

Relativistic quantum measurement

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(Received 10 April 2002; published 15 July 2002)

Does the measurement of a quantum system necessarily break Lorentz invariance? We present a simple model of a detector that measures the spacetime localization of a relativistic particle in a Lorentz invariant manner. The detector does not select a preferred Lorentz frame as a Newton-Wigner measurement would do. The result indicates that there exists a Lorentz invariant notion of quantum measurement and sheds light on the issue of the localization of a relativistic particle. The framework considered is that of single-particle mechanics as opposed to field theory. The result may be taken as support for the interpretation postulate of the spacetime-states formulation of single-particle quantum theory.

DOI: 10.1103/PhysRevD.66.023510

PACS number(s): 98.80.Hw, 04.60.Kz

I. INTRODUCTION

Does the measurement of a quantum system necessarily break Lorentz invariance? Does a state prepared by a quantum measurement necessarily know about the Lorentz frame in which the measurement was performed? Of course, the center of mass of any measurement apparatus selects a Lorentz frame, and the prepared state may well depend on this frame. But can we use a fully Lorentz covariant description of the system and the apparatus, and formulate a Lorentz covariant measurement theory, including the projection postulate, such that all probabilities computed are Lorentz invariant?

The naive Copenhagen-style answer is that a quantum measurement does break Lorentz invariance: a measurement happens at a certain time T , namely on a specific simultaneity surface. Therefore it selects a Lorentz frame. As a consequence, for instance, the localization of a quantum relativistic particle is only defined after the choice of a frame. One often discusses the Newton-Wigner position operators [1], which are not covariant. According to the Newton-Wigner theory, we cannot simply measure whether or not the particle is at or around a spacetime point x . We can only measure whether or not the particle is around x in a certain Lorentz frame. This is reflected in the fact that the Newton-Wigner operators in different frames do not commute. Accordingly, a (generalized) quantum state prepared by a Newton-Wigner measurement does not depend only on the spacetime point—it also depends on a Lorentz frame at that point.

In the context of field theory, it is of course clear that localized and covariant measurements can be associated with *fields* smeared over regions of spacetime. Our concern here is rather with single-particle mechanics.

In particular, we are interested in the possible implications for quantum cosmology through its well-known mathematical analogy with single-particle relativistic mechanics. Although such quantum cosmology settings need not possess Lorentz invariant dynamics, the issue is often raised of whether quantum cosmology requires that some degree of

freedom be chosen as an “intrinsic time” [2,3] and taken to play a special role. Preserving Lorentz invariance in a Lorentz invariant model provides an elegant way to show that, despite appearances at intermediate steps in our procedure, no such special role has in fact been assigned to any degree of freedom. While there is no exact symmetry to preserve when similar methods are applied to models without Lorentz invariant dynamics, our results here suggest that these methods will continue to treat all degrees of freedom on an equal conceptual footing.

Consider then the single-particle setting. In this framework, can we not just measure whether or not the particle is around x , with no reference to a simultaneity surface? More precisely, is it not possible to compute a well defined probability $P_{y,x}$ of detecting the particle around y if it was previously detected around x , such that $P_{y,x}$ would not depend on a preferred Lorentz frame?

In this paper we argue that there is at least one limit in which this is possible, contrary to what is often assumed (but see [4]). Toward this aim, we present a simple model of a detector for a relativistic particle, and show that the probabilities of its outcomes are Lorentz invariant. More precisely, we consider two detectors. The first detects the particle in a region R_x around a point x in order to prepare the state for the second measurement. Assuming the particle has been detected (and therefore that its wave function has “collapsed”¹), we calculate the probability that the particle is then detected in a region R_y around y . We find this probability to be given by a Lorentz invariant function of R_x and R_y .

The two key ingredients for the definition of the detector are as follows. The first is the observation [5–9] that any realistic detector interacts with the system during a time interval which cannot be null. Thus, we shall not neglect the

¹We follow tradition and use such language, though it is not necessarily our intention to endorse a Copenhagen interpretation of quantum mechanics.

finite duration of the interaction. Therefore the detection of the particle “around” the point x does not mean here in small *space* region around x , but rather is a small *spacetime* region around x .

The second ingredient is the observation that any physical interaction, including the one between the system and the measuring device, must be Lorentz invariant. Thus we shall choose a Lorentz invariant interaction Hamiltonian describing the system/apparatus interaction. We take these two observations into account and perform a standard analysis of a measurement using first order perturbation theory and the standard Copenhagen theory of the wave function collapse at the time of the measurement. Our main tool is the standard trick of exploiting the freedom, pointed out by Von Neumann, of moving the boundary between the quantum system and the classical world. Thus, we describe the apparatus quantum mechanically, and assume that the Copenhagen measurements happen on the detector.

The intermediate steps of the calculation are highly non-covariant: the wave function collapses on a certain simultaneity surface and so on. Rather surprisingly the various factors that depend on the Lorentz frame cancel out at the end. The result suggests that, at least in the limit we consider, there exists a Lorentz invariant notion of quantum measurement and quantum collapse. One may also choose to take this as an indication that such an interpretation exists more generally.

