

Note on inflation and trans-Planckian physics

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In this paper we consider the influence of trans-Planckian physics on the CMBR anisotropies produced by inflation. We consider a simple toy model that allows for analytic calculations and argue on general grounds, based on ambiguities in the choice of vacuum, that effects are expected with a magnitude of the order of H/Λ , where H is the Hubble constant during inflation and Λ the scale for new physics, e.g., the Planck scale.

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I. INTRODUCTION

In recent years it has been realized that much can be learned about the highest energies and the smallest scales by studying cosmology and in particular the very early universe. An especially intriguing idea in this context is inflation. For some nice introductions to inflation with references, see [1,2]. Inflation successfully solves several problems of the standard big bang scenario, and also makes a number of new predictions. Of particular interest is the cosmic microwave background radiation (CMBR) anisotropies which currently is measured with higher and higher precision. Inflation magnifies tiny quantum fluctuations generated a fraction of a second after the big bang into seeds that eventually cause the formation of galaxies and clusters of galaxies. The fluctuations leave an imprint on the CMBR that can be used to test inflation at high precision.

Recently a tantalizing possibility has been discussed in the literature that suggests that inflation might provide a window towards physics beyond the Planck scale [3–24]. Since inflation works by magnifying microscopic quantum fluctuations into cosmic size, it is reasonable to worry about the initial linear size of the fluctuations. Were they ever smaller than the Planck scale? Typically inflation is discussed from a purely field theoretic perspective, and the only scale in the problem is, basically, the vacuum energy that generates inflation. As a consequence the quantum fluctuations are supposed to originate in the infinite past with an infinitely short wavelength. But in the real world we know that fundamentally new physics is to be expected at the Planck scale, and this simple picture cannot be correct. The key question, then, is whether modifications of the high energy behavior can change the predictions of inflation. So far no real consensus has been reached in the literature, and there are at least two competing estimates of the size of the corrections to the CMBR spectrum. In, e.g., Ref. [12] the corrections are argued to be of size $(H/\Lambda)^2$, while in, e.g., Refs. [13,16], one is dealing with substantially larger corrections of order H/Λ . Λ is the energy scale of new physics, e.g., the Planck scale or the string scale, and H is the Hubble constant during inflation. Very recently it was argued in [25], using a low energy effective field theory, that local physics imply that the effects cannot be larger than $(H/\Lambda)^2$. This conclusion has

been criticized in [26], where it was pointed out that trans-Planckian physics can effectively provide the low energy theory with an excited vacuum, thereby circumventing the arguments of [25].

The purpose of the present paper is to discuss the trans-Planckian problem from the point of view of an extremely simple modification of the standard scenario where we focus on the choice of vacuum. In the usual model of inflation, the initial state is assumed to be the empty vacuum in the infinite past when all scales that have a finite linear size today have a size infinitely smaller than the Planck scale. Even though this does not make too much sense—after all, we have no idea of how the physics at these scales work—it is interesting that this naive approach seems to give sensible results. To test how robust the predictions are, one has, in the various works mentioned above, changed by hand the high energy behavior in different ways to see whether and how much the result is influenced. In this paper we will follow a more conservative approach. We will note when a given mode reaches a certain minimum scale and encode our ignorance in the choice of state at that time. For each mode we will choose a time when the mode is of a specific size comparable to the Planck scale and impose, at that time, some reasonable initial conditions. A fixed scale for imposing the initial condition implies that the larger the mode is today, the further back in time we need to go to impose the initial condition. In fact, one can think of a semi-eternal inflation where modes appear out of nowhere, always at the same minimum scale, after which inflation takes over and makes them grow larger. This happens all the time until, for some reason, inflation turns off. In this way there is a continuous creation of fluctuations with no unique moment in time (not even the infinite past); it is instead the unknown small scale physics that produce a certain state at the minimum scale.

But what is the state created at the minimum scale? The standard proposal says that it is “the state that would have been produced if no new physics occurs on small scales and we start off with an empty vacuum in the infinite past.” Clearly there is *a priori* no justification for such a claim. It might be a reasonable first guess, but it should be the subject of criticism and discussion. Another proposal, no less plausible or implausible, would be that the state produced at the minimum scale is the vacuum determined by some principle of naturalness at *that* time. The idea would then be that the span provided by the two choices gives a reasonable estimate of how uncertain our prediction of the fluctuation spectrum is

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due to trans-Planckian physics. That is, how sensitive inflation is to reasonable variations in the initial conditions.

