Hard scattering factorization from effective field theory

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In this paper we show how gauge symmetries in an effective theory can be used to simplify proofs of factorization formulas in highly energetic hadronic processes. We use the soft-collinear effective theory, generalized to deal with back-to-back jets of collinear particles. Our proofs do not depend on the choice of a particular gauge, and the formalism is applicable to both exclusive and inclusive factorization. As examples we treat the π-γ form factor ($\gamma\gamma^* \to \pi^0$), light meson form factors ($\gamma^* M \to M$), as well as deep inelastic scattering ($e^- p \to e^- X$), the Drell-Yan process ($p\bar{p} \to X l^+ l^-$), and deeply virtual Compton scattering ($\gamma^* p \to \gamma^{(*)} p$).

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I. INTRODUCTION

The principle of factorization underlies all theoretical predictions for hadronic processes. Simply put, factorization is the statement that short and long distance contributions to physical processes can be separated, up to corrections suppressed by powers of the relevant large scale in the process. The predictive power gained from this result stems from the fact that the incalculable long distance effects are universal, defined in an unambiguous way in terms of matrix elements. As a consequence, the nonperturbative long distance effects can be extracted in one process and then used in another. In general, proving factorization is a difficult task [1]. The proof of factorization in Drell-Yan processes, for instance, took several years to sort out [2] (for reviews on factorization see [3–5]). Indeed, there are still some processes such as $B \to \pi\pi$ where a proof of factorization only exists at one loop [6].

Given that we would like to retain our predictive power over the largest possible range of energies, we are compelled to understand power corrections to the factorized rates. These corrections are not necessarily universal, and as such, the relevant size of the power corrections is process dependent. In processes for which there exists an operator product expansion (OPE), there is a systematic way in which to include power corrections. However, for a majority of observables we do not have an OPE at our disposal, and the nature of the power correction is not always known. For instance, in the case of shape variables there is still some ongoing discussion about the form of subleading corrections [7,8].

The purpose of this paper is to show that an effective theory framework can be used to simplify proofs of factorization and describe processes with an operator formalism. To do this we extend the soft-collinear effective theory (SCET) developed in Refs. [9–13], to high energy processes. It should be emphasized that there are several other useful advantages in using an effective field theory (EFT). For instance, the EFT makes any symmetries which emerge in the $Q \to \infty$ limit manifest in the Lagrangian and operators, and allow statements to be made to all orders in perturbation theory. The calculation of hard coefficients reduces to simple matching calculations, where subtracting the EFT graphs automatically removes all infrared divergences from the QCD calculation. Perhaps most importantly, it provides a framework for systematically investigating power corrections. Finally, the EFT framework allows standard renormalization group techniques to be used for the resummation of logarithms that are often necessary in calculating rates for certain high energy scattering events [9,10,14,15]. The factorization formulas that we prove in this paper are not new, but serve to illustrate our approach in familiar settings. The results are valid to all orders in α_s and leading order in the power expansion. The simplicity of our approach lies in the fact that factorization occurs at the level of the SCET Lagrangian and operators, and is facilitated by gauge symmetry in the EFT. This provides the advantage that our proofs do not rely on making use of Ward identities and induction, or on specifying a particular gauge.[1] Furthermore, it becomes rather simple to derive factorization formulas for a myriad of processes, since many results are universal. The examples given here serve to illustrate these simplifications. Developments on the issues of power corrections and resummations are left to future publications.[2]

In Sec. II we review the construction of the SCET. The formalism developed in Refs. [9–12] is extended to include two types of collinear particles moving in opposite directions in Sec. II B, and factorization for γ^* to two collinear states is discussed as an example. In Sec. II C we define the nonperturbative matrix elements such as the parton distribution functions that will be needed for the processes presented in the paper, and in Sec. II D we discuss some of the symmetries in SCET that may be used to place restrictions on ma-

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[1]In fact our factorization proofs rely heavily on the gauge symmetry structure of SCET. When a gauge fixing term is required for explicit calculations we use general covariant gauges.

[2]For recent work on subleading corrections in SCET for heavy-to-light transitions see Ref. [16].

trix elements. In the remaining sections we give various examples on how factorization theorems emerge in the effective theory language. In Sec. III we prove factorization theorems for two exclusive processes, namely the π-γ form factor, and meson form factors ($\gamma^* M \to M$) for arbitrary spin and isospin structure. In Sec. IV two inclusive processes are treated, namely deep inelastic scattering (DIS) ($e^- p \to e^- X$) and the Drell-Yan process. ($p\bar{p} \to X l^+ l^-$), and we also give results for deeply virtual Compton scattering ($\gamma^* p \to \gamma^{(*)} p$). In these processes we include all leading power contributions in the factorization proofs (even if the operators are only matched onto at higher orders in perturbation theory such as for the gluon distribution functions). Our conclusions are given in Sec. V. In the Appendix we show how auxiliary fields can be used to prove the simultaneous factorization of soft fields from collinear fields for particles in back-to-back directions.

II. FORMALISM

Effective field theories provide a simple and elegant way of organizing physics in processes containing widely disparate energy scales. In constructing an EFT, some degrees of freedom are eliminated, and the remaining degrees of freedom must reproduce all the infrared physics of the full theory in the domain where the EFT is valid. The EFT is organized by an expansion in λ, defined as the ratio of small to large energy scales. As a useful guideline the following steps are used to identify the infrared degrees of freedom: (1) Determine the relevant scales in a problem from the size of the momenta and masses of all particles that can make up the initial and final states, (2) Construct all momenta from these scales whose components correspond to propagating degrees of freedom, and which have off-shellness less than the large scale, i.e. $p^2 - m^2 \lesssim Q^2$. Effective theory fields are then constructed for each unique set of these momenta.

We will be interested in an EFT with particles of energy Q much greater than their mass. The dynamics of these particles can be described by constructing a soft-collinear effective theory (SCET). This theory is organized as an expansion in powers of $\lambda \sim p_\perp / Q$, and off-shell fluctuations with $p^2 \gg (Q\lambda)^2$ are integrated out. In Sec. II A we begin by describing this procedure and comparing the construction to other EFT's. We then give a brief review of the soft-collinear effective theory developed in Refs. [9–13]. We do not attempt to give a comprehensive treatment, but instead emphasize the main results and refer the reader to the literature for details. In Sec. II B we extend the formulation of SCET to describe processes with collinear particles moving in back to back directions, and prove the factorization formula for γ^* to two collinear states as an example. In Sec. II C we define the nonperturbative matrix elements that are needed for our examples, then in Sec. II D we discuss some of the symmetry properties of collinear fields and currents.

