2\alpha}q^2$
Electrostatic	Kerr ($\theta\neq 0$)	$\frac{M}{2\alpha}q^2$	?

$$\mathcal{E}^{(1)}_{self}\geqslant\frac{M}{2\alpha}q^2. \tag{17}$$

We note that an exact expression for the self-energy of a charged particle is available only for the specific case in which the particle is placed along the *symmetry* axis ($\theta=0$) of the Kerr black hole [28,29], $\mathcal{E}^{(1)}_{self}=Mq^2/2\alpha$. Note that this result coincides exactly with the bound Eq. (17). Furthermore, the exact result (available only in the $\theta=0$ case) yields $\mathcal{E}^{(1)}_{self}=(q^2/4M)(1-\varepsilon/M)$ for a near extremal Kerr black hole. Thus, taking cognizance of Eq. (16) we find that cosmic censorship is respected provided one takes into account the electrostatic self-energy of the particle in the background of the black hole.

In summary, in this paper we have analyzed gedanken experiments in which particles carrying angular momentum or electric charge are assimilated by a black hole. The gedanken experiments are considered from the point of view of Penrose's cosmic censorship conjecture and Hawking's area theorem. It was shown that first-order interaction effects (the self-energy of the particle) *must* be taken into account in order to preserve the black-hole integrity and to insure the validity of the cosmic censorship conjecture.

Moreover, exact calculations of the self-energy are available in the literature only for a limited number of cases. Using Hawking's area theorem, we derived bounds on the self-energy of a particle in the vicinity of a black hole. The resulting bounds are summarized in Table I.

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