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Relativistic magnetized star with poloidal and toroidal fields

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We study the inner magnetohydrodynamic structure of a general relativistic magnetized star, with poloidal and toroidal fields. The star is taken to be differentially rotating, stationary, axisymmetric, and made from perfect, infinitely conducting fluid. Strong toroidal fields of up to 10^{17} G can be created from the initial poloidal field by a variation of the mechanism proposed by Meier *et al.* and Kluźniak and Ruderman. It is also found that the redshifted toroidal field and the redshifted chemical potential are constants along a magnetic surface. We prove that a spacetime containing an ideal magnetohydrodynamic fluid which flows only azimuthally is circular in the sense of Carter if, and only if, the magnetic field has only poloidal or only toroidal components. Further, we show through post Newtonian analysis that, even when this criterion is breached, spacetime inside astrophysical compact objects where the magnetic field is less than 10^{19} G can be considered circular. In both cases the metric inside the star assumes a simple form, with only one nonvanishing off diagonal term. It is shown that imposing chemical equilibrium forces the magnetic field to assume a force-free configuration. We derive the form of the electric 4-current in force-free relativistic magnetohydrodynamics. The connection between both field components is then given through the vector potential and is used to rule out some field configurations. We derive a new separability condition on the metric which shows that not every pure fluid metric can be dressed with a frozen in, force-free field.

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I. INTRODUCTION

A wide range of stars where general relativistic effects are important is currently known. Of special importance are neutron stars, as today it is recognized that they are at the core of many of the most intriguing astrophysical phenomena, from gamma ray bursts to supernovae to binary mergers and pulsars. Since in such compact stars the gravitational binding energy per particle can be about a tenth of the particle's rest mass, general relativistic effects are important. In addition, the combination of high angular momentum per particle and small spatial size leads the surfaces of these stars to rotate at velocities which can reach a substantial fraction ($\sim 10-20\%$) of the speed of light. Many of these phenomena (e.g. the theory of stellar pulsations and the accompanying gravitational waves) can be explained in terms of pure general relativistic hydrodynamics. However, there is an even larger class of important phenomena, foremost among which are pulsars, which cannot be explained without including electromagnetic effects through general relativistic plasma physics. A convenient approximation to this is magnetohydrodynamics (MHD), which provides a relatively simple tool for describing macroscopic continuum phenomena. Its attractiveness is enhanced by the high conductivity and low viscosity characterizing neutron star material, which increase its validity. The general relativistic version of magnetohydrodynamics may be found through the works of Lichnerowicz [1], Novikov and Thorne [2], Bekenstein and Oron [3], etc.

It is widely agreed that pulsars and related compact objects are rotating magnetized neutron stars. One still has to determine the properties and evolution of the star's electromagnetic field. The model dominating the literature for the last four decades is that of a *poloidal* magnetic field which is misaligned by a constant angle to the star's symmetry axis and corotates with it. Most works consider a dipole field for simplicity. In addition, the landmark paper by Goldreich and Julian [4] demonstrated the existence of a charged magnetosphere around the star. This model has been successful in explaining many of the observed phenomena in neutron stars. In this paper we would like to consider a wider range of magnetic field morphology, including *toroidal* components. Toroidal fields are very well known in the nonrelativistic electromagnetic theory of stars [5]; even an ordinary star such as the Sun has a toroidal field (e.g. [6]). It is also known that the so-called "live" pulsars model can have a toroidal field which extends from the magnetosphere into the star [5]. Several well established mechanisms are capable of creating strong toroidal fields in compact stars. Perhaps the best known is the winding of poloidal field lines by differential rotation of the star. The first needed ingredient is high conductivity to anchor the field lines to the fluid elements. This condition is met by most astrophysical plasma and by neutron star matter. The second ingredient is strong differential rotation. Differential rotation can be supplied by core collapse during a supernova, certainly if the progenitor star had such a rotation curve, but even if it was only mildly rotating [7,8]. Differential rotation can be understood on the grounds of angular momentum conservation and the small size of the collapsed object. The relativistic simulations by Shibata and Uryu [9] reveal that binary mergers of neutron stars can also lead to differentially rotating remnants, and also accretion induced collapse of white dwarfs [10]. Meier *et al.* [11] and Wheeler *et al.* [12] estimated that this process can lead to fields of up to 10^{17} G. Stronger fields will be expelled from the star by magnetic buoyancy. Another mechanism which can generate large toroidal fields is a dynamo process which sets in during the first few seconds of

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stellar collapse [13,14]. Despite all of the above, toroidal fields have attracted relatively little attention in the literature, perhaps because they were not needed for explaining the fundamental pulsar phenomena.

Although the need for a relativistic treatment of neutron star electrodynamics was early recognized, relatively few analytic and semianalytic works exist. This is largely due to the immense difficulty of simultaneously solving the Maxwell equations and the highly nonlinear Einstein equations. This hurdle is overcome by two main approaches. The first is to assume that the magnetic field, or alternatively the electric 4-current or vector potential, is known and then use it to calculate the metric. This approach will work for weak and simple fields as it requires that the pressure and density distribution do not contain high order multipoles. It was used by Konno *et al.* [15,16] to compute the deformation of neutron stars with a dipole field. Konno [17] computed the moment of inertia of rotating stars with a dipole field. The second approach is to assume that the metric is known (usually some pure fluid metric), and use it to solve Maxwell's equations for the electromagnetic field. Muslimov and Harding [18] used this method to describe general relativistic magnetospheres; Bekenstein and Oron [19] studied the interior structure of general relativistic fluid stars with poloidal force-free fields. Rezzola *et al.* [20,21] used this approach to study misaligned rotators with a dipole field. Inclusion of a toroidal field adds one further complication: it can break down the circularity of spacetime [22]. This acts to complicate the metric by increasing the number of nonvanishing metric elements (see e.g. [23]). This paper discusses ways to circumvent this difficulty

The purpose of this paper is to study rotating relativistic stars with mixed poloidal and toroidal fields through use of general relativistic MHD. We present the star model and assumptions in Sec. II. In Sec. III we justify the inclusion of toroidal fields. In Sec. IV we obtain the criterion for spacetime circularity in MHD. In Sec. V we tackle the problem of circularity breakdown when the field has mixed components, and we show that a simple metric can be obtained for all relevant cases. In Sec. VI we write down the magnetic field components and obtain the toroidal field distribution. In Sec. VII we demonstrate that a one-fluid model in chemical equilibrium implies a force-free field; we derive the current of this field and the chemical potential distribution. In Sec. VIII we find the dependence between vector potential components for a force-free field. Section IX shows that not every pure fluid metric can be dressed with a force-free magnetic field.

II. STAR MODEL AND ASSUMPTIONS

We consider a stationary axisymmetric star which rotates differentially about its symmetry axis. We work in coordinates $x^\alpha=(t,x^1,x^2,\phi)$. t is the time measured by an observer which is stationary with respect to far objects. ϕ is the azimuthal angle. x^1 and x^2 are some spatial coordinates whose integral curves are symmetric about the star's rotation axis and orthogonal to one another. Our signature is $+2$. Greek indices run from 0 to 3. Latin indices run from 1 to 3. We use units with $G=c=1$. We assume that the fluid interior of the star is made up of three constituents: neutral baryons, and positive and negative charge carriers. We further assume all three constituents flow together as one fluid. We denote by u^α the fluid bulk 4-velocity. The angular velocity of the star is $\Omega=d\phi/dt=u^\phi/u^t$; due to the symmetries it is a function of x^1, x^2 only. The motion of free charges induces an electric current J^α; this current is a source for a magnetic field configuration which is symmetric about the rotation axis.