The result also sheds light on the controversial issue of the localization of a relativistic particle. The states prepared and detected by the detector are different from the Newton-Wigner states. They were first introduced by Philips [10], though without a measurement interpretation. The result of this paper therefore shows that these states do correspond to a rather well defined measurement. Unlike the Newton-Wigner states, the Philips states are defined in a fully covariant manner.

Finally, a covariant interpretation of quantum theory based on the so-called spacetime states has been proposed in [9] (see also [11]). This interpretation is based on a covariant interpretation postulate on the extended configuration space. In [9] it was shown that in the context of nonrelativistic quantum mechanics this postulate is equivalent to the standard interpretation. The postulate was then assumed to be true, by inference, in more general contexts. The problem was raised of whether the postulate could be reconciled with the predictions of relativistic quantum particle mechanics. The result that we obtain here using standard quantum theory and taking a certain limit is precisely the postulate of [9]. Therefore the result presented here provides some support to the covariant formulation of quantum theory considered in [9].

II. A NONRELATIVISTIC PARTICLE DETECTOR

We begin by describing a related detector in the nonrelativistic context, following [9] (see also [6] for an earlier discussion of the same detector). This serves to set the stage for our relativistic (and Lorentz invariant) treatment in Sec. IV. For definiteness, we work in the familiar 3 + 1 dimensions in

both the relativistic and nonrelativistic cases, though the generalization to d dimensions is trivial.

We want to measure the position of the particle at a certain time. That is, we want to check whether the particle is present at a certain space point $\vec{x}=0$ at a certain time $t=0$. We thus set up a physical apparatus that interacts with the particle. This apparatus will have a pointer that tells us whether or not the particle has been detected. We exploit the freedom in choosing the boundary between the quantum system under observation and the measuring apparatus: we treat the particle and the detector as the quantum system, and assume that the Von Neumann measurement is realized when the position of the pointer is observed. This trick allows us to better understand which aspect of the particle state is probed by an apparatus measuring the localization of the particle.

Consider a pointer which has two possible states: a state $|0\rangle$, which corresponds to no detection, and a state $|1\rangle$, which corresponds to detection. We represent the state space of the coupled particle-detector system by the Hilbert space $H_{PD}=H\otimes C^2$, where H is the Hilbert space of the particle and C^2 is the state space of a two-state system. We write a state of the combined system at time t as [we use the notation $x=(\vec{x},t)$]

$$\Psi(t)=\Psi_0(t)\otimes|0\rangle+\Psi_1(t)\otimes|1\rangle. \quad (1)$$

At any time after the interaction, one may describe the two terms in Eq. (1) as “branches” of the state corresponding to detection ($|1\rangle$) and nondetection ($|0\rangle$) of the particle.

We write the spacetime wave function of the particle’s states $\Psi_0(t)$ and $\Psi_1(t)$ as $\psi_0(\vec{x},t)=\langle\vec{x}|\Psi_0(t)\rangle$ and $\psi_0(\vec{x},t)=\langle\vec{x}|\Psi_0(t)\rangle$. The free Hamiltonian of the particle is $\vec{p}^2/2m$, and we take the free Hamiltonian of the detector to be zero. Note that here we are in a standard nonrelativistic setting so that the norm of Ψ_0 is $\int_{t=t_0}d^d x|\psi_0(x,t_0)|$, where d is the dimension of a $t=t_0$ slice and, as usual, unitarity guarantees the norm to be independent of t_0 .

We need an interaction Hamiltonian H_{int} , representing the interaction that gives rise to the measurement. H_{int} must have the following properties. First, it must cause the transition $|0\rangle\rightarrow|1\rangle$. Second, the particle should interact only at or around the spacetime position $\vec{x}=0,t=0$. Thus the interaction Hamiltonian must be time dependent, and vanish for late and early times. We have to concentrate the interaction around $t=0$. However, we cannot have a perfectly instantaneous interaction because this would require infinite force to have a finite effect. We must therefore assume that the interaction is nonvanishing for a finite period of time. Putting these requirements together, and requiring that the Hamiltonian is self-adjoint, we arrive at an interaction Hamiltonian of the form

$$H_{int}=\alpha V(\vec{x},t)(|1\rangle\langle 0|+|0\rangle\langle 1|) \quad (2)$$

where $\alpha V(\vec{x},t)$ is the potential acting on the particle in the interaction (with α a coupling constant). The potential $V(\vec{x},t)$ is concentrated in a small but finite spacetime region

\mathcal{R} , around $\vec{x}=0$ and $t=0$. For simplicity, we take V to be the characteristic function of the region \mathcal{R} (one on \mathcal{R} and zero elsewhere). Nothing substantial changes in the discussion if one uses a different function V .