In agreement with [26] we will find corrections larger than those discussed in [25]. Similar ideas as those presented in this paper can also be found in [13,16], but our analysis will be made in a simpler model that allows for analytic results and provides some further insight into what the nature of the effect is. In particular, we will discuss the role of the adiabatic vacuum. In fact, our main result will be that *the expected magnitude of the trans-Planckian corrections will be given by the magnitude of the first order corrections to the zeroth order adiabatic vacuum.*

II. A SIMPLE MODEL

A. A Heisenberg setup

Let us consider an inflating background with metric

$$ds^2 = dt^2 - a(t)^2 dx^2, \quad (1)$$

where the scale factor is given by $a(t) = e^{Ht}$. The equation for a scalar field in this background is given by

$$\ddot{\phi} + 3H\dot{\phi} - \nabla^2\phi = 0. \quad (2)$$

In terms of the conformal time $\eta = -1/aH$, and the rescaled field $\mu = a\phi$, we find

$$\mu_k'' + \left(k^2 - \frac{a''}{a}\right)\mu_k = 0 \quad (3)$$

in Fourier space. Prime refers to derivatives with respect to conformal time. Note that we have $k = ap$, where p is the physical momentum which is redshifting away with the expansion (k is fixed). We will also need the conjugate momentum to μ_k which is given by

$$\pi_k = \mu' - \frac{a'}{a}\mu_k. \quad (4)$$

When quantizing the system it turns out that the Heisenberg picture is the most convenient one to use. A nice discussion of this approach can be found in [27], see also [2]. In terms of time dependent oscillators we can write

$$\mu_k(\eta) = \frac{1}{\sqrt{2k}}(a_k(\eta) + a_{-k}^\dagger(\eta)) \quad (5)$$

$$\pi_k(\eta) = -i\sqrt{\frac{k}{2}}(a_k(\eta) - a_{-k}^\dagger(\eta)).$$

The oscillators can be conveniently expressed in terms of their values at some fixed time η_0 ,

$$\begin{aligned} a_k(\eta) &= u_k(\eta)a_k(\eta_0) + v_k(\eta)a_{-k}^\dagger(\eta_0) \\ a_{-k}^\dagger(\eta) &= u_k^*(\eta)a_{-k}^\dagger(\eta_0) + v_k^*(\eta)a_k(\eta_0), \end{aligned} \quad (6)$$

which is nothing but the Bogolubov transformations which describe the mixing of the creation and annihilation operators as time goes by. Plugging this back into the expressions for $\mu_k(\eta)$ and $\pi_k(\eta)$ we find

$$\begin{aligned} \mu_k(\eta) &= f_k(\eta)a_k(\eta_0) + f_k^*(\eta)a_{-k}^\dagger(\eta_0) \\ \pi_k(\eta) &= -i(g_k(\eta)a_k(\eta_0) - g_k^*(\eta)a_{-k}^\dagger(\eta_0)), \end{aligned} \quad (7)$$

where

$$\begin{aligned} f_k(\eta) &= \frac{1}{\sqrt{2k}}(u_k(\eta) + v_k^*(\eta)) \\ g_k(\eta) &= \sqrt{\frac{k}{2}}(u_k(\eta) - v_k^*(\eta)). \end{aligned} \quad (8)$$

$f_k(\eta)$ is a solution of the mode equation (3). We are now in the position to start discussing the choice of vacuum. A reasonable candidate for a vacuum is

$$a_k(\eta_0)|0, \eta_0\rangle = 0. \quad (9)$$

In general this corresponds to a class of different vacua depending on the parameter η_0 . At this initial time it follows from Eq. (6) that $v_k(\eta_0) = 0$, and the relation between the field and its conjugate momentum is particularly simple:

$$\pi_k(\eta_0) = ik\mu_k(\eta_0). \quad (10)$$

This choice of vacuum has a simple physical interpretation. Following [27] it is easy to show that it corresponds to a state which minimizes the uncertainty at $\eta = \eta_0$. Using $\langle\mu_k\rangle = \langle\pi_k\rangle = 0$ it follows that

