Semiclassical zero temperature black holes in spherically reduced theories

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We numerically integrate the semiclassical equations of motion for spherically symmetric Einstein-Maxwell theory with a dilaton coupled scalar field and look for zero temperature configurations. The solution we find is studied in detail close to the horizon and comparison is made with the corresponding one in the minimally coupled case.

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I. INTRODUCTION

The most attractive feature of zero temperature black holes is that they are the natural candidates for the end state of the evaporation process. Indeed, they represent the ideal setting where one can address the various issues connected with the quantum evolution of black holes, such as, for instance, the problem of information loss (see, e.g., [1]).

In spherically symmetric Einstein-Maxwell theory the only solution with this property is the extremal Reissner-Nordström (RN) black hole. Turning to semiclassical theory, quantum corrections induced by the vacuum expectation value of the stress energy tensor due to matter fields modify the spacetime geometry, and it is very important to check whether the resulting solution still has a zero temperature. Perturbative corrections $O(\hbar)$ to the classical geometry evaluated close to the horizon in four dimensions do not appear to answer the above question unambiguously [2,3]. It is clear that more information will come only if one knows the exact analytical solution to the semiclassical equations of motion. For the simple case of spherically reduced Einstein-Maxwell theory coupled with 2D minimal scalar fields, Trivedi [4] was able to prove the existence of zero temperature solutions which reduce, as $\hbar\to 0$, to the extreme RN black hole. He also showed that, although the energy density measured by an infalling observer close to the horizon diverges for the classical solution, the semiclassical configuration is regular there (only a mild singularity emerges in the second derivative of the scalar curvature). The drawback of this analysis is that, due to the special type of matter fields used, these results do not have an obvious four-dimensional interpretation. In order to improve this study, we consider here a more realistic 2D model that recently has received a lot of attention. We employ a 2D conformal scalar field nonminimally coupled to the dilaton field, which classically corresponds to the s-wave sector of a 4D minimal scalar field (this model was first studied in [5]). We perform a numerical integration of the semiclassical equations of motion and show good evidence that zero temperature black holes exist in this theory. In particular, we inspect in detail the spacetime geometry in the region close to the horizon and compare it with the results one gets by numerical integration of the minimally coupled case.

The outline of this article is the following. In Sec. II we briefly review the spherically reduced Einstein-Maxwell theory and its zero temperature solution, the extreme RN black hole. In Sec. III the matter model we shall use is introduced and the expression for $\langle T_{ab}\rangle$ in the extreme RN background derived. In Sec. IV we numerically solve the back reaction equations, and finally Sec. V contains a discussion of our results and a comparison with the case analyzed in [4].

II. EINSTEIN-MAXWELL THEORY IN $D=2$

Let us start with Einstein-Maxwell theory in four dimensions

$$S=S_G+S_{e.m.}, \tag{1}$$

where S_G is the Einstein-Hilbert action[1]

$$S_G=\frac{1}{8\pi}\int d^4x\sqrt{-g^{(4)}}R^{(4)}, \tag{2}$$

and $S_{e.m.}$ denotes the action associated with the electromagnetic field

$$S_{e.m.}=-\frac{1}{8\pi}\int d^4x\sqrt{-g^{(4)}}F^2. \tag{3}$$

$R^{(4)}$ is the four-dimensional scalar curvature and F^2 the strength of the electromagnetic field $F_{\mu\nu}$. Assuming spherical symmetry, the 4D metric can be written

$$ds^2=g^{(2)}_{a\,b}dx^adx^b+e^{-2\phi(x_a)}d\Omega^2, \tag{4}$$

where $g^{(2)}_{a\,b}(x^a)$ $(a,b=1,2)$ is the two-dimensional metric in the $(r\text{-}t)$ plane, $\phi(x_a)$ the dilaton, and $d\Omega^2=d\theta^2+\sin^2\theta d\phi^2$ the line element of the unit two-sphere. Dimen-

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[1]We use units where $\hbar=G=c=k_B=1$.

sional reduction of the Einstein-Hilbert action (2) can be performed by integrating over the angles θ and ϕ:

$$S_G^{(2)}=\frac{1}{2}\int d^2x\sqrt{-g^{(2)}}e^{-2\phi}\times[R^{(2)}+2(\nabla\phi)^2+2e^{2\phi}]. \tag{5}$$