The Schrödinger equation for the spacetime wave functions $\psi_0(\vec{x}, t)$ and $\psi_1(\vec{x}, t)$ reads

$$i\hbar \frac{\partial \psi_0}{\partial t} = -\frac{\hbar^2}{2m} \vec{\nabla}^2 \psi_0 + \alpha V \psi_1 \quad (3)$$

$$i\hbar \frac{\partial \psi_1}{\partial t} = -\frac{\hbar^2}{2m} \vec{\nabla}^2 \psi_1 + \alpha V \psi_0. \quad (4)$$

Assume that at some early time $t_{in} \ll 0$ the particle is in some arbitrary normalized initial state and the pointer is in the state $|0\rangle$:

$$\Psi(t_{in}) = \Psi_0(t_{in}) \otimes |0\rangle. \quad (5)$$

What is the state of the system at a later time $t_f \gg 0$? It is straightforward to integrate the evolution equations to first order in α . One obtains

$$\psi_0(\vec{x}, t) = \int d\vec{x}' W(\vec{x}, t; \vec{x}', t_{in}) \psi_0(\vec{x}', t_{in}), \quad (6)$$

$$\psi_1(\vec{x}, t_f) = \frac{\alpha}{i\hbar} \int d^4x' W(\vec{x}, t_f; x') V(x') \psi_0(x') \quad (7)$$

$$= \frac{\alpha}{i\hbar} \int_{\mathcal{R}} d^4x' W(\vec{x}, t_f; x') \psi_0(x'), \quad (8)$$

where W is the propagator

$$\begin{aligned} W(\vec{x}, t; \vec{x}', t') &= \int \frac{d\vec{p}}{4\pi^2\hbar^2} dE e^{i/\hbar [p(\vec{x}-\vec{x}') - E(t-t')]} \delta\left(E - \frac{\vec{p}^2}{2m}\right) \\ &= \int \frac{d\vec{p}}{4\pi^4\hbar^2} e^{i/\hbar [p(\vec{x}-\vec{x}') - (\vec{p}^2/2m)(t-t')]} \\ &= \left(\frac{2\pi m}{i\hbar(t-t')}\right)^{3/2} \exp\left[-\frac{m(\vec{x}-\vec{x}')^2}{2i\hbar(t-t')}\right]. \end{aligned} \quad (9)$$

The probability $P_{\mathcal{R}}$ that the pointer is observed in the state $|1\rangle$ after the interaction is the norm of $\Psi_1(t_f)$. Using the well known properties of the propagator

$$\overline{W(x; y)} = W(y; x) \quad (10)$$

and

$$W(\vec{x}, t; \vec{x}', t') = \int d\vec{x}'' W(\vec{x}, t; \vec{x}'', t'') W(\vec{x}'', t''; \vec{x}', t'), \quad (11)$$

this probability is easily computed

$$\begin{aligned} P_{\mathcal{R}} &= \int d\vec{x} |\psi_1(\vec{x}, t_f)|^2 \\ &= \frac{\alpha^2}{\hbar^2} \int_{\mathcal{R}} d^4x \int_{\mathcal{R}} d^4x' W(x; x') \overline{\psi_0(x)} \psi_0(x'). \end{aligned} \quad (12)$$

Since we have assumed that \mathcal{R} is small, we can take the lowest order terms in the size of \mathcal{R} and assume that $\psi_0(\vec{x}, t)$ is constant over \mathcal{R} . If $x_{\mathcal{R}}$ is an arbitrary spacetime point in \mathcal{R} we have then

$$P_{\mathcal{R}} = \frac{\alpha^2}{C_{\mathcal{R}}^2 \hbar^2} |\psi_0(x_{\mathcal{R}})|^2 \quad (13)$$

where

$$\frac{1}{C_{\mathcal{R}}^2} = \int_{\mathcal{R}} d^4x \int_{\mathcal{R}} d^4y W(x; y) \quad (14)$$

is a normalization factor that plays an important role in what follows.

Is the result that we have obtained reasonable? In order to test it, let us assume that the region \mathcal{R} has a finite but very small time extension. Then the measurement we consider can be identified with a position measurement at a fixed time, and we must recover the usual interpretation of the modulus of the wave function as a *spatial* probability density. If the temporal size Δt of \mathcal{R} is very small ($m\Delta t \ll \hbar \Delta V^{2/3}$) compared with its spatial volume ΔV , the normalization factor $C_{\mathcal{R}}^{-2}$ is easy to compute (see [9]). It turns out to be given by

$$C_{\mathcal{R}}^{-2} = \Delta V \Delta t^2. \quad (15)$$

Therefore the detection probability for this region is

$$P_{\mathcal{R}} = \gamma \Delta V |\psi_0(x_{\mathcal{R}})|^2. \quad (16)$$

Here

$$\gamma^2 = \frac{\alpha^2 \Delta t^2}{\hbar^2} \quad (17)$$

is a dimensionless parameter that characterizes the efficiency of the detector. On the other hand, $\Delta V |\psi_0(x_{\mathcal{R}})|^2$ is the probability for the particle to be detected in a small spatial region of volume ΔV at time t_f . Therefore $|\psi_0(x_{\mathcal{R}})|^2$ is the *spatial* probability density and the result is fully consistent with the standard interpretation of the wave function. The factor γ^2 is interpreted as the intrinsic efficiency of our detector. Note that some such parameter is necessarily present as our perturbative analysis assumes that the interaction is weak.

After the measurement, we may consider the state of the system collapse to $\Psi_1 / \|\Psi_1\| \otimes |1\rangle$. Namely after the measurement, the state of the particle may be described by the wave function

$$\psi_{after}(x) = C_{\mathcal{R}} \int_{\mathcal{R}} d^4y W(x; y). \quad (18)$$

Notice that the dependences on both the initial wave function and on the coupling constant α disappear with the normalization. We denote this state of the particle as $|\mathcal{R}\rangle$. That is

$$\langle x|\mathcal{R}\rangle = \psi_{after}(x). \quad (19)$$

Explicitly, after the interaction we have

$$|\mathcal{R}\rangle = C_{\mathcal{R}} \int_{\mathcal{R}} d^4y |x\rangle, \quad (20)$$

where $|x\rangle = |\vec{x}, t\rangle$ is the eigenstate of the Heisenberg position operator $\vec{x}(t)$ with eigenvalue \vec{x} . This is an example of a spacetime-smearred state associated to a region, as defined in [9].

