# Wave tails in time-dependent backgrounds

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It is well known that waves propagating under the influence of a scattering potential develop ‘‘tails.’’ However, the study of late-time tails has so far been restricted to time-independent backgrounds. In this paper, we explore the late-time evolution of spherical waves propagating under the influence of a *time-dependent* scattering potential. It is shown that the tail structure is modified due to the temporal dependence of the potential. The analytical results are confirmed by numerical calculations.

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The phenomenon of wave tails has fascinated many physicists and mathematicians from the early explorations of wave theories. Wave tails have found various applications from the first studies in light propagation [1] to the theory behind the proposed experiments to detect gravitational waves [2–4]. In fact, tail-free propagation seems to be the exception rather than the rule [5,6]. For instance, it is well established that scalar, electromagnetic, and gravitational waves in curved spacetimes propagate not only along light cones, but also spread inside them. This implies that waves do not cut off sharply after the passage of the wave front, but rather leave a tail or wake at late times.

From a physical point of view, the most interesting mechanism for the production of late-time tails is the back scattering of waves off a potential (or a spacetime curvature) at asymptotically far regions [7,8]. This can be described as follows. Consider a wave from a source point $y$. The late-time tail observed at a fixed spatial location $x$ and at time $t$ is a consequence of the wave first propagating to a distant point $x' \gg y,x$, being scattered by $V(x',t')$ at time $t' \simeq t/2$, and then returning to $x$ at a time $t \simeq (x'-y)+(x'-x) \simeq 2x'$ [9]. Hence, the scattering amplitude (and thus the late-time tail itself) is expected to be proportional to $V(x',t') \simeq V(t/2,t/2)$. (However, in a previous paper [10] we have shown that this picture is somewhat naive, and requires some important modifications.)

The propagation of spherical waves in curved spacetimes or in optical cavities is often governed by the Klein-Gordon (KG) equation [11]

$$\left[\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}+\frac{1}{x_s^2}V(x,t)\right]\Psi=0, \tag{1}$$

where $V(x,t)$ is an effective curvature potential which determines the scattering of the waves by the background geometry (we henceforth take $x_s=1$ without loss of generality). It was first demonstrated by Price [8] that a (nearly spherical) collapsing star leaves behind it a ‘‘tail’’ which decays asymptotically as an inverse power of time.

The analysis of Price has been extended by many authors. Gundlach, Price, and Pullin [12] showed that power-law tails are a genuine feature of gravitational collapse—the existence of these tails was demonstrated in full *nonlinear* numerical simulations of the collapse of a self-gravitating scalar field (this was later reproduced in [13]). Moreover, since the late-time tail is a direct consequence of the scattering of the waves at asymptotically far regions, it has been pointed out that the same power-law tails would develop independently of the existence of a horizon [14]. This implies that tails should also be formed when the collapse fails to produce a black hole, or even in the context of stellar dynamics (e.g., in perturbations of neutron stars). In recent years, there has been a flurry of activity in the field of wave tails; see, e.g., [15–47], and references therein.

Yet, in spite of the numerous works addressing the problem of wave tails, a thorough understanding of this fascinating phenomenon is not complete. In particular, most of the previous analyses were restricted to the specific class of (time-independent) ‘‘logarithmic potentials’’ of the form $V(x) \sim \ln^\beta x/x^\alpha$ (where $\alpha>2$ and $\beta=0,1$ are parameters) [9]. Recently, we have given a systematic analysis of the tail phenomenon for waves propagating under the influence of a *general* time-independent scattering potential [10].

It should be realized, however, that a realistic gravitational collapse produces a *time-dependent* spacetime geometry, on which the tails are developing. This fact calls for a systematic exploration of the general properties of wave tails in *dynamical* (time-dependent) backgrounds. This is the aim of the present paper, in which we present our main results.

We consider the evolution of a wave field whose dynamics is governed by a KG-type equation $\Phi^{;\nu}_{;\nu}+V(r,t)\Phi=0$. Substituting $\Phi=\Psi(t,r)/r$ ($r$ being the circumferential radius), one obtains a wave equation of the form Eq. (1) [48].

It proves useful to introduce the double-null coordinates $u \equiv t-x$ and $v \equiv t+x$, which are a retarded time coordinate and an advanced time coordinate, respectively. The initial data are in the form of some compact outgoing pulse in the range $u_0 \leq u \leq u_1$, specified on an ingoing null surface $v=v_0$.

The general solution to the wave equation (1) can be written as a series depending on two arbitrary functions $F$ and $G$ [8]:

$$\Psi=G^{(0)}(u)+F^{(0)}(v)+\sum_{k=0}^{\infty}[B_k(u,v)G^{(-k-1)}(u)+C_k(u,v)F^{(-k-1)}(v)]. \tag{2}$$

For any function $H$, $H^{(k)}$ is its $k$th derivative; negativeorder derivatives are to be interpreted as integrals (we shall also denote $\partial_u^m \partial_v^n H$ by $H^{(m,n)}$). The functions $B_k(u,v)$ satisfy the recursion relation

$$B_{k,v} = -B_{k-1,uv} - \tfrac{1}{4} V B_{k-1} \tag{3}$$

for $k \geqslant 1$ and

$$B_{0,v} = -V/4. \tag{4}$$

For the first Born approximation to be valid, the scattering potential $V$ should approach zero faster than $1/v^2$ as $v \to \infty$; see, e.g., [9,24]. Otherwise, the scattering potential cannot be neglected at asymptotically far regions [see Eq. (6) below]. The recursion relation, Eq. (3), yields $B_k(u,v) = (-1)^{k+1} V^{(k,-1)}/4$.