Proceeding similarly and considering $F_{\mu\nu}=F_{\mu\nu}(x^a)$, the Maxwell action becomes

$$S_{e.m.}^{(2)}=-\frac{1}{2}\int d^2x\sqrt{-g^{(2)}}e^{-2\phi}\tilde{F}^2, \tag{6}$$

where $\tilde{F}^2$ represents the field strength of a two-dimensional gauge field. Black hole solutions of the theory defined by

$$S^{(2)}=S_G^{(2)}+S_{e.m.}^{(2)} \tag{7}$$

are given by the Reissner-Nordström solution

$$ds^2=-f(r)dt^2+\frac{1}{f(r)}dr^2,\quad e^{-2\phi}=r^2, \tag{8}$$

with

$$f(r)=1-\frac{2M}{r}+\frac{Q^2}{r^2}. \tag{9}$$

The parameter M is the Arnowitt-Deser-Misner (ADM) mass and Q the electric charge (the corresponding field strength is $\tilde{F}_{rt}=Q/r^2$). The equation $f=0$ has two solutions for $M>|Q|$ given by $r_\pm=M\pm\sqrt{M^2-Q^2}$. $r_+\equiv r_h$ and r_- are, respectively, the event horizon and the inner horizon. The Hawking temperature is

$$T_H=\frac{\sqrt{M^2-Q^2}}{2\pi r_h^2}. \tag{10}$$

Vanishing of T_H, i.e., $M=|Q|$, defines the extremal configuration for which $r_+=r_-\equiv r_h=M$.

III. MATTER FIELDS

In order to investigate the existence of zero temperature solutions in the semiclassical theory we must couple $S^{(2)}$ in Eq. (7) to quantized free matter fields. Reference [4] considered a 2D minimally coupled scalar field described by the classical action[2]

$$S_M=-\frac{1}{4}\int d^2x\sqrt{-g^{(2)}}(\nabla\tilde{f})^2 \tag{11}$$

which, after quantization, yields the well-known Polyakov effective action [6]

$$S_{eff}=-\frac{1}{96\pi}\int d^2x\sqrt{-g^{(2)}}R^{(2)}\frac{1}{\Box}R^{(2)}. \tag{12}$$

This action can be formally obtained by functional integration of the trace anomaly

$$\langle T\rangle=\frac{R^{(2)}}{24\pi}. \tag{13}$$

As it was mentioned in [4], due to the 2D origin of this field the results one gets using $S_{tot}=S^{(2)}+S_M+S_{eff}$ do not have an obvious four-dimensional interpretation. To start with, we shall instead consider a 4D minimally coupled scalar field

$$S_M^{(4)}=-\frac{1}{16\pi}\int d^4x\sqrt{-g^{(4)}}(\nabla f)^2. \tag{14}$$

In a spherically symmetric spacetime, the matter fields can be expanded into spherical harmonics, the s-wave sector $\tilde{f}$ of the scalar field f depending only on t and r. For the s-wave field $\tilde{f}$ dimensional reduction gives the 2D action:

$$S_M^{(2)}=-\frac{1}{4}\int d^2x\sqrt{-g^{(2)}}e^{-2\phi}(\nabla\tilde{f})^2. \tag{15}$$

Comparison with the scalar field in Eq. (11) shows that the field $\tilde{f}$, though still 2D conformal, has acquired a nontrivial coupling with the dilaton field ϕ. The corresponding trace anomaly has additional ϕ-dependent terms [5]:

$$\langle T\rangle=\frac{1}{24\pi}[R^{(2)}-6(\nabla\phi)^2+6\Box\phi]. \tag{16}$$

Performing a functional integration of this expression we get the following effective action [5,7,8]:

$$S_{eff}^{(2)}=-\frac{1}{2\pi}\int d^2x\sqrt{-g^{(2)}}\left[\frac{1}{48}R^{(2)}\frac{1}{\Box}R^{(2)}-\frac{1}{4}(\nabla\phi)^2\frac{1}{\Box}R^{(2)}+\frac{1}{4}\phi R^{(2)}\right], \tag{17}$$

where the first nonlocal term is the same as in Eq. (12). It is important to point out that, unlike Eq. (12), this effective action is not exact. Unphysical results obtained for the evaporation of Schwarzschild black holes [5,8] suggest that, at least at finite temperature, $S_{eff}^{(2)}$ must be modified by the addition of conformally invariant (local and nonlocal) terms [9,10]. Considering instead zero temperature configurations, $S_{eff}^{(2)}$ gives physically meaningful results [10].