Coming back to the state $|\psi_1\rangle \in H$, for which $\langle x|\psi_1\rangle = \psi_1(x)$ and which represents the branch of the wave function in which the particle is detected in \mathcal{R} , we see that this state may be written

$$|\psi_1\rangle = \gamma \langle \mathcal{R}|\Psi_0\rangle |\mathcal{R}\rangle. \quad (21)$$

This result is the key to the standard measurement interpretation that the interaction “measures” some projection associated with the normalized state $|\mathcal{R}\rangle$ with some efficiency γ^2 . It is of course important that the detector efficiency γ^2 be independent of the initial state $|\Psi_0\rangle$. It immediately follows that the detection probability $P_{\mathcal{R}}$ can be written as

$$P_{\mathcal{R}} = \gamma^2 |\langle \mathcal{R}|\Psi_0\rangle|^2. \quad (22)$$

Summarizing, Eq. (21) allows us to say that the detector we have described prepares the state $|\mathcal{R}\rangle$ defined in Eq. (20); the amplitude to detect an arbitrary Ψ state is $\gamma \langle \mathcal{R}|\Psi\rangle$, and the efficiency of the detector is γ^2 , given in Eq. (17).

It is convenient to denote $\langle \mathcal{R}|\Psi\rangle$ as the amplitude for a particle in the state $|\Psi\rangle$ to be detected in \mathcal{R} . This is the theoretical amplitude of a hypothetical detector with efficiency 1. (“Hypothetical” since above we have used perturbation theory and therefore assumed α , and therefore γ , to be small.)

Finally, consider two detectors: the detector 1 in the region \mathcal{R}_1 , and the detector 2 in the region \mathcal{R}_2 . We take \mathcal{R}_2 (entirely) in the past of \mathcal{R}_1 . Assume that the detector 1 has detected the particle. What is then the probability $P_{\mathcal{R}_2\mathcal{R}_1}$ that the detector 2 detects the particle? Applying the results of the previous section it is immediate to conclude that the (theoretical: $\gamma=1$) probability is

$$P_{\mathcal{R}_2\mathcal{R}_1} = |\langle \mathcal{R}_2|\mathcal{R}_1\rangle|^2. \quad (23)$$

That is

$$P_{\mathcal{R}_2\mathcal{R}_1} = C_{\mathcal{R}_1}^2 C_{\mathcal{R}_2}^2 \left| \int_{\mathcal{R}_1} d^4x \int_{\mathcal{R}_2} d^4y W(x;y) \right|^2. \quad (24)$$

Equivalently,

$$P_{\mathcal{R}_2\mathcal{R}_1} = \frac{|W(\mathcal{R}_2, \mathcal{R}_1)|^2}{W(\mathcal{R}_1, \mathcal{R}_1)W(\mathcal{R}_2, \mathcal{R}_2)}, \quad (25)$$

where we have defined

$$W(\mathcal{R}_2, \mathcal{R}_1) = \int_{\mathcal{R}_1} d^4x \int_{\mathcal{R}_2} d^4y W(x;y). \quad (26)$$

Of course, this result is in no way Lorentz invariant.

We close this section with two brief comments on these results. First, note that this result could also have been achieved by simply coupling two copies of our detector to the system. One would then consider the branch $|\psi_{12}\rangle$ of the state in which both detectors (the one in \mathcal{R}_1 and the one in \mathcal{R}_2) are excited. Each detector has some efficiency γ_1, γ_2 given by the appropriate form of Eq. (17). From Eq. (21) it follows that

$$\begin{aligned} \langle \psi_{12}|\psi_{12}\rangle &= \gamma_1^2 \gamma_2^2 (|\langle \mathcal{R}_1|\Psi_0\rangle|^2) |\langle \mathcal{R}_1|\mathcal{R}_2\rangle|^2 \\ &= \gamma_2^2 \mathcal{P}(\mathcal{R}_1) |\langle \mathcal{R}_1|\mathcal{R}_2\rangle|^2, \end{aligned} \quad (27)$$

as desired.

Finally, we remind the reader that in order to reach the conclusion (26) we must ask that \mathcal{R}_1 and \mathcal{R}_2 have a large separation in time (relative to some scale set by the size of \mathcal{R}_2) so that dispersion does indeed guarantee that the wave function of the state $|\mathcal{R}_1\rangle$ is indeed nearly constant over the region \mathcal{R}_2 .

