Ermakov-Pinney equation in scalar field cosmologies

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It is shown that the dynamics of cosmologies sourced by a mixture of perfect fluids and self-interacting scalar fields are described by the nonlinear, Ermakov-Pinney equation. The general solution of this equation can be expressed in terms of particular solutions to a related, linear differential equation. This characteristic is employed to derive exact cosmologies in the inflationary and quintessential scenarios. The relevance of the Ermakov-Pinney equation to the braneworld scenario is discussed.

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I. INTRODUCTION

Observations of the cosmic microwave background (CMB) power spectrum [1] now provide strong support for the inflationary scenario [2] (for recent reviews, see, e.g., [3]). The simplest mechanism for inflation utilizes the potential energy associated with the self-interaction of a scalar inflaton field to drive the accelerated expansion. High redshift observations of type Ia supernovae suggest that the universe is experiencing another phase of accelerated expansion at the present epoch [4]. This, combined with the CMB data, presents a picture of the universe dominated by a dark energy component [5,6]. One possible source of this dark energy is a scalar quintessence field that interacts with baryonic and nonbaryonic matter in such a way that its potential energy is currently dominating the cosmic dynamics [7]. The observations favor a model of structure formation where 70% of the energy density of the universe is presently in the form of quintessence [6]. The remaining fraction of the energy density is comprised of visible and cold dark matter which collectively act as a pressureless perfect fluid.

The ekpyrotic scenario has recently been proposed as an alternative to the standard inflationary cosmology [8]. In this scenario the big bang is interpreted as the collision of two domain walls or branes traveling through a fifth dimension. Before the collision, the effective dynamics on the four dimensional branes is described by Einstein's gravity minimally coupled to a self-interacting scalar field. The field parametrizes the separation between the branes and at early times slowly rolls down a *negative* potential. This results in an accelerated collapse of the universe and since the field is minimally coupled, its energy density is related to the Hubble parameter by the standard Friedmann equation. Thus, self-interacting scalar fields play a central role in modern cosmology and in view of the above developments, it is important to investigate cosmologies that contain both a scalar field and a perfect fluid.

In this paper, we develop an analytical approach to models of this type by expressing the cosmological field equations in terms of an Ermakov system [9–11]. In general, an Ermakov system is a pair of coupled, second-order, nonlinear ordinary differential equations (ODEs) [11] and such systems often arise in studies of nonlinear optics [12], nonlinear elasticity [13], molecular structures [14], quantum field theory in curved spaces [15] and quantum cosmology [16]. (For further references, see, e.g., Refs. [17].)

In the one-dimensional case, the two equations decouple and the system reduces to a single equation known as the Ermakov-Pinney equation [9,18]. This is given by[1]

$$\frac{d^2b}{d\tau^2}+Q(\tau)b=\frac{\lambda}{b^3}, \tag{1}$$

where Q is an arbitrary function of τ and λ is a constant. Although Eq. (1) is a nonlinear ODE, it exhibits a remarkable superposition property [10,20] that implies that its general solution can be expressed directly in terms of particular solutions to the related *linear*, second-order ODE, where $\lambda=0$ [18].

The paper is organized as follows. We discuss the Ermakov-Pinney equation in a cosmological setting in Sec. II and proceed in Sec. III to develop algorithms for solving the field equations. A number of models are considered in Sec. IV. Section V considers the role of the Ermakov-Pinney equation within the context of the recently developed braneworld scenario and we conclude with a discussion in Sec. VI.

II. THE ERMAKOV-PINNEY EQUATION IN COSMOLOGY

We begin by deriving the Ermakov-Pinney equation in a cosmological context. The field equations for a spatially flat, Friedmann-Robertson-Walker (FRW) universe with a scalar field and perfect fluid matter source are given by

$$H^2=\frac{\kappa^2}{3}\left(\frac{1}{2}\dot{\phi}^2+V(\phi)+\frac{D}{a^n}\right) \tag{2}$$

$$\ddot{\phi}+3H\dot{\phi}+\frac{dV}{d\phi}=0, \tag{3}$$

where $\rho_\phi\equiv\dot{\phi}^2/2+V(\phi)$ is the energy density of the scalar field with potential $V(\phi)$, $\rho_{\rm mat}\equiv Da^{-n}$ is the energy density

[1]Equation (1) is sometimes referred to as the Milne-Pinney equation [19].

of the barotropic perfect fluid with equation of state $p_{\rm mat}=[(n-3)/3]\rho_{\rm mat}$, a dot denotes differentiation with respect to cosmic time, t, $H\equiv\dot{a}/a$ represents the Hubble parameter, a is the scale factor of the universe, D is an arbitrary, positive constant, $0\leqslant n\leqslant 6$, $\kappa^2\equiv 8\pi m_P^{-2}$ and m_P is the Planck mass. Equation (3) represents the conservation of the energy momentum of the scalar field and can be expressed in the form $\dot{\rho}_\phi=-3H\dot{\phi}^2$.