$$\begin{aligned} \langle\Delta\mu_k\Delta\mu_{k'}\rangle &= |f_k|^2\delta^{(3)}(\mathbf{k}-\mathbf{k}') \\ \langle\Delta\pi_k\Delta\pi_{k'}\rangle &= |g_k|^2\delta^{(3)}(\mathbf{k}-\mathbf{k}'), \end{aligned} \quad (11)$$

where

$$|f_k|^2|g_k|^2 = \frac{1}{4}(1 + |uv - u^*v^*|^2). \quad (12)$$

The latter expression is indeed minimized at $\eta = \eta_0$ where $v_k(\eta_0) = 0$.

We will now show that the vacuum defined in this way can be referred to as the zeroth order adiabatic vacuum.

B. The role of the adiabatic vacuum

In a time dependent background the notion of a vacuum is a tricky issue. The ideal situation is if there is only some transitional time dependence, in which case there is a unique definition of the vacuum in the infinite past as well as in the infinite future. The time evolution of the initial vacuum will, however, not necessarily generate the final vacuum, a phenomenon which we interpret as the creation of particles. With a time dependence that never shuts off, the situation is less clear. One possibility is to use the adiabatic vacuum, where the solution of the wave equation is, formally, as-

sumed to be of WKB form. A nice discussion of the adiabatic vacuum can be found in [28]. Often the exact solution is expanded to some finite order in the adiabatic parameter that determines the slowness of the process. The idea is to approximate the field equations, at some moment in time, with their time independent counterparts (possibly with corrections to some finite order) and define positive and negative energy using solutions to these approximative equations. Even though the adiabatic vacuum obtained in this way in general does not correspond to a solution of the exact field equation, it certainly corresponds to *some* specific choice of vacuum. What one should remember, however, is that the adiabatic vacuum (to some finite order in the adiabatic parameter) is not unique but depends on what moment in time one uses for its definition. In de Sitter space, however, it happens that the finite order adiabatic vacuum obtained in the infinite past actually corresponds to an exact solution of the exact field equations, and therefore in some sense is distinguished. After all, when the modes are small enough they do not care about the expansion of the universe.

Which vacuum should we choose? One possibility is to use the adiabatic vacuum of arbitrary order—corresponding to an exact solution—but there are also other choices like the one of minimum uncertainty discussed in the previous section. As we will argue below the minimum uncertainty vacuum agrees with the adiabatic one only to zeroth order. In fact, it is only at zeroth order, where the expansion of the universe can be ignored, that ambiguities in the definition of the vacuum are removed. It is important to observe that these distinctions between various vacua only become important since we insist on imposing the choice of vacua at a finite time corresponding to some specific finite wavelength, e.g. the Planck scale. Any claim about the structure of the vacuum beyond the zeroth order requires knowledge of physics on this scale. Since such knowledge is currently not available, we can only list various alternatives. The vacuum choice of the previous section represents one such viable alternative, besides the standard one, and can be used to indicate the natural span of possibilities. Let us now proceed with a more detailed analysis.

In the zeroth order adiabatic approximation, the solution of a mode equation of the form

$$\mu_k'' + (k^2 - C(\eta))\mu_k = 0, \quad (13)$$

is given by

$$\mu_k = \frac{1}{\sqrt{2\omega}} e^{\pm i\omega\eta}, \quad (14)$$

where

$$\omega = \sqrt{k^2 - C(\eta)}. \quad (15)$$

For the approximation to make sense we must have an ω that varies slowly enough (i.e., adiabatically). A necessary condition for this to be the case is that

$$\frac{d}{d\eta} \ln C \ll \omega, \quad (16)$$

which for us [where $C(\eta) = 2/\eta^2$] typically leads to

$$k\eta \gg 1. \quad (17)$$

With the help of this the zeroth order solution simply degenerates into

$$\mu_k = \frac{1}{\sqrt{2k}} e^{\pm ik\eta}, \quad (18)$$

and one finds a conjugate momentum given by

$$\pi_k = ik\mu_k. \quad (19)$$

This is precisely what our choice in the previous subsection led to, and we can therefore refer to the vacuum that we will analyze as the zeroth order adiabatic vacuum. As pointed out above, a finite order adiabatic mode is in general not an exact solution of the field equations, but the vacuum that it corresponds to is nevertheless an honest proposal for a vacuum. One should view Eq. (19) as initial conditions with a subsequent time evolution given by the exact solution.

