Heavy quark action on anisotropic lattices

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We investigate an $O(a)$ improved quark action on anisotropic lattices as a potential framework for heavy quark physics, which may enable high precision computations of hadronic matrix elements for heavy-light mesons. The relativity relations of heavy-light mesons as well as of heavy quarkonium are examined on a quenched lattice with a spatial lattice cutoff $a_\sigma^{-1} \simeq 1.6$ GeV and a (renormalized) anisotropy $\xi = 4$. We find that the corresponding bare anisotropy parameter, once tuned for the massless quark, describes both heavy-heavy and heavy-light mesons within 2% accuracy for a quark mass $a_\sigma m_Q < 0.8$, a range which covers the charm quark. This bare anisotropy parameter also successfully describes the heavy-light mesons alone even in the quark mass region $a_\sigma m_Q \leqslant 1.2$ within the same accuracy. Beyond this region, the discretization effects seem to grow gradually. The renormalized anisotropy ξ turns out to be roughly equal to the factor stretching the quark mass region in which the parameters in the action are applicable for heavy-light systems with well controlled systematic errors.

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I. INTRODUCTION

Recent progress in experimental heavy flavor physics exploring the effect of new physics calls for adequately precise theoretical predictions within the standard model. However, model independent calculations of hadronic matrix elements are difficult because of the nonperturbative nature of QCD. Lattice QCD simulation is the most promising approach, in which the systematic uncertainties can be reduced systematically [1]. The ultimate goal of the present paper is to construct a framework for lattice calculations of hadronic matrix elements on a few percent accuracy level, which would be adequate to the experiments in progress. For example, the accuracy requested from the CLEO-c experiment [2] as well as from the B factories [3,4] is about 2%. This paper proposes a project (and starts to systematically pursue it) that aims to achieve this accuracy by taking advantage of the anisotropic formulation of lattice QCD.

In lattice calculations of heavy quark systems such as mesons containing charm or bottom quarks, one needs to control the large discretization error of $O[(am_Q)^n]$. Extensive studies in various approaches have already achieved remarkable progress in understanding heavy quark systems. However, for several reasons it still seems difficult to reach the necessary systematic accuracy with the current techniques. Let us recall the previous approaches which fall into three groups and point out their respective advantages and disadvantages from the point of view of a high precision study of matrix elements for heavy flavor physics.

(1) Effective theories. This approach uses descriptions of the heavy quark degrees of freedom based on heavy quark effective theory using nonrelativistic QCD [5]. The advantage is that it is possible to remove the mass-dependent errors at the tree or one-loop level. However, since this theory does not have a well-defined continuum limit, one cannot remove the discretization errors by extrapolating the results obtained on lattices with finite spacings. Another disadvantage is that the nonperturbative renormalization is difficult to perform due to the strong mass dependence. More precise measurements below 10% uncertainty of the weak matrix elements for the heavy-light mesons are therefore hardly possible.

(2) Relativistic framework. The most straightforward approach is to use the $O(a)$ improved Wilson action in a natural way for the lighter ones among the heavy quarks, and to extrapolate the results in $1/m$ according to the heavy quark effective theory. Since the theory has a sound continuum limit, the discretization errors can be removed by an extrapolation. The perturbative errors can also be avoided by employing nonperturbative renormalization. In practice, however, lattice artifacts of $O((am_Q)^2)$ give dominant errors [6] so that high precision results will remain difficult to achieve for the next few years, even within the quenched approximation. Brute force improvement by using finer and larger lattices will quickly increase the simulation cost. Therefore, this way is not a feasible solution for the calculation of matrix elements.

(3) Fermilab approach. The Fermilab approach links the above two approaches [7,8]. In the heavy quark mass region, the $O(a)$ improved Wilson action with asymmetric parameters is reinterpreted as an effective theoretical description just like the nonrelativistic QCD action. Since the action reduces to the conventional $O(a)$ improved Wilson action for small masses $am_Q \leqslant 1$, it has in principle a smooth continuum limit. The disadvantage is that it is not known how to extrapolate the results obtained on lattices with $am_Q > 1$, which is currently unavoidable, in particular for systems containing a b-quark. To cover such a quark mass region, a mass-dependent tuning of parameters in the action is absolutely required for proper improvement, although a systematic tuning prescription beyond perturbation theory is still representing a theoretical challenge [9]. Therefore, precise calculations below 10% uncertainty for the weak matrix elements of the heavy-light mesons are hampered by lack of necessary ingredients.

These observations are based on currently available techniques and computational resources. Future progress within the three approaches is not excluded. However, for the time being it is desirable to develop an alternative framework for heavy quark systems which is characterized by the following features. (i) The continuum limit can be taken; (ii) a systematic improvement program, such as the nonperturbative renormalization technique [10], can be applied not only to the parameters in the action but also to the operators; (iii) a modest size of computational cost is sufficient for the systematic computation of matrix elements.

The anisotropic lattice, on which the temporal lattice spacing a_τ is finer than the spatial one, a_σ, is a good candidate to provide such a framework [11,12]. Our whole program basically follows the Fermilab approach, however formulated on the anisotropic lattice. In particular, the large temporal momentum cutoff is expected to drastically ameliorate the above problems. The most crucial one is that the mass dependence of the parameters in the action may become so mild in the region of practical interest that one can adopt the $O(a)$ improving clover coefficients determined by the nonperturbative renormalization technique. On the other hand, the standard size of the spatial lattice spacing helps to keep the total computational cost modest. Therefore, the extrapolation of the simulation results to the continuum limit may be possible while the systematic uncertainties can be kept under control. Whether these promises practically will be satisfied should be examined numerically, as well as in perturbation theory.

Our form of the quark action on an anisotropic lattice has been founded in Refs. [12,13], where perturbative results for light and heavy quarks have been presented and simulation results for light quark systems have been reported. In this paper we focus on systems that include heavy quarks and study the mass dependence of the breaking of relativity in order to understand the mass range for which a consistent description is possible. This analysis confirms that the anisotropic lattice formulation is applicable for heavy quark systems with the expected advantages. We have investigated heavy-heavy and heavy-light mesons on a quenched lattice with a (renormalized) anisotropy $\xi=4$ and a spatial lattice spacing $a_\sigma^{-1}\simeq 1.6$ GeV. The heavy quark mass has been varied from the charm quark mass up to about 6 GeV to examine the applicability over that mass range.

This paper is organized as follows. In Sec. II, we first recall our quark action, which has been discussed in detail in Ref. [12]. Then we formulate our conjecture concerning the lattice spacing dependence of the anisotropy parameter toward the continuum limit and discuss the advantage of the anisotropic lattice compared to the isotropic one. In Sec. III we observe the tree-level expectation of the mass dependence of the $O(a^2)$ terms in the quark dispersion relation and study how the anisotropy parameter or the breaking of relativity behave as functions of the heavy quark mass. Section IV describes the results of our numerical simulation. We compute the heavy-light and heavy-heavy meson spectra and dispersion relations with two sets of heavy quark parameters. In one set (*Set I*), the bare anisotropy is set to the value for the massless quark. The other set (*Set II*) adopts the result of the mass-dependent tuning using the heavy-heavy meson dispersion relation as obtained in Ref. [13]. We observe the mass dependence of the renormalized anisotropy in order to probe the breaking of relativity for heavy quarks. We also observe how the inconsistency among the binding energies of heavy-heavy, heavy-light and light-light mesons [14,15] grows as a function of mass. The last part of this section discusses the hyperfine splitting of the heavy-light meson. In Sec. V we summarize the results of our simulations and sketch the perspectives for further development of the proposed framework.