One method of reducing this second-order, nonlinear pair of ODEs (2),(3) is to write them as a first-order system. This can be done if the scalar field varies monotonically with cosmic time. It then follows that Eq. (3) can be expressed as [21]

$$\frac{d\rho_\phi}{d\phi}=-3H\dot{\phi} \tag{4}$$

and the Friedmann equation (2) then takes the form

$$\frac{d\chi}{d\phi}\frac{d\rho_\phi}{d\phi}+(n\kappa^2)\chi\rho_\phi=-n\kappa^2 D, \tag{5}$$

where $\chi\equiv a^n$. The perfect fluid source leads to a nontrivial right-hand side in Eq. (5). The general solution to Eq. (5) can be expressed in terms of quadratures if the functional form of $\rho_\phi(\phi)$ is known:

$$\begin{aligned} a(\phi)=\exp\left[-\kappa^2\int^{\phi}d\tilde{\phi}\rho_\phi(\tilde{\phi})\left(\frac{d\rho_\phi}{d\tilde{\phi}}\right)^{-1}\right] \\ \times\left\{\Pi-n\kappa^2 D\int^{\phi}d\tilde{\phi}\left(\frac{d\rho_\phi}{d\tilde{\phi}}\right)^{-1}\right. \\ \left.\times\exp\left[n\kappa^2\int^{\tilde{\phi}}d\varphi\rho_\phi(\varphi)\left(\frac{d\rho_\phi}{d\varphi}\right)^{-1}\right]\right\}^{1/n}, \end{aligned} \tag{6}$$

where Π is a constant of integration. The effects of the fluid source are contained in the second term.

The disadvantage of this change of variables is that it is not valid if the scalar field exhibits oscillatory behavior at any time. We therefore consider a different approach by differentiating Eq. (2) and substituting Eq. (3). This implies that

$$\frac{\ddot{a}}{a}-\frac{\dot{a}^2}{a^2}=-\frac{\kappa^2}{2}\left(\dot{\phi}^2+\frac{nD}{3a^n}\right). \tag{7}$$

We now define an effective scale factor, b:

$$a\equiv b^{2/n} \tag{8}$$

and a new time parameter, τ:

$$\frac{d}{dt}\equiv b\frac{d}{d\tau} \tag{9}$$

and since b is positive-definite, τ is a monotonic increasing function of cosmic time, t. Equation (7) then transforms into the second-order, nonlinear Ermakov-Pinney equation

$$\frac{d^2b}{d\tau^2}+\frac{n\kappa^2}{4}\left(\frac{d\phi}{d\tau}\right)^2 b=-\frac{Dn^2\kappa^2}{12}\frac{1}{b^3} \tag{10}$$

and comparison with Eq. (1) then implies that

$$Q=\frac{n\kappa^2}{4}\left(\frac{d\phi}{d\tau}\right)^2,\quad \lambda=-\frac{Dn^2\kappa^2}{12}. \tag{11}$$

The change of variables (8) and (9) is interesting because Pinney [18] has shown that two linearly independent solutions, $(b_1(\tau),b_2(\tau))$, of the time-dependent oscillator equation

$$\frac{d^2b}{d\tau^2}+Q(\tau)b=0 \tag{12}$$

can be combined to give the general solution to the Ermakov-Pinney equation (1):

$$b_P=[Ab_1^2+Bb_2^2+2Cb_1b_2]^{1/2}, \tag{13}$$

where $\{A,B,C\}$ are constants satisfying the constraint equation

$$AB-C^2=\frac{\lambda}{W^2} \tag{14}$$

and the Wronskian

$$W\equiv b_1\frac{db_2}{d\tau}-b_2\frac{db_1}{d\tau} \tag{15}$$

is a constant due to Abel's theorem. It can be verified by direct differentiation that Eq. (13) does indeed satisfy Eq. (1). An important property of Eq. (12) is that if a particular solution, $b_1(\tau)$, is known, a second solution can be written in terms of a quadrature:

$$b_2(\tau)=Wb_1(\tau)\int^{\tau}\frac{d\tilde{\tau}}{b_1^2(\tilde{\tau})}. \tag{16}$$

Thus, the general solution to the Ermakov-Pinney equation (10) is found in terms of a particular solution to Eq. (12).