In the conformal gauge

$$ds^2=-f\,du\,dv \tag{18}$$

this action becomes local [i.e., $(1/\Box)R^{(2)}=-\ln f$], and for static configurations

[2]Usually, in order to make physical sense of the semiclassical approximation one considers N matter fields and considers the large N limit while keeping $N\hbar$ fixed. In this way the quantum corrections due to the other fields can be neglected.

$$f=f(r) \tag{19}$$

where

$$u=t-r_*, \quad v=t+r_*, \quad r_*=\int \frac{dr}{f(r)}, \tag{20}$$

the components of the 2D stress energy tensor read

$$\begin{aligned}\langle T_{uu}^{(2)}\rangle=\langle T_{vv}^{(2)}\rangle &= \frac{1}{96\pi}\left[ff''-\frac{1}{2}(f')^2\right]+\frac{1}{64\pi}f^2 \\ &\quad\times\left[\left(\frac{k'}{k}\right)^2\ln f-\left(\frac{k'}{k}\right)^2+2\frac{k''}{k}\right],\end{aligned} \tag{21}$$

$$\begin{aligned}\langle T_{uv}^{(2)}\rangle &= \frac{1}{96\pi}ff'' \\ &\quad+\frac{1}{32\pi}f\left[f'\frac{k'}{k}+f\frac{k''}{k}-\frac{1}{2}f\left(\frac{k'}{k}\right)^2\right],\end{aligned} \tag{22}$$

where the notation

$$k=e^{-2\phi} \tag{23}$$

has been introduced and the prime denotes derivative with respect to r. Considering now the dependence on the dilaton field, another relation can be deduced by functional differentiation of the effective action (17) with respect to the dilaton:

$$\begin{aligned}\frac{1}{\sqrt{-g^{(2)}}}\frac{\delta S_{eff}^{(2)}}{\delta\phi} &= \frac{1}{4\pi}\left\{\left[f'\frac{k'}{k}-f\left(\frac{k'}{k}\right)^2+f\frac{k''}{k}\right]\ln f \right.\\ &\quad\left.+f'\frac{k'}{k}-f''\right\}.\end{aligned} \tag{24}$$

This term is specific to the effective action considered and does not appear in the Polyakov theory. From a 4D viewpoint it is related to the tangential pressure $\langle P\rangle\equiv\langle T_\theta^\theta\rangle=\langle T_\phi^\phi\rangle$ through the relation ([5,8])

$$\langle P\rangle=\frac{1}{8\pi e^{-2\phi}\sqrt{-g^{(2)}}}\frac{\delta S_{eff}^{(2)}}{\delta\phi}. \tag{25}$$

For the particular case of the extremal Reissner-Nordström black hole $f=(1-M/r)^2, k=r^2$ we obtain the following results (see also [11]):

$$\begin{aligned}\langle T_{uu}^{(2)}\rangle=\langle T_{vv}^{(2)}\rangle &= -\frac{1}{24\pi}\frac{M}{r^3}\left(1-\frac{M}{r}\right)^3+\frac{1}{16\pi r^2}f^2\ln f,\end{aligned} \tag{26}$$

$$\begin{aligned}\langle T_{uv}^{(2)}\rangle &= -\frac{1}{48\pi}\frac{M}{r^3}\left(1-\frac{M}{r}\right)^2\left(2-3\frac{M}{r}\right) \\ &\quad+\frac{1}{8\pi}\frac{M}{r^3}\left(1-\frac{M}{r}\right)^3,\end{aligned} \tag{27}$$

where the first term on the right-hand side of these equations comes from the Polyakov contribution to the effective action. Also, from Eqs. (24) and (25) we obtain the 4D tangential pressure:

$$\begin{aligned}\langle P\rangle &= -\frac{1}{16\pi^2 r^4}\left(1-\frac{M}{r}\right)\left(1-3\frac{M}{r}\right)\ln f \\ &\quad+\frac{M}{16\pi^2 r^5}\left(4-5\frac{M}{r}\right).\end{aligned} \tag{28}$$

Another important physical quantity is

$$F=\frac{(T_r^r-T_t^t)}{f}=\frac{4\langle T_{uu}\rangle}{f^2}, \tag{29}$$

which is proportional to the energy density measured by an infalling observer [13]. Equation (26) leads to

$$F=-\frac{1}{6\pi}\frac{M}{r^2(r-M)}+\frac{1}{2\pi r^2}\ln\left|1-\frac{M}{r}\right|. \tag{30}$$