II. FORMULATION

A. Quark action

We adopt the following $O(a)$ improved Wilson quark action in the hopping parameter form [12,16]:

$$S_F=\sum_{x,y}\bar{\psi}(x)K(x,y)\psi(y), \tag{1}$$

$$\begin{aligned}K(x,y)=\delta_{x,y}-\kappa_\tau[&(1-\gamma_4)U_4(x)\delta_{x+\hat{4},y}\\&+(1+\gamma_4)U_4^\dagger(x-\hat{4})\delta_{x-\hat{4},y}]\\&-\kappa_\sigma\sum_i[(r-\gamma_i)U_i(x)\delta_{x+\hat{i},y}\\&+(r+\gamma_i)U_i^\dagger(x-\hat{i})\delta_{x-\hat{i},y}]\\&-\kappa_\sigma c_E\sum_i\sigma_{4i}F_{4i}(x)\delta_{x,y}\\&-r\kappa_\sigma c_B\sum_{i>j}\sigma_{ij}F_{ij}(x)\delta_{x,y},\end{aligned} \tag{2}$$

where κ_σ and κ_τ are the spatial and temporal hopping parameters which are related to the bare quark mass and bare anisotropy as given below. The parameter r is the spatial Wilson coefficient, and the parameters c_E and c_B are the respective clover coefficients for the $O(a)$ improvement. Although the explicit Lorentz symmetry is not manifest due to the anisotropy in lattice units, it can be restored in principle for physical observables in physical units at long distances up to errors of $O(a^2)$ by properly tuning $\kappa_\sigma/\kappa_\tau$, r, c_E and c_B for a given κ_σ. The action is constructed in accord with the Fermilab approach [7] and hence applicable to an arbitrary quark mass, although a mass-dependent tuning of parameters is difficult beyond perturbation theory. This may be circumvented by taking $a_\tau^{-1}\gg m_Q$, with which the mass dependence of parameters are expected to be small so that the $O(a)$ Symanzik improvement program for the heavy quark can be applied. To check whether this expectation really holds true or not is the main subject of this paper.

In the present study, we vary only two parameters κ_σ and κ_τ with fixed other parameters. We put the Wilson parameter as $r=1/\xi$ and take the clover coefficients equal to the tadpole-improved tree-level values, $c_E=1/u_\sigma u_\tau^2$, and $c_B=1/u_\sigma^3$. The tadpole improvement [17] is achieved by rescaling the link variable as $U_i(x)\to U_i(x)/u_\sigma$ and $U_4(x)$

$\to U_4(x)/u_\tau$, with the mean-field values of the spatial and temporal link variables, u_σ and u_τ, respectively. The definitions of the mean-field values are given in Sec. IV. Instead of κ_σ and κ_τ, we introduce κ and $\gamma_F(=1/\zeta)$ as

$$\frac{1}{\kappa}\equiv\frac{1}{\kappa_\sigma u_\sigma}-2(\gamma_F+3r-4)=2(m_0\gamma_F a+4),$$
$$\gamma_F\equiv 1/\zeta\equiv\kappa_\tau u_\tau/\kappa_\sigma u_\sigma. \tag{3}$$

The former controls the bare quark mass and the latter corresponds to the bare anisotropy.

B. Mass dependence of anisotropy parameter

On an anisotropic lattice, one must tune the parameters so that the hadronic states satisfy the relativistic dispersion relations. In general, the lattice dispersion relation for arbitrary values of ζ is described in lattice units as

$$E^2(\boldsymbol{p})=m^2+\boldsymbol{p}^2/\xi_F^2+O(p^4). \tag{4}$$

In the above expression the energy E and the rest mass m are in temporal lattice units while the momentum $\boldsymbol{p}$ is in spatial units. The parameter ξ_F introduced in this equation characterizes the anisotropy of the quark fields. The difference between the quark field anisotropy ξ_F and the gauge field anisotropy ξ probes the breaking of relativity. Therefore, the calibration is nothing but the tuning of $\zeta(=1/\gamma_F)$ for a given κ so that ξ_F equals ξ and hence the relativistic dispersion relations are satisfied. Let us call the tuned parameter as $\zeta^*(=1/\gamma_F^*)$. ζ^* depends on the quark mass and can in general be different from unity even for the case of isotropic lattice. The relativity relation automatically enforces that the rest and kinetic masses are equal to each other. In this sense, our calibration procedure of anisotropic lattice action is a natural generalization of the Fermilab approach.

Now we consider the tuning of the anisotropy parameter $\zeta=1/\gamma_F$, either on an isotropic or anisotropic lattice, for a fixed physical quark mass. It can be tuned using the dispersion relation of either a heavy-heavy or a heavy-light meson in a mass-dependent way. Alternatively, one could also adopt the value tuned for the light-light mesons neglecting the mass dependence. In principle, these procedures of calibration can give different results due to the discretization errors. What we would like to know at this stage is (1) the lattice spacing $a_\sigma^{hh\text{-}hl}$ above which the calibrated parameters ζ^* using the dispersion relations of heavy-heavy and heavy-light mesons start to differ from each other by more than a certain accuracy ϵ_{acc} (say, 2%), and (2) the lattice spacing $a_\sigma^{hl\text{-}ll}$ above which the calibrated parameters ζ^* coming from the dispersion relations of heavy-light and light-light mesons start to be different by more than a certain accuracy ϵ'_{acc}.

Naively speaking, the difference between ζ^*'s derived from heavy-heavy and heavy-light mesons originates from the $O((a_\sigma p)^2)$ effects, where p is the typical quark momentum in a heavy-heavy meson, $p\sim\alpha m_Q$. On the other hand, the deviation of ζ^* originating from heavy-light mesons from the ζ^* based on light-light mesons is due to effects of order $O((a_\tau m_Q)^2)=O((a_\sigma m_Q/\xi)^2)$. Demanding that these discrepancies are bounded by the required accuracies, we obtain

$$a_\sigma^{hh\text{-}hl}\sim\sqrt{\epsilon_{acc}}/(\alpha m_Q),$$
$$a_\sigma^{hl\text{-}ll}\sim\sqrt{\epsilon'_{acc}}\;\;\xi/m_Q. \tag{5}$$