The perfect fluid component ($D\neq 0$) results in the nonlinear sector on the right-hand side of Eq. (10) and the effects of the scalar field are parametrized entirely in terms of the function, Q, defined in Eq. (11). We refer to this function as the ''oscillator potential.'' It is worth noting that Eq. (10) is independent of the specific functional form of the scalar field interaction potential, $V(\phi)$.

For the pure scalar field model ($D=0$), Eq. (12) may also be interpreted as a classical analogue of the zero-energy Schrödinger equation, where the scale factor, b, plays the role of the ''wave function'' and the ''wave vector,'' $Q(\tau)$, is determined by the kinetic energy of the scalar field. An equation of this form has arisen previously in studies of FRW cosmologies containing only a perfect fluid source with a constant equation of state $\omega=p_{\rm mat}/\rho_{\rm mat}$ [22] and for the

special case of a self-interacting scalar field that satisfies this condition [23]. This analogy may be extended further by considering an alternative reparametrization, where a new scale factor and time variable are defined by $c \equiv a^{-n/2}$ and $\sigma \equiv \int dtc$, respectively. It follows that Eq. (7) may then be expressed as a one-dimensional Schrödinger equation:

$$\frac{d^2c}{d\sigma^2} + [E - P(\sigma)]c = 0, \tag{17}$$

where $E = -(n^2\kappa^2 D)/12$ represents the total energy of the system, and is determined by the momentum of the perfect fluid, and the potential energy is given by $P = (n\kappa^2/4)(d\phi/d\sigma)^2$ in terms of the scalar field.

There is a close relationship between the Schrödinger equation (17) and the Ermakov-Pinney equation (1), as discussed by Milne [19], who developed a method for solving the former equation in full generality in terms of a particular solution of the latter, in such a way that the oscillatory behavior of the wavefunction is manifest. In this approach, the general solution to Eq. (12) is expressed in the form

$$b_H = U\psi(\tau)\cos(\sqrt{\lambda}\,\theta(\tau) + \epsilon), \tag{18}$$

where $\{U, \epsilon\}$ are arbitrary constants and $\{\psi(\tau), \theta(\tau)\}$ are functions that need to be determined. Substituting this ansatz into the homogeneous equation (12) reveals that Eq. (18) represents the general solution if these functions satisfy the coupled system of ODE's:

$$\frac{d^2\psi}{d\tau^2} + Q(\tau)\psi = \frac{\lambda}{\psi^3} \tag{19}$$

$$\frac{d\theta}{d\tau} = \frac{1}{\psi^2} \tag{20}$$

and it follows, therefore, that the general solution to the Schrödinger equation (12) can be deduced, at least with respect to the quadrature, $\theta = \int^{\tau} d\tilde{\tau}\psi^{-2}(\tilde{\tau})$, if a solution to the Ermakov-Pinney equation (19) is known.

Furthermore, two linearly independent parametric solutions to Eq. (12) are given by $b_1 = \psi\cos(\sqrt{\lambda}\,\theta)$ and $b_2 = \psi\sin(\sqrt{\lambda}\,\theta)$, where the Wronskian is $W^2 = \lambda$, $\theta(\tau)$ satisfies Eq. (20) and $\psi(\tau)$ is a particular solution to Eq. (19). Substituting these two solutions into Eq. (13) gives

$$b_P = \psi[A\cos^2(\sqrt{\lambda}\,\theta) + B\sin^2(\sqrt{\lambda}\,\theta) + (AB-1)^{1/2}\sin(2\sqrt{\lambda}\,\theta)]^{1/2}, \tag{21}$$

where we have employed Eq. (14). This implies that the general solution to the Ermakov-Pinney equation (1) can be determined in terms of a particular solution, $\psi(\tau)$, to this equation [20].