As in the minimally coupled case F diverges when $r\to M$. The term $\sim 1/(r-M)$ is the same as that found in the Polyakov theory [4], but despite its presence it was shown in [4] that the corresponding semiclassical zero temperature solution is regular at the horizon (only a mild divergence is present in the second derivative of the scalar curvature R). In our case, in addition to this term there appears a subleading logarithmic divergence $\sim\ln(r-M)$, which is present also in the analytic approximations in four dimensions proposed in [12], as well as a nontrivial pressure $\langle P\rangle$ [Eq. (28)]. As stressed in the first of Refs. [2], the divergence of F on the horizon of the classical extreme black hole causes the perturbative expansion in powers of $\hbar$ to break down there. The calculations performed in [2] are motivated by the fact that F has been proven to be finite at $r=M$ numerically in $D=4$ [13], but due to the result (30) reliable near horizon calculations for zero temperature black holes performed using the effective action (17) must be nonperturbative in $\hbar$. Similarly, the $O(\hbar)$ results presented in [14] (obtained by considering near-extreme black holes in the near horizon region) do not appear to have much physical meaning.

IV. BACK REACTION

We now come to the main question addressed in this paper: Do self-consistent zero temperature black holes exist in the semiclassical theory? For this purpose, we need first of all to write down the semiclassical Einstein equations, which can be derived by differentiation of the action $S_{tot}^{(2)}=S^{(2)}$

$+S_M^{(2)}+S_{eff}^{(2)}$ [see Eqs. (7), (15), and (17)] with respect to the 2D metric $g_{ab}^{(2)}$ and the dilaton field ϕ. In the conformal gauge (18) and considering the static configurations (19), (20) the relevant expressions concerning the matter part of the action were derived in Eqs. (21), (22), and (24). The corresponding quantities coming from the gravity and electromagnetic actions (where $F^2=-2Q^2/k^2$) can be easily obtained by differentiation of Eqs. (5) and (6).

The uu (or vv) constraint reads

$$0=f^2k''-\frac{1}{2}\left(f\frac{k'}{k}\right)^2k+\xi\left(f''f-\frac{1}{2}(f')^2\right)+3\xi\left[\frac{1}{2}\left(f\frac{k'}{k}\right)^2\ln f-\frac{1}{2}\left(f\frac{k'}{k}\right)^2+3f^2\frac{k''}{k}\right], \tag{31}$$

where the coefficient $\xi=\hbar/12\pi$ has been introduced (we have reintroduced $\hbar$ in the formulas in order to make the distinction between classical and quantum terms clearer). The equation obtained by varying the trace of the metric (i.e., g_{uv}) reads

$$0=-2+f'k'+fk''+2\frac{Q^2}{k}+\xi f''+3\xi\left[f'\frac{k'}{k}+f\frac{k''}{k}-\frac{1}{2}f\left(\frac{k'}{k}\right)^2\right]. \tag{32}$$

Finally, differentiation with respect to ϕ gives

$$0=f''-\frac{1}{2}f\left(\frac{k'}{k}\right)^2+f'\frac{k'}{k}+f\frac{k''}{k}-\frac{2Q^2}{k^2}-3\xi\left[\frac{f'k'}{k^2}-\frac{f''}{k}+\ln f\left(\frac{f'k'}{k^2}-f\frac{(k')^2}{k^3}+f\frac{k''}{k^2}\right)\right]. \tag{33}$$

For $\xi=0$ Eqs. (31), (32), and (33) are the classical equations of motion, for which the only zero temperature configuration is the extremal Reissner-Nordström black hole $f=(1-M/r)^2, k=r^2$. In the quantum terms, we have separated the ones multiplying ξ, coming from the Polyakov contribution to the effective action and present also in [4], and those proportional to 3ξ representing the additional contributions in the effective action $S_{eff}^{(2)}$ [Eq. (17)].