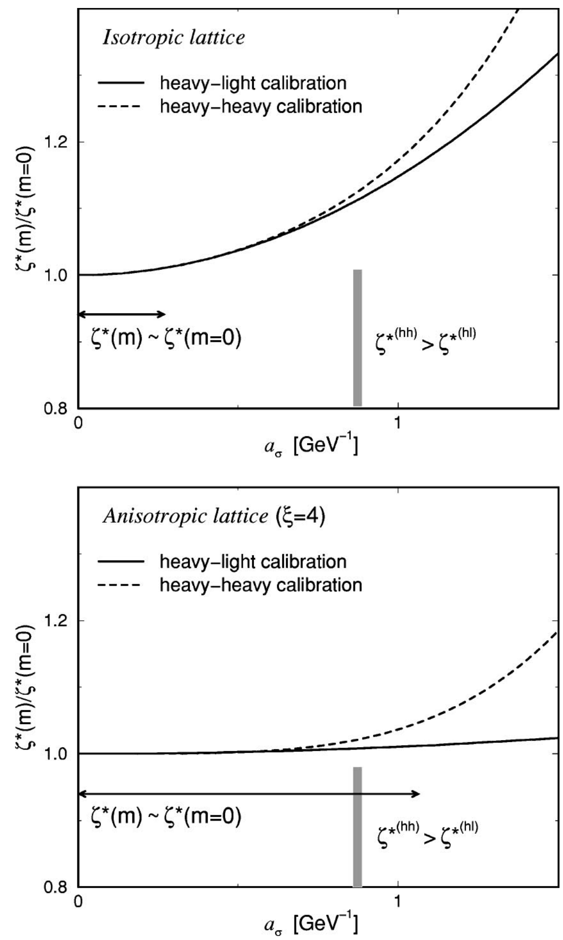

FIG. 1. The conjectured lattice spacing dependence of the anisotropy parameter for a fixed quark mass for isotropic (top) and anisotropic (bottom) lattices. Horizontal lines with arrows roughly represent the region $a_\sigma<a_\sigma^{hl\text{-}ll}$, while vertical thick lines correspond to $a_\sigma^{hh\text{-}hl}$, above which the ζ^*'s determined from heavy-heavy and heavy-light mesons are no longer equal. In this figure a heavy quark with roughly the charm quark mass is considered as an example.

Figure 1 depicts the expected behavior of the tuned anisotropy parameter ζ^* for isotropic $(\xi=1)$ and anisotropic $(\xi=4)$ lattices. The tuned anisotropy normalized by the value in the massless limit is displayed for a fixed physical quark mass around the charm quark mass as an example. In the case of an isotropic lattice, as shown in the top panel of Fig. 1, $a_\sigma^{hl\text{-}ll}<a_\sigma^{hh\text{-}hl}$ is expected to hold. In this case, for a sufficiently small lattice spacing $a_\sigma<a_\sigma^{hl\text{-}ll}$, both the heavy-heavy and heavy-light mesons can be successfully described by fixing the anisotropy parameter to its value in the massless limit. For a coarser lattice spacing $a_\sigma^{hl\text{-}ll}<a_\sigma<a_\sigma^{hh\text{-}hl}$, a mass-dependent tuning, as in the Fermilab approach, is nec-

essary. In the result, the simultaneous description of both heavy-heavy *and* heavy-light mesons is possible. For an even coarser lattice spacing $a_\sigma^{hh\text{-}hl} < a_\sigma$, a simultaneous, consistent description of heavy-heavy and heavy-light mesons is no longer warranted, due to the severe discretization effect becoming manifest in the heavy quarkonia.

Now let us turn to the anisotropic lattice case with $\xi = 4$. The expected mass dependence of ζ^* is schematically represented in the bottom panel of Fig. 1. As it is obvious in Eq. (5) and will be studied in the following sections, the lattice anisotropy does not improve the situation for heavy-heavy meson states. Therefore $a_\sigma^{hh\text{-}hl}$ remains roughly the same size as in the isotropic lattice case. On the other hand, as one can see from Eq. (5) and as the studies in the following sections will demonstrate, $a_\sigma^{hl\text{-}ll}$ is expected to increase for $\xi = 4$, and $a_\sigma^{hh\text{-}hl} < a_\sigma^{hl\text{-}ll}$ occasionally holds. This latter situation is particularly likely for quarks with not very large mass, and is expected to apply for masses up to the bottom quark mass region. In this case, for sufficiently small lattice spacing $a_\sigma < a_\sigma^{hh\text{-}hl}$, both heavy-heavy and heavy-light mesons can be successfully described by fixing the anisotropy parameter to its value in the massless limit. For a coarser lattice spacing $a_\sigma^{hh\text{-}hl} < a_\sigma < a_\sigma^{hl\text{-}ll}$, a consistent description of heavy-heavy and heavy-light mesons is no longer possible. Nevertheless the heavy-light meson is successfully described with the anisotropy parameter tuned in the massless limit. Finally for an even coarser lattice spacing $a_\sigma^{hl\text{-}ll} < a_\sigma$, the genuine Fermilab approach, a mass-dependent tuning of ζ is indispensable in order to describe the heavy-light mesons.

In the range of lattice spacing $a_\sigma < a_\sigma^{hl\text{-}ll}$, if we abandon the description of heavy quarkonia and content ourselves with the heavy-light systems, we can successfully work with the value of ζ tuned in the massless limit. This is the most important case for our approach to be applied, since in this mass region the mass dependence of other parameters in the action, for instance c_E and c_B, is also expected to be small, and hence the result of tuning at the massless limit is also considered to be valid. In order to tune the clover coefficients for the case of massless quarks, it may be possible to apply the nonperturbative renormalization technique [10], which is currently the most efficient procedure beyond perturbation theory to remove the $O(a)$ effect. The most advantageous feature of the anisotropic lattice is that this region $a < a_\sigma^{hl\text{-}ll}$ is expected to be enlarged by a factor ξ as suggested by Eq. (5). In the result, this would cover the heavy quark mass region being of practical interest. Therefore the anisotropic lattice is a preferable framework for the description of heavy-light systems, which possesses the attractive features (i), (ii), and (iii) mentioned in the Introduction. Also for heavy-heavy systems, it would be important to study quantitatively from which mass of the heavy quark the action would be doomed to fail to produce a correct description.

The following sections are dedicated to a study to what extent these expectations are true. Using Eq. (5), a similar expectation can be deduced concerning the heavy quark mass dependence with a fixed lattice spacing. In this paper we work at only one lattice spacing. Therefore, instead of studying the lattice spacing dependence with a fixed heavy quark mass, we will study the mass dependence with a fixed lattice spacing. This means to specify in which of the regions considered above the present a_σ is located for a given value of the quark mass. This question is discussed in more detail within the tree level analysis in the next section before we turn to the results of numerical simulations in Sec. IV.

III. EXPECTATION FROM TREE LEVEL ANALYSIS

In this section we review the tree level analysis of the heavy quark action in the Fermilab formulation. In the following, we express the energy and the momenta in physical units so that the dependence on the lattice spacings a_τ and a_σ is explicitly shown. The dispersion relation of the heavy quark on anisotropic lattice is given as [7,12]

$$\cosh(a_\tau E) = 1 + \frac{\left[\bar{m}_0 + 2r\zeta \sum_i \sin^2(a_\sigma p_i/2)\right]^2 + \zeta^2 \sum_i \sin^2(a_\sigma p_i)}{2\left[1 + \bar{m}_0 + 2r\zeta \sum_i \sin^2(a_\sigma p_i/2)\right]}, \tag{6}$$

where $\bar{m}_0 = a_\tau m_0$. This leads to the following dispersion relation:

$$E^2 = M_1^2 + \left(\frac{\xi^{\rm tree}}{\xi_F^{\rm tree}}\right)^2 \boldsymbol{p}^2 + A_1 a_\sigma^2 (\boldsymbol{p}^2)^2 + A_2 a_\sigma^2 \sum_i p_i^4 + \cdots, \tag{7}$$

where M_1 is the pole mass,

$$M_1 = \log(1+\bar{m}_0) a_\tau^{-1}, \tag{8}$$

and the anisotropy of the quark field at the tree level is

$$\left(\frac{\xi^{\rm tree}}{\xi_F^{\rm tree}}\right)^2 = \log(1+\bar{m}_0)\xi^2 \left[\frac{2\zeta^2}{\bar{m}_0(2+\bar{m}_0)} + \frac{r\zeta}{1+\bar{m}_0}\right]. \tag{9}$$