To summarize thus far, the dynamics of a pure scalar field cosmology is determined by a one-dimensional oscillator equation with a time-dependent frequency. Depending on the parametrization chosen, the inclusion of a perfect fluid source results in an effective constant shift in the total energy of the system, as in Eq. (17), or equivalently introduces a nonlinear term as is the case in Eq. (10). In particular, the problem of solving Eq. (10) for a coupled perfect fluid and scalar field cosmology has been reduced to solving Eq. (12) for a given functional form of $Q(\tau)$. This is interesting because this latter equation has been studied extensively in the literature. For example, algorithms that extend the Wentzel, Kramers and Brilluen (WKB) technique have recently been developed to find power series solutions to Eq. (12) [24]. For related work, see also Ref. [25].

III. ALGORITHMS FOR SOLVING THE FIELD EQUATIONS

We now proceed to outline an algorithm for deriving exact cosmological models when both a scalar field and perfect fluid are dynamically significant. Attention to date has primarily focused on pure scalar field models [21,26–28]. Some specific models with a perfect fluid were also recently analyzed [29]. From a particle physics perspective the potential of the scalar field is the fundamental quantity that defines the model. In general, however, Eqs. (2) and (3) are very difficult, if not impossible, to solve unless the potential has a specific form, as in the exponential case [26]. There are several possible routes to take when solving the Ermakov-Pinney equation (1). Given its properties as discussed above, one possibility is to specify the time dependence of the scalar field, $\phi = \phi(\tau)$, since the functional form of $Q = Q(\tau)$ is unaltered when deriving the solution (13) from the corresponding solutions to the homogeneous equation (12).

From the cosmological point of view, however, it is the time-dependence of the scale factor that is important. In particular, the effective equation of state and the potential of the scalar field can in principle be determined directly from high redshift observations if the dependence of the scale factor on redshift is known to a sufficiently high accuracy [30,31]. This question has recently attracted renewed interest in light of the proposed Supernova/Acceleration Probe (SNAP) [31,32].

Instead of choosing the potential, therefore, an alternative approach is to invert the problem by first specifying the dependence of the scale factor on cosmic time [26,27]. One could begin by substituting an ansatz for the scale factor directly into Eq. (19). The disadvantage of such an approach, however, is that a square root must be taken in order to express the resulting equation as a first-order, nonlinear ODE in the scalar field and, in general, the resulting integral can not be performed analytically. In practice, it is more convenient to begin with an ansatz for the scale factor, $a_1(t)$, or equivalently, the rescaled function, $b_1(t)$, and employ Eq. (12) to find $Q(\tau)$. Integrating this function then yields the time dependence of the scalar field. The general solution to the Ermakov-Pinney equation can be deduced immediately from Eqs. (13) and (16). The scale factor for the Ermakov-Pinney equation in parametric form then follows from Eq. (8). Its dependence on cosmic time may be evaluated by integrating Eq. (9) and inverting the result. Finally, the scalar field potential is reconstructed directly from the Friedmann

