Positronium hyperfine splitting in noncommutative space at order α^6

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(Received 22 September 2001; published 30 July 2002)

We obtain positronium hyperfine splitting owing to the noncommutativity of space and show that, to leading order, it is proportional to $\theta\alpha^6$ where θ is the parameter of noncommutativity. It is also shown that spatial noncommutativity splits the spacing between $n=2$ triplet excited levels $E(2^3S_1)\rightarrow E(2^3P_2)$, which provides an experimental test of the noncommutativity of space.

DOI: 10.1103/PhysRevD.66.016005 PACS number(s): 11.10.St, 36.10.Dr

I. INTRODUCTION

The question of measuring the spatial noncommutativity effects in physical processes is of intense interest. Noncommutative QED (NCQED) seems to be a straightforward method to examine such effects. For this purpose, one needs precise experimental data such as positronium hyperfine splitting (HFS) among other processes. The basic difference between NCQED and QED is the existence of new interactions (three-photon and four-photon vertices) which complicate the calculations in NCQED. Although the Feynman rules of this theory are given in [1,2], to apply these rules to bound states, one needs special treatments such as the Bethe-Salpeter (BS) approach [3] or nonrelativistic QED (NRQED) [4]. In our previous Letter [5], using the BS equation, we showed that up to order α^4 no spin-dependent correction owing to the spatial noncommutativity appears in the positronium spectrum. Therefore one should calculate the higher order corrections. In this article we calculate the corrections to positronium using the NRQED method. In Sec. II, we introduce NRQED vertices in the NC space. Consequently, in Sec. III, we use modified NRQED to determine the HFS at lowest order. In this section, we show that our calculations at leading order lead to corrections at the order of $\theta\alpha^6$, where θ is the parameter of noncommutativity. At the end, we summarize our results.

II. NRQED IN NONCOMMUTATIVE SPACE

NRQED is an effective field theory which simplifies bound state calculations. To apply this technique in noncommutative space one should modify the NRQED vertices by performing a p/m_e expansion on the NCQED scattering amplitude. In doing this, one obtains an effective theory of nonrelativistic particles which permits the direct application of well tested techniques based on Schrödinger's equation. Now, by comparing NCQED scattering amplitudes with NRQED we can completely determine the matching coefficients. Some of the vertices with their appropriate matching coefficients are shown in Fig. 1. They contribute to the tree level matching to get the leading order bound state energy shift. One should note that these coefficients, apart from a phase factor, are very similar to those in the standard NRQED [6,7]. This similarity is owing to the fact that the scattering amplitude of e^+e^- in NCQED is independent of the parameter of noncommutativity of space [8]. The other vertices, which are not shown in Fig. 1 and have no counterpart in the standard NRQED, due to the existence of the three- and four-photon vertices, give contributions to higher order corrections to the energy shift. Now, by using the first graph of Fig. 1 and expanding the vertices up to order θ, one can easily verify the results of Refs. [5,9] at order $\theta\alpha^4$ as

$$\Delta E=\left\langle \alpha\frac{\mathbf{\Theta}\cdot\mathbf{L}}{r^3}\right\rangle=\theta\alpha^4\frac{P_{n,l}}{l\left(l+\frac{1}{2}\right)(l+1)}. \tag{1}$$

Such an energy shift is spin independent and therefore has no contribution to HFS.

III. POSITRONIUM HFS AT THE LEADING ORDER

By using the modified NRQED we can determine the diagrams that contribute to the lowest order of HFS (Fig. 2). We can now calculate each diagram separately as follows:

$$\triangle E_a=\int\frac{d^3p\,d^3p'}{(2\pi)^6}\psi^*(\mathbf{p}')\Gamma_a(\mathbf{p},\mathbf{p}')\psi(\mathbf{p}), \tag{2}$$

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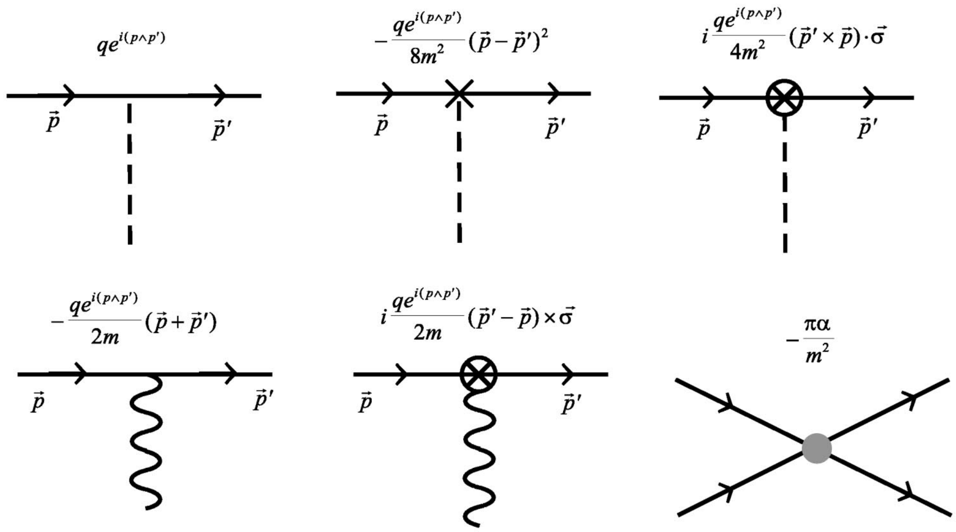