The star's electromagnetic field is described by the antisymmetric Faraday tensor $F_{\alpha\beta}$ which obeys the relativistic Maxwell equations

$$F_{[\alpha\beta;\gamma]}=0 \tag{2.1}$$

$$F^{\alpha\beta}{}_{;\beta}=4\pi J^\alpha \tag{2.2}$$

where the [. . .] denotes all the antisymmetric permutations of $\alpha\beta\gamma$, '',''' denotes an ordinary derivative and '';'' a covariant derivative. The first set of equations, Eq. (2.1), dictates that $F_{\alpha\beta}$ is the curl of the electromagnetic vector potential A_α, $F_{\alpha\beta}=A_{\beta,\alpha}-A_{\alpha,\beta}$. As we have a preferred velocity field u^α, it is convenient to follow the scheme developed by Lichnerowicz [1] and Novikov and Thorne [2] and construct two 4-vectors describing the electric and magnetic fields correspondingly:

$$E_\alpha=F_{\alpha\beta}u^\beta \tag{2.3}$$

$$B_\alpha={}^*F_{\beta\alpha}u^\beta\equiv\frac{1}{2}\epsilon_{\beta\alpha\gamma\delta}F^{\gamma\delta}u^\beta, \tag{2.4}$$

where $\epsilon_{\alpha\beta\gamma\delta}$ is the Levita-Civita totally antisymmetric tensor. Equations (2.3),(2.4) can be inverted to yield $F_{\alpha\beta}$

$$F_{\alpha\beta}=u_\alpha E_\beta-u_\beta E_\alpha+\epsilon_{\alpha\beta\gamma\delta}u^\gamma B^\delta. \tag{2.5}$$

By multiplying Eq. (2.4) with u^α one finds the important and well known relation

$$B_\alpha u^\alpha=0. \tag{2.6}$$

We take our star to have infinite conductivity; therefore, the electric field vanishes anywhere inside the star. This is the famous ideal MHD condition

$$F_{\alpha\beta}u^\beta=0. \tag{2.7}$$

The magnetic field has no way of dissipating itself and is ''frozen'' into the fluid, its flux through a closed loop moving with fluid remaining constant with time. To these equations we add the general relativistic version of Ferraro's theorem [24] developed by Soderholm [25]

$$B^1\Omega_{,1}+B^2\Omega_{,2}=0. \tag{2.8}$$

This theorem states that magnetic field lines are tangent to fluid surfaces rotating with equal angular velocity. Otherwise the field lines which are frozen into the fluid would be stretched by the differential rotation, converted into a toroidal field, and thus destroy the stationarity.

The evolution of the fluid's 4-velocity is governed by the magnetic Euler equation

$$(\rho+p)a_{\alpha}=-h_{\alpha}^{\beta}p_{,\beta}+f_{\alpha} \tag{2.9}$$

where ρ is the proper energy density, including rest mass, and the internal energy, p is the scalar pressure as measured in the fluid's local rest frame (LRF), and $h_{\alpha}^{\beta}=\delta_{\alpha}^{\beta}+u^{\beta}u_{\alpha}$ is the projection tensor. a_{α} is the fluid's 4-acceleration $a_{\alpha}=u_{\alpha;\beta}u^{\beta}$. The term $f_{\alpha}=F_{\alpha\beta}J^{\beta}$ is the Lorentz 4-force per unit volume acting on the fluid. The effects of gravitation are automatically incorporated by the use of the covariant derivative in Eq. (2.9) and Eqs. (2.1),(2.2).

III. THE CASE FOR A TOROIDAL FIELD

We now turn to the star's magnetic field structure. Most works of magnetic fields inside relativistic compact objects assume that the magnetic field is poloidal (having only x^1,x^2 components) and ignore toroidal fields completely. Many also take that field to be a dipole field due to its simplicity. This is because solving the Einstein field equations is difficult enough for a pure fluid star. Adding the electromagnetic effects needed to explain the rich phenomena in white dwarfs, and in the various types of neutron stars, within the frame of general relativity makes this already arduous task an almost impossible one. Only the simplest cases can usually be treated analytically. Moreover, it is well known that adding a toroidal field can complicate the metric considerably, adding terms which are absent for pure poloidal fields as we shall elaborate in the next section. For now we shall put this aside and concentrate on nonrelativistic considerations.

Ignoring toroidal fields can be too limiting. It was shown by Flowers and Ruderman [26], and later by Eichler [27], that isolated spherical fluid stars with purely poloidal fields extending outside the star are unstable, as the magnetic field tends to acquire a domain-like structure, similar to that of a ferromagnet, to reduce the exterior magnetic energy to a minimum while leaving the magnitude of the internal field almost unchanged. This argument is independent of the conductivity of the medium and stems from an energy principle [27]. Flowers and Ruderman [26] have further shown that including a toroidal field can stabilize the magnetic field by preventing the various "magnets" from flipping as this will twist the toroidal field and increase the total magnetic energy. This argument favors a toroidal field but does not necessitate one as there are other stabilizing mechanisms such as fast enough crust crystallization which can halt the field flipping [26].

How can a toroidal field be created in a compact object, and how can it be maintained over time? The simplest mechanism is linear amplification of the magnetic field by differential rotation [11,28]. If the star acquires a differential rotation profile during collapse and has an initial purely poloidal field, the rotating fluid of the star will stretch the frozen-in poloidal field lines and wind them up around the star thus creating a toroidal field. With each turn of the star the field lines are increasingly wound and stretched, thus amplifying the field continuously. This is a very efficient mechanism that can generate huge toroidal fields. But the process cannot continue perpetually. The fluid threaded with magnetic field has lower density because magnetic pressure $B^2/8\pi$ replaces some of the fluid pressure. The fluid density is usually higher inside the star than at the surface. When the magnetic pressure is strong enough, the buoyancy force on the magnetized fluid suffices to bouy the denser inner fluid to the surface of the star and expel it.

This critical value of the toroidal field is given by [12,28]

$$B_f\approx 2f^{0.5}\rho_{13}^{0.5}\times 10^{18}\ \text{G} \tag{3.1}$$

where $f\approx 0.01$ is the fractional difference in density at the origin of the floating fluid and the stellar surface, and $\rho_{13}=\rho/10^{13}$ g cm^{-3}. This give maximum values of $10^{16}-10^{17}$ G for the toroidal fields of neutron stars. According to Wheeler *et al.* [12], the number of revolutions it takes a star with an initial poloidal field B_0 to reach this maximum field assuming the poloidal field is wound once per revolution is

$$n_f\approx 3\times 10^3\left(\frac{B_0}{10^{12}\ \text{G}}\right)^{-1} \tag{3.2}$$

and the time it takes a protoneutron star with $B_0\approx 10^{12}$ G and rotation period of $P\approx 25$ ms to reach that critical field is

$$t_f\approx n_f P\approx 75\ \text{s}. \tag{3.3}$$

This is an extremely short time scale considering the lifetime of neutron stars. It tells us that if the actual mechanism is orders of magnitude less efficient, we can still get very strong toroidal fields, and as long as they are below B_f, they will stay anchored within the star.

Note that the critical field value is independent of the original poloidal field. If B_0 is less than 10^{12} G it will simply take more revolutions to reach the critical field. This mechanism was proposed by Kluźniak and Ruderman [28] to be the central engine of γ ray bursters. Wheeler *et al.* [12] adopted it to explain asymmetric supernova initiated by jets ejected from the protoneutron star. In both processes the expelled material provides the energy source. We, on the other hand, are interested in the phase where this mechanism fails and we are left with a less than critical toroidal field. After the material threaded with the critical field is expelled from the star, linear amplification can start all over again producing a series of ejected toroids. Whenever matter is expelled the star loses energy [28]: work must be done to stretch the field lines and is converted to magnetic energy of the toroidal field. The rotating toroid also carries angular momentum away. In addition, as the toroid breaks out of the star, the surface magnetic field reconnects as it settles down. This process is powered by energy and angular momentum stored in the star's differential rotation. After the ejection of each toroid, the star's differential rotation diminishes as it is brought closer to a uniform rotation. The process terminates either when the star is brought to a uniform rotation or when the magnetic poloidal field orients itself along surfaces of equal angular velocity [see Ferraro's theorem, Eq. (2.8)]. Either of these endings can leave us with a wide range of toroidal fields weaker than the critical field 10^{17} G.