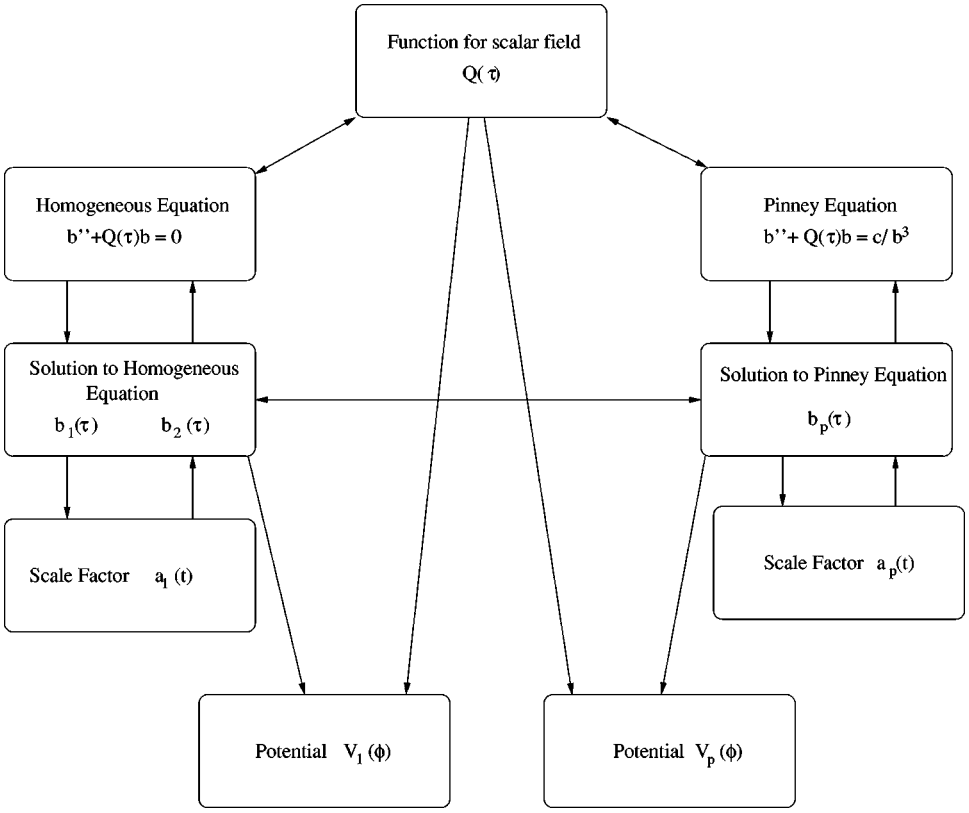

FIG. 1. Illustrating the possible routes for finding the scalar field potential in a pure scalar field system and in a system comprised of both a scalar field and a perfect fluid. The cosmological Ermakov-Pinney equation (10) plays a central role. The two potentials can be found from a given functional form of the oscillator potential, $Q(\tau)$, or from the scale factors, $a(t)$. In principle, it is consistent to begin at any point on this figure. From a cosmological perspective, the scale factor is the primary function of interest, and specifying this parameter from the outset is physically well motivated.

equation (2), rewritten in terms of the new variables (8) and (9):

$$V(\tau)=\frac{12}{n^2\kappa^2}\left(\frac{db_P}{d\tau}\right)^2-\frac{1}{2}b_P^2\left(\frac{d\phi}{d\tau}\right)^2-\frac{D}{b_P^2}. \tag{22}$$

The form of $V(\phi)$ then follows by inverting $\phi(\tau)$ and substituting $\tau(\phi)$ into Eq. (22).

The various possible algorithms are illustrated in Fig. 1.

IV. EXACT COSMOLOGIES SOURCED BY SCALAR FIELDS AND PERFECT FLUIDS

We now illustrate this procedure with a number of examples. The simplest case is that of the scale factor as an exponential function of time $a_1(t)=e^{2t/n}$ where the power has been chosen so that $b_1(\tau)=\tau$ and therefore $b_2(\tau)=1$. Such a choice produces a static scalar field, $d\phi/d\tau=0$, corresponding to a pure cosmological constant. The solution for a cosmological constant with a perfect fluid is already known [33], but this example illustrates the generality of the technique. Thus, from Eqs. (8) and (13), the general solution for the scale factor when a perfect fluid is present is given by

$$a_P=(A\tau^2+2C\tau+B)^{1/n}, \tag{23}$$

where $AB-C^2=-Dn^2\kappa^2/12$. Since the scalar field remains independent of time, the potential is also a constant, $V=12A/(\kappa^2n^2)$, from Eq. (22). Finally, the dependence of the scale factor on cosmic time is deduced by using Eq. (8) and integrating Eq. (9):

$$a_P=\left(\frac{-\lambda}{A}\right)^{1/n}\sinh^{2/n}(\sqrt{A}t). \tag{24}$$

A more general ansatz for the scale factor is given by

$$a_\pm(t)=Gt^{q_\pm}, \tag{25}$$

where $q_\pm$ are constants and $G\equiv[2/(2+nq_\pm)]^{q_\pm}$. Using the relationship between a and b given in Eq. (8), it is possible to express the two solutions to the homogeneous equation as

$$b_\pm=\tau^{p_\pm}, \tag{26}$$

where $p_\pm=nq_\pm/(2+nq_\pm)$. Solving the homogeneous equation (12) then gives the form of the scalar field $\phi(\tau)$,

$$\phi=F\ln\tau, \tag{27}$$

where the constant F is related to the power $p_\pm$ via the quadratic equation

$$p_\pm=\frac{1\pm\sqrt{1-n\kappa^2F^2}}{2}. \tag{28}$$

The Wronskian is $W^2=1-n\kappa^2F^2$ and, for consistency, we assume that $n\kappa^2F^2<1$ in what follows. The general solution to Eq. (10) is then found by substituting Eq. (26) into Eq. (13) and is

$$a_P=(A\tau^{2p_+}+B\tau^{2p_-}+2C\tau)^{1/n}. \tag{29}$$

The analytical form of the potential deduced from Eq. (22) is a combination of exponential functions of the scalar