FIG. 1. NRQED vertices in noncommutative space.

with

$$\Gamma_a(\mathbf{p},\mathbf{p}')=\left[\frac{-ie(\mathbf{p}'-\mathbf{p})\times\boldsymbol{\sigma}_1}{2m_e}e^{ip\wedge p'}\right]_i\frac{-1}{(\mathbf{p}-\mathbf{p}')^2}\times\left[\delta_{ij}-\frac{(\mathbf{p}-\mathbf{p}')_i(\mathbf{p}-\mathbf{p}')_j}{(\mathbf{p}-\mathbf{p}')^2}\right]\times\left[\frac{e(\mathbf{p}'+\mathbf{p})}{2m_e}e^{ip\wedge p'}\right]_j, \tag{3}$$

where $\mathbf{p}\wedge\mathbf{p}'=\frac{1}{2}\theta_{\mu\nu}p_\mu p'_\nu$ and $\theta_{\mu\nu}$, the parameter of noncommutativity is given as

$$\theta_{\mu\nu}=i[x_\mu,x_\nu]. \tag{4}$$

It is shown that $\theta_{0i}\neq 0$ leads to some problems with the unitarity of field theory and the concept of causality [10,11]; therefore in our calculations we consider $\theta_{0i}=0$.

After some algebra Eq. (2) yields

$$\begin{aligned}\triangle E_a&=\frac{ie^2}{2m_e^2}\int\frac{d^3p\,d^3p'}{(2\pi)^6}\psi^*(\mathbf{p}')\frac{\boldsymbol{\sigma}_1\cdot\mathbf{p}\times\mathbf{p}'}{(\mathbf{p}-\mathbf{p}')^2}e^{i\theta_{ij}p_ip'_j}\psi(\mathbf{p})\\&=\frac{e^2}{8\pi m_e^2}\int d^3r\left[\psi^*(\mathbf{r}+i\theta\cdot\nabla)\frac{\mathbf{r}\times\mathbf{p}}{r^3}\cdot\boldsymbol{\sigma}_1\right]\psi(\mathbf{r})\\&=\frac{\alpha}{2m_e^2}\left\langle\frac{\mathbf{S}_1\cdot\mathbf{L}}{r^3}\right\rangle-\frac{3\alpha}{m_e^2}\int d^3r(\Theta\cdot\mathbf{L}\psi^*)\frac{\mathbf{S}_1\cdot\mathbf{L}}{r^5}\psi\\&\quad+O(\alpha^7),\end{aligned} \tag{5}$$

where $(\theta\cdot\nabla)_i=\theta_{ij}\partial_j$ and $\Theta=(\theta_{23},\theta_{31},\theta_{12})$. In the third equality we used

$$\psi^*(\mathbf{r}+i\theta\cdot\boldsymbol{\nabla})=\psi^*(\mathbf{r})+i[\boldsymbol{\nabla}\psi^*(\mathbf{r})\cdot\theta\cdot\boldsymbol{\nabla}]+O(\theta^2). \tag{6}$$

One should note that the first term in Eq. (5) is the usual term in NRQED which is of order α^4. But the second term that appears in Eq. (5), owing to the spatial noncommutativity, is of order $\theta\alpha^6$. The nonexistence of terms of order of α^4