A. Soft-collinear effective theory

In the standard construction of an EFT one removes the short distance scales and massive fields by integrating them out one at a time. A classical example is integrating out the W boson to obtain the effective electroweak Hamiltonian with 4-fermion operators. However, in some situations we are interested in integrating out large momentum fluctuations without fully removing the corresponding field. The simplest example of this is heavy quark effective theory (HQET) [17], which is constructed to describe the low energy properties of mesons with a heavy quark. Here the heavy anti-quarks are integrated out and only heavy quarks with fluctuations close to their mass-shell are retained. This is accomplished by removing fluctuations of order the heavy quark mass m_Q with a field redefinition [18]

$$\psi(x) = \sum_v e^{-i m_Q v \cdot x} h_v(x), \tag{1}$$

where v is the heavy quark velocity and h_v is the field in the EFT. While $\partial^\mu \psi(x) \sim m_Q \psi(x)$, the effective field has $\partial^\mu h_v(x) \sim \Lambda_{\rm QCD} h_v(x)$, indicating that it no longer describes short-distance fluctuations about the perturbative scale m_Q. Instead these effects are encoded in calculable Wilson coefficients. The HQET degrees of freedom with off-shellness $p^2 \sim \Lambda_{\rm QCD}^2$ are the heavy quarks, soft gluons, and soft quarks.

Similarly, for collinear particles with energy $Q \gg m$, one needs to remove momentum fluctuations $\sim Q$ while retaining effective theory fields to describe smaller momenta. However, unlike heavy quarks the collinear particles have two low energy scales. Consider the light-cone momenta, $p^+ = n \cdot p$ and $p^- = \bar{n} \cdot p$ where $n^2 = \bar{n}^2 = 0$ and $n \cdot \bar{n} = 2$. Here n parametrizes a light-cone direction close to that of the collinear particle and $\bar{n}$ the opposite direction [e.g. for motion in the z direction $n_\mu = (1,0,0,1)$ and $\bar{n}_\mu = (1,0,0,-1)$]. For a particle of mass $m \lesssim p_\perp \ll Q$, we have $p^- \sim Q$, and a small parameter $\lambda \sim p_\perp / Q$. The scaling of the p^+ component is then fixed by the equations of motion $p^+ p^- + p_\perp^2 = m^2$, so that $(p^+, p^-, p_\perp) \sim Q(\lambda^2, 1, \lambda)$.

The appearance of two small scales, $Q\lambda^2 \ll Q\lambda \ll Q$, is similar to the situation in nonrelativistic QCD (NRQCD), which is an EFT for systems of two heavy quarks with an expansion in their relative velocity β. In a nonrelativistic bound state the momentum of a heavy quark is $\mathbf{p} \sim m_Q \beta$, but the equations of motion $E = \mathbf{p}^2/(2m_Q)$ make the energy $E \sim m_Q \beta^2$, giving scales $m_Q \beta^2 \ll m_Q \beta \ll m_Q$. The two low energy scales can be distinguished by following Eq. (1) with a further field redefinition [19] $h_v(x) = \Sigma_{\mathbf{p}} e^{i \mathbf{p} \cdot \mathbf{x}} \psi_{\mathbf{p}}(x)$, so that derivatives on $\psi_{\mathbf{p}}$ only pick out the $m\beta^2$ scale. The on-shell degrees of freedom are then the heavy quarks, soft quarks and gluons with $p^2 \sim (m_Q v)^2$, and ultrasoft quarks and gluons with $p^2 \sim (m_Q v^2)^2$.

SCET fields

For collinear particles the analogous field redefinitions are [10,11]

$$\phi(x) = \sum_n \sum_p e^{-ip \cdot x} \phi_{n,p}(x), \tag{2}$$

where the collinear fields $\phi_{n,p}$ are labeled by light-cone vectors n and label momentum p. Here p contains the $\bar{n}\cdot p \sim Q$ and $p_\perp \sim Q\lambda$ momenta so that $\partial^\mu \phi_{n,p} \sim (Q\lambda^2)\phi_{n,p}$. The field ϕ_p can either be a quark or gluon field. Similar to $\psi_{\mathbf{p}}$, the missing $\sim Q$ fluctuations are described by Wilson coefficients and the $\sim Q\lambda$ labels simplify the power counting by distinguishing the $Q\lambda$ and $Q\lambda^2$ scales. Now

$$\phi_{n,p} \equiv \phi^+_{n,p} + \phi^-_{n,-p}, \tag{3}$$

so collinear particles and antiparticles are contained in the same effective theory field, but have momentum labels with the opposite sign. In the large energy limit the four component fermion spinors contain two large and two small components. One therefore defines collinear quark fields $\xi_{n,p}$ which only retain the large components for motion in the n direction and satisfy $\not{n}\xi_{n,p}=0$. For these fields $\xi^+_{n,p}/\xi^-_{n,p}$ destroy/create the particles/antiparticles with large momentum $\bar{n}\cdot p>0$ [11]. For collinear gluons $A^{\mu\dagger}_{n,q}=A^\mu_{n,-q}$, and $(A^\mu_{n,q})^+/(A^\mu_{n,q})^-$ destroy/create gluons with $n\cdot q>0$.

For simplicity we will ignore quark masses and only consider massless u and d quarks. For the processes considered here SCET then requires three types of degrees of freedom: collinear, soft, and ultrasoft (usoft) fields. These are distinguished by the scaling of the light cone components (p^+, p^-, $p^\perp$) of their momenta: (λ^2,1,λ) for collinear modes in the n direction ($A_{n,q}$, $\xi_{n,p}$), (λ,λ,λ) for the soft modes (A^s_q, q^s_p), and (λ^2,λ^2,λ^2) for the usoft modes (A_{us}, q_{us}). The soft modes are labeled by their order $Q\lambda$ momenta, so A^s_q and q^s_p are essentially just momentum space fields. The usoft fields have no labels and depend only on the coordinate x. The fields are assigned a scaling with λ to make the action for their kinetic terms order λ^0 [9,10,12]. For instance $\xi_{n,p} \sim \lambda$, $A^\mu_{n,q} \sim (\lambda^2,1,\lambda)$, $A^s_q \sim \lambda$, and $A^\mu_{us} \sim \lambda^2$. At leading order only order λ^0 vertices are necessary to correctly account for all order λ^0 Feynman diagrams.