The third and fourth terms in Eq. (7) represent the $O((ap)^2)$ errors, where A_1 and A_2 are given as

$$A_1 = \frac{\xi^2}{4}\left[\left(\frac{2\zeta^2}{\bar{m}_0(2+\bar{m}_0)} + \frac{r\zeta}{1+\bar{m}_0}\right)^2 - \log(1+\bar{m}_0) \times \left(\frac{8\zeta^4}{\bar{m}_0^3(2+\bar{m}_0)^3} + \frac{4\zeta^3[\zeta + 2r(1+\bar{m}_0)]}{\bar{m}_0^2(2+\bar{m}_0)^2} + \frac{r^2\zeta^2}{(1+\bar{m}_0)^2}\right)\right], \tag{10}$$

$$A_2 = -\frac{\xi^2}{3}\log(1+\bar{m}_0)\left[\frac{2\zeta^2}{\bar{m}_0(2+\bar{m}_0)} + \frac{r\zeta}{4(1+\bar{m}_0)}\right]. \tag{11}$$

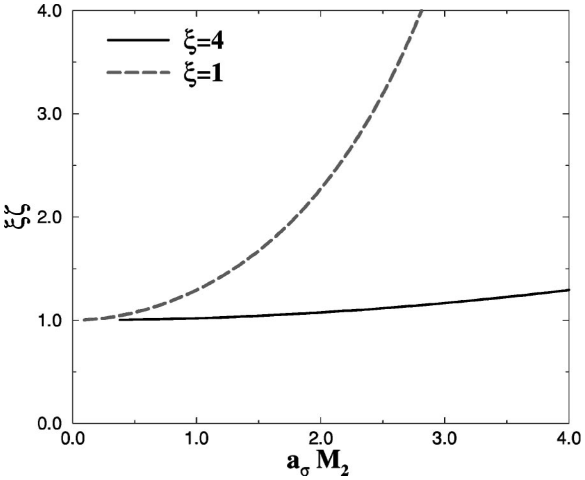

FIG. 2. The mass dependence of $\xi\zeta$ tuned at the tree level for the isotropic lattice and anisotropic lattice with $\xi=4$. The horizontal axis is the kinetic quark mass in spatial lattice units.

The above coefficients are derived using the M_1, M_2, M_4 and w_4 defined in Ref. [7], now extended to anisotropic lattice by replacing certain parameters as explained in Ref. [12].

Figure 2 shows the mass dependence of ζ tuned by requiring $\xi_F=\xi$, both on the isotropic lattice and on the anisotropic lattice with $\xi=4$. In the latter case, it is clear that the mass dependence is drastically reduced so that taking the value of ζ at the massless limit is a good approximation over a wide range of quark mass, in contrast to the case of the isotropic lattice.

In the following let us consider either the case with mass-dependent tuning of ζ (denoted by "Fermilab" in the figures) or with the ζ at the massless limit ("massless"). Figure 3 shows the breaking of the relativity relation, ξ_F/ξ, according to Eq. (9). The kinetic mass M_2 of quark is related to ξ_F/ξ as $M_2=M_1\xi_F^2/\xi^2$. For ζ tuned in a mass dependence way, ξ_F trivially equals ξ. Without explicit tuning, the mass dependence of ξ_F is drastically reduced in the case of $\xi=4$. This is understandable since the deviation of ξ_F/ξ from unity is an $O(a_\tau m_Q)$ error. In Fig. 4 one can see that the coefficient A_1, whose deviation from zero signals the $O(a^2)$ effect, is also drastically reduced on the anisotropic lattice, either with or without mass-dependent tuning of ζ. On the other hand, the $O(a^2)$ error from A_2 is not reduced but has become somewhat worse on the anisotropic lattice as is seen in Fig. 5. However, the coefficient A_2 is not larger than the value at the massless limit. As a general feature for all of the quantities ξ_F/ξ, A_1 and A_2, the mass dependences are drastically reduced on the anisotropic lattice. On the other hand, mass-dependent tuning of ζ on an isotropic lattice does not reduce the coefficients A_1 and A_2, while it completely removes the discrepancy between ξ_F and ξ. In order to reduce these breaking of relativity one has to introduce higher order terms, as is pointed out in Ref. [7].

From these results one can predict the following systematic errors in the description of heavy quark systems. In

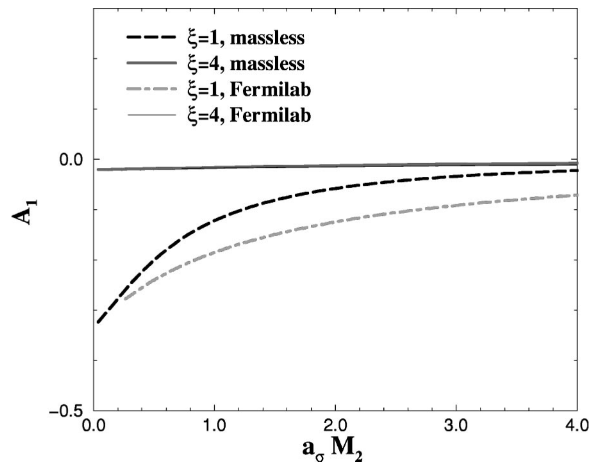

FIG. 4. The kinetic mass dependence of the $O((ap)^2)$ coefficient A_1 in the dispersion relation (7). "Massless" denotes the result of tuning of ζ in the massless limit. "Fermilab" denotes the result of mass-dependent tuning of ζ by requiring $\xi_F=\xi$.

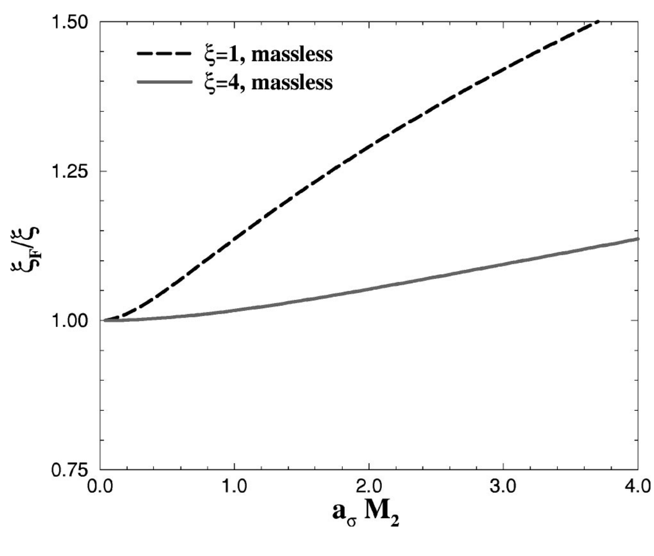

FIG. 3. The kinetic mass dependence of ξ_F/ξ at the tree level. We do not show the result of mass-dependent tuning of ζ since $\xi_F/\xi=1$ is satisfied by definition.

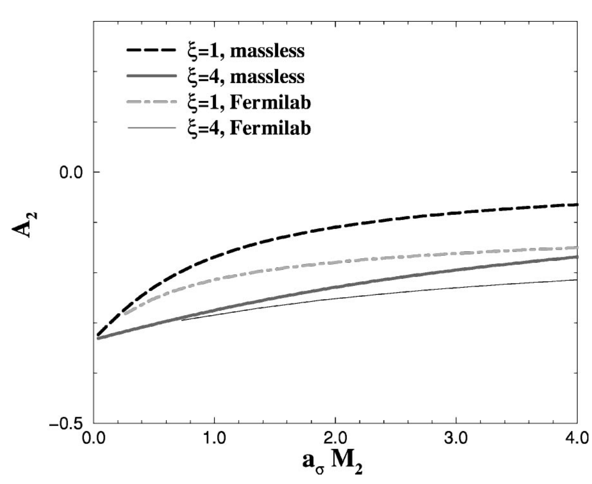

FIG. 5. The kinetic mass dependence of the $O((ap)^2)$ coefficient A_2 in the dispersion relation (7).