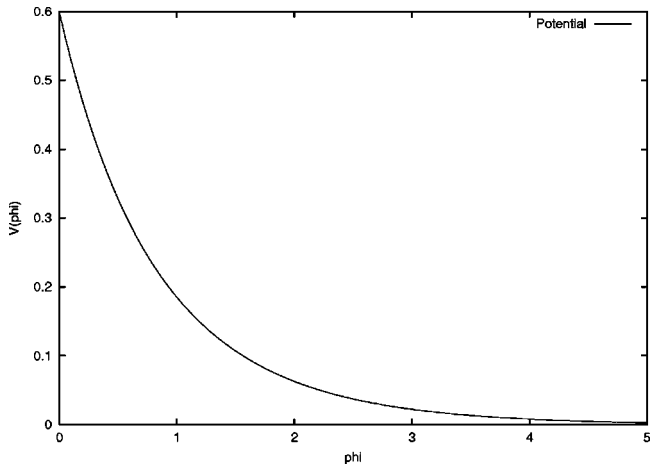

FIG. 2. The potential (32) for $A=1$, $B=0$, $C=\sqrt{3}$, $D=1$, $F=1/2$, $\kappa=1$ and $n=3$. The potential asymptotes to an exponential form at high and low energies.

field. The simplest case arises for $A=B=0$. Equations (11) and (14) then imply a relationship between the constants:

$$D=\frac{12C^2}{n^2\kappa^2}(1-n\kappa^2F^2) \tag{30}$$

and consequently, the scale factor, the scalar field and its potential are given by

$$a_P(t)=(Ct)^{2/n}$$

$$\phi(t)=F\ln\left(\frac{C}{2}\right)+2F\ln t$$

$$V(\phi)=\frac{CF^2(6-n)}{n}\exp\left(-\frac{\phi}{F}\right). \tag{31}$$

The solution (31) describes a cosmology where the energy density of the scalar field tracks that of the perfect fluid in a way that leaves the dependence of the scale factor on cosmic time unaltered from the pure perfect fluid model [34]. The self-interaction coupling, F, of the scalar field is constrained only by the condition that the fluid integration constant, D, be real.

Equation (31) represents a past or future attractor for the more general classes of solutions given by Eq. (29). For example, if $A\neq 0$ and $B=0$ the potential (shown in Fig. 2 for the case of a pressureless fluid) is given by

$$V=\frac{1}{Y}\left[\frac{12}{n^2\kappa^2}Z^2-\frac{F^2Y^2}{2}e^{-2\phi/F}-D\right], \tag{32}$$

where

$$Y\equiv Ae^{2p_+\phi/F}+2Ce^{\phi/F}$$

$$Z\equiv Ap_+e^{[(2p_+-1)\phi/F]}+C. \tag{33}$$

In the early-time limit ($\tau\to 0$), Eq. (32) asymptotes to Eq. (31). On the other hand, the potential reduces to

$$V_\infty=\frac{A}{2}\left(\frac{24p_+^2}{n^2\kappa^2}-F^2\right)\exp\left[-\frac{2p_-}{F}\phi\right] \tag{34}$$

at late times ($\tau\to\infty$), since $p_+>1/2$. In this limit, the scale factor asymptotes to $a\propto t^p$, where $p\equiv 2p_+/(np_-)$ and it is straightforward to verify that this power law behavior is the late-time attractor for a universe dominated by a single scalar field with an exponential potential, $V\propto\exp(\sqrt{2}\kappa\phi/\sqrt{p})$. Since $p_+>p_-$, the power of the expansion in this limit, p, exceeds that of the pure perfect fluid model ($a\propto t^{2/n}$) and it is greater than unity if $p_+>n\kappa F/\sqrt{8}$. In this region of parameter space, therefore, the solution given by Eqs. (29) and (32) describes a universe that behaves as a perfect fluid cosmology at early times. However, the perfect fluid becomes negligible as the expansion proceeds and the universe subsequently enters a phase of power law, inflationary expansion driven by the exponential potential of a scalar field. For $n=3$, the fluid is pressureless and current observations indicate that the universe is undergoing such a transition at the present epoch [4–6]. Thus, it has been demonstrated in the example outlined here that the algorithm for solving the cosmological field equations (2) and (3) using the Ermakov-Pinney equation (10) has generated a new exact solution describing the quintessence scenario. Moreover, exponential potentials often arise in superstring-inspired models through nonperturbative effects [35]. The asymptotic form of the general solution (29) is determined by the numerical values of the constants defined in Eq. (14). These in turn are determined by the initial conditions in the early universe. In particular, as we have shown, different late-time behavior arises if the constant A is nontrivial. Hence, the approach developed in Sec. II provides direct insight into the question of how the conditions for tracking and quintessence at late times depend on conditions in the early universe.