There is no lack of scenarios in which stars evolve differential rotation. We know the Sun exhibits a differential rotation (e.g. [6]) and we expect many other stars to have such a profile prior to collapse. We would expect from considerations of angular momentum conservation that such a profile would develop as different shells infall with different radial velocity during collapse, even if the progenitor star had a uniform rotation, and would be enhanced if it existed prior to the collapse. The nonrelativistic simulations by Zwerger and Müller [7] and Rampp *et al.* [8] indeed show this is the case. It was also noted by Shapiro [10] that the ratio $\beta = E_r/|E_p|$ of rotational to potential energy in the star can grow dramatically during the collapse. As uniformly rotating compressible stars can only support very small values of β without shedding mass (e.g. [6,29]), collapsed cores of fast rotating progenitor would have to acquire some differential rotation as they settle into equilibrium. Shibata and Uryu [9] have shown through a fully relativistic simulation that binary mergers of neutron stars can result in a differentially rotating compact remnant.

Another mechanism capable of generating strong toroidal fields is the dynamo process. It was argued by several authors (Duncan and Thompson [13], Thompson and Duncan [14], Wheeler *et al.* [12]) that during the first few seconds of collapse, convection ensues at different stages of the collapse and paves the way for the action of various dynamo processes such as the $\alpha-\Omega$ or α^2 dynamos. Although the main role of these dynamos is to produce a strong poloidal field, they can produce toroidal fields as a by-product.

We conclude then that strong toroidal fields can be generated in compact stars either by linear amplification or by a dynamo process. This toroidal field will help in stabilizing the poloidal field and vice versa. As we assume the star to be a very good conductor, the toroidal field can be sustained indefinitely. We will therefore include such a field in our magnetic configuration.

IV. METRIC AND CIRCULARITY

Our next step is to determine the effects of toroidal fields on the metric. Before we can go any further we must define what a toroidal 4-vector is. While in Newtonian mechanics there is no difference between the covariant and contravariant components of vectors, this is not so in relativity. Through the scope of this work, a toroidal 4-vector will be one whose contravariant components in the x^1, x^2 direction vanish. This does not imply the vanishing of the corresponding covariant components: For this to happen the metric components $g_{t1}, g_{t2}, g_{\phi 1}, g_{\phi 2}$ must vanish. This, in general, is not guaranteed. Similarly, we define a poloidal 4-vector as one whose contravariant t,ϕ components vanish.

Magnetic fields can affect the metric in two main ways: through the magnetic energy-momentum density which enters Einstein's equations, and through the Lorentz force which acts to redistribute the mass density and hydrostatic pressure throughout the star. The rest mass density of a compact star is about $\rho c^2 \approx 10^{27}$ erg cm^{-3}. The magnetic energy density inside a star is, therefore, negligible if the magnetic field is smaller than 10^{17} G, which is true for all compact objects, including magnetars; the same is true for the momentum density. The Lorentz force, on the other hand, can greatly alter the metric as it causes a redistribution of the pressure and, therefore, the mass density and angular momentum in the star. The magnetic pressure can, for instance, replace some of the hydrostatic pressure thus supporting the same mass with less fluid pressure; this in turn allows for smaller mass densities. As a result, a magnetic star can be expected to have a larger radius than its nonmagnetic, equally massive, counterpart. Shapiro and Teukolsky [29] used a nonrelativistic version of the virial theorem to show that for white dwarfs with extreme relativistic degeneracy, the maximum mass limit is expected to grow due to the magnetic pressure by a factor of $1+\frac{3}{2}\delta$ where δ is the ratio of magnetic to gravitational energy of the star.

Another feature of the Lorentz force is that it is usually not spherically symmetric. The assumption of a slowly rotating star which deviates slightly from spherical symmetry is ubiquitous in analytic treatments of relativistic stars. If the Lorentz force is significant, it could easily invalidate this assumption, giving way to more complicated patterns of mass, pressure and angular momentum distributions. More so, the task of consistently solving the Einstein and Maxwell equations is a formidable one, usually accomplished fully only through numerical analysis (e.g. see Bocquet *et al.* [30]). The way around these problems is usually to assume that the field is of simple known form (dipole) so the Einstein equations can be solved by multipole expansion (e.g. [15]), or that it is weak enough so the metric corresponds to a pure fluid metric, which is then used to solve the Maxwell equations for the field (e.g. [20]). In either case the metric is assumed to have five nonzero coefficients: four on the diagonal and one off-diagonal, $g_{t\phi}$, responsible for frame dragging. This form relies on a theorem originally by Papapetrou [31] and later by Carter [22] regarding circular spacetimes. We now recapitulate some of the essentials of these results as they are crucial to our analysis.

Consider an axisymmetric stationary spacetime. We can define on this spacetime two Killing vector fields: k^α, the stationary Killing vector and m^α, the azimuthal Killing vector. Such a spacetime is said to be circular if it satisfies [22]

$$k^\delta T_{\delta[\alpha} k_\beta m_{\gamma]} = 0 \tag{4.1}$$

$$m^\delta T_{\delta[\alpha} k_\beta m_{\gamma]} = 0. \tag{4.2}$$

The circularity of spacetime is a sufficient and necessary condition for the the Killing vectors to satisfy the following identities:

$$k_{[\alpha;\beta} k_\gamma m_{\delta]} = 0 \tag{4.3}$$

$$m_{[\alpha;\beta} k_\gamma m_{\delta]} = 0. \tag{4.4}$$

If Eqs. (4.3),(4.4) are satisfied the metric can assume a simpler form. This is best seen when the coordinates t,ϕ are taken along the integral curves of k^α, m^α,

$$k^\alpha = \delta_t^\alpha; \quad k_\alpha = g_{t\alpha} \tag{4.5}$$

$$m^\alpha = \delta_\phi^\alpha; \quad m_\alpha = g_{\phi\alpha}. \tag{4.6}$$

Equations (4.3),(4.4) together with Frobenius theorem indicate the existence of a family of 2-surfaces which are everywhere orthogonal to the plane of the Killing vectors k^α, m^α. Choosing the remaining two coordinates x^1, x^2 so their integral curves reside inside these 2-surfaces makes the metric attain the following form [22]:

$$d\tau^2 = g_{ab}dx^a dx^b + g_{tt}dt^2 + g_{\phi\phi}d\phi^2 + 2g_{t\phi}dtd\phi \tag{4.7}$$

where a,b run from 1 to 2 and from Eqs. (4.5),(4.6) $g_{t\phi} = k^\alpha m_\alpha$. Thus, by choosing x^1, x^2 properly one can get rid of the g_{12} cross term and obtain a metric with five nonvanishing components. Papapetrou [31] has shown that in order for the energy tensor of a perfect fluid to satisfy the circularity condition, Eqs. (4.1),(4.2), the 4-velocity must be purely toroidal i.e.

$$u^{[\alpha}k^\beta m^{\gamma]} = 0. \tag{4.8}$$

Carter [22] extended this result to include electromagnetic fields: for the energy-momentum tensor of perfect fluid plus an electromagnetic field

$$T^{\alpha\beta} = pg^{\alpha\beta} + (p+\rho)u^\alpha u^\beta + (F^{\alpha\gamma}F^\beta{}_\gamma - \tfrac{1}{4}F^{\gamma\delta}F_{\gamma\delta}g^{\alpha\beta})/(4\pi), \tag{4.9}$$

a sufficient condition for circularity is for the electric 4-current to be purely toroidal

$$J[^\alpha k^\beta m^\gamma] = 0 \tag{4.10}$$

in addition to the toroidal 4-velocity. This poses a problem because toroidal currents can sustain only poloidal fields. On the face of it a star with a toroidal field component would seem to require a metric more complicated than Eq. (4.7). A partial way out is by remembering that Carter's result relates to a general electromagnetic field. Consider an ideal magnetohydrodynamic flow, the energy momentum of which is [2]

$$T^{\alpha\beta} = (\rho+p)u^\alpha u^\beta + pg^{\alpha\beta} + \frac{1}{8\pi}(2B^2u^\alpha u^\beta + B^2 g^{\alpha\beta} - 2B^\alpha B^\beta). \tag{4.11}$$