TABLE I. Quark parameters and the result of calibration. The result for $\kappa=0.1100$, 0.1020, and 0.0930 are taken from Ref. [13]. The number of configurations is 200 for each value of the hopping parameter, except for $\kappa=0.1100$ for which the simulation is carried out with 300 configurations.

κ	m_0	input γ_F	$\gamma_F^{*(PS)}$	$\gamma_F^{*(V)}$	γ_F^*	$\delta\gamma_F^*$
0.1100	0.1437	3.90,4.00	3.945(28)	3.946(36)	3.946(29)	-0.001(20)
0.1020	0.2328	3.90,4.00	3.848(26)	3.847(34)	3.847(28)	0.001(15)
0.0930	0.3514	3.70,3.80	3.688(23)	3.687(29)	3.688(25)	0.000(11)
0.0840	0.4954	3.39,3.48	3.471(20)	3.462(23)	3.467(21)	0.0092(60)
0.0760	0.6521	3.04,3.15	3.205(20)	3.191(23)	3.199(21)	0.0139(45)
0.0700	0.7930	2.85,2.95	2.946(23)	2.932(23)	2.939(23)	0.0134(52)
0.0630	0.9914	2.50,2.60	2.584(22)	2.561(24)	2.573(23)	0.0237(35)

heavy-light systems, the heavy quarks have momenta typically of the hadronic scale, $\Lambda_{QCD}\simeq 200$–500 MeV. Then the $O((ap)^2)$ errors are well under control even for large $a_\sigma M_2$, if one keep $a_\sigma^{-1}\gg\Lambda_{QCD}$. While the largest error originates from A_2, it is of the same order as that in the light quark systems. For the heavy-heavy system, the situation is quite different. In this case the heavy quark momenta are typically $p\sim\alpha m_Q$. The $O((ap)^2)$ errors are expected to be as large as $A_1(\alpha a_\sigma m_Q)^2, A_2(\alpha a_\sigma m_Q)^2$. Again, A_2 gives the largest contribution, but the size of the error is actually out of control when $a_\sigma m_Q$ is large.

To summarize, the anisotropic lattice largely reduces the discretization effects represented by $\xi_F/\xi-1$ and A_1, while it does not improve the A_2. For the heavy-light systems, this suffices for computations with discretization effects kept under control. On the other hand, when $a_\sigma M_2$ is of order of unity, the anisotropic lattice does not improve the situation for heavy quarkonia because of the severe effect of A_2 in these systems. Although, by mass-dependent tuning of ζ, one is able to remove the deviation of ξ_F from ξ, a ζ parameter simply tuned to the massless quark case also provides a good approximation as long as $a_\tau m_0\ll 1$. This is the biggest advantage of the anisotropic lattice approach compared with the isotropic Fermilab approach, where the mass dependence of the parameter is much stronger. These observations are in accord with our conjecture formulated in Sec. II.

IV. NUMERICAL SIMULATION

In this section we describe the numerical examination of the ideas outlined in the previous sections. For this purpose, we perform simulations with two series of heavy quark parameters. In set I the anisotropy parameter is set to the value at the chiral limit, $\gamma_F^*(m_q=0)$, just the same as applied for light quarks. In set II the fully tuned anisotropy parameter γ_F^* is adopted as obtained using heavy-heavy mesons. This calibration is done in Sec. IV A. For the heavy-heavy and heavy-light mesons, the rest and kinetic masses are obtained with two sets of parameters. Two quantities are used to probe the breaking of relativity: the fermionic anisotropy ξ_F for heavy-heavy and heavy-light mesons, and the inconsistency among the binding energies of heavy-heavy, heavy-light, and light-light mesons. The behavior of these quantities allows us to overlook the regions in which our framework can be applied. Section IV B treats the heavy-light meson spectrum, and we observe in particular how the hyperfine splitting behaves with the meson kinetic mass.

A. Calibration in heavy quark region

The simulation parameters used in this paper are forming the second set in Ref. [13]: the quenched anisotropic lattice of the size $16^3\times 128$ generated with the standard plaquette gauge action with $(\beta,\gamma_G)=(5.95,3.1586)$, which correspond to the renormalized anisotropy $\xi=4$ [18], and the spatial lattice cutoff $a_\sigma^{-1}=1.623(9)$ GeV is fixed by reference to the hadronic radius r_0 [19]. The mean-field values in the quark action are set to the mean values of link variables in the Landau gauge, $u_\sigma=0.7917$ and $u_\tau=0.9891$.

In Ref. [13] the optimum bare anisotropy γ_F^* is determined using the dispersion relation of mesons with degenerate quark masses, and the resultant values of γ_F^* are well represented by a linear function

$$\frac{1}{\gamma_F^*}=\zeta_0+\zeta_2 m_q^2, \quad m_q=\frac{1}{2\xi}\left(\frac{1}{\kappa}-\frac{1}{\kappa_c}\right), \tag{12}$$

where for the present lattice $\zeta_0=0.2490(8)$, $\zeta_2=0.189(15)$, and $\kappa_c=0.12592(6)$.

In this paper we use seven values of κ for heavy quark (κ_h) covering the mass of 1–6 GeV. Three of them have already been analyzed in Ref. [13]. We start with the calibration for the remaining four values of κ in the heavy quark region in the same manner as in Ref. [13]. The values of κ used are listed in Table I together with the result of calibration. The second column is the naive estimate of bare quark mass according to Eq. (12). For the heaviest case, m_0 is almost unity in temporal lattice units, and therefore the breaking of relativity might already become visible. Here we notice that for the heavier quark masses the difference of γ_F^* for pseudoscalar and vector mesons,

$$\delta\gamma_F^*=\gamma_F^{*(V)}-\gamma_F^{*(PS)}, \tag{13}$$

is sizable beyond the statistical fluctuations. This can be understood as a warning that the quarkonium system is not properly described within the present framework at this lat-

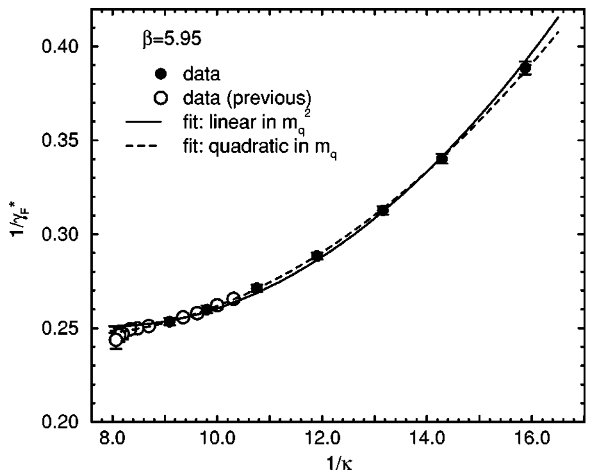

FIG. 6. The result of calibration in the heavy quark mass region (filled circles), together with the previous result in Ref. [13]. The fits are performed with all available data, including previous results.

tice spacing. This problem will again be discussed later in terms of the fermionic anisotropies for heavy-heavy and heavy-light mesons.