C. Imposing the initial conditions

Let us now consider the standard treatment of fluctuations in inflation. In this case we have

$$f_k = \frac{1}{\sqrt{2k}} e^{-ik\eta} \left(1 - \frac{i}{k\eta} \right) \quad (20)$$

and

$$g_k = \sqrt{\frac{k}{2}} e^{-ik\eta}. \quad (21)$$

The logic behind the choice is that the mode at early times (when $\eta \rightarrow -\infty$) is of positive frequency and corresponds to what one would naturally think of as the vacuum. It is nothing but the state obeying Eq. (9) for $\eta_0 \rightarrow -\infty$ and is therefore the zeroth order adiabatic vacuum of the infinite past. Note that the zeroth order adiabatic vacuum in this case is actually an exact solution (for $\eta \rightarrow -\infty$). For later times (when $\eta \rightarrow 0$ and the second term of f_k becomes important) we see how the initial vacuum leads to particle creation thereby providing the fluctuation spectrum.

But what if the initial conditions are chosen differently? In general we could have

$$f_k = \frac{A_k}{\sqrt{2k}} e^{-ik\eta} \left(1 - \frac{i}{k\eta} \right) + \frac{B_k}{\sqrt{2k}} e^{ik\eta} \left(1 + \frac{i}{k\eta} \right) \quad (22)$$

$$g_k = A_k \sqrt{\frac{k}{2}} e^{-ik\eta} - B_k \sqrt{\frac{k}{2}} e^{ik\eta},$$

with a nonzero B_k . If we then work backwards, we can calculate what this corresponds to in terms of u_k and v_k . The result is

$$\begin{aligned} u_k &= \frac{1}{2} \left(A_k e^{-ik\eta} \left(2 - \frac{i}{k\eta} \right) + B_k e^{ik\eta} \frac{i}{k\eta} \right) \\ v_k^* &= \frac{1}{2} \left(B_k e^{ik\eta} \left(2 + \frac{i}{k\eta} \right) - A_k e^{-ik\eta} \frac{i}{k\eta} \right). \end{aligned} \quad (23)$$

At this point we should also remember that

$$|u_k|^2 - |v_k|^2 = 1 \quad (24)$$

from which we find

$$|A_k|^2 - |B_k|^2 = 1. \quad (25)$$

As we have seen, the choice of vacuum that we make requires that we put $v_k^*(\eta_0) = 0$ at some initial moment η_0 . This implies that

$$B_k = \frac{ie^{-2ik\eta_0}}{2k\eta_0 + i} A_k, \quad (26)$$

from which we conclude that

$$|A_k|^2 = \frac{1}{1 - |\alpha_k|^2}, \quad (27)$$

where

$$\alpha_k = \frac{i}{2k\eta_0 + i}. \quad (28)$$

We next move to the calculation of the fluctuation spectrum given by

$$\begin{aligned} P_\phi &= \frac{1}{a^2} P_\mu = \frac{k^3}{2\pi^2 a^2} |f_k|^2 \sim \frac{1}{4\pi^2 \eta^2 a^2} \\ &\times (|A_k|^2 + |B_k|^2 - A_k^* B_k - A_k B_k^*) \\ &= \left(\frac{H}{2\pi} \right)^2 (1 + |\alpha_k|^2 - \alpha_k e^{-2ik\eta_0} - \alpha_k^* e^{2ik\eta_0}) \frac{1}{1 - |\alpha_k|^2}, \end{aligned} \quad (29)$$

where we have used $\eta = -1/aH$ in the prefactor and considered the leading term at late times when $\eta \rightarrow 0$. If we impose the initial condition at $\eta_0 \rightarrow -\infty$ we get $\alpha = 0$ and recover the standard result $P_\phi = (H/2\pi)^2$. But let us now do something different following the discussion in the Introduction. For a given k we choose a finite η_0 such that the physical momentum corresponding to k is given by some fixed scale Λ . Λ is the energy scale of new physics, e.g., the Planck scale or the string scale. From