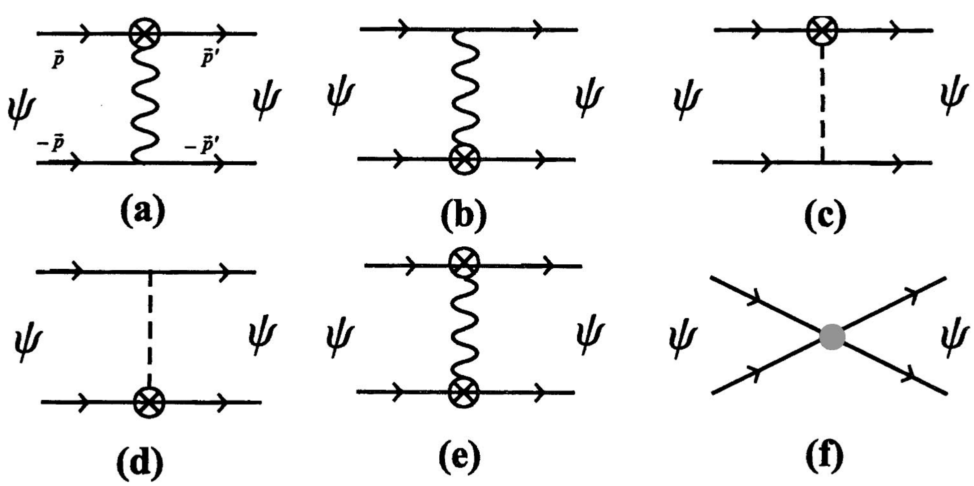

FIG. 2. All the bound state diagrams at lowest order.

which carry θ dependence is a remarkable result which happens in all diagrams of the HFS. Indeed, this fact is due to the appearance of $\psi^*(\mathbf{r}+i\theta\cdot\boldsymbol{\nabla})$ instead of $\psi^*(\mathbf{r})$ in all energy-correction expressions. Therefore, to obtain the energy corrections for HFS at order α^6 one should calculate once the corrections up to lowest order of NRQED (i.e., Fig. 2). In other words, the α^6 corrections in NRQED calculations of commutative space lead to a higher order of α in noncommutative space.

Now we work out Figs. 2(b)–2(f) as follows:

$$\triangle E_b=\triangle E_a(\mathbf{S}_1\to\mathbf{S}_2),$$

$$\triangle E_c+\triangle E_d=\frac{1}{2}(\triangle E_a+\triangle E_b), \tag{7}$$

$$\triangle E_e=\int\frac{d^3p\,d^3p'}{(2\pi)^6}\psi^*(\mathbf{p}')\Gamma_e(\mathbf{p},\mathbf{p}')\psi(\mathbf{p}), \tag{8}$$

with

$$\Gamma_e(\mathbf{p},\mathbf{p}')=\left[\frac{-ie(\mathbf{p}'-\mathbf{p})\times\boldsymbol{\sigma}_1}{2m_e}e^{ip\wedge p'}\right]_i\frac{-1}{(\mathbf{p}-\mathbf{p}')^2}\times\left[\delta_{ij}-\frac{(\mathbf{p}-\mathbf{p}')_i(\mathbf{p}-\mathbf{p}')_j}{(\mathbf{p}-\mathbf{p}')^2}\right]\times\left[\frac{-ie(\mathbf{p}'-\mathbf{p})\times\boldsymbol{\sigma}_2}{2m_e}e^{ip\wedge p'}\right]_j, \tag{9}$$

which results in

$$\triangle E_e=\frac{e^2}{4m_e^2}\int d^3r\,\psi(\mathbf{r})\psi^*(\mathbf{r}+i\theta\cdot\boldsymbol{\nabla})\times[-\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\boldsymbol{\nabla}^2+(\boldsymbol{\sigma}_1\cdot\boldsymbol{\nabla})(\boldsymbol{\sigma}_2\cdot\boldsymbol{\nabla})]\frac{1}{4\pi r}=(\cdots)+\frac{3e^2}{16\pi m_e^2}\int d^3r\,\psi(\mathbf{r})\tilde{\Gamma}_e\psi^*(\mathbf{r}), \tag{10}$$

where $(\cdots)$ means the usual part of the energy shift and

$$\tilde{\Gamma}_e=\left[\frac{\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2}{r^5}\Theta\cdot\mathbf{L}-\frac{\boldsymbol{\sigma}_2\cdot\mathbf{r}}{r^5}\boldsymbol{\sigma}_1\cdot(\Theta\times\hat{\mathbf{p}})-\frac{\boldsymbol{\sigma}_1\cdot\mathbf{r}}{r^5}\boldsymbol{\sigma}_2\cdot(\Theta\times\hat{\mathbf{p}})-\frac{5}{r^7}(\boldsymbol{\sigma}_1\cdot\mathbf{r})(\boldsymbol{\sigma}_2\cdot\mathbf{r})\Theta\cdot\mathbf{L}\right], \tag{11}$$

where $\hat{\mathbf{p}}=-i\boldsymbol{\nabla}$. The final diagram [Fig. 2(f)] has no contribution at the order of our interest. For $S=1$ one can easily find

$$\triangle E_a^{\rm NC}+\triangle E_b^{\rm NC}=\frac{-3e^2}{4\pi m_e^2}\int d^3r\left[\frac{\Theta\cdot\mathbf{L}}{r^5}\psi^*(\mathbf{r})\right]l\psi(\mathbf{r}),$$

$$\triangle E_a^{\rm NC}+\triangle E_b^{\rm NC}=\frac{1}{2}(\triangle E_c^{\rm NC}+\triangle E_d^{\rm NC}),$$