Figure 6 shows the result of calibration. The result is well fitted to a linear form in m_q^2 or a quadratic form in m_q, in spite of large quark mass. The fits, including previous data in Ref. [13], result in

$$\text{linear:}\quad \zeta_0 = 0.2510(6),\quad \zeta_2 = 0.1437(26), \tag{14}$$

$$\text{quadratic:}\quad \zeta_0 = 0.2473(12),\quad \zeta_1 = 0.0267(72),$$

$$\zeta_2 = 0.1151(81). \tag{15}$$

In both cases the values of ζ_0 are close to the value at the mean-field tree level, $\xi^{-1} = 0.25$. Since there is no guarantee that the small quark mass dependence persists up to $m_q \simeq a_\tau^{-1}$, the above result just shows that the quark mass dependence of $1/\gamma_F^*$ is still tractable at this mass region. Although we found a difference between ζ_2's in the linear fits with the present data and with the old subset of data, two fits show no significant difference in the lighter quark mass region.

B. Breaking of relativity

In this section we compute the heavy-heavy and heavy-light meson spectra and the dispersion relations using two sets of heavy quark parameters. For the first one, set I, the bare anisotropy is set to the value of γ_F^* in the chiral limit obtained in Ref. [13], namely $\gamma_F = 4.016$ at $\beta = 5.95$, for all quark masses. The second one, set II, adopts the results of Sec. IV A of mass-dependent calibration. We use the same hopping parameter values κ_h for the heavy quark as given in Sec. IV A. For the light quark, we use the single value $\kappa_l = 0.1235$. The value of γ_F at this κ_l is set to the value in the chiral limit, the same as in the case of light hadron spectroscopy treated in Ref. [13]. We regard that at $\kappa = \kappa_l$ the quark mass is sufficiently light for our present purposes, and we do not extrapolate the results to the chiral limit. The numbers of configurations used are 200 and 500 for heavy-heavy and heavy-light meson masses, respectively.

The lowest pseudoscalar and vector meson masses are listed in Tables II and III for heavy-heavy and heavy-light mesons, respectively. In these tables we also list the results of ξ_F for each meson channel, and the difference between them, $\delta\xi_F = \xi_F^{(V)} - \xi_F^{(PS)}$. If the anisotropic lattice action does not describe the quarks inside mesons in a way respecting the relativity relation, the breaking of relativity appears in the dispersion relations of the respective mesons. Therefore the deviation of fermionic anisotropy ξ_F from ξ signals the breaking effect of relativity. In the following, we first discuss the result of ξ_F for set I parameters, namely with γ_F tuned for the massless limit, and then briefly summarize the result for set II parameters.

TABLE II. Heavy-heavy meson spectrum for set I and set II parameters obtained with 200 configurations.

	κ_h	γ_F	m_{PS}	m_V	$m_V - m_{PS}$	$\xi_F^{(PS)}$	$\xi_F^{(V)}$	$\xi_F^{(V)} - \xi_F^{(PS)}$
Set I	0.1100	4.016	0.42468(23)	0.43755(33)	0.01288(18)	4.069(34)	4.068(46)	-0.001(24)
	0.1020	4.016	0.58409(25)	0.59358(34)	0.00949(13)	4.148(27)	4.143(35)	-0.005(16)
	0.0930	4.016	0.76947(24)	0.77692(31)	0.00745(10)	4.292(22)	4.294(28)	0.002(10)
	0.0840	4.016	0.96746(25)	0.97358(31)	0.00612(8)	4.527(24)	4.537(29)	0.010(8)
	0.0760	4.016	1.15894(27)	1.16411(33)	0.00517(8)	4.821(30)	4.839(35)	0.018(10)
	0.0700	4.016	1.31453(27)	1.31896(31)	0.00443(7)	5.146(33)	5.187(38)	0.041(8)
	0.0630	4.016	1.51262(27)	1.51617(31)	0.00355(5)	5.650(39)	5.726(43)	0.076(7)
Set II	0.1100	3.946	0.42942(23)	0.44248(33)	0.01306(18)	4.005(29)	4.003(41)	-0.002(22)
	0.1020	3.847	0.60172(25)	0.61146(33)	0.00975(14)	4.002(24)	4.000(32)	-0.003(14)
	0.0930	3.688	0.81797(24)	0.82574(31)	0.00777(10)	3.995(19)	4.000(24)	0.005(9)
	0.0840	3.467	1.07512(25)	1.08167(30)	0.00655(7)	3.996(18)	4.005(22)	0.009(6)
	0.0760	3.199	1.35939(26)	1.36509(31)	0.00571(7)	4.002(22)	4.011(26)	0.010(6)
	0.0700	2.939	1.62650(27)	1.63164(31)	0.00514(6)	3.993(23)	4.009(26)	0.016(5)
	0.0630	2.573	2.02101(28)	2.02550(31)	0.00449(5)	3.988(23)	4.013(26)	0.025(4)

TABLE III. Heavy-light meson spectrum for set I and set II parameters obtained with 500 configurations.

	κ_h	m_{PS}	m_V	$m_V - m_{PS}$	$\xi_F^{(PS)}$	$\xi_F^{(V)}$	$\xi_F^{(V)} - \xi_F^{(PS)}$
Set I	0.1100	0.29108(27)	0.30997(51)	0.01889(39)	3.985(36)	4.006(49)	0.020(44)
	0.1020	0.37705(30)	0.39106(50)	0.01401(34)	3.994(39)	4.031(51)	0.036(42)
	0.0930	0.47595(36)	0.48643(57)	0.01048(36)	4.032(51)	4.101(73)	0.070(55)
	0.0840	0.58123(46)	0.58123(46)	0.00805(43)	4.088(73)	4.21(11)	0.119(77)
	0.0760	0.68279(55)	0.68916(81)	0.00638(44)	4.148(99)	4.30(15)	0.147(87)
	0.0700	0.76546(64)	0.77079(89)	0.00534(45)	4.21(13)	4.38(18)	0.173(98)
	0.0630	0.87080(78)	0.8751(10)	0.00430(47)	4.30(18)	4.51(25)	0.21(12)
Set II	0.1100	0.29323(27)	0.31229(52)	0.01906(39)	3.952(36)	3.970(48)	0.018(44)
	0.1020	0.38526(30)	0.39961(50)	0.01435(34)	3.905(37)	3.936(49)	0.031(40)
	0.0930	0.49889(35)	0.50990(57)	0.01101(37)	3.846(46)	3.904(66)	0.058(51)
	0.0840	0.63258(40)	0.64135(60)	0.00877(36)	3.757(54)	3.829(76)	0.071(54)
	0.0760	0.77918(49)	0.78647(75)	0.00729(43)	3.666(73)	3.77(11)	0.108(71)
	0.0700	0.91609(54)	0.92246(80)	0.00637(44)	3.571(84)	3.69(12)	0.118(73)
	0.0630	1.11749(62)	1.12294(87)	0.00544(45)	3.45(10)	3.58(14)	0.130(77)

Figure 7 displays the heavy quark mass dependence of ξ_F for set I. The horizontal axis is the bare quark mass m_q in temporal lattice units. The behaviors of ξ_F's are well in accord with the expectation in Sec. II. For quantitative discussion, let us consider the case where the required accuracies which define the $a_\sigma^{hh\text{-}hl}$ and $a_\sigma^{hl\text{-}ll}$ are 2%, namely $\epsilon_{acc} = \epsilon_{acc} = 0.02$. The ξ_F's from the heavy-heavy and heavy-light mesons disagree beyond this accuracy at $m_q > 0.2$ ($a_\sigma m_Q > 0.8$). Therefore, one must keep $m_q < 0.2$ to avoid large systematic uncertainty in the heavy quarkonia. In contrast to this, the ξ_F from the heavy-light mesons are rather close to ξ, and $\gamma_F^*(m_q = 0)$ can be applied up to $m_q \simeq 0.3$ within the presently required accuracy. For quark mass larger than this value, the discrepancy between ξ_F's from the pseudoscalar and vector mesons gradually grows beyond the statistical error. This signals the growth of systematic error. In the region of $m_q < 0.3$ this effect is sufficiently small.