Another class of solvable models arises when the scale factor is a trigonometric function of τ. The two solutions to Eq. (12) are $b_1=\sin\gamma\tau$ and $b_2=\cos\gamma\tau$, where γ is a constant and the Wronskian is $W=-\gamma$. Given this ansatz for the scale factor, the scalar field varies linearly with τ, such that

$$\phi=\frac{2\gamma}{\sqrt{n}\kappa}\tau. \tag{35}$$

The solution to Eq. (10) is then given by

$$a_P=[A\sin^2\gamma\tau+B\cos^2\gamma\tau+C\sin 2\gamma\tau]^{1/n}. \tag{36}$$

In general, the potential is a complicated expression of trigonometric functions. However, for the particular choice $A=B=0$, Eq. (22) takes the simple form

$$V=-\frac{2C\gamma^2}{n^2\kappa^2}(n+6)\sin(\sqrt{n}\kappa\phi) \tag{37}$$

and it follows from Eq. (36) that $V\propto a_P^n$. Consequently, the potential energy of the scalar field is inversely proportional to the energy density of the perfect fluid.

So far in this section it has been demonstrated that once a solution to the pure scalar field system given in Eq. (12) is known, a solution to the Ermakov-Pinney equation (10) describing a scalar field and perfect fluid cosmology can be found. Before concluding this section, we employ the correspondence summarized in Eqs. (18)–(20) to show the reverse procedure, i.e., to derive a pure scalar field cosmology from a known solution to the Ermakov-Pinney equation (19). We invoke the ansatz $\psi(t)=\Lambda e^{\Lambda t}$ which in terms of the rescaled time variable τ is $\psi=\Lambda\tau$, where Λ is a constant. Substituting this ansatz into Eq. (19) and solving Eq. (11) for the scalar field gives

$$\phi=\mp\left(\frac{4\lambda}{n\kappa^2}\right)^{1/2}\frac{1}{\Lambda^2\tau}, \tag{38}$$

where the integration constant is specified to be zero without loss of generality. Integrating Eq. (20) implies that $\theta=-1/(\Lambda^2\tau)$ and after specifying $U=1$ and $\epsilon=0$ for simplicity, the scale factor of the universe and the potential for such a pure scalar field model are given by

$$a=\left[\Lambda\tau\cos\left(\frac{\sqrt{\lambda}}{\Lambda^2\tau}\right)\right]^{2/n} \tag{39}$$

$$V=\frac{2\Lambda^2}{n^2\kappa^2}[6(\cos\varphi+\varphi\sin\varphi)^2-n\varphi^2\cos^2\varphi], \tag{40}$$

where $\varphi\equiv\sqrt{n}\kappa\phi/2$. In the small field limit, $\varphi\ll 1$, the potential (40) approximates to an inverted harmonic oscillator and for sufficiently small values of the scalar field, the universe undergoes inflation [36]. Potentials of this form generically produce a spectrum of density perturbations in the postinflationary universe that increases in amplitude on larger scales without producing a detectable signal of gravitational waves [37]. Observations of CMB anisotropies presently favor such a spectrum [38].

V. BRANEWORLD SCENARIOS

It is also of interest to consider whether the approach outlined above can be applied to other cosmological scenarios. Considerable attention has been focused recently on the possibility that the observable universe may be viewed as a domain wall or brane [39] that is moving along a timelike geodesic in a five-dimensional, static, bulk spacetime [40–43], where the brane equations of motion are determined by the Israel junction conditions [44]. An observer confined to the domain wall interprets such motion in terms of cosmic expansion or contraction. The junction conditions relate the second fundamental form (extrinsic curvature) of the induced metric on the brane to the energy-momentum tensor of matter confined on the brane. For the case of a spatially flat brane moving through a Schwarzschild–anti–de Sitter space, the effective Friedmann equation on the brane is, after some simplifying assumptions [41,43,45,46], given by

$$H^2=\frac{\kappa^2}{3}\rho\left(1+\frac{\rho}{2\lambda}\right)+\frac{\mu_n}{a^n}, \tag{41}$$

where λ is the brane tension, μ_n is related to the mass of the black hole in the bulk and $n=4$. The standard, linear dependence of the Friedmann equation on the energy density, ρ, is recovered in the low-energy regime, $\rho\ll\lambda$. Further modifications to the Friedmann equation (41) may also arise in more general braneworld settings. For example, a term of the form $\Sigma_6=\mu_6 a^{-6}$, where $\mu_6<0$, arises if the black hole carries an electric charge [43,47].