$$k = ap = -\frac{p}{\eta H} \quad (30)$$

with $p = \Lambda$ we find

$$\eta_0 = -\frac{\Lambda}{Hk}. \quad (31)$$

It is important to note that η_0 depends on k . If we assume $\Lambda/H \gg 1$ we get

$$P_\phi = \left(\frac{H}{2\pi} \right)^2 \left(1 - \frac{H}{\Lambda} \sin\left(\frac{2\Lambda}{H} \right) \right), \quad (32)$$

which is our final result.¹

D. Some comments on the result

There are several comments one can make. First, one verifies that the size of the correction ($\sim H/\Lambda = |1/k\eta_0|$) is precisely what is to be expected from a higher order correction to the zeroth order adiabatic vacuum. If the vacuum is imposed in the infinite past, the vacuum is exact, but if it is imposed at a later time it is natural to expect nonvanishing corrections. Corrections of precisely this order of magnitude have been found in, e.g., Refs. [13,16] and as we have seen this can be expected on quite general grounds. These corrections are in general larger than those discussed in [25] which went like $(H/\Lambda)^2$.

Second, it is interesting to note that if we have a model of inflation where H is slowly changing, leading to a spectrum which is not exactly scale invariant, the correction term will be very sensitive to k through the dependence of H on k . That is, there will be a modulation of P_ϕ . In fact, one could expect this to be a rather general phenomena in models where the initial conditions are set at a particular scale. The modulation that we have found is precisely of the same form as in the numerical work of [16] which considered a specific example of slow roll. It would be interesting to study this phenomenon in a more systematic way for various models.

III. SUMMARY AND CONCLUSIONS

In this paper we have studied the possible influence of trans-Planckian physics on the fluctuation spectrum predicted by inflation. We have made use of an extremely natural initial condition: we require that the modes are created in a state of minimized uncertainty. If this is imposed in the infinite past there is no difference between this choice and the usual choice of an adiabatic vacuum. But contrary to the standard treatment we have imposed the initial condition not in the infinite past, but at a mode dependent time determined by when a particular mode reaches a size of the order of the fundamental scale (e.g. the Planck scale). As a consequence our analysis agrees with the standard choice only to zeroth order in an adiabatic expansion with corrections at first order. This should be viewed as a conservative approach appropriate

¹If the field that we are considering is a gravitational mode, P_ϕ directly gives the density fluctuations. For a scalar field, on the other hand, one needs to take an extra factor $(H/\dot{\phi})^2$ into account. See [2] for further details.

ate for estimating how well the fluctuation spectrum can be predicted without any knowledge of high energy physics. To phrase it differently: if measurements can be done at the accuracy required, trans-Planckian physics will be within reach. The size of the prediction is not large² but it would be interesting to further analyze under what circumstances it might be observable.

In this context one should also consider the bound found in [29]. There it is argued that present day physics severely limits how far from the standard vacuum choice one can deviate. The problem is that with a large deviation too many particles (e.g., gravitons) will be produced which could contribute to the present energy density. According to this estimate the coefficient in front of the wrong mode can be at most of the order H_0/Λ , where H_0 is the Hubble constant now. But, as argued in [13], when the coefficient is traced back in time it might very well correspond to a considerably larger ratio in the past. In fact, a natural expectation is that it becomes of the order H/Λ , meaning that it is really a tricky

²If Λ is the Planck scale, H/Λ is at most 10^{-4} . If Λ is the string scale, H/Λ could possibly be 10^{-2} in a very optimistic scenario.

question involving numbers of order not too far from one. Similar comments apply to the work of [30], which discusses back reaction due to particle production during inflation itself. More detailed discussions on these issues can be found in [26]. Actually, one way to view the argument of [29] is as yet another example of how sensitive inflation is to trans-Planckian physics.

The conclusion is, therefore, that effects of trans-Planckian physics are possibly within the reach of cosmological observations even though much more detailed calculations are required to make a definite statement. But even this is much more optimistic than the usual expectations in standard particle physics, and could imply a very exciting future for cosmology as well as string theory.

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