It is useful to classify the scattering potentials into two groups, according to their asymptotic behavior: group I, $|V_{,u}|$ approaches zero *faster* than $|V|$ as $v \to \infty$; group II, $|V_{,u}|$ approaches zero at the *same* rate as $|V|$ as $v \to \infty$.

*Group I*. The first stage of the evolution is the scattering of the field in the region $u_0 \leqslant u \leqslant u_1$. The first sum in Eq. (2) represents the primary waves in the wave front (i.e., the zeroth-order solution, with $V \equiv 0$), while the second sum represents back scattered waves. The interpretation of these integral terms as back scatter comes from the fact that they depend on data spread out over a *section* of the past light cone, while outgoing waves depend only on data at a fixed $u$ [8].

After the passage of the primary waves there is no outgoing radiation for $u > u_1$, aside from back scattered waves. This means that $G(u_1) = 0$. Hence, at $u = u_1$ and for $v \gg u_1$ (where $t \simeq x \simeq v/2$), the dominant term in Eq. (2) is

$$\Psi(u = u_1, v) = B_0(u = u_1, v) G^{(-1)}(u_1). \tag{5}$$

This is the dominant back scatter of the primary waves.

With this specification of characteristic data on $u = u_1$, we shall next consider the asymptotic evolution of the field. We confine our attention to the region $u > u_1, \quad x \gg x_s$. To a *first* Born approximation, the spacetime in this region is approximated as flat [8,14]. Thus, to first order in $V$ (that is, in a first Born approximation) the solution for $\Psi$ can be written as

$$\Psi = g^{(0)}(u) + f^{(0)}(v). \tag{6}$$

Comparing Eq. (6) with the initial data on $u = u_1$, Eq. (5), one finds

$$f(v) = -G^{(-1)}(u_1) V^{(0,-1)}(u = u_1, v)/4. \tag{7}$$

For late times $t \gg x$, one can expand $g(u) = \Sigma_{n=0}^{\infty} (-1)^n g^{(n)}(t) x^n/n!$ and similarly for $f(v)$. With these expansions, Eq. (6) can be rewritten as

$$\Psi = \sum_{n=0}^{\infty} K_0^n x^n [f^{(n)}(t) + (-1)^n g^{(n)}(t)], \tag{8}$$

where the coefficients $K_0^n$ are those given in [8].

Using the boundary conditions for small $r$ [regularity as $x \to -\infty$, at the horizon of a black hole, or at $x = 0$, for a nonsingular model (e.g., a stellar model)], one finds that at late times $g(t) = -f(t)$ to first order in the scattering potential $V$ (see, e.g., [8,14] for additional details). That is, the incoming and outgoing parts of the tail are equal in magnitude at late times. This almost total reflection of the ingoing waves at small $r$ can easily be understood on physical grounds—it simply manifests the impenetrability of the barrier to low-frequency waves [8] (which are the ones to dominate the late-time evolution [16]). We therefore find that the late-time behavior of the field at a fixed radius ($x \ll t$) is dominated by [see Eq. (8)]

$$\Psi \simeq 2 K_0^1 f^{(1)}(t) x, \tag{9}$$

which implies

$$\Psi \simeq -2^{-1} K_0^1 G^{(-1)}(u_1) x V(u = u_1, v = t), \tag{10}$$

or equivalently

$$\Psi(x,t) \simeq -2^{-1} K_0^1 G^{(-1)}(u_1) x V(t/2, t/2). \tag{11}$$

*Group II*. The dominant back scatter of the primary waves is $\Psi(u = u_1, x) = \Sigma_{k=0}^{\infty} B_k(u = u_1, v) G^{(-k-1)}(u_1)$. Using an analysis along the same lines as before, one finds

$$\Psi \simeq \sum_{n=1,3,\ldots}^{\infty} 2^{-1} K_0^n x^n \sum_{k=0}^{\infty} (-1)^{k+1} G^{(-k-1)}(u_1) V^{(k,n-1)} \times (u = u_1, v = t) \tag{12}$$

at late times. Note that Eq. (12) is merely a generalization of Eq. (10), and reduces to it if $|V_{,x}|$ or $|V_{,t}|$ approaches zero faster than $|V|$ (in which case $V^{(0,0)}$ dominates at late times).