As the three previous equations involve only two independent functions $f(r)$ and $k(r)$, one is redundant. Indeed, they are related through the Bianchi identities combined with the "nonconservation" equations for the matter part [8]:

$$\nabla_a\langle T_b^{(2)a}\rangle+8\pi e^{-2\phi}\langle P\rangle\nabla_b\phi=0. \tag{34}$$

Also, as these nonlinear differential equations involve the second order derivatives of f and k, two boundary conditions on these functions are required to determine them uniquely. For a zero temperature black hole, natural boundary conditions can be imposed on the function f at the horizon. First of all, f has to vanish there. Moreover, in the gauge used the temperature of the black hole takes the simple expression

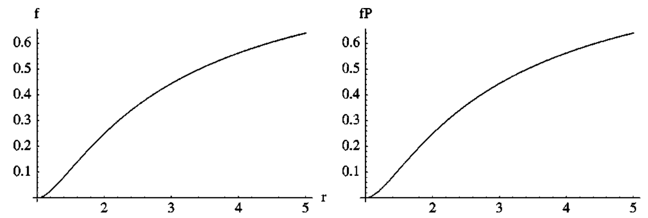

FIG. 1. Plot of the function f for our model (left) and the corresponding f for the minimally coupled case (right). We have set $Q=1$.

$$T_H=\frac{\kappa}{2\pi}, \tag{35}$$

where

$$\kappa=\frac{1}{2}f'\bigg|_{r=r_h} \tag{36}$$

is the surface gravity and r_h denotes the radius of the horizon. So $T_H=0$ means $f'=0$ at $r=r_h$.

In the Polyakov case, starting from these boundary conditions Trivedi [4] has found the form of the exact solution (nonperturbative in $\hbar$) close to the horizon as a (nonanalytic) expansion in powers of the coordinate distance from the horizon $r-r_h$. In our case, however, the terms proportional to $\ln f$ complicate this analysis exceedingly and seem to prevent an expansion in closed form of the solution for small values of $r-r_h$ analogous to that proposed in [4].

A numerical resolution was then undertaken and the boundary conditions were imposed at infinity [15], where the solution is to a good approximation the extreme black hole R.N. In this region apart from a finite but very small renormalization of the classical mass [following [3] it is $M_R/Q=1+O(\xi^2/Q^4)$] the first quantum corrections to the spacetime metric are of the order $O(1/r^3)$.

To start with, we introduced dimensionless variables and functions in the differential equations that we have to integrate:

$$x=r/Q, \quad \tilde{k}(x)=k(r)/Q^2, \quad \tilde{F}(x)=f(r).$$

This means choosing the black hole charge to be the natural unit of length.

Our numerical integrations were performed using the uu-constraint equation (31) and the ϕ equation (33) for the value $\xi/Q^2=10^{-5}$. Throughout our calculations, the solutions of these equations have been checked to be compatible with Eq. (32) as well with a precision less than 10^{-7}.

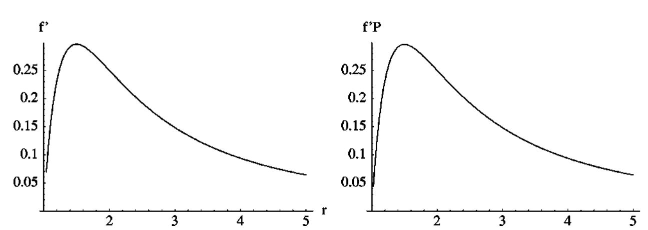

FIG. 2. Comparison of the values of f' for the two theories.

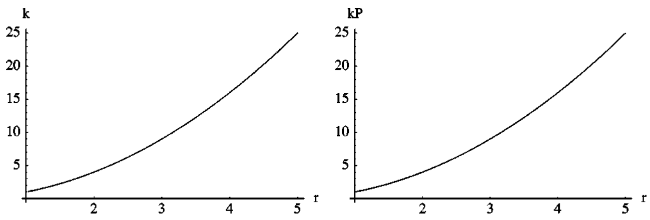

FIG. 3. Plots of the function k.

In order to probe the accuracy of our numerical simulations, we first considered the integration of the semiclassical equations for the minimally coupled case [i.e., discarding the terms proportional to 3ξ in Eqs. (31), (32), and (33)] and compared the numerical results with the form of the exact solution close to the horizon given by Trivedi [4]. We find that, for a value of the horizon $x_P=1.0229$ (in units where $Q=1$), i.e., with a deviation of about 2.3% from the classical value, the functions f and f' behave like those in [4] with a precision of about 5×10^{-4} when $x\to x_P$ and that k is accurate with a precision 5×10^{-5} and k' with the accuracy 2×10^{-4}. We can then expect the global precision of our simulation to be at least about $5\times10^{-2}\%$.