$$\triangle E_e^{\rm NC}=\frac{3e^2}{16\pi m_e^2}\int d^3r\,\psi(\mathbf{r})\bar{\Gamma}\psi^*(\mathbf{r}), \tag{12}$$

where

$$\bar{\Gamma}=\frac{\Theta\cdot\mathbf{L}}{r^5}-\frac{2}{r^5}\left\{\begin{array}{c}z(\Theta\times\hat{\mathbf{p}})_3\\ \mathbf{r}\cdot(\Theta\times\hat{\mathbf{p}})-2z(\Theta\times\hat{\mathbf{p}})_3\\ z(\Theta\times\hat{\mathbf{p}})_3\end{array}\right\}-\frac{5}{r^7}\left\{\begin{array}{c}z^2\\ r^2-2z^2\\ z^2\end{array}\right\}\Theta\cdot\mathbf{L}. \tag{13}$$

The superscript NC in Eq. (12) means the noncommutative part of the energy shift and the three lines in Eq. (13) are related to $S_z=1,0,-1$, respectively. Meanwhile, for the spin zero state ($S=0$), all contributions to the energy shift are zero.

The average of $\triangle E_e$ over the triplet is zero, which means the spin-spin interaction part carries no correction on average and therefore the hyperfine splitting due to the noncommutativity becomes

$$\delta E^{\rm NC}=\frac{9e^2}{8\pi m_e^2}\int d^3r\left[\frac{\Theta\cdot\mathbf{L}}{r^5}\psi^*_{nlm}(\mathbf{r})\right]l\psi_{nlm}(\mathbf{r}), \tag{14}$$

where ψ_{nlm} is the wave function of positronium in commutative space with the Coulomb potential and we have defined $\delta E^{\rm NC}=\triangle E^{\rm NC}(S=1)-\triangle E^{\rm NC}(S=0)$. If the z axis is chosen parallel to the vector Θ, the above result simplifies to

$$\delta E^{\rm NC}=\frac{9e^2}{8\pi m_e^2}|\Theta|lm\left\langle\frac{1}{r^5}\right\rangle=\frac{|\Theta|}{\lambda_e^2}\alpha^6 m_e lmf(n,l), \tag{15}$$

where λ_e is the Compton wavelength of the electron and $f(n,l)$ is defined as

$$f(n,l)=\frac{P_{n,l}^{(1)}}{l\left(l+\frac{1}{2}\right)(l+1)\left(l+\frac{3}{2}\right)(l+2)}+\frac{P_{n,l}^{(2)}}{(l-1)\left(l-\frac{1}{2}\right)l\left(l+\frac{1}{2}\right)(l+1)}. \tag{16}$$

One should note that the divergence of $\delta E^{\rm NC}$ at $l=1$ is owing to the singularity of $\langle 1/r^5\rangle$ at $r=0$, the region where θ expansion is not well defined. Actually, it is shown that θ-expanded NCQED is not renormalizable [12].

The θ expansion implies a cutoff $\Lambda \sim 1/\sqrt{|\Theta|}$ while the validity of NRQED requires $p \leqslant m_e = 1/\lambda_e$. Since $\sqrt{|\Theta|} \leqslant \lambda_e$, the appropriate cutoff is $\Lambda = 1/\lambda_e$. Therefore the energy shift for $n=2$, $l=1$ can be obtained as

$$\delta E^{\mathrm{NC}} = \frac{3}{512} m_e \left(\frac{|\Theta|}{\lambda_e^2} \right) [\ln 2 - \gamma - \ln \alpha] \alpha^6. \qquad (17)$$

The above result should be added to the values of HFS derived in NRQED at order α^6. The reported uncertainties on the experimental values of $E(2^3S_1) \to E(2^3P_2)$ are about 0.1 MHz [13], which gives an upper bound $|\Theta|/\lambda_e^2 \sim 10^{-1}$. Therefore determining the value of $|\Theta|$ requires more accurate experiments.

IV. SUMMARY

Using the NRQED method in noncommutative space, we have obtained the result that there is no correction at order α^4 for the HFS of positronium; the order α^4 corrections are spin independent. The correction to the energy shift starts at order α^6, Eqs. (14),(15), and it depends on the l and m quantum numbers. Therefore it has no contribution to $E(1^3S_1) \to E(1^1S_0)$ (in the spectroscopic notation $n^{2S+1}L_j$), while for $l \neq 0$ there are $2l+1$ different shifts. Consequently, a closer look at the spacing between the $n=2$ triplet excited levels $[E(2^3S_1) \to E(2^3P_2)]$, which has already been measured [14–17], can provide an experimental test on the noncommutativity of space.

ACKNOWLEDGMENTS

S.M.Z. gratefully acknowledges research funding from Shiraz University and IPM. The research of M.H. and F.L. was supported by Isfahan University of Technology (IUT) and IPM.

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