III. RELATIVISTIC DYNAMICS

The quantum theory of a single relativistic particle is not a realistic theory since it neglects the physical phenomenon of particle creation which is described by quantum field theory. Nevertheless it is interesting to ask whether there exists a logically consistent quantum theory, or several, whose classical limit is the dynamics of a single relativistic particle and which respects the Lorentz invariance of the classical theory. Two such quantizations appear natural: one which contains only positive frequency solutions of the Klein-Gordon equation and one with both frequencies. For simplicity, we consider here only the first,² though adding the negative frequency modes should not cause undue complications. We start from the classical theory defined by

$$p^2 = m^2, \quad (28)$$

$$E > 0, \quad (29)$$

where $p = (\vec{p}, E)$ and $p^2 = -\vec{p}^2 + E^2$. We use here $\hbar = c = 1$. Upon quantization, the constraint (28) becomes the Klein-Gordon equation

$$\left(\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 + m^2 \right) \psi(\vec{x}, t) = 0 \quad (30)$$

²However, our energies will no longer be entirely positive once we add an interaction with the detector.

and the positive energy condition (29) becomes the restriction to positive frequencies. Equivalently, we can write the relativistic Schrödinger equation

$$i \frac{\partial}{\partial t} \psi(\vec{x}, t) = \sqrt{-\vec{\nabla}^2 + m^2} \psi(\vec{x}, t) \equiv H_0 \psi(\vec{x}, t), \quad (31)$$

where the square root is defined by Fourier transform (i.e., by spectral methods). The general solution $\psi(x)$ of these equations is the Fourier transform of a function supported on the upper mass- m hyperboloid in momentum space,

$$\psi(x) = \int \frac{d^4 p}{4 \pi^2} \delta(E - \sqrt{\vec{p}^2 + m^2}) \tilde{\psi}(p) e^{ipx}. \quad (32)$$

Given the wave function $\psi(\vec{x}, 0)$ on an initial time surface, we obtain a solution of the Schrödinger equation by

$$\Psi(t) = e^{-iH_0 t} \Psi(0) \equiv W_0(t) \Psi(0). \quad (33)$$

Explicitly, we have

$$\psi(x) = \int d\vec{x}' W_0(x; \vec{x}', 0) \psi(\vec{x}', 0), \quad (34)$$

where the kernel of the evolution operator $W_0(t)$ is the propagator

$$\begin{aligned} W_0(x; x') &= \int \frac{d^4 p}{4 \pi^2} \delta(E - \sqrt{\vec{p}^2 + m^2}) e^{-ip(x-x')} \\ &= \int \frac{d\vec{p}}{4 \pi^2} e^{i\vec{p}(\vec{x}-\vec{x}') - iE(\vec{p})(t-t')}, \end{aligned} \quad (35)$$

with $E(\vec{p}) = +\sqrt{\vec{p}^2 + m^2}$. This propagator is not a Lorentz invariant object. For later purposes, we can consider also the Lorentz invariant propagator

$$\begin{aligned} W(x; x') &= \int \frac{d^4 p}{4 \pi^2} \delta(p^2 - m^2) \theta(E) e^{-ip(x-x')} \\ &= \int \frac{d\vec{p}}{4 \pi^2} \frac{1}{2E(\vec{p})} e^{iP(\vec{x}-\vec{x}') - iE(\vec{p})(t-t')}. \end{aligned} \quad (36)$$

Notice that

$$W_0 = 2H_0 W = (2H_0)^{1/2} W (2H_0)^{1/2}, \quad (37)$$

where $W(x; y)$ is the kernel of W .

IV. RELATIVISTIC PARTICLE DETECTOR

Let us now couple a particle detector of the kind considered in Sec. II to the relativistic theory described in Sec. III. One may be tempted to simply add the interaction Hamiltonian

$$U = \alpha V(\vec{x}, t) (|1\rangle\langle 0| + |0\rangle\langle 1|) \quad (38)$$

to the free positive frequency Hamiltonian H_0 . But the resulting theory is not Lorentz invariant (even at the classical level). This can easily be seen from the relation

$$E = \sqrt{\vec{p}^2 + m^2} + U. \quad (39)$$

We have (keeping only the linear term in the perturbation)

$$p^2 = m^2 + 2EU. \quad (40)$$

To get a Lorentz invariant theory, we must add a local interaction to the constraint (28). That is, we consider instead the interaction between the particle and the detector defined by

$$p^2 = m^2 + U. \quad (41)$$

$$E > 0. \quad (42)$$

To first order in the coupling we have

$$E = +\sqrt{\vec{p}^2 + m^2} + \frac{U}{2\sqrt{\vec{p}^2 + m^2}}. \quad (43)$$

We order the corresponding Schrödinger equation symmetrically, obtaining the total Hamiltonian

$$H = H_0 + (2H_0)^{-1/2} U (2H_0)^{-1/2} \equiv H_0 + H_{int}. \quad (44)$$

Therefore,

$$H_{int} = (2H_0)^{-1/2} U (2H_0)^{-1/2}. \quad (45)$$

Even here the quantum system fails to be manifestly invariant. Indeed, we have

$$\begin{aligned} -\frac{\partial^2}{\partial t^2} \psi &= H^2 \psi \\ &= H_0^2 \psi + \frac{1}{2} (H_0^{1/2} U H_0^{-1/2} + H_0^{-1/2} U H_0^{1/2}) \psi + O(U^2). \end{aligned} \quad (46)$$

However, it turns out that U and $H_0^{-1/2}$ commute in the limit that we consider. To see this, note that at the semi-classical level the commutator is a sum of terms involving derivatives of the characteristic function V with respect to the spatial coordinates x^i . But, we will act only on states $|\psi_0\rangle$ that are approximately constant over \mathcal{R} , so that expectation values involving $\partial_i V$ vanish. The vanishing of this commutator may also be checked by a longer but fully quantum calculation. As a result, our interaction is effectively Lorentz invariant.