Inserting Eqs. (4.5),(4.6) into Eqs. (4.1),(4.2) we finally get

$$T_{t[\alpha}k_\beta m_{\gamma]} = 0 \tag{4.12}$$

$$T_{\phi[\alpha}k_\beta m_{\gamma]} = 0. \tag{4.13}$$

The contributions to those of the first two terms in Eq. (4.11), the pure fluid parts of $T_{\alpha\beta}$, vanish because the fluid's 4-velocity in the star is purely toroidal. We are left with the remaining three magnetic terms. The contribution of the $B^2u_\alpha u_\beta$ term vanishes bacause $u_{[\alpha}k_\beta m_{\gamma]} = 0$ from Eq. (4.8). The contribution of the $B^2 g_{\alpha\beta}$ vanishes because according to Eqs. (4.5),(4.6) $g_{t\alpha} = k_\alpha$; $g_{\phi\alpha} = m_\alpha$ and obviously $k_{[\alpha}k_\beta m_{\gamma]} = m_{[\alpha}k_\beta m_{\gamma]} = 0$. We finally derive that a necessary and sufficient condition for the circularity of the energy momentum tensor of an ideal infinitely conducting fluid carrying a magnetic field is

$$B_t B_{[\alpha}k_\beta m_{\gamma]} = 0 \tag{4.14}$$

$$B_\phi B_{[\alpha}k_\beta m_{\gamma]} = 0. \tag{4.15}$$

There are two field configurations satisfying these equations. The first is a purely poloidal field: this is seen by examining Eq. (2.6), which gives us the relation between the toroidal field components

$$B_t = -\frac{B_\phi}{\Omega}. \tag{4.16}$$

We see that for $\Omega \neq 0$, if B_ϕ vanishes so does B_t and Eqs. (4.14),(4.15) are both satisfied leaving only nonvanishing poloidal field components. The second circular configuration is a purely toroidal field which is automatically in the plane spanned by k_α, m_β and, therefore, also satisfies the circularity condition. These results can be summarized into the following theorem: *A spacetime, containing a stationary axisymmetric purely toroidal flow of a perfect infinitely conducting fluid carrying a magnetic field, will be circular if, and only if, the magnetic field will be either purely poloidal, or purely toroidal*. For stars exhibiting these field configurations a "diagonal plus one" metric can be obtained. A problem arises, however, when the field is a mixed one having both poloidal and toroidal components. Then, Eqs. (4.14),(4.15) are not satisfied and a simple metric seems to be beyond our reach. A different approach is then needed to show it is still possible to reach a simple metric.

V. MIXED FIELD AND CIRCULARITY

As shown in the previous section although ideal MHD allows also for pure toroidal fields to induce circular spacetime, mixed poloidal and toroidal fields do not. It is obvious that if the fields are weak enough, the deviation from circularity is negligible, but how "weak" is enough? We answer this question by turning to the post Newtonian formalism. It is well known that the gravitational potentials and velocities of matter in neutron stars, and all less relativistic stars make them adequate candidates for the post Newtonian formalism. We will adopt the formalism presented by Weinberg [32]. In the post Newtonian (PN) approximation all tensors and tensor densities are obtained as a power series in $\overline{v}$, the typical Newtonian velocity in the star. Since we are considering an axisymmetric, stationary fluid, it is sufficient to concentrate on a metric of order 1PN. In this order g_{tt} is obtained to order $O(\overline{v}^4)$ and all other metric components to order $O(\overline{v}^3)$ or $O(\overline{v}^2)$, as appropriate. Although it is more natural to formulate our problem in spherical or cylindrical coordinates, we choose to work in Cartesian coordinates and transform to more convenient coordinates afterwards since we wish to deal with metric terms which converge asymptotically. We choose the z axis to be along the symmetry axis. In the 1PN approximation the metric reads

$$g_{tt} = -1 - 2\phi - 2\phi^2 - 2\psi \tag{5.1}$$

$$g_{ij} = \delta_{ij} - 2\delta_{ij}\phi \tag{5.2}$$

$$g_{it} = \zeta_i, \tag{5.3}$$

where ϕ is the Newtonian gravitational potential

$$\phi(\mathbf{x},t) = -\int \frac{\overset{0}{T}{}^{tt}(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|} d^3x' \tag{5.4}$$

and is $O(\overline{v^2})$ and where $\mathbf{x},\mathbf{x}'$ are the radius 3-vectors. $\boldsymbol{\zeta}$ is a triplet of potentials of $O(\overline{v^3})$, defined by

$$\zeta_i(\mathbf{x},t) = -4\int \frac{\overset{1}{T}{}^{it}(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|} d^3x' \tag{5.5}$$

and ψ is

$$\psi(\mathbf{x},t) = -\int \frac{\overset{2}{T}{}^{tt}(\mathbf{x}',t) + \overset{2}{T}{}^{ii}(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|} d^3x' \tag{5.6}$$

and is $O(\overline{v^4})$.

To compute the different potentials one must find $\overset{n}{T}{}^{\mu\nu}$, the terms of order n when $T^{\mu\nu}$ is written as a series in powers of $(\overline{M}/\overline{r}^3)\overline{v}^n$, where $\overline{M},\overline{r}$ are the typical mass and length of the problem (in our case the star's mass and radius):

$$T^{tt} = \overset{0}{T}{}^{tt} + \overset{2}{T}{}^{tt} + \overset{4}{T}{}^{tt} + \dots \tag{5.7}$$

$$T^{it} = \overset{1}{T}{}^{it} + \overset{3}{T}{}^{it} + \dots \tag{5.8}$$

$$T^{ij} = \overset{2}{T}{}^{ij} + \overset{4}{T}{}^{ij} + \dots . \tag{5.9}$$

Post Newtonian hydrodynamics [32] yields for the pure fluid contributions to $\overset{n}{T}{}^{\mu\nu}$ of order 1PN

$$\overset{0}{T}{}^{tt} = \rho \tag{5.10}$$

$$\overset{1}{T}{}^{it} = \rho v_i \tag{5.11}$$

$$\overset{2}{T}{}^{ij} = p\,\delta_{ij} + \rho v_i v_j \tag{5.12}$$

$$\overset{2}{T}{}^{tt} = \rho(v^2 - 2\phi) \tag{5.13}$$

where v_i is the Newtonian velocity defined by

$$v_i \equiv \frac{dx^i}{dt} = \frac{u^i}{u^t};\quad v^2 = v_i v_i, \tag{5.14}$$

u^t is given by

$$u^t = \frac{dt}{d\tau} = 1 - \phi + \frac{v^2}{2} + O(\overline{v^4}). \tag{5.15}$$

Equation (5.12) assumes that the hydrostatic pressure is of the same order as the typical ram pressure $\overline{\rho v^2}$, $\overline{\rho} = \overline{M}/\overline{r}^3$.