For the charm quark mass, the present lattice spacing a_σ is already safely smaller than $a_\sigma^{hl\text{-}ll}$ so that the γ_F tuned for massless quark is applicable to the charmed hadron systems. In contrast to this, for the bottom quark mass this is not the case, and one needs a finer lattice spacing or a larger anisotropy ξ. The present lattice would be also sufficient for the charmonium states, since the region $m_q < 0.2$ also covers the charm quark mass. Another striking feature is that the observed ξ_F's for heavy-light mesons are close to the tree level expectation. This implies that the deviation of ξ_F from ξ may be largely removed by a *tree level tuning* of γ_F. Such an approach would work for the spectroscopy of hadrons containing a single bottom quark. This procedure is a good alternative to the mass-dependent calibration using a heavy-light meson, since the statistical error of ξ_F from heavy-light mesons rapidly grows with the heavy quark mass.

Now we summarize the result for the set II. The heavy-heavy meson satisfies the relativity relation by definition. However, the heavy-light meson dispersion relation violates the relativity relation so that ξ_F/ξ deviates from unity towards a smaller value with increasing heavy quark mass.

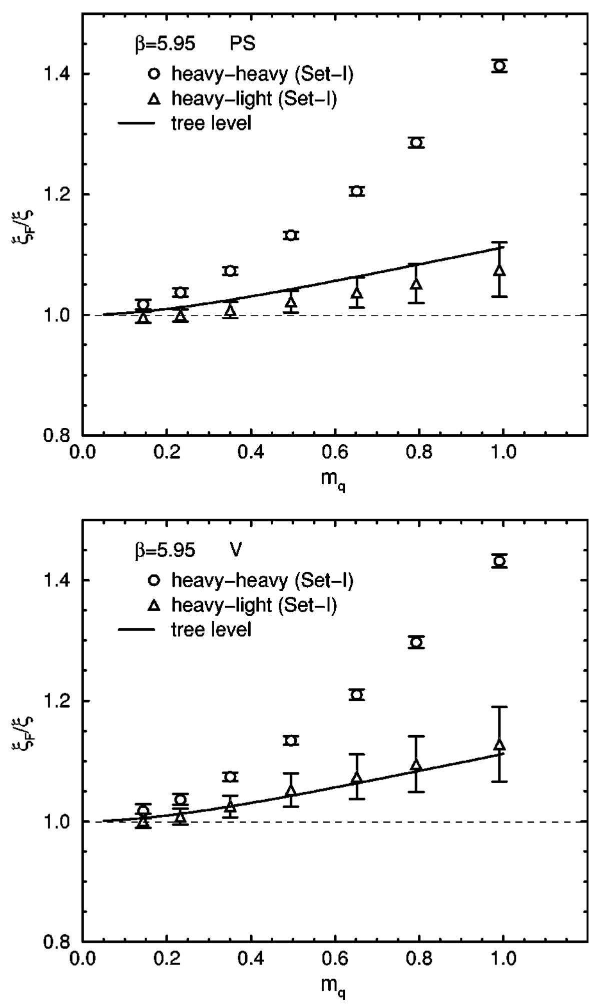

FIG. 7. Fermionic anisotropy determined nonperturbatively from the dispersion relations of heavy-heavy and heavy-light mesons. The top and bottom panels show the results for the pseudoscalar and vector channels, respectively. The solid lines represent the tree level values according to Eq. (9).

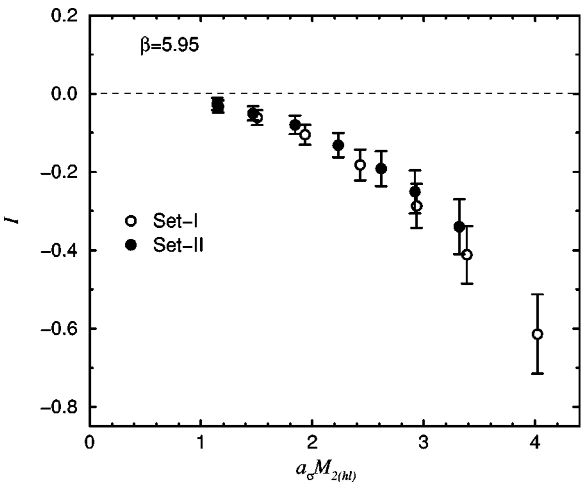

FIG. 8. The inconsistency among the binding energies of heavy-heavy, heavy-light and light-light mesons. The horizontal axis is in spatial lattice units.

This implies that the mass-dependent tuning cannot absorb the discrepancy between the ξ_F's from the heavy-heavy and heavy-light meson systems. For the quark mass of $m_q < 0.2$, the result for set II are consistent with the result for set I. In this region the heavy-heavy meson can be successfully described, and therefore the result of calibration performed with heavy-heavy mesons remains also valid.

We now observe the inconsistency among the binding energies of heavy-heavy, heavy-light, and light-light mesons discussed in Refs. [14,15]. The inconsistency is measured by

$$I \equiv \frac{2\,\delta M_{hl} - (\delta M_{hh} + \delta M_{ll})}{2 M_{2hl}}, \tag{16}$$

where $\delta M = M_2 - M_1$, M_1 and M_2 are the rest and kinetic masses of the meson, respectively. The subscripts hh, hl, and ll represent the quark contents of each meson (h for heavy and l for light quarks). We neglect the last term in the numerator since the calibration of the light quark mass region requires that δM_{ll} vanishes. Since the rest and kinetic quark masses cancel in each kind of mass, nonvanishing I represents the inconsistency in the binding energy, namely the dynamical effect. The anomalous behavior of I in the large kinetic mass region was first reported in Ref. [14] for the $O(a)$ improved quark action on isotropic lattice. It has been pointed out in Ref. [15] that this behavior originates from the $O((ap)^2)$ discretization effect in the heavy quarkonium system, and an estimate of the size of I with the help of a potential model analysis has been given. The result was $I \simeq -0.5$ at $aM_{2hl} \simeq 3.2$, which is in good agreement with the result in [14].

Figure 8 displays the results of I for set I and II for the pseudoscalar channel. The behavior of I is quite similar to that in Refs. [14,15], as is expected because it originates from the $O((a_\sigma p)^2)$ error and therefore cannot be improved by the anisotropy. The behavior of I for the two sets is very similar to each other. This means that the inconsistency cannot be eliminated by tuning the anisotropy parameter with either the heavy-heavy or heavy-light system. Therefore I is a kind of universal quantity to signal the failure in a consistent description of heavy quarkonium systems. The values of I rapidly deviate from zero for any heavy quark mass larger than our lightest one. This is consistent with the above observations concerning ξ_F.