Let us now consider a setting analogous to that of Sec. II, with the same sort of initial state $|\Psi_0\rangle$ evolving into a state with two branches, $|\psi_1\rangle$ and $|\psi_0\rangle$ corresponding to the detection of the particle in \mathcal{R} and to the lack of such detection.

If we take V to represent the Heisenberg operator $V = \int dt V(\vec{x}, t)$, the branch corresponding to detection may be written

$$|\psi_1\rangle = \frac{\alpha}{\hbar} \frac{1}{\sqrt{2H_0}} V \frac{1}{\sqrt{2H_0}} |\Psi_0\rangle. \quad (47)$$

Note that the associated wave function at a time after the interaction would contain a factor of W_0 (representing time evolution) and would thus take the same form as in Eq. (8). In this case, it will turn out that the detection probability is proportional not to $\Psi_0(x_{\mathcal{R}})$, but to $\langle x_{\mathcal{R}} | 1/\sqrt{2H_0} |\Psi_0\rangle$. As a result, it is useful to introduce the state

$$|\tilde{\Psi}_0\rangle = \frac{1}{\sqrt{2H_0}} |\Psi_0\rangle \quad (48)$$

and the associated wave function $\tilde{\Psi}_0(x) = \langle x | \tilde{\Psi}_0\rangle$.

Much as in Sec. II, we now assume that $\tilde{\Psi}_0(x)$ is roughly constant over \mathcal{R} . In this case, we have $V|\tilde{\Psi}_0\rangle \approx \tilde{\Psi}_0(x_{\mathcal{R}}) \int_{\mathcal{R}} d^4x |x\rangle$ so that we may write Eq. (47) as

$$|\psi_1\rangle = \frac{\alpha}{\hbar} \frac{1}{\sqrt{2H_0}} \tilde{\Psi}_0(x_{\mathcal{R}}) \int_{\mathcal{R}} d^4x |x\rangle. \quad (49)$$

Recall that our goal is to express this in the form (21) of a product of a state-independent detector efficiency γ , a normalized state $|\mathcal{R}\rangle$, and an inner product $\langle \mathcal{R} | \Psi_0\rangle$ of the initial state with the same normalized state $|\mathcal{R}\rangle$:

$$|\psi_1\rangle = \gamma \langle \mathcal{R} | \Psi_0\rangle |\mathcal{R}\rangle. \quad (50)$$

Introducing the spacetime volume $\text{Vol}_4(\mathcal{R})$ of \mathcal{R} and the normalization factor

$$C_{\mathcal{R}}^{-2} = W(\mathcal{R}, \mathcal{R}) \equiv \int_{\mathcal{R}} d^4x \int_{\mathcal{R}} d^4y W(x, y), \quad (51)$$

we now make the identifications:

$$\gamma = \frac{\alpha}{\hbar C_{\mathcal{R}}^2 \text{vol}_4(\mathcal{R})},$$

$$|\mathcal{R}\rangle = C_{\mathcal{R}} \int_{\mathcal{R}} d^4x (2H_0)^{-1/2} |x\rangle,$$

$$\langle \mathcal{R} | \Psi_0\rangle = C_{\mathcal{R}} \text{vol}_4(\mathcal{R}) \tilde{\Psi}_0(x_{\mathcal{R}}). \quad (52)$$

The last of these identifications is of course not independent, but instead follows directly from the identification of $|\mathcal{R}\rangle$. Note that the efficiency (17) of the detector in Sec. II can also be written in the above form.

As pointed out in Sec. II, the form (50) immediately implies that when two detectors (associated with regions \mathcal{R}_1 and \mathcal{R}_2) are considered, the probability of detecting the particle in both regions is

$$\gamma_2^2 \mathcal{P}_{\mathcal{R}_1} |\langle \mathcal{R}_1 | \mathcal{R}_2\rangle|^2, \quad (53)$$

where $\mathcal{P}_{\mathcal{R}_1}$ is the probability of detecting the particle in \mathcal{R}_1 . Thus, idealizing to a perfect ($\gamma=1$) detector of this sort, we may say that the probability for a particle prepared in $|\mathcal{R}_1\rangle$ to arrive in \mathcal{R}_2 is

$$\mathcal{P}_{\mathcal{R}_2 \mathcal{R}_1} = \frac{|W(\mathcal{R}_2, \mathcal{R}_1)|^2}{W(\mathcal{R}_1, \mathcal{R}_1) W(\mathcal{R}_2, \mathcal{R}_2)}. \quad (54)$$

Here

$$W(\mathcal{R}_2, \mathcal{R}_1) = \int_{\mathcal{R}_1} d^4x \int_{\mathcal{R}_2} d^4y W(x; y) \quad (55)$$

where $W(x; y)$ is the Lorentz invariant propagator, defined in Eq. (36).