We next consider the implications of our results to the case of the gravitational collapse of a star. It should be emphasized that in all previous analyses it has been assumed that the fields propagate on a *fixed* background. In particular, the leading-order scattering potential is taken to be $M/r^3$, with the parameter $M$ being a *constant*. However, it is well known that the mass parameter approaches a constant value only asymptotically. Namely, $M(t) = M_0 + M_1 t^{-\alpha}$ at asymptotically late times, $M_0$ being the mass of the formed black hole (the value of $\alpha$ depends on the multipole order of the field [8]). Thus, the results of the present paper reveal that one should expect to find higher-order corrections to the tail of gravitational collapse (as compared with the predictions of the *fixed*-background analyses). These higher-order corrections "contaminate" the leading-order tail, and vanish only asymptotically.

Moreover, for *subcritical* initial data [49] (in which case the collapsing matter fails to produce a black hole or a star) the spacetime geometry is highly *time-dependent*. Thus, in these cases the central mass of the spacetime, $M(t)$, cannot be taken as a constant. Rather, it evolves with time and eventually falls to zero. A fixed background analysis would therefore be inappropriate in these situations, and one should consider the scattering by a time-dependent scattering potential (with a decreasing mass parameter). One such example is given in Fig. 1.

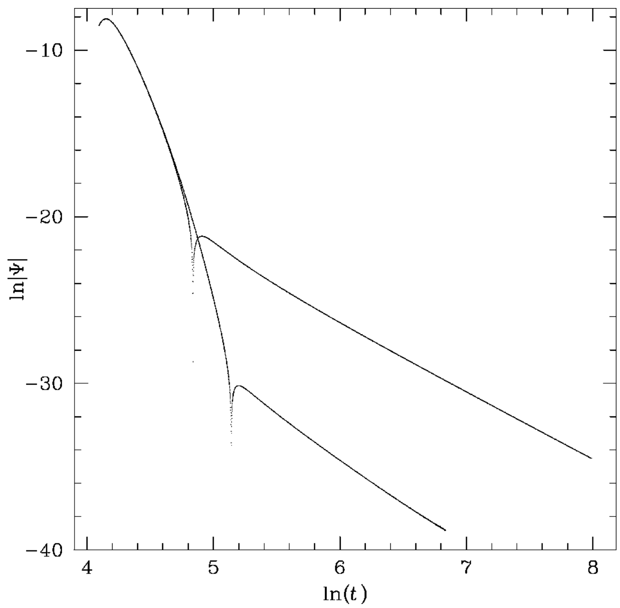


FIG. 1. Temporal evolution of the field for time-dependent scattering potentials of the form $V(x,t)=1/x^{\alpha}t^{\beta}$ (the results presented here are for $\alpha=3$). The power-law indices are $-4.04$ and $-5.08$ for $\beta=1$ (upper graph) and $\beta=2$, respectively. These values should be compared with the *analytically* predicted values of $-4$ and $-5$, respectively.

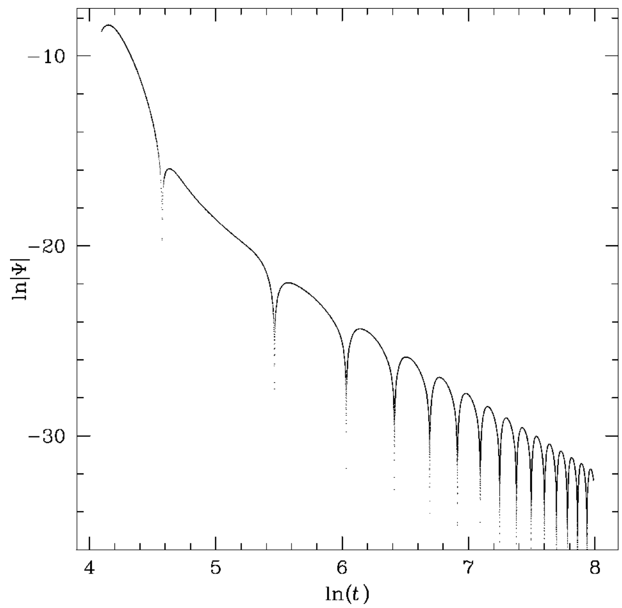


FIG. 2. Temporal evolution of the field for time-dependent scattering potentials of the form $V(x,t)=\sin(\omega t)/x^{\alpha}$ (the results presented here are for $\alpha=4$ and $\omega=\pi/100$). The slope (determined from the maxima of the oscillations) is $-4.07$, in excellent agreement with the *analytically* predicted value of $-4$. The frequency of the oscillations is $\omega$ to within 1%.

*Numerical calculations*. It is straightforward to integrate Eq. (1) using the methods described in [14,27]. The *late-time* evolution of the field is independent of the form of the initial data used. The results presented here are for a Gaussian pulse.

The temporal evolutions of the waves (under the influence of the various scattering potentials) are shown in Figs. 1 and 2. (We have studied other potentials as well, which are not shown here.) We find an excellent agreement between the *analytical* results and the *numerical* calculations.

In summary, we have explored the tail phenomena for spherical waves propagating under the influence of a general *time-dependent* scattering potential. It was shown that the late-time tail at a fixed spatial location is governed by the scattering potential itself, and by its derivatives (both the spatial and the temporal ones). The analytical results are in agreement with numerical calculations.

We are at present extending the analysis to include scattering potentials that lack spherical symmetry (in which case the scattering problem is of $2+1$ dimensions).

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