In HQET only external currents with momenta of order m_b can change the label v. Thus the Lagrangian has a superselection rule forbidding changes in the four-velocity of the heavy quark [17,18]. In NRQCD the v labels are also conserved, but the smaller momentum labels $\mathbf{p}$ are changed by operators in the effective theory such as the Coulomb potential. A novel feature of SCET is that interactions in the leading action can change both the large and small parts of the momentum labels p^μ. However, only external currents can change the direction n of a collinear particle, so this label is conserved. Thus, for each distinct direction n a separate set of collinear fields is needed. In the remainder of this section we will restrict ourselves to collinear particles with a single n. We will generalize the discussion to the case of two back-to-back directions and discuss the factorization of collinear particles with different n's in Sec. II B.

Since in SCET interactions can change the order Q label momenta it turns out to be very useful to introduce a label operator, $\mathcal{P}^\mu$ [11], for which the collinear fields satisfy $\mathcal{P}^\mu \xi_{n,p} = p^\mu \xi_{n,p}$. More generally, $\mathcal{P}^\mu$ acts on a product of labeled fields as

$$\begin{aligned} f(\mathcal{P}^\mu)(\phi^\dagger_{q_1}\cdots\phi^\dagger_{q_m}\phi_{p_1}\cdots\phi_{p_n}) \\ = f(p_1^\mu + \ldots + p_n^\mu - q_1^\mu - \ldots - q_m^\mu) \\ \times(\phi^\dagger_{q_1}\cdots\phi^\dagger_{q_m}\phi_{p_1}\cdots\phi_{p_n}) \end{aligned} \tag{4}$$

so conjugate field labels come with a minus sign. The operator $\mathcal{P}_\mu$ acts to the right, while the conjugate operator $\mathcal{P}^\dagger_\mu$ acts to the left. As explained in Ref. [11] the label operator allows all large phases to be moved to the front of operators with a factor $\exp(-ix\cdot\mathcal{P})$. This phase and the label sums can then be suppressed if we impose that interactions conserve label momenta and that the momentum indices on fields are implicitly summed over. Basically, for labels p and p' and residual momenta k and k',

$$\int d^4x e^{i(p'-p+k'-k)\cdot x} = \delta(p-p')\int d^4x e^{i(k'-k)\cdot x}, \tag{5}$$

so that the label and residual momenta are individually conserved. [Although technically the label momenta are discrete we abuse notation and use $\delta(p-p')$ rather than $\delta_{p,p'}$ because it makes the subscripts easier to read.] For convenience we define the operator $\bar{\mathcal{P}}$ to pick out only the order λ^0 labels on collinear fields, and the operator $\mathcal{P}^\mu$ to pick out only the order λ labels. For the matrix element of any collinear operator $\mathcal{O}$, momentum conservation constrains the sum of field labels [11], giving

$$\langle M_{n,p_1}|[f(\bar{\mathcal{P}})\mathcal{O}]|M_{n,p_2}\rangle = f(\bar{n}\cdot(p_2-p_1))\langle M_{n,p_1}|\mathcal{O}|M_{n,p_2}\rangle, \tag{6}$$

for any function f.

For a single n the Lagrangian can be broken up into three sectors: collinear, usoft, and soft. We therefore write

$$\mathcal{L} = \mathcal{L}_{c,n}[\xi_{n,p}, A^\mu_{n,q}, A^\mu_{us}] + \mathcal{L}_{us}[q_{us}, A^\mu_{us}] + \mathcal{L}_s[q_{s,p}, A^\mu_{s,q}], \tag{7}$$

where we have made the field content of each sector explicit. We will discuss each of these terms separately.

Collinear sector

As explained in detail in Ref. [12], gauge invariance in SCET restricts the Lagrangian and allowed form of operators. Only local gauge transformations whose action is closed on the effective theory fields need to be considered. These include collinear, soft, and usoft transformations. Each of these vary over different distance scales, with collinear gauge transformations satisfying $\partial^\mu U_n(x) \sim Q(\lambda^2,1,\lambda)U_n(x)$, soft satisfying $\partial^\mu V_s(x) \sim Q\lambda V_s(x)$, and usoft transformations with $\partial^\mu V_{us}(x) \sim Q\lambda^2 V_{us}(x)$. All particles transform under $V_{us}(x)$ and usoft gluons act like background fields for collinear particles. Invariance under $U_n(x)$ requires a collinear Wilson line built out of the order λ^0 gluon fields [10,11]

$$W_n(x) = \left[\sum_{\text{perms}} \exp\left(-g\frac{1}{\bar{\mathcal{P}}}\bar{n}\cdot A_{n,q}(x)\right)\right]. \tag{8}$$

Here the operator $\overline{\mathcal{P}}$ acts only inside the square brackets, the n on W_n refers to the direction of the collinear quanta, and W_n is local with respect to x (corresponding to the residual momenta). Taking the Fourier transform of $\delta(\omega-\overline{\mathcal{P}})W_n(0)$ with respect to ω gives the more familiar path-ordered Wilson line $W_n(y,-\infty)=\mathrm{P}\exp[ig\int_{-\infty}^{y}\mathrm{d}s\,\overline{n}\cdot A_n(s\overline{n})]$. Under a collinear gauge transformation W_n transforms as $W_n \to U_n W_n$. An invariant under collinear gauge transformations can therefore be formed by combining a collinear fermion $\xi_{n,p}$ and the Wilson line W_n in the form

$$W_n^\dagger(x)\xi_{n,p}(x). \tag{9}$$

This combination still transforms under a usoft gauge transformation, $W_n^\dagger\xi_{n,p}\to V_{us}(x)W_n^\dagger\xi_{n,p}$. We will often suppress the x dependence of the combination $W_n^\dagger\xi_{n,p}$.

Integrating out hard fluctuations gives Wilson coefficients in the effective theory that are functions of the large $\overline{n}\cdot p_i$ collinear momenta, $C(\overline{n}\cdot p_i)$. However, collinear gauge invariance restricts these coefficients to only depend on the linear combination of momenta picked out by the order λ^0 operator $\overline{\mathcal{P}}$ [11]. In general the Wilson coefficients are then functions $C(\overline{\mathcal{P}},\overline{\mathcal{P}}^\dagger)$ which must be inserted between gauge invariant products of collinear fields. In general the Wilson coefficients also depend on the large momentum scales in a process such as Q and the renormalization scale μ.