We are now equipped to assess the contribution of the magnetic part of $T^{\mu\nu}$, Eq. (4.9). We will take the square of the magnetic field $B^2 = B^\mu B_\mu$ to be also of $O(\overline{\rho v^2})$. This will make the calculation valid for extremely strong fields where the magnetic pressure is a significant part of the hydrostatic pressure. We do not have to calculate the magnetic contribution to $\overset{2}{T}{}^{\mu\nu}$ as these terms contribute only to g_{tt} which is unchanged when transforming from the Cartesian spatial coordinates to more convenient ones leaving the time coordinate unchanged. We also do not have to evaluate $\overset{0}{T}{}^{tt}$ as it only affects ϕ and is therefore irrelevant to off diagonal metric elements. Thus, we are charged only with estimating the magnetic contribution to $\overset{1}{T}{}^{it}$. We use Eqs. (5.1)–(5.3) to tie the covariant and contravariant components of the magnetic field:

$$B_i = g_{i\alpha}B^\alpha \approx \zeta_i B^t + \delta_{ij}B^j = B^i + \zeta_i B^t. \tag{5.16}$$

B_t can be tied to the spatial field components by Eqs. (2.6),(5.14)

$$B_t = -B_i v_i. \tag{5.17}$$

On the other hand we have

$$B_t = g_{t\alpha}B^\alpha = g_{tt}B^t + g_{ti}B^i. \tag{5.18}$$

With Eq. (5.17) for B_t we have, to lowest order,

$$B^t = B^i v_i. \tag{5.19}$$

We see that B^t is small relative to B^i as we deal with relatively low velocities and from Eq. (5.16) it follows that

$$B_i \approx B^i. \tag{5.20}$$

From Eq. (5.18) we now find

$$B_t \approx -B^t. \tag{5.21}$$

We can now use these relations in conjunction with the magnitude of $B^2 = B^\mu B_\mu$ to find the size of the components of B^α

$$B^i B^i \approx B^2 = O(\overline{\rho v^2}) \tag{5.22}$$

$$B^i \approx O(\sqrt{\overline{\rho v}}) \tag{5.23}$$

$$B^t \approx O(\sqrt{\overline{\rho v^2}}). \tag{5.24}$$

These last three equations allow us to evaluate magnetic contributions of various orders in the energy momentum tensor Eq. (4.9). To do this we substitute for u^α from Eqs. (5.14),(5.15). We must also supply the contravariant metric terms $g^{\alpha\beta}$ appearing in Eq. (4.9) from [32]

$$g^{ij} = \delta_{ij}(1+2\phi); \quad g^{it} = \zeta_i. \tag{5.25}$$

Following the above procedure we conclude that the lowest order magnetic contribution to T^{it},

$$\frac{1}{4\pi}(B^2 v_i - B^t B^i) = O(\overline{\rho v^3}), \tag{5.26}$$

leaves $\overset{1}{T}{}^{it}$ unchanged and contributes only to $\overset{3}{T}{}^{it}$. Another simplification comes from remembering that for azimuthal flow $v_z = 0$ and therefore ζ_z vanishes. The Cartesian 1PN metric for a magnetic star therefore acquires the form

$$g_{\mu\nu} = \begin{pmatrix} -1-2\phi-2\phi^2-2\psi & \zeta_x & \zeta_y & 0 \\ \zeta_x & 1-2\phi & 0 & 0 \\ \zeta_y & 0 & 1-2\phi & 0 \\ 0 & 0 & 0 & 1-2\phi \end{pmatrix}. \tag{5.27}$$

When such a metric is cast into t, x^1, x^2, ϕ coordinates, it yields a "diagonal + one" metric (of course not if the fluid motion is other than purely rotational) where

$$g_{t\phi} = -4R(x^1,x^2)\int \frac{\rho\Omega R(x'^1,x'^2)\cos(\phi-\phi')}{|\mathbf{x}-\mathbf{x}'|} d^3x'. \tag{5.28}$$

R is the projection of the radius vector on a 2-plane passing through the origin of the axes and perpendicular to the symmetry axis. We can now conclude that for any axisymmetric, stationary, differentially rotating magnetized star made of infinitely conducting ideal fluid, the metric can always be cast in a "diagonal + one" form, regardless of the magnetic configuration as long as the magnetic pressure does not exceed the typical hydrostatic pressure. For neutron star $\overline{\rho} \approx 10^{13}$ g cm^{-3}, $\overline{v} \approx 0.1c$, and this conclusion is valid for $B < 10^{19}$ G. Thus, from now on we shall adopt the following metric:

$$\begin{aligned} ds^2 = &-g_{tt}dt^2 + 2g_{t\phi}dt\,d\phi \\ &+ g_{\phi\phi}d\phi^2 + g_{11}(dx^1)^2 + g_{22}(dx^2)^2 \end{aligned} \tag{5.29}$$

where x^1, x^2 can be some two orthogonal coordinates for e.g. r, θ for spherical coordinates or ρ, z for cylindrical, and $g_{\mu nu} = g_{\mu\nu}(x^1,x^2)$. It is useful to express $g^{\alpha\beta}$, the reciprocal of the metric using the $g_{\alpha\beta}$,

$$g^{tt} = g_{\phi\phi}/\Delta \tag{5.30}$$

$$g^{\phi\phi} = g_{tt}/\Delta \tag{5.31}$$

$$g^{11} = 1/g_{11} \tag{5.32}$$

$$g^{22} = 1/g_{22} \tag{5.33}$$

$$g^{t\phi} = -g_{t\phi}/\Delta \tag{5.34}$$

where $\Delta = g_{tt}g_{\phi\phi} - g_{t\phi}^2$, $-g = g_{11}g_{22}\Delta$ is the determinant of $g_{\mu\nu}$.

For further reference, the temporal component of the velocity can be expressed using the metric $g_{\alpha\beta}$, Ω and by requiring the fluid 4-velocity normalization $u^\alpha u_\alpha = -1$

$$u^t = (-g_{tt} - 2g_{t\phi}\Omega - g_{\phi\phi}\Omega^2)^{-1/2}. \tag{5.35}$$

The covariant velocity is also toroidal

$$u_t = (g_{tt} + g_{t\phi}\Omega)u^t; \quad u_\phi = (g_{t\phi} + g_{\phi\phi}\Omega)u^t;$$
$$u_1 = 0; \quad u_2 = 0. \tag{5.36}$$

VI. THE MAGNETIC FIELD CONFIGURATION

The Maxwell equation Eq. (2.1), $F_{[t\phi;a]} = 0$, shows that $F_{t\phi}$ is constant throughout the star. This constant is determined to vanish by the vanishing of the azimuthal component of the electric field at the star's center. Thus, out of the six independent components of the electromagnetic tensor $F_{\alpha\beta}$, one is zero as we have seen and four are determined by the poloidal field components. Those and their contravariant counterparts are

$$F_{1t} = \sqrt{-g}\,u^\phi B^2; \quad F^{1t} = -\frac{1}{\sqrt{-g}}u_\phi B_2 \tag{6.1}$$

$$F_{1\phi} = -\sqrt{-g}\,u^t B^2; \quad F^{1\phi} = \frac{1}{\sqrt{-g}}u_t B_2 \tag{6.2}$$

$$F_{2t} = -\sqrt{-g}\,u^\phi B^1; \quad F^{2t} = \frac{1}{\sqrt{-g}}u_\phi B_1 \tag{6.3}$$

$$F_{2\phi} = \sqrt{-g}\,u^t B^1; \quad F^{2\phi} = -\frac{1}{\sqrt{-g}}u_t B_1. \tag{6.4}$$

The last component of $F_{\alpha\beta}$ is determined by the toroidal field

$$F_{12} = -\frac{\sqrt{-g}B^\phi}{u^t}; \quad F^{12} = \frac{B_\phi}{\sqrt{-g}u^t}. \tag{6.5}$$