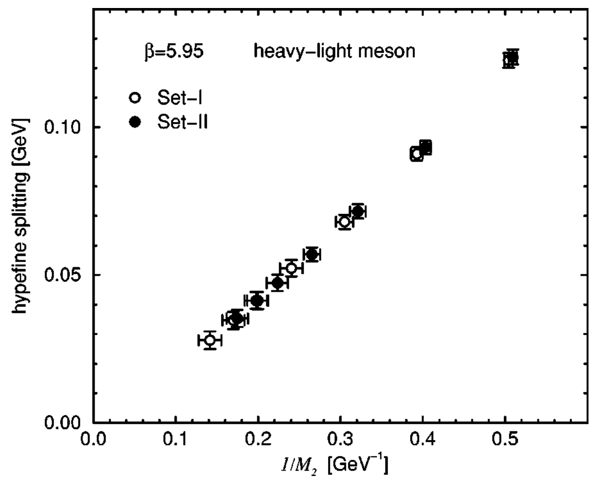

FIG. 9. The hyperfine splitting of heavy-light meson versus the inverse of the heavy-light meson kinetic mass. The physical scale is set by the hadronic radius r_0. The error in $1/M_2$ is the systematic one estimated from the discrepancy of ξ_F's from pseudoscalar and vector mesons.

Our results indicate that for $a_\tau m_Q > 0.2$ the present action cannot be applied to describe the dynamics of heavy quarkonium, even with the aid of reinterpretation of the meson masses. Thus we have to abandon the attempt to apply the present framework to quarkonia in this quark mass region, or we have to improve the action by incorporating higher order correction terms. Therefore in the remaining part of this section we focus on the heavy-light meson spectrum.

C. Heavy-light meson spectrum

Now we turn our attention to the heavy-light meson sector, which is our main interest. According to the heavy quark expansion, the spin flipping interaction of a heavy quark in the heavy-light systems is of $O(1/m_Q)$, and hence the hyperfine splitting of mesons is proportional to the inverse of the meson mass in leading order. In heavy-light systems, the large mass of the heavy quark is not important for the dynamics. Employing the Fermilab formulation, this circumstance is taken into account and the correct heavy quark expansion is in terms of kinetic mass [7]. Therefore, the hyperfine splitting, which is measured as the difference of rest masses, is expected to be reciprocally proportional to the kinetic meson mass. If this is the case, set I and set II should show a similar behavior, up to the $O(a^2)$ systematic uncertainty.

The hyperfine splitting $m_V - m_{PS}$ is displayed in Fig. 9. The horizontal axis shows the inverse of the spin-averaged kinetic meson mass in physical units (if lattice units are defined via the hadronic radius r_0). Since the most serious uncertainty in the kinetic mass stems from the systematic uncertainty in ξ_F, we estimate this error as

$$\delta M_2 = 2M_2|\delta\xi_F|/\xi_F, \tag{17}$$

where $\delta\xi_F = \xi_F^{(V)} - \xi_F^{(PS)}$. Hence it does not include the statistical errors of ξ_F and M_2. On the other hand, the error associated with the hyperfine splitting is the statistical one. Although the data show a linear dependence in the heavy quark mass region, there appears a small negative intercept. We consider this small discrepancy with the heavy quark expansion as a result of the breaking of relativity in the meson dispersion relation, an effect of order $O(a^2)$. For a fixed physical quark mass, this effect is expected to vanish linearly in a^2 towards the continuum limit. The results for the parameter sets, set I and set II, clearly show a similar behavior, and therefore the above interpretation of the Fermilab formulation is correct up to the violation of relativity represented by $\delta\xi_F$. For a more quantitative analysis, one has to quantify the size of the $O(a^2)$ systematic errors and needs to inspect how they disappear towards the continuum limit. This is beyond the scope of this paper.

V. CONCLUSION

In this paper we have investigated the applicability of the anisotropic lattice quark action in the heavy quark mass region, for quenched lattice QCD with $a_\sigma^{-1} \simeq 1.6$ GeV and a renormalized anisotropy $\xi = 4$. The intended effect of anisotropy is to extend the region in which the parameters in the action, if they are tuned for massless quarks, are applicable to high precision computations of heavy-light matrix elements. In order to check this feature, we have measured the heavy-heavy and heavy-light meson masses and dispersion relations which enables us to monitor the breaking of relativity. The calculation has been carried out for two sets of parameters, set I and set II. Set I adopts the values at the massless limit, while in set II the bare anisotropy is tuned using the heavy-heavy mesons. Our main results are summarized in the following.

(a) In the quark mass region $a_\tau m_Q < 0.2$, the observed fermionic anisotropies ξ_F's are consistent for heavy-heavy and heavy-light mesons within 2% accuracy for both set I and set II parameters. This implies that the proposed framework is applicable to both kinds of systems even without the tuning of the anisotropy parameter. Beyond this region, the action fails to describe the heavy quarkonium states correctly. This could have been expected from the tree level analysis of the quark dispersion relation.

(b) The mass dependence of the renormalized anisotropy ξ_F for the heavy-light mesons with γ_F^* at the massless limit (set I) is so small that one can exploit the parameters obtained by massless tuning in the region of $a_\tau m_Q < 0.3$ with less than 2% errors. This result is particularly important for our strategy, because it implies that the parameters tuned at the massless limit are directly applicable for this mass region, which already covers the case of the charm quark with a lattice of the present size. This strategy can be extended to the bottom quark with the development of computational resources in the coming decade.

(c) For $a_\tau m_Q > 0.3$, the breaking of relativity in the heavy-light mesons seems to grow as a function of the heavy quark mass. This is witnessed by the discrepancy of ξ_F's from the pseudoscalar and vector mesons, although the present statistics is not sufficient to give a quantitative estimate of this effect. In the scaling of hyperfine splitting, we also found a small discrepancy with the expectations from the heavy quark expansion, which can be considered as a $O(a^2)$ systematic effect. It is important to quantify these effects and to study how they vanish towards the continuum limit, as preparation for future high precision computations in this approach.

We conclude that the anisotropic lattice quark action actually possesses the features (i)–(iii) required for high precision calculations of heavy-light matrix elements which have been listed in the Introduction. Among the above results it is the small mass dependence of the anisotropy parameter, in particular, which is very encouraging for further development of the framework based on the anisotropic lattice formulation. One of the promising strategies is to calibrate the parameters in the action at the massless limit, including the clover coefficients, using nonperturbative renormalization technique [11], and to use them for all masses. The result of this paper suggests that these parameters with improved accuracy should be applicable to the quark mass region $a_\tau m_Q < 0.3$, while a numerical confirmation is necessary. This quark mass region is already sufficient to describe the D meson systems with the present lattice size.

If one wants to treat B meson systems, the heavy quark mass is larger, $a_\tau m_Q > 0.3$, in which case the mass-dependent errors cannot be neglected anymore. However, since the mass dependence is small, it can be interpreted as an $O(a^2)$ error and can be removed by taking the continuum limit. Alternatively, one can also apply the genuine Fermilab approach for the bottom quark. In this case, the mass dependences of the renormalization coefficients are the source of systematic errors. Nevertheless, as long as one obtains such coefficients in a nonperturbative way first in the massless limit and then uses one-loop perturbation theory only to compute the mass-dependent corrections, the perturbative error can be much better controlled in the anisotropic case compared to the Fermilab approach used on the isotropic lattice. The nice agreement between the observed ξ_F and the tree level expectation suggests that this idea is promising in the bottom quark mass region, too.

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