To construct the collinear Lagrangian one can match full QCD onto operators with collinear fields that are invariant under usoft and collinear gauge transformations. The collinear Lagrangian at order λ^0 is [10–12]

$$\begin{aligned}\mathcal{L}_{c,n}=\overline{\xi}_{n,p'}\Bigg\{ & in\cdot D+gn\cdot A_{n,q} \\ & +(\mathcal{P}_\perp+g\mathcal{A}^\perp_{n,q})W\frac{1}{\overline{\mathcal{P}}}W^\dagger(\mathcal{P}_\perp+g\mathcal{A}^\perp_{n,q'})\Bigg\}\frac{\overline{n}}{2}\xi_{n,p} \\ & +\frac{1}{2g^2}\mathrm{tr}\{[iD^\mu+gA^\mu_{n,q},iD^\nu+gA^\nu_{n,q'}]^2\}+\mathcal{L}_c^{g.f.},\end{aligned} \tag{10}$$

where $\mathcal{L}_c^{g.f.}$ are gauge fixing terms, $iD^\mu=i\partial^\mu+gA^\mu_{us}$, and

$$i\mathcal{D}^\mu=\frac{n^\mu}{2}\overline{\mathcal{P}}+\mathcal{P}_\perp^\mu+\frac{\overline{n}^\mu}{2}in\cdot D. \tag{11}$$

Since usoft gluons act as background fields in the collinear gauge transformation the couplings, $g(\mu)$, for both types of gluons must be identical.

Usoft and soft sectors

The usoft and soft Lagrangians for gluons and massless quarks are the same as those in QCD. From Eq. (7) we see that collinear quarks and gluons interact with usoft gluons, however at order λ^0 only the $n\cdot A_{us}$ component appears in Eq. (10). In order to prove factorization formulas it is essential to disentangle the collinear and usoft modes. This can be done by introducing a usoft Wilson line

$$Y_n(x)=\mathrm{P}\exp\left(ig\int_{-\infty}^{x}\mathrm{d}s\,n\cdot A_{us}(sn)\right), \tag{12}$$

where the subscript n on Y_n labels the direction of the Wilson line [we emphasize that this is different from the meaning of the subscript on W_n in Eq. (8)]. A usoft gauge transformation takes $Y_n\to V_{us}Y_n$. In Ref. [12] it was shown that the field redefinitions

$$\xi_{n,p}=Y_n\xi^{(0)}_{n,p},\quad A^\mu_{n,p}=Y_nA^{(0)\mu}_{n,p}Y_n^\dagger, \tag{13}$$

imply $W_n=Y_nW_n^{(0)}Y_n^\dagger$ and decouple the usoft gluons from the collinear particles in the leading order Lagrangian

$$\mathcal{L}_{c,n}[\xi_{n,p},A^\mu_{n,q},n\cdot A_{us}]=\mathcal{L}_{c,n}[\xi^{(0)}_{n,p},A^{(0)\mu}_{n,q},0]. \tag{14}$$

Thus, the new collinear fields with superscript (0) no longer interact with usoft gluons or transform under a usoft gauge transformation. Since the field redefinitions do not change physical S matrix elements, the new fields give an equally valid parametrization of the collinear modes. The leading SCET Lagrangian therefore factors into separate collinear and usoft sectors. This alone does not guarantee factorization in operators and currents, since after the field redefinition these operators may still contain both usoft and collinear fields. However, the field redefinition makes factorization transparent since identities such as $Y_n^\dagger Y_n=1$ may be applied directly to the operators. This will become clear in the examples in Secs. III and IV.

The coupling of soft gluons to collinear particles differs from the usoft-collinear interactions. Interactions of a soft gluon with a collinear particle results in a particle with momentum $p\sim Q(\lambda,1,\lambda)$, so soft gluons cannot appear in the collinear Lagrangian. These off-shell particles have $p^2\sim Q^2\lambda$ and since $Q^2\lambda\gg(Q\lambda)^2$ these off-shell quarks and gluons can be integrated out. At leading order in λ it was shown in Ref. [12] that in operators with collinear fields this simply builds up factors of a soft Wilson line S_n involving the $n\cdot A_s$ component of the soft gluon field,

$$S_n=\left[\sum_{\text{perms}}\exp\left(-g\frac{1}{n\cdot\mathcal{P}}n\cdot A_{s,q}\right)\right]. \tag{15}$$

The factors of S_n appear outside gauge invariant products of collinear fields, and their location is restricted by soft gauge invariance.

B. SCET for n and $\overline{n}$ collinear fields

In this section we extend SCET to include the possibility of collinear fields moving in different light-cone directions: n_1, n_2, n_3, ... These directions can be considered to be distinct provided that $n_i\cdot n_j\gg\lambda^2$ for $i\neq j$. This follows from the fact that if $n_1\cdot n_2\sim\lambda^2$ then the directions n_1 and n_2 are too close to be distinguished. For example, a momentum $p_2=Qn_2$ can be considered to be collinear in the n_1 direction if $n_1\cdot p_2=Qn_1\cdot n_2\sim Q\lambda^2$, since this is the correct scaling for the small momentum component of an n_1-collinear particle.

For simplicity we will only consider the case of back-to-back jets corresponding to collinear particles moving in the n and $\bar{n}$ directions. These are clearly distinct since $n\cdot\bar{n}=2$. Collinear particles in the $\bar{n}$ direction have $(+,-,\perp)$ momenta $\sim Q(1,\lambda^2,\lambda)$, and the $n\cdot p\sim 1$ and $p_\perp\sim\lambda$ momenta appear as labels on the corresponding fields: $\xi^-_{n,p}$ and $A^\mu_{n,p}$. Emission of a collinear particle moving in the n direction from a collinear particle in the $\bar{n}$ direction results in a particle with momentum $k\sim Q(1,1,\lambda)$ and off-shellness $k^2\sim Q^2$. These off-shell modes are integrated out to construct the SCET, so collinear modes in the n direction do not directly couple to collinear modes in the $\bar{n}$ direction. A distinct set of collinear gauge transformations is associated with each of n and $\bar{n}$, and fields in one direction do not transform under the gauge symmetry associated with the opposite direction. Two Wilson lines $W_n(x)$ and $W_{\bar{n}}(x)$ are necessary [defined as in Eq. (8)], and they appear in a way that makes collinear operators gauge invariant. For instance the combinations