For this model, the energy-momentum tensor of matter on the brane is covariantly conserved, and consequently, Eq. (3) must also be satisfied for the case of a single, self-interacting scalar field. Equations (3) and (41) are therefore relevant to a number of braneworld scenarios. By differentiating Eq. (41) with respect to cosmic time, substituting for the scalar field from Eq. (3), and using the change of variables (8) and (9), it can be shown that the dynamics of this system can be written in the Ermakov-Pinney form

$$\frac{d^2b}{d\tau^2}+\tilde{Q}b=-\frac{n^2\mu_n}{4b^3}, \tag{42}$$

where

$$\tilde{Q}\equiv\frac{n\kappa^2}{4}\left(1+\frac{\rho}{\lambda}\right)\left(\frac{d\phi}{d\tau}\right)^2. \tag{43}$$

In Eq. (42), the effects that lead to the quadratic dependence of the Friedmann equation (41) on the energy density of the scalar field modify only the oscillator potential (43), whereas the nonlinear term on the right-hand side arises through the bulk black hole contribution. Thus, in principle, a similar approach to that outlined above in Sec. III may be developed for analytically investigating the importance of these terms on braneworld cosmologies. For example, if a given solution is known for the case where there is no black hole in the bulk [48], it may serve as a seed for generating a corresponding cosmology when the black hole is present. This is interesting since it has recently been shown that such terms can significantly alter the qualitative dynamics of braneworld cosmologies [49].

VI. SUMMARY

In this paper it has been demonstrated that after a suitable redefinition of variables, Eqs. (2),(3) can be related to the simplest Ermakov system, thus reducing the problem of solving these equations to solving a single second-order, linear ODE. Moreover, the nature of the Ermakov-Pinney equation implies that there exists a correspondence between a spatially flat FRW universe containing a scalar field and a cosmology containing both a scalar field and a perfect fluid. The functional form of the scalar field, $\phi(\tau)$, is identical for the two models, but the potentials, $V(\phi)$, are different in each case. We have proposed an algorithm for finding exact solutions for a scalar field and perfect fluid model by employing a pure

scalar field cosmology as a seed. The algorithm is founded on the property that a particular solution to the homogeneous equation (12) leads to the general solution of Eq. (1). It has also been shown that a model containing both a scalar field and perfect fluid can act as a seed for finding more general pure scalar field cosmologies, since a particular solution to Eq. (1) allows the general solution of Eq. (12) to be deduced [19]. Exact solutions are important because they allow one to gain insight into the generic nature of cosmologies of this type and they also provide a framework for classifying the different behaviors that may arise.

The emphasis of the present work has focused on the effect of introducing a perfect fluid source. However, the approach we have developed may also be employed to consider pure scalar field cosmologies in more general spacetimes. In particular, the last term on the right-hand side of Eq. (2) may be interpreted as arising from the spatial curvature of the universe if $n=2$, where $D>0$ $(D<0)$ for negative (positive) curvature. On the other hand, the shear of the spatially flat, anisotropic Bianchi type I metric leads to a term where $n=6$.

Finally, it would be of interest to investigate whether other classes of scalar field cosmologies can be analyzed in terms of an Ermakov system. Scalar-tensor cosmologies provide one such example, where the scalar field plays the role of the gravitational coupling [50]. Many higher-order and higher-dimensional theories of gravity, including the string effective actions, may be expressed in a scalar-tensor form [51]. Since these theories are conformally equivalent to Einstein gravity that is minimally coupled to a scalar field, it is to be expected that a similar approach can be employed. A related, but distinct, class of theories has recently been developed to allow for a possible variation in the fine-structure constant [52]. Observational evidence for such an effect has been growing [53] and this behavior can again be parametrized in terms of a scalar field that is coupled to the matter fields in an appropriate fashion.

In conclusion, therefore, Ermakov systems have applications in many branches of mathematics and physics. In this paper, we have found that the simplest such system is relevant to scalar field cosmological models that are favored by astrophysical observations.

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