We can now determine the electric current. We use Eq. (2.2) and the identity $F^{\alpha\beta}_{;\beta} = (1/\sqrt{-g})(\sqrt{-g}F^{\alpha\beta})_{,\beta}$, which is valid for any antisymmetric tensor, to establish the 4-current components:

$$J^t = \frac{1}{\sqrt{-g}}[(u_\phi B_2)_{,1} - (u_\phi B_1)_{,2}] \tag{6.6}$$

$$J^\phi = \frac{1}{\sqrt{-g}}[(u_t B_1)_{,2} - (u_t B_2)_{,1}] \tag{6.7}$$

$$J^1 = \frac{1}{\sqrt{-g}}(\sqrt{-g}F^{12})_{,2} \tag{6.8}$$

$$J^2 = -\frac{1}{\sqrt{-g}}(\sqrt{-g}F^{12})_{,1}. \tag{6.9}$$

The next step is to combine Eqs. (6.1)–(6.5) and Eqs. (6.6)–(6.9) and obtain the toroidal components of the Lorentz force

$$f_t = -u^\phi[B^2(\sqrt{-g}F^{12})_{,2} + B^1(\sqrt{-g}F^{12})_{,1}] \tag{6.10}$$

$$f_\phi = u^t[B^1(\sqrt{-g}F^{12})_{,2} + B^1(\sqrt{-g}F^{12})_{,1}]. \tag{6.11}$$

Bekenstein and Oron [19] have shown that due to the symmetries, for any star exhibiting only toroidal flows and a metric of the form Eq. (5.29), the 4-acceleration a_α takes the much simpler form

$$a_\alpha = -\frac{(u^t)^2}{2}(g_{tt,\alpha} + 2g_{t\phi,\alpha}\Omega + g_{\phi\phi,\alpha}\Omega). \tag{6.12}$$

Inserting this expression into Eq. (2.9) and calculating h_α^β explicitly we get

$$-(\rho+p)\frac{(u^t)^2}{2}(g_{tt,\alpha} + 2g_{t\phi,\alpha}\Omega + g_{\phi\phi,\alpha}\Omega)$$
$$= -p_{,\alpha} + f_\alpha. \tag{6.13}$$

Looking at the toroidal components of Eq. (6.13) we immediately arrive at the conclusion that $f_t = f_\phi = 0$, that is, the magnetic field is force-free in the toroidal direction. This is the relativistic analogue of the nonrelativistic condition $\mathbf{J_p} \times \mathbf{B_p} = 0$ (e.g. [5]) for stationary axisymmetric field MHD configurations, where $\mathbf{J_p}, \mathbf{B_p}$ are the nonrelativistic poloidal current and field. The field cannot apply any force in the toroidal direction as this would imply a poloidal torque acting on the star thus changing Ω and violating the assumed stationarity. Applying this to Eqs. (6.10),(6.11) we arrive at the equation

$$B^2(\sqrt{-g}F^{12})_{,2} + B^1(\sqrt{-g}F^{12})_{,1} = 0. \tag{6.14}$$

This equation tells us that $\sqrt{-g}F^{12}$ is constant along the poloidal field lines just like Ω. By replacing F^{12} with the explicit expression we find the distribution of the toroidal field

$$B_\phi = Ku^t \tag{6.15}$$

where K is constant along a magnetic surface (MS).

VII. FORCE-FREE FIELD

As we have seen in the last section the magnetic field is force-free in the toroidal direction but not automatically in the poloidal direction, where the Lorentz force does not vanish. We now prove that if in addition to assuming one fluid velocity we assume that the fluid is in chemical equilibrium, the field must be completely force-free. This is done by essentially repeating a nonrelativistic argument by Easson [33] in relativisitc language. In chemical equilibrium the number of neutral baryons turning into positive and negative charge carriers equals the number of charge carriers recombining back to form neutral baryons. Such processes can be facilitated, for example, by ionization and recombination for hydrogen and other plasmas in mildly relativistic stars or by β and inverse β decay and similar processes for neutrons in neutron stars and white dwarfs. We denote by μ_n, μ_e and μ_p the chemical potentials of the neutrals, negative charge carriers and positive charge carriers respectively; chemical equilibrium then implies

$$\mu_n = \mu_p + \mu_e. \tag{7.1}$$

In GTR μ for a fluid takes the following form [2]:

$$\mu = \frac{(\rho+p)}{n} \tag{7.2}$$

where n is the proper baryon number density. Bekenstein and Oron [3] have shown that the following dynamic relation holds for a perfect fluid:

$$p_{,\alpha} = n\mu_{,\alpha}. \tag{7.3}$$

Inserting the above relation into Euler's equation Eq. (2.9) and setting $f_\alpha = 0$ we get for the neutral fluid the following equation:

$$\mu_n a_\alpha^n = -\mu_{n,\alpha} \tag{7.4}$$

where a_α^n denotes the 4-acceleration of the neutral fluid.

Now let us look at the Euler equation of the charge carrying fluid,

$$(\rho_c + p_c)a_\alpha^c = -p_{c,\alpha} + f_\alpha \tag{7.5}$$

where ρ_c, p_c are the energy density and pressure of the charged fluid. These can be broken up into the sum of their constituents $\rho_c = \rho_e + \rho_p$, $p_c = p_e + p_p$ where ρ_e, p_e, ρ_p, p_p are correspondingly the energy density and partial pressure of the negative and positive charge carrying fluids. This, together with Eq. (7.2) allows us to write Eq. (7.5) as

$$(n_e\mu_e + n_p\mu_p)a_\alpha^c = -p_{e,\alpha} - p_{p,\alpha} + f_\alpha \tag{7.6}$$

where n_e, n_p are the proper number densities of negative and positive charge carriers. The quasineutrality of ideal MHD dictates $n_e \approx n_p$ which with the help of Eq. (7.3) puts Eq. (7.6) into the form

$$(\mu_e+\mu_p)a^c_\alpha=-\mu_{e,\alpha}-\mu_{p,\alpha}+\frac{f_\alpha}{n_e}. \tag{7.7}$$

We now use the equilibrium condition Eq. (7.1) accompanied by the relation $a^c_\alpha=a^n_\alpha$ which stems from the one fluid velocity assumption to derive for the charged fluid

$$\mu_n a^n_\alpha=-\mu_{n,\alpha}+\frac{f_\alpha}{n_e}. \tag{7.8}$$

Comparing Eq. (7.8) with Eq. (7.4) immediately yields

$$f_\alpha=0; \tag{7.9}$$

hence the field is force-free everywhere inside the star. A force-free field puts severe constraints on the possible current and magnetic field configuration. Let us calculate the current distribution. The current J^α should be such that $F_{\alpha\beta}J^\beta=0$, meaning it is a linear combination of the eigenvectors of $F_{\alpha\beta}$ with eigenvalue zero. One can try and solve explicitly for those eigenvectors, we use a different method. Examine the Lorentz force $f_\alpha=F_{\alpha\beta}J^\beta$. Replacing $F_{\alpha\beta}$ with Eq. (2.5) yields

$$f_\alpha=\epsilon_{\alpha\beta\gamma\delta}u^\gamma B^\delta J^\beta. \tag{7.10}$$

However, Eq. (7.10) shows that f_α is simply the 4-volume enclosed between the three vectors $u^\alpha, B^\alpha, J^\alpha$. It can vanish for a given u^α, B^α only if J^α resides in the plane spanned by u^α and B^α (which can never be parallel as B^α is spacelike while u^α is timelike), that is, J^α is a linear combination of the two

$$J^\alpha=\epsilon u^\alpha+\xi B^\alpha. \tag{7.11}$$

ϵ is the charge density in the LRF, as can be verified by projecting J^α on the 4-velocity u^α. The convection current ϵu^α is sufficient to support a purely poloidal field. This can be seen by looking at the poloidal components of the current Eqs. (6.8),(6.9). In the absence of a toroidal field ($F^{12}=0$) these components vanish, implying $\xi=0$ for a poloidal field. At first glance it seems that something is wrong with Eq. (7.11): In the nonrelativistic limit the spatial components of the current become

$$\mathbf{J}=\epsilon\mathbf{v}+\xi\mathbf{B} \tag{7.12}$$

where $\mathbf{v},\mathbf{B}$ are the nonrelativistic fluid velocity and magnetic field 3-vectors. This differs from the well known relation $\mathbf{J}=\xi\mathbf{B}$. The solution to that problem lies in the fact that the relativistic expression $F_{\alpha\beta}J^\beta$ accounts for all the electromagnetic interaction both Lorentz and Coulomb. Consider a nonrelativistic flow of an infinitely conducting fluid with velocity field $\mathbf{v}$. The field freezing condition then reads