$$W_n^\dagger\xi_{n,p}, \quad W_{\bar{n}}^\dagger\xi^-_{\bar{n},p} \tag{16}$$

are invariant under collinear gauge transformations in the n and $\bar{n}$ directions, respectively. We also require two types of label operators, $\bar{\mathcal{P}}$ as before, and an operator $\mathcal{P}$ to pick out $n\cdot p$ labels that are order λ^0. Thus, $\bar{\mathcal{P}}$ and $\mathcal{P}$ act only on the n and $\bar{n}$ collinear fields respectively. (The label operator $\mathcal{P}^\mu$ still picks out order λ momentum components and therefore acts on both n and $\bar{n}$ fields.) With two collinear directions, decoupling usoft gluons requires introducing both Y_n and $Y_{\bar{n}}$ Wilson lines, defined as in Eq. (12), but along the n or $\bar{n}$ paths respectively. Finally, integrating out $\sim Q^2\lambda$ fluctuations at leading order induces both S_n and $S_{\bar{n}}$ soft Wilson lines defined analogous to Eq. (15). This is discussed in greater detail in the Appendix where we show explicitly to all orders in g that integrating out the $Q^2\lambda$ fluctuations causes

$$\begin{aligned} &W_n^\dagger\xi_{n,p}\to S_nW_n^\dagger\xi_{n,p}, \quad \bar{\xi}_{n,p}W_n\to\bar{\xi}_{n,p}W_nS_n^\dagger,\\ &W_{\bar{n}}^\dagger\xi^-_{\bar{n},p}\to S_{\bar{n}}W_{\bar{n}}^\dagger\xi^-_{\bar{n},p}, \quad \bar{\xi}^-_{\bar{n},p}W_{\bar{n}}\to\bar{\xi}^-_{\bar{n},p}W_{\bar{n}}S_{\bar{n}}^\dagger. \end{aligned} \tag{17}$$

Relations for operators with collinear gluon fields are also derived in the Appendix.

Note that we have not included ''Glauber gluons'' with momenta $p^\mu\sim(\lambda^2,\lambda^2,\lambda)$, which are kinematically allowed in t-channel ''Coulomb'' exchange between n and $\bar{n}$ collinear quarks. In determining the relevant degrees of freedom we have assumed that Glauber gluons are not necessary to describe the infrared for the processes considered in this paper. Intuitively, this can be seen from the fact these gluons are instantaneous in both time and longitudinal separation, and only could contribute when the n and $\bar{n}$ jets overlap for a duration of order $1/Q\lambda^2$ in a space-time diagram. In processes with a hard interaction the overlap scale is always much shorter than this (however this need not be the case in processes such as forward scattering). For the Drell-Yan process more quantitative arguments can be found in Refs. [2,35].

At order λ^0 it is not possible to construct a gauge invariant kinetic Lagrangian with terms that involve both n and $\bar{n}$ fields. Thus, the n and $\bar{n}$ collinear modes are described by independent Lagrangians (however n and $\bar{n}$ modes may still both appear in an external operator). The collinear sector of the SCET Lagrangian is therefore

$$\mathcal{L}_{c,n}[\xi_{n,p},A^\mu_{n,q},n\cdot A_{us}]+\mathcal{L}_{c,\bar{n}}[\xi^-_{\bar{n},p},A^\mu_{\bar{n},q},\bar{n}\cdot A_{us}]. \tag{18}$$

Making the field redefinitions

$$\begin{aligned} \xi_{n,p}&=Y_n\xi^{(0)}_{n,p}, \quad A^\mu_{n,p}=Y_nA^{(0)\mu}_{n,p}Y_n^\dagger,\\ \xi^-_{\bar{n},p}&=Y_{\bar{n}}\xi^{(0)}_{\bar{n},p}, \quad A^\mu_{\bar{n},p}=Y_{\bar{n}}A^{(0)\mu}_{\bar{n},p}Y_{\bar{n}}^\dagger, \end{aligned} \tag{19}$$

gives $W_n=Y_nW_n^{(0)}Y_n^\dagger$, $W_{\bar{n}}=Y_{\bar{n}}W_{\bar{n}}^{(0)}Y_{\bar{n}}^\dagger$, and usoft degrees of freedom once again decouple from the collinear modes since

$$\begin{aligned} &\mathcal{L}_{c,n}[\xi_{n,p},A^\mu_{n,q},n\cdot A_{us}]+\mathcal{L}_{c,\bar{n}}[\xi^-_{\bar{n},p},A^\mu_{\bar{n},q},\bar{n}\cdot A_{us}]\\ &\quad=\mathcal{L}_{c,n}[\xi^{(0)}_{n,p},A^{(0)\mu}_{n,q},0]+\mathcal{L}_{c,\bar{n}}[\xi^{(0)}_{\bar{n},p},A^{(0)\mu}_{\bar{n},q},0]. \end{aligned} \tag{20}$$

Thus, usoft gluons are removed from the collinear Lagrangian at the expense of inducing Y_n and $Y_{\bar{n}}$ factors in operators with collinear fields. In certain cases the identities $Y_n^\dagger Y_n=1$ and $Y_{\bar{n}}^\dagger Y_{\bar{n}}=1$ can be used in these operators to cancel usoft gluon interactions. Perturbatively these cancellations would occur by adding an infinite set of Feynman diagrams.

To see in more detail how this works consider the simple example of the γ^*-production of back-to-back collinear states X_n and $X_{\bar{n}}$. The full theory current $\bar{\psi}(x)\Gamma\psi(x)$ matches onto an effective theory operator $O_{n\bar{n}}$. Naively one might guess that the SCET operator mediating this process is

$$O_{n\bar{n}}=\bar{\xi}_{n,p_1}\Gamma\xi^-_{\bar{n},p_2}. \tag{21}$$

However, this operator is not invariant under the collinear gauge transformations U_n and $U_{\bar{n}}$, so the process is instead mediated by the invariant operator

$$O_{n\bar{n}}=\bar{\xi}_{n,p_1}W_n\Gamma W_{\bar{n}}^\dagger\xi^-_{\bar{n},p_2}. \tag{22}$$

A hard matching coefficient $C(\bar{\mathcal{P}},\bar{\mathcal{P}}^\dagger,\mathcal{P},\mathcal{P}^\dagger)$ can be inserted in any location in the operator that does not break apart the gauge invariant combinations of fields in Eq. (16). The operators $\bar{\mathcal{P}}$ and $\mathcal{P}$ in the coefficient only pick out momenta that are order λ^0 in the power counting. Thus, $\mathcal{P}$ does not act on fields in the n direction and $\bar{\mathcal{P}}$ does not act on fields in the $\bar{n}$ direction, and the most general result is

$$O_{n\bar{n}}=\bar{\xi}_{n,p_1}W_n\Gamma C(\bar{\mathcal{P}}^\dagger,\mathcal{P})W_{\bar{n}}^\dagger\xi^-_{\bar{n},p_2}. \tag{23}$$