That is, despite the appearance of $(2H_0)^{-1/2}$ in the definition of $|\mathcal{R}\rangle$, the probability amplitude to detect the particle in \mathcal{R}_2 if it was detected in \mathcal{R}_1 is Lorentz invariant. In the next section we will come to understand this factor of $(2H_0)^{-1/2}$ as merely compensating for writing $|\mathcal{R}\rangle$ in terms of states $|x\rangle$ whose inner product singles out a preferred Lorentz frame. This factor will disappear when $|\mathcal{R}\rangle$ is written in terms of the truly Lorentz invariant ‘‘Philips states.’’

V. PHILIPS STATES

Historically, two types of (generalized) states have been associated to spacetime points $x = (\vec{x}, t)$ in relativistic quantum mechanics. Recall from Eq. (32) that we can write $\psi(\vec{x}, t)$ as the Fourier transform of a function supported on the upper mass- m hyperboloid in momentum space

$$\psi(x) = \int \frac{d^4p}{4\pi^2} \delta(E - \sqrt{\vec{p}^2 + m^2}) \tilde{\psi}(p) e^{ipx}. \quad (56)$$

We can also write the equivalent but more covariant looking expression

$$\psi(x) = \int d^4p \delta(p^2 - m^2) \theta(E) \tilde{\phi}(p) \sqrt{2p_0} e^{ipx}. \quad (57)$$

We remind the reader that $\sqrt{2p_0} e^{ipx}$ gives plane waves with the Lorentz invariant normalization $(2p_0) \delta^{(3)}(\vec{p} - \vec{p}')$ on the mass shell, corresponding to the Lorentz invariant measure $d^3p/(2p_0)$. The relation between Eqs. (56) and (57) is obviously

$$\tilde{\psi}(p) = \left(\frac{1}{2\sqrt{\vec{p}^2 + m^2}} \right)^{1/2} \tilde{\phi}(p). \quad (58)$$

Now, pick a point y in Minkowski space and consider the two generalized states associated to this point defined, respectively, by

$$\tilde{\psi}_y(p) = e^{ipy} \quad (59)$$

and by

$$\tilde{\phi}_y(p) = e^{ipy}. \quad (60)$$

Explicitly, the two states are given by the following two solutions of the relativistic Schrödinger equation:

$$\psi_y^{(NW)}(x) = W_0(x, y), \quad (61)$$

and

$$\psi_y^{(PH)}(x) = W_{1/2}(x, y), \quad (62)$$

where, in operator form, $W_{1/2} = W_0 / (2H_0)^{1/2} = W(2H_0)^{1/2}$. The Lorentz invariance of the Philips states is now manifest from the inner product

$$\langle \psi_y^{(PH)} | \psi_x^{(PH)} \rangle = W(y, x). \quad (63)$$

If $y = (\vec{y}, t)$, the states are given at fixed time t by

$$\psi_y^{(NW)}(\vec{x}) = \delta(\vec{x}, \vec{y}) \quad (64)$$

and

$$\psi_y^{(PH)}(\vec{x}) = \int d\vec{p} \left(\frac{1}{\vec{p}^2 + m^2} \right)^{1/2} e^{i\vec{p}(\vec{x} - \vec{y})}. \quad (65)$$

Therefore the states $\psi_y^{(NW)}$ form a (generalized) orthonormal basis

$$\langle \psi_{\vec{y}, t}^{(NW)} | \psi_{\vec{y}', t}^{(NW)} \rangle = \delta^3(\vec{y}, \vec{y}') \quad (66)$$

while the states $\psi_y^{(PH)}$ do not. The states $\psi_y^{(NW)}$ are the well known Newton-Wigner states [1]: they diagonalize the Newton-Wigner position operator at time t . They are noncovariantly defined. That is, they depend not only on the spacetime point y , but also on the choice of a preferred Lorentz frame at y .

What about the states $\psi_y^{(PH)}$? They are associated to the spacetime point y and are invariantly defined. That is, they only depend on the point, not on any choice of reference frame at the point. These states were first considered by Philips [10], shortly after the appearance of the Newton-Wigner paper. In spite of the virtue of being covariantly defined, the Philips states have not been very popular. The reason is that so far their physical interpretation has not been clear. In particular it was not clear what kind of measurement would produce a Philips state. The discussion in the previous section shows that the spacetime detector considered there does indeed prepare states of this sort. In particular,

$$|\mathcal{R}\rangle = \int_{\mathcal{R}} d^4x |\psi_x^{(PH)}\rangle. \quad (67)$$

An immediate consequence is the property

$$\langle \psi_y^{(PH)} | \psi \rangle = \tilde{\psi}(y), \quad (68)$$

where $|\tilde{\psi}\rangle$ is the state introduced in Eq. (48). Intuitively, in the limit in which the region \mathcal{R} shrinks to a point y , the states $|\mathcal{R}\rangle$ approach $|\psi_y^{(PH)}\rangle$. Thus, the detector we have described is a “detector of Philips states.”

Of course, all propagators that we have considered “propagate faster than light,” as is well known. They do not vanish at spacelike separations. The leakage out of the light cone is small: it is exponentially damped with the Compton wavelength of the particle. In particular, the Philips states associated to the different spatial points on a given simultaneity surface are not orthogonal to each other. This feature of the special relativistic quantum dynamics of a particle is sometimes regarded as a defect of the theory, which could compromise its consistency or its classical limit. We do not think this is the case. Simply, the quantum particle has an intrinsic Compton “extension” that allows it to excite two spacelike separated (but close) detectors.³ In the classical limit, the trajectories stay inside the light cone.