Considering now the nonminimally coupled case, we have integrated Eqs. (31) and (33). The results of this simulation are illustrated by the plots on the left of Figs. 1–4 where the functions f and k and their first derivatives have been shown for x varying from the horizon $x_D=1.0378$ to 5 (in units where $Q=1$). To facilitate the comparison, the same functions in the minimal case (the plots on the right of Figs. 1–4) have been reported for x varying from $x_P=1.0229$ to 5.

V. DISCUSSION AND CONCLUSIONS

Our numerical simulations presented in Figs. 1–4 and the comparison with the corresponding solution of the Polyakov theory appear to give good evidence that zero temperature configurations exist in this theory. To get some insights from these results, we compared the numerical values close to the horizon of these solutions with those obtained in the minimally coupled case. It turns out that the differences between the values of the functions f and f' for the two theories are less than 8×10^{-4} (as deduced previously, the numerical precision is about 5×10^{-4}). The same reasoning applies to the function k with a difference less than 8×10^{-5}. The first noticeable difference between the two theories appears at the first order derivative of the function k: the value of k' on the horizon for our model is estimated at -18.094 compared to $+2.075$ for the Polyakov case. Going further in the derivatives of the function f we have that the third derivative of f

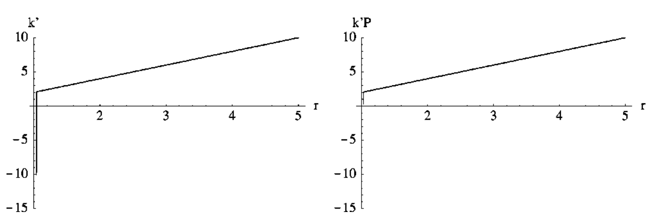

FIG. 4. Plots of the functions k'.

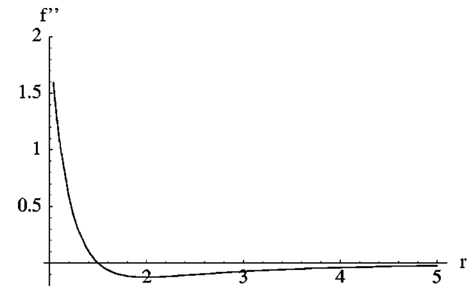

FIG. 5. Values of $f''=-R^{(2)}$ for our model.

blows up at the horizon for the nonminimally coupled field compared to the divergence of the fourth derivative in the minimal case. Moreover, it is interesting to stress that the value of the coordinate x at the horizon $x_D=1.0378$ (in units where $Q=1$) differs from $x_P=1.0229$ by about 1.5% (up to a numerical precision of about $5\times10^{-2}\%$).

Curvature invariants can be easily calculated starting from the results presented here. The 2D Ricci scalar $R^{(2)}=-f''$ is finite on the horizon, as shown in the plot of Fig. 5. Moreover, the finiteness of k and k' are enough to prove that the corresponding four-dimensional scalar curvature $R^{(4)}$ is also regular as the horizon is approached. A similar conclusion, despite the divergence of F at the horizon of the extreme RN black hole, was found in the minimally coupled case [4]. In our case, since the leading term in Eq. (30) is the same as in the Polyakov theory, it is reasonable that the main conclusion about the regularity of the geometry at the horizon is unchanged. The difference with respect to the case analyzed in [4] is that the "mild" singularity appearing at the horizon in the second derivative of the curvature close to the horizon ($\sim f''''$) is replaced by a "stronger" divergence in its first derivative (i.e., f'''). This is due to the logarithmic divergence ln(f) appearing in Eq. (30) as well as in Eqs. (31) and (33).

In conclusion, by numerical integration of the two-dimensional semiclassical equations of motion for the case of the spherically reduced Einstein-Maxwell theory and a scalar field nonminimally coupled to the dilaton we found solutions describing zero temperature black holes. Similarities and differences with respect to the simpler minimally coupled case were studied in detail. Due to the intrinsic four-dimensional nature of the matter field used, our results could be relevant in order to address the same issue in the physical world $D=4$. Finally, following Refs. [2], an interesting extension of this work would be to check whether the solution found here does indeed represent the end point of the evaporation process (for the minimally coupled case this problem has been addressed both in the near-horizon approximation [16] and in the whole spacetime numerically [17]).

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