Next, we integrate out the off-shell $Q^2\lambda$ fluctuations which induces additional soft Wilson lines in the operator. This is discussed in detail in the Appendix and from Eq. (17) gives

$$O_{n\bar{n}} = \bar{\xi}_{n,p_1} W_n S_n^\dagger \Gamma C(\bar{\mathcal{P}}^\dagger, \mathcal{P}) S_{\bar{n}} W_{\bar{n}}^\dagger \xi_{\bar{n},p_2}. \quad (24)$$

Note that $\bar{\mathcal{P}}$ and $\mathcal{P}$ do not act on the fields in the soft Wilson lines since soft gluons carry only order λ momenta. Finally, we can make the usoft gluon couplings explicit by switching to the (0) fields using Eq. (19):

$$O_{n\bar{n}} = \bar{\xi}^{(0)}_{n,p_1} W_n^{(0)} Y_n^\dagger S_n^\dagger \Gamma C(\bar{\mathcal{P}}^\dagger, \mathcal{P}) S_{\bar{n}} Y_{\bar{n}} W_{\bar{n}}^{(0)\dagger} \xi^{(0)}_{n,p_2}. \quad (25)$$

This operator is manifestly invariant under collinear gauge transformation in the n and $\bar{n}$ direction, as well as under soft and usoft gauge transformations.

To separate the short distance Wilson coefficient from the long-distance operator one introduces convolution variables ω and ω' to give

$$O_{n\bar{n}} = \int d\omega d\omega' C(\omega,\omega') O_{n\bar{n}}(\omega,\omega'),$$

$$O_{n\bar{n}}(\omega,\omega') = [\bar{\xi}^{(0)}_{n,p_1} W_n^{(0)} \delta(\bar{\mathcal{P}}^\dagger - \omega) Y_n^\dagger S_n^\dagger \Gamma \times S_{\bar{n}} Y_{\bar{n}} \delta(\mathcal{P} - \omega') W_{\bar{n}}^{\dagger(0)} \xi^{(0)}_{n,p_2}]. \quad (26)$$

The function $C(\omega,\omega')$ contains all the short distance physics and is determined by matching the full theory onto this effective theory operator. $O_{n\bar{n}}(\omega,\omega')$ contains all the infrared long-distance QCD contributions at leading order in λ.

Now consider the matrix element of the production current between $\langle X_n X_{\bar{n}}|$ and the vacuum. Taking the γ^* to have large time-like momentum $q^\mu = (Q,0,0,0)$ (and zero residual momentum) we have

$$\begin{aligned} &\int d^4x e^{-iq\cdot x} \langle X_n X_{\bar{n}}| \bar{\psi}(x) \Gamma \psi(x)|0\rangle \\ &= \int d^4x \langle X_n X_{\bar{n}}| O_{n\bar{n}}(x)|0\rangle \\ &= \int d^4x e^{ik\cdot x} \langle (X_n X_{\bar{n}})(k)| O_{n\bar{n}}(x=0)|0\rangle \\ &= \langle (X_n X_{\bar{n}})(0)| O_{n\bar{n}}(0)|0\rangle. \end{aligned} \quad (27)$$

In the first step the conservation of the large label momentum q was made implicit in the matrix element [cf. Eq. (5)]. Since we are in the center-of-mass frame the X_n has large momentum $\bar{n}\cdot p = Q$, and the $X_{\bar{n}}$ has momentum $n\cdot p' = Q$. Now using translation invariance, we see that the remaining x integral forces the $|X_n X_{\bar{n}}\rangle$ state to have zero residual momentum. Using Eq. (26) this matrix element is equal to

$$\begin{aligned} &\int d\omega d\omega' C(\omega,\omega') \langle X_n \bar{X}_{\bar{n}}| O_{n\bar{n}}(\omega,\omega')|0\rangle \\ &= \int d\omega d\omega' C(\omega,\omega') J_n(\omega) \Gamma S_{n\bar{n}} J_{\bar{n}}(\omega'), \end{aligned} \quad (28)$$

where the functions C, J_n, $J_{\bar{n}}$, and S also depend on the renormalization point μ. Here we have used the fact that both X_n and $X_{\bar{n}}$ can be described entirely by collinear particles in the n and $\bar{n}$ directions respectively. Since the Lagrangians for the collinear, soft, and usoft fields are factorized the remaining matrix element splits into distinct matrix elements for each class of modes. These matrix elements are

$$\begin{aligned} J_n(\omega) &= \langle X_n(Q/2)| \bar{\xi}^{(0)}_{n,p_1} W_n^{(0)} \delta(\bar{\mathcal{P}}^\dagger - \omega)|0\rangle, \\ J_{\bar{n}}(\omega') &= \langle X_{\bar{n}}(Q/2)| \delta(\mathcal{P} - \omega') W_{\bar{n}}^{\dagger(0)} \xi^{(0)}_{n,p_2}|0\rangle, \\ S_{n\bar{n}} &= \langle 0| Y_n^\dagger S_n^\dagger S_{\bar{n}} Y_{\bar{n}}|0\rangle \end{aligned} \quad (29)$$

(and are matrices whose color, spin, and flavor indices are suppressed). Note that J_n, $J_{\bar{n}}$, and $S_{n\bar{n}}$ are explicitly invariant under the collinear, soft, and usoft gauge transformations [12] of SCET, but still transform globally under a color rotation. Now using the momentum conservation identity in Eq. (6), the large momentum of the X_n and $X_{\bar{n}}$ states sets $\mathcal{P} \to Q$ and $\bar{\mathcal{P}}^\dagger \to Q$. Label conservation also implies that the total perpendicular momentum of each of J_n, $J_{\bar{n}}$, and $S_{n\bar{n}}$ is zero. The sum over ω and ω' can then be performed to give the final factorized form

$$C(Q,Q) J_n(Q) \Gamma S_{n\bar{n}} J_{\bar{n}}(Q). \quad (30)$$

Although rather idealized, the above example illustrates the main steps needed to derive a factorization formula. Taking X_n and $X_{\bar{n}}$ to be single quark states, the result in Eq. (30) also agrees with the factorization formula for the on-shell production form factor for $q\bar{q}$ [20,21].[3] In the above example the factors of S and Y in the operator in Eq. (25) do not cancel. In the examples we will consider in Secs. III and IV there are several operators at leading order, however the factors of S and Y cancel in observable matrix elements. The collinear matrix elements of long-distance operators, such as those in Eq. (29), are the ones that have interpretations as structure functions or wave functions. In the next section we give the operator definitions for these functions that will be needed in the remainder of the paper.