Of course, this acausal feature makes the theory less attractive than quantum field theory (in which such effects do not occur). Note that this observation implies that the above detectors cannot be constructed from quantum field theoretic local measuring devices in any limit. As a result, they presumably do not correspond to “real” particle detectors any more than do the Newton-Wigner detectors (which share this acausal property). Instead, these detectors exist in a “relativistic particle” system that is best thought of as a toy model for quantum cosmology.

VI. EXACTLY LORENTZ INVARIANT DETECTOR

We found the detector above to be effectively Lorentz invariant due to the fact that U and H_0 commute in the limit that we have taken. One might ask about a truly Lorentz invariant notion of a spacetime localized detector. We shall not discuss this issue in detail, but we sketch here a possible answer.

Consider the following manifestly Lorentz invariant algorithm. By fixing boundary conditions in the past as we did above, one can compare solutions of the *quadratic* constraint $p^2 + m^2 = 0$ with solutions of the perturbed quadratic constraint $p^2 + m^2 + U = 0$. One simply imposes that the two solutions agree on any Cauchy surface to the past of the support of U . To the future of the support of U , the perturbed and unperturbed constraints again agree and the perturbed solution can be written as a sum of two unperturbed solutions as in Eq. (1). One would then associate “probabilities” for detection/nondetection with the norms of these two unperturbed states. One needs only a Lorentz-invariant definition of this norm to complete the discussion.

In general, one cannot restrict consideration to positive frequency states, as negative frequencies may be introduced by the interaction U . However, the technique known as “group averaging” (see e.g. [12–15]) allows one to define a positive definite manifestly Lorentz invariant inner product

³Nevertheless, one wonders if this might be improved in a single-particle formalism which allows negative frequency states.

on all solutions of any constraint of the form $p^2 + m^2 + U = 0$ where U is a localized disturbance. In fact, it defines such an inner product in much more general circumstances as well. See in particular the recent work of [16–18] for the connection to BRST techniques. The fact that it is positive definite is a strong advantage over the historically more popular Klein-Gordon inner product.

We do not pursue here a detailed treatment of this manifestly Lorentz-invariant approach, because of the distance from the familiar von Neumann measurement theory of non-relativistic quantum mechanics. Furthermore, due to the assignment of positive norms to negative frequency states, such a scheme can be physically appropriate only in the quantum cosmology setting. In that context, negative frequency states can be interpreted simply as collapsing universes and not as particles traveling backwards in time. Perhaps there is some general lesson in this last observation, in that one must decide at the outset whether one wishes to discuss something approximating the relativistic particles of the real world (which are of course properly described by excitations of a field theory) or whether one really wishes to discuss a simplified model of quantum cosmology. While the two systems seem rather similar mathematically, the radically different conceptual status of the associated causal structures on the configuration space may in the end require radically different foundations for the corresponding notions of measurement theory and detectors.

VII. CONCLUSIONS

Does a real particle detector detect a Newton-Wigner state or a Philips state? Is a real detector better represented by the interaction that we have described or by a Newton-Wigner operator? As noted above, the proper answer is “neither,” as a real particle detector is a local construction in quantum field theory. However, taking the relativistic particle as a toy model for quantum cosmology, one may still ask which detector is the most useful. In this context, the Philips detector has the interesting property of being associated with Lorentz invariant probabilities.

The interest of the model we have presented, however, is not in the realism of the model detector considered. Rather, it is in the fact that the construction shows that it is possible to

think about quantum measurement in a fully covariant way, at least in a certain limit. By covariant, we mean not only that Lorentz invariance can be preserved in models with Lorentz invariant dynamics, but more generally that measurement models exist which treat all degrees of freedom on an equal conceptual footing. This result is close in spirit with Hartle’s generalized quantum mechanics [4]. See also [19–21].

In particular, the results presented here support the legitimacy of the particular postulate proposed in [9,11] for a covariant spacetime-states formulation of (canonical) quantum theory.⁴ According to this postulate, the probability for detecting a system in a small region \mathcal{R}' of the extended configuration space if it was detected in a small region \mathcal{R} is given by

$$P_{\mathcal{R}_2\mathcal{R}_1} = \frac{|W(\mathcal{R}_2, \mathcal{R}_1)|^2}{W(\mathcal{R}_1, \mathcal{R}_1)W(\mathcal{R}_2, \mathcal{R}_1)}, \quad (69)$$

where

$$W(\mathcal{R}_2, \mathcal{R}_1) = \int_{\mathcal{R}_1} dx \int_{\mathcal{R}_2} dy W(x; y) \quad (70)$$

where dx is a measure on the extended configuration space and $W(x; y)$ is the covariant propagator that defines the quantum theory. This postulate is assumed to replace and generalize the usual interpretation of the wave function, in which measurements happen at fixed time. Here we have shown that this postulate is true in a certain limit of relativistic quantum particle mechanics, provided that the interaction producing the measurement is described in a covariant manner.

⁴One can also imagine a fully covariant interpretation without a class of preferred measurements such as those associated with spacetime regions above. In such a framework, our Philips measurements would stand on an equal footing with the (frame-dependent) Newton-Wigner measurements. This is essentially the formulation used in [12].

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