$$\mathbf{E}+\mathbf{v}\times\mathbf{B}=0 \tag{7.13}$$

where $\mathbf{E}$ is the electric field 3-vector. The total electromagnetic force acting on the fluid is the sum of the Columb force and the Lorentz force. The Coulomb force is simply $\mathbf{F}_c=\epsilon\mathbf{E}$, to that we add the Lorentz force acting on the convection current $\mathbf{F}_l=\epsilon\mathbf{v}\times\mathbf{B}$. These two forces come out due to the existence of the small net charge density which is the source of the convection current. To those two we must add the Lorentz force acting on the conduction current $\mathbf{J}_{con}$, $\mathbf{F}_{con}=\mathbf{J}_{con}\times\mathbf{B}$, there is no Coulomb force associated with this current as it is not derived from a charge denstiy. Adding all three terms together we find the total force acting on the fluid

$$\mathbf{F}=\mathbf{F}_c+\mathbf{F}_l+\mathbf{F}_{con}. \tag{7.14}$$

The first two terms in Eq. (7.14) add up to $\epsilon(\mathbf{E}+\mathbf{v}\times\mathbf{B})$ and vanish by virtue of Eq. (7.13). Thus, the convection current is always force-free because the Lorentz force acting on the fluid is always countered by the Coulomb force, and we are left with

$$\mathbf{F}=\mathbf{J}_{con}\times\mathbf{B}. \tag{7.15}$$

The total electromagnetic force acting on the fluid then is simply the Lorentz force due the conduction current, and for it to vanish we must choose $\mathbf{J}_{con}=\xi\mathbf{B}$. However, the total current will now be $\mathbf{J}=\epsilon\mathbf{v}+\xi\mathbf{B}$ as in Eq. (7.11). In nonrelativistic MHD the convection current contribution to the total current can usually be ignored as $\epsilon\approx 0$ due to the smallness of $\mathbf{E}$ as reflected by Eq. (7.13). In GTR that is not so; the large velocities and effects like length contraction can make the convection current significant. Still, as it is proportional to u^α, the GTR field freezing condition, Eq. (2.7), makes sure it is force-free.

The behavior of ξ can be deduced by taking the divergence of Eq. (7.11) which must vanish on account of charge conservation. The divergence of the convection current is zero due to the symmetries and the purely toroidal nature of the velocity field. Thus we are left with the equation

$$(\xi B^\alpha)_{;\alpha}=0. \tag{7.16}$$

This equation can be further simplified by using the relation discovered by Bekenstein and Oron [19] for ideal MHD

$$(\mu B^\alpha)_{;\alpha}=0. \tag{7.17}$$

This equation derives from the Euler equation, and ties the dynamics of the magnetic field with that of the fluid. Thus,

$$\left(\frac{\xi}{\mu}\right)_{,\alpha}B^\alpha=0. \tag{7.18}$$

Hence ξ equals the chemical potential multiplied by a function which is constant over the MS. This is only partially helpful as we have yet no knowledge of the distribution of the chemical potential itself.

We now go on to determine μ. We start with Eq. (7.17) and substitute for $B^\alpha_{;\alpha}$ the relation $B^\alpha_{;\alpha}=B^\alpha a_\alpha$ from Bekenstein and Oron [19]. After inserting Eq. (6.12) we have the following:

$$\mu_{,\alpha}B^\alpha-\mu\frac{(u^t)^2}{2}(g_{tt,\alpha}+2g_{t\phi,\alpha}\Omega+g_{\phi\phi,\alpha}\Omega)B^\alpha=0. \tag{7.19}$$

We now add to the term in parentheses $2g_{t\phi}\Omega_{,\alpha}+2g_{\phi\phi}\Omega_{,\alpha}$. This is possible due to Ferraro's theorem, Eq. (2.8). Comparing with Eq. (5.36) we notice that Eq. (7.19) has become

$$\left(\frac{\mu}{u^t}\right)_{,\alpha}B^\alpha=0. \tag{7.20}$$

The redshifted chemical potential is therefore constant over a MS.

VIII. THE VECTOR POTENTIAL

So far the poloidal and toroidal fields were considered as independent entities. The toroidal field was determined solely by the 4-velocity distribution and some function $K(\Omega)$ which is a function of the 4-velocity and the metric. The poloidal field, on the other hand, was determined separately. However, the force-free nature of the configuration removes this independence. The demand for a vanishing Lorentz force constrains the possible form of the currents which are the sources of the fields. How are the fields now connected and how is K affected by the poloidal field, and vice versa?

We start by considering Eq. (7.11) for a poloidal current. As we have shown, the poloidal current appears only if a toroidal field is present. On the other hand the poloidal current contains a term proportional to the poloidal field. Hence, it links between the two configurations. Remembering Eqs. (6.8),(6.9) and (6.5) and the constancy of F^{12} along a MS we have

$$B^2=\frac{1}{\xi\sqrt{-g}}K_{,1} \tag{8.1}$$

$$B^1=-\frac{1}{\xi\sqrt{-g}}K_{,2}. \tag{8.2}$$

Bekenstein and Oron [19] have shown that by choosing an appropriate gauge, the poloidal components of A_α, the electromagnetic vector potential can be made independent of t and ϕ. From this they showed that the poloidal field is given by

$$B^2=\frac{1}{u^t\sqrt{-g}}A_{\phi,1} \tag{8.3}$$

$$B^1=-\frac{1}{u^t\sqrt{-g}}A_{\phi,2}. \tag{8.4}$$

By comparing those last four equations and remembering from Eq. (7.20),(7.18) that both $\mu/u^t=-D$, $\xi/\mu=\alpha$ where D,α are constants over a MS, we find that

$$A_{\phi,a}=\frac{1}{-\alpha D}K_{,a}. \tag{8.5}$$

In other words with t and ϕ derivatives trivial

$$\frac{dK}{dA_\phi}=-\alpha D. \tag{8.6}$$

Thus K, the function which determines the strength of the toroidal field, is a function of the azimuthal vector potential A_ϕ, whose derivative is determined by the two functions α,D. While the form of B_ϕ and the emergence of K originate from the absence of force in the toroidal direction only, which is a must for any stationary axisymmetric flow, Eq. (8.6) is true only for a fully force-free configuration configuration.

We now use the above results to rule out some field configurations: On the face of it, the configuration characterized by constant K throughout the star is a legitimate one. However, from Eqs. (8.1),(8.2), the poloidal field components vanish, given $\xi\neq0$. According to Eq. (7.11), this leaves only a toroidal current, but from Eqs. (6.6),(6.7) if B^1 and B^2 vanish so do J^t and J^ϕ, and the current vanishes as a whole, leaving us with no magnetic field. The only possible choice for K that would yield a nontrivial field is $K=0$; this causes ξ to vanish leaving us with with a purely poloidal field sustained by the toroidal convection current.

IX. CONDITIONS ON THE METRIC

In many works the common procedure for determining the magnetic field relies on it being small, or force-free, so its impact on the star's metric is negligible. This way, one can avoid a full solution of Einstein's equations with the full effect of the magnetic field. One usually takes a familiar well known metric for a pure fluid star, and uses it in conjunction with suitable boundary conditions to solve Maxwell's equations. Naively one might think that any metric can be dressed with a force-free magnetic field. However, as we shall show now, this is not so.