C. Non-perturbative matrix elements

Predictions for hadronic processes depend on universal matrix elements that are not computable in perturbation theory. For exclusive processes these include light-cone

[3] In this case depending on the choice of infrared regulator(s), it may not be possible to distinguish the Y_n and S_n Wilson lines in $S_{n\bar{n}}(\mu)$. For instance if one chooses $\Lambda_{\rm IR} \sim Q\lambda$ then the usoft gluons give scaleless loop integrals and can be dropped, so that $Y_n^\dagger S_n^\dagger S_{\bar{n}} Y_{\bar{n}} \to S_n^\dagger S_{\bar{n}}$. If instead one chooses $\Lambda_{\rm IR} \sim Q\lambda^2$ then the soft gluons give scaleless loop integrals (they simply act to pull-up the ultraviolet divergences in the usoft integrals to the hard scale [22,23]), so the soft Wilson lines can be suppressed. This is why one only finds $S_n^\dagger S_{\bar{n}}$ for this operator in the literature. For typical regulator choices the other gluons are simply not required to reproduce the infrared structure of the full theory result.

wave functions and form factors, while for inclusive processes they include parton distribution functions and fragmentation functions. In this section we define matrix elements in SCET that are needed for our examples. All the collinear operators considered here decouple from usoft gluons since they are local in the residual coordinate x and because $Y^\dagger(x)Y(x)=1$. Thus

$$\bar{\xi}_{n,p'} W\Gamma W^\dagger \xi_{n,p} = \bar{\xi}^{(0)}_{n,p'} W^{(0)}\Gamma W^{(0)\dagger}\xi^{(0)}_{n,p}, \tag{31}$$

and expressions with and without the (0) superscript are equal. For convenience we will write the collinear fields without the superscript in the remainder of this section.

Consider first the light-cone wave functions. For the pion isotriplet π^a, the wave function $\phi_\pi(x)$ is conventionally defined by [24]

$$\langle \pi^a(p)|\bar{\psi}(y)\gamma^\mu\gamma^5\frac{\tau^b}{\sqrt{2}}Y(y,x)\psi(x)|0\rangle = -if_\pi\delta^{ab}p^\mu\int_0^1 dz e^{i[zp\cdot y+(1-z)p\cdot x]}\phi_\pi(\mu,z). \tag{32}$$

Here $f_\pi \simeq 131$MeV and the QCD field ψ denotes the isospin doublet $\{\psi^{(u)},\psi^{(d)}\}$. The coordinates satisfy $(y-x)^2=0$, which ensures that the path from y^μ to x^μ is along the light-cone, and $Y(y,x)$ is a Wilson line along this path. In SCET we require the matrix elements of highly energetic pions, which therefore have collinear constituents. Boosting the matrix element in Eq. (32) and letting $y^\mu = y\bar{n}^\mu$, $x^\mu = x\bar{n}^\mu$ we have

$$\langle \pi^a_{n,p}|\bar{\xi}_{n,y}\Gamma^b_\pi W(y,x)\xi_{n,x}|0\rangle = -if_\pi\bar{n}\cdot p\,\delta^{ab}\int_0^1 dz e^{i\bar{n}\cdot p[zy+(1-z)x]}\phi_\pi(\mu,z), \tag{33}$$

where $\Gamma^b_\pi = \bar{\not{n}}\gamma_5\tau^b/\sqrt{2}$, and $\xi_{n,z}$ is a collinear field with position space label $z\bar{n}^\mu$. For our purposes it is more useful to use the operator with momentum space labels [13]

$$\begin{aligned}\langle \pi^a_{n,p}|\bar{\xi}_{n,p_1} W\Gamma^b_\pi \delta(\omega-\bar{\mathcal{P}}_+)W^\dagger\xi_{n,p_2}|0\rangle &= \int\frac{dy}{2\pi}e^{-i\omega y}\langle \pi^a_{n,p}|\bar{\xi}_{n,y}\Gamma^b_\pi W(y,-y)\xi_{n,-y}|0\rangle \\ &= -if_\pi\bar{n}\cdot p\,\delta^{ab}\int_0^1 dz\,\delta[\omega-(2z-1)\bar{n}\cdot p]\phi_\pi(\mu,z),\end{aligned} \tag{34}$$

where $\bar{\mathcal{P}}_\pm = \bar{\mathcal{P}}^\dagger \pm \bar{\mathcal{P}}$. In Eq. (34) the delta function fixes ω to the sum of labels picked out by the $\bar{\mathcal{P}}_+$ operator. The combination picked out by $\bar{\mathcal{P}}_-$ is equivalent to $(-\bar{\mathcal{P}})$ acting on the entire operator, and using Eq. (6) is fixed to the $\bar{n}\cdot p$ momentum of the pion state.

In some situations it is convenient to have delta functions which fix the labels of both $W^\dagger\xi_{n,p}$ and $\bar{\xi}_{n,p}W$. In this case a useful field is

$$\chi^{(i)}_{n,\omega} \equiv [\,\delta(\omega-\bar{\mathcal{P}})W^\dagger_n\xi^{(i)}_{n,p}]. \tag{35}$$

Here i is the flavor index and will be omitted if the flavor doublet field is implied. Note that unlike the p in $\xi_{n,p}$ the label ω on $\chi_{n,\omega}$ is not summed over. A matrix element with $\chi_{n,\omega}$ fields is related to a matrix element like the one in Eq. (34) through

$$\begin{aligned}&\langle M_{n,p}|\bar{\chi}_{n,\omega}\Gamma\chi_{n,\omega'}|M_{n,p'}\rangle \\ &\quad = 2\,\delta(\omega_- - \bar{n}\cdot p_-)\langle M_{n,p}|\bar{\xi}_{n,p_1}W \\ &\quad\quad \times\Gamma\,\delta(\omega_+ - \bar{\mathcal{P}}_+)W^\dagger\xi_{n,p_2}|M_{n,p'}\rangle,\end{aligned} \tag{36}$$

where $\omega_\pm = \omega \pm \omega'$ and $p_- = p - p'$. Thus, with the χ notation the momentum conserving delta functions become explicit. The factor of 2 appears from treating the ω's as continuous variables, and in the final results cancels with a factor of 1/2 from a Jacobian.