We start with a purely poloidal field of a uniformly rotating star. We choose x^1 to vary along the poloidal field lines while x^2 is constant along them. As x^1,x^2 are orthogonal, the metric retains the form in Eq. (5.29). In such coordinates only $B^1\neq0$ and Eq. (7.17) becomes

$$(\sqrt{-g}\mu B^1)_{,1}=0. \tag{9.1}$$

This equation can be readily integrated, and with help of Eq. (7.20) we get

$$B^1=\frac{1}{\sqrt{-g}h(x^2)u^t} \tag{9.2}$$

where $h(x^2)$ is an undetermined function. On the other hand we have Maxwell's equations for the current which for our force-free poloidal field take the form

$$\epsilon u^t=\frac{(u_\phi B_1)_{,2}}{\sqrt{-g}} \tag{9.3}$$

$$\epsilon u^\phi=-\frac{(u_t B_1)_{,2}}{\sqrt{-g}}. \tag{9.4}$$

The equations for the x^1,x^2 components vanish identically as there is no toroidal field. By multiplying Eq. (9.3) by $-\Omega$ and adding the two equations together we find

$$[(u_t+\Omega u_\phi)g_{11}B^1]_{,2}=0. \tag{9.5}$$

Using the normalization $u^\alpha u_\alpha=-1$ to write

$$(u_t+\Omega u_\phi)=-\frac{1}{u^t}, \tag{9.6}$$

we can integrate Eq. (9.5) to get

$$B^1=\frac{f(x^1)u^t}{g_{11}} \tag{9.7}$$

where $f(x^1)$ is another undetermined function. Inserting Eq. (9.7) into Eq. (9.2) and writing $\sqrt{-g}$ explicitly we finally have

$$\frac{\sqrt{g_{11}}}{\sqrt{g_{22}}\Delta(u^t)^2}=f(x^1)h(x^2). \tag{9.8}$$

A similar condition was found by Bekenstein and Oron [19]. We see that only metrics that can be written in the separated form (u^t and Δ are both functions of metric elements) Eq. (9.8) can support force-free frozen-in poloidal fields. What is the origin of this condition on the metric? If we examine the pair (quartet in the general case) of Maxwell equations for the currents, we find that they depend on the metric since the current of the force-free configuration is proportional to the 4-velocity components which are determined by the metric. In addition, the field is expressed by Eqs. (8.1),(8.2) which also contain the metric. We could solve these equations for the magnetic field if the metric were known without needing Eq. (7.17). However, in addition to satisfying the equations for the current, the magnetic field must also be frozen in. This ties the evolution of the magnetic field to the 4-velcoity field and hence to the metric. This connection culminates into Eq. (7.17). Thus we can find a magnetic field which is derived from a convection current, but if we also want it to be part of a MHD solution the underlying metric must satisfy Eq. (7.17). This condition is independent of Einstein's equations.

How would Eq. (9.8) be modified if a toroidal field were added? In this case we have three Maxwell equations (the x^1 component of the equations vanishes identically)

$$\epsilon u^t+\xi B^t=\frac{(u_\phi B_1)_{,2}}{\sqrt{-g}} \tag{9.9}$$

$$\epsilon u^\phi+\xi B^\phi=-\frac{(u_t B_1)_{,2}}{\sqrt{-g}} \tag{9.10}$$

$$\xi B^1=\frac{-K_{,2}}{\sqrt{-g}}. \tag{9.11}$$

Again we multiply Eq. (9.9) by $-\Omega$ and add it to Eq. (9.10) to get rid of the convection current

$$\xi(-\Omega B^t+B^\phi)=-\frac{[(\Omega u_\phi+u_t)B_1]_{,2}}{\sqrt{-g}}. \tag{9.12}$$

We utilize Eqs. (6.15),(4.16) and (5.30)–(5.34) to write

$$-\Omega B^t+B^\phi=-\frac{K}{\Delta u^t}. \tag{9.13}$$

Inserting into Eq. (9.12) and rearranging terms we have

$$-\frac{\xi K}{\Delta u^t}=\frac{-1}{\sqrt{-g}}\left(\frac{g_{11}B^1}{u^t}\right)_{,2}. \tag{9.14}$$

Now we can replace ξ by using Eq. (9.11) and B^1 by Eq. (9.2) and finally get

$$-\frac{\sqrt{-g}h(x^2)(K^2)_{,2}}{2\Delta}=\left(\frac{g_{11}}{\sqrt{-g}h(x^2)(u^t)^2}\right)_{,2} \tag{9.15}$$

where we have used $K=K(x^2)$ from Eq. (6.15). Thus, there is again a separability condition on the metric. We see that if K is constant over all field lines we retrieve the condition for purely poloidal fields, which coincides with the previous result in Sec. VIII that it must be zero.

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[1] A. Lichnerowicz, *Relativistic Hydrodynamics and Magnetohydrodynamics* (Benjamin, New York, 1967).
[2] I. D. Novikov and K. S. Thorne, in *Black Holes*, edited by B. S. DeWitt and C. M. DeWitt (Gordon and Breach, New York, 1973).
[3] J.D. Bekenstein and E. Oron, Phys. Rev. D **18**, 1809 (1978).
[4] P. Goldreich and W.H. Julian, Astrophys. J. **157**, 869 (1969).
[5] L. Mestel, *Stellar Magnetism* (Clarendon, Oxford, 1999).
[6] J. L. Tassoul, *Theory of Rotating Stars* (Princeton University Press, Princeton, NJ, 1978).
[7] T. Zwerger and E. Müller, Astron. Astrophys. **320**, 209 (1997).
[8] M. Rampp, E. Müller, and M. Ruffert, Astron. Astrophys. **332**, 969 (1997).
[9] M. Shibata and K. Uryu, Phys. Rev. D **61**, 064001 (2000).
[10] S.L. Shapiro, Astrophys. J. **544**, 397 (2000).
[11] D.L. Meier, R.L. Epstein, W.D. Arnett, and D.N. Schramm,

Astrophys. J. **204**, 869 (1976).
[12] J.C. Wheeler, I. Yi, P. Hoflich, and L. Wang, Astrophys. J. **537**, 810 (2000).
[13] R.C. Duncan and C. Thompson, Astrophys. J. Lett. **392**, L9 (1992).
[14] C. Thompson and R.C. Duncan, Astrophys. J. **408**, 194 (1993).
[15] K. Konno, T. Obata, and Y. Kojima, Astron. Astrophys. **352**, 211 (1999).
[16] K. Konno, T. Obata, and Y. Kojima, Astron. Astrophys. **356**, 234 (2000).
[17] K. Konno, Astron. Astrophys. **372**, 594 (2001).
[18] A. Muslimov and K. Harding, Astrophys. J. **485**, 735 (1997).
[19] J.D. Bekenstein and E. Oron, Phys. Rev. D **19**, 2827 (1979).
[20] L. Rezzola, B.J. Ahmedov, and J.C. Miller, Found. Phys. **31**, 1051 (2001).
[21] L. Rezzola, B.J. Ahmedov, and J.C. Miller, Mon. Not. R. Astron. Soc. **322**, 723 (2001).
[22] B. Carter, in *Black Holes*, edited by B. S. DeWitt and C. M. DeWitt (Gordon and Breach, New York, 1973).
[23] E. Gourgoulhon and S. Bonazzola, Phys. Rev. D **48**, 2635 (1993).
[24] V.C.A. Ferraro, Mon. Not. R. Astron. Soc. **97**, 458 (1937).
[25] L.H. Soderholm, Lett. Nuovo Cimento Soc. Ital. Fis. **16**, 210 (1976).
[26] E. Flowers and M.A. Ruderman, Astrophys. J. **215**, 302 (1977).
[27] D. Eichler, Astrophys. J. **254**, 683 (1981).
[28] W. Kluźniak and M.A. Ruderman, Astrophys. J. Lett. **505**, L113 (1998).
[29] S. L. Shapiro and S. A. Teukolsky, *Black Holes, White Dwarves and Neutron Stars* (Wiley, New York, 1983).
[30] M. Bocquet, S. Bonazzola, E. Gourgoulhon, and J. Novak, Astron. Astrophys. **301**, 757 (1995).
[31] A. Papapetrou, Ann. Inst. Henri Poincare, Sect. A **4**, 83 (1966).
[32] S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972).
[33] I. Easson, Nature (London) **263**, 486 (1976).