For inclusive processes such as DIS it is the proton parton distribution functions for quarks of flavor i, $f_{i/p}(z)$, antiquarks $\bar{f}_{i/p}(z)$, and gluons, $f_{g/p}(z)$ which are needed. The standard coordinate space definitions [25] are ($y^\mu = y\bar{n}^\mu$)

$$\begin{aligned}f_{i/p}(z) &= \int\frac{dy}{2\pi}e^{-i2z\bar{n}\cdot py}\langle p|\bar{\psi}^{(i)}(y)Y(y,-y) \\ &\quad\times\frac{\bar{\not{n}}}{2}\psi^{(i)}(-y)|p\rangle|_{\text{spin avg.}}, \\ f_{g/p}(z) &= \frac{2}{z\bar{n}\cdot p}\int\frac{dy}{2\pi}e^{-i2z\bar{n}\cdot py}\bar{n}_\mu\bar{n}^\nu \\ &\quad\times\langle p|G^{\mu\lambda}_a(y)Y^{ab}(y,-y) \\ &\quad\times G^b_{\lambda\nu}(-y)|p\rangle|_{\text{spin avg.}},\end{aligned} \tag{37}$$

$\bar{f}_{i/p}(z) = -f_{i/p}(-z)$. Here $G^a_{\mu\lambda}(y)$ is the gluon field strength, $Y(y,-y)$ and $Y^{ab}(y,-y)$ are path-ordered Wilson lines in the fundamental and adjoint representations, and $|p\rangle$ is the proton state with momentum p. In SCET these distribution functions can be defined by the matrix elements of collinear fields with collinear proton states

$$\begin{aligned}\frac{1}{2}\sum_{\text{spin}}\langle p_n|\bar{\chi}^{(i)}_{n,\omega}\frac{\bar{\not{n}}}{2}\chi^{(i)}_{n,\omega'}|p_n\rangle &= 4\bar{n}\cdot p\int_0^1 dz\,\delta(\omega_-)\,\delta(\omega_+ - 2z\bar{n}\cdot p)f_{i/p}(z) \\ &\quad - 4\bar{n}\cdot p\int_0^1 dz\,\delta(\omega_-)\,\delta(\omega_+ + 2z\bar{n}\cdot p)\bar{f}_{i/p}(z),\end{aligned} \tag{38}$$

$$\frac{1}{2}\sum_{\text{spin}}\langle p_n|\mathrm{Tr}[B^{\mu}_{n,\omega}B^{n,\omega'}_{\mu}]|p_n\rangle = -\frac{\omega_+\bar{n}\cdot p}{2}\int_0^1 dz\,\delta(\omega_-)\,\delta(\omega_+-2z\bar{n}\cdot p)f_{g/p}(z),$$

where $\omega_\pm=\omega\pm\omega'$, and $B^{\mu}_{n,\omega}\equiv\bar{n}_\nu(\mathcal{G}_{n,\omega})^{\nu\mu}$ with the collinear gauge invariant field strength

$$(\mathcal{G}_{n,\omega})^{\mu\nu}=-\frac{i}{g}[\,\delta(\omega-\bar{\mathcal{P}})W^\dagger[i\mathcal{D}^{\mu}_n+gA^{\mu}_{n,q},\; i\mathcal{D}^{\nu}_n+gA^{\nu}_{n,q'}]W]. \tag{39}$$

Both operators in Eq. (38) are order λ^2 since $\xi_{n,\omega}\sim B^{\perp}_{n,\omega}\sim\lambda$. Note that the matrix element of a single operator $(\bar{\chi}^{(i)}_{n,\omega}\bar{n}\!\!\!/\chi^{(i)}_{n,\omega'})$ contains both the quark and antiquark distributions. This is due to Eq. (3), from which we see that for $\omega=\omega'>0$ $(\omega=\omega'<0)$ this operator reduces to the number operator for collinear quarks (antiquarks) with momentum ω.

Processes other than DIS sometimes depend on more complicated distribution functions. In deeply virtual Compton scattering (DVCS) we will need to parametrize the matrix element of an operator between proton states with different momenta. In terms of QCD fields ψ the nonforward parton distribution function (NFPDF) defined by Radyushkin in Eq. (4.1) of Ref. [26] is (up to a trivial translation)

$$\langle p',\sigma'|\bar{\psi}^{(i)}(y)Y(y,-y)\bar{n}\!\!\!/\psi^{(i)}(-y)|p,\sigma\rangle = e(\sigma',\sigma)\int_0^1 dz[e^{i\bar{n}\cdot p(2z-\zeta)y}\mathcal{F}^{(i)}_\zeta(z;t)-e^{-i\bar{n}\cdot p(2z-\zeta)y}\bar{\mathcal{F}}^{(i)}_\zeta(z;t)] + h(\sigma',\sigma)\int_0^1 dz[e^{i\bar{n}\cdot p(2z-\zeta)y}\mathcal{K}^{(i)}_\zeta(z;t)-e^{-i\bar{n}\cdot p(2z-\zeta)y}\bar{\mathcal{K}}^{(i)}_\zeta(z;t)], \tag{40}$$

where $t=(p-p')^2$, and $\zeta=1-\bar{n}\cdot p'/\bar{n}\cdot p$. Here $e(\sigma,\sigma')$ and $h(\sigma,\sigma')$ are matrix elements which respectively preserve or flip the proton spin. They are defined in terms of the proton spinors

$$e(\sigma',\sigma)=\bar{u}(p',\sigma')\bar{n}\!\!\!/u(p,\sigma),\quad h(\sigma',\sigma)=\frac{1}{2m_p}\bar{u}(p',\sigma')[\bar{n}\!\!\!/,p\!\!\!/-p\!\!\!/']u(p,\sigma), \tag{41}$$

where m_p is the proton mass. The NFPDF for gluons is similarly given by [26]

$$\bar{n}_\mu\bar{n}^\nu\langle p',\sigma'|G^{\mu\lambda}_a(y)Y^{ab}(y,-y)G^b_{\lambda\nu}(-y)|p,\sigma\rangle = \frac{\bar{n}\cdot p}{2}e(\sigma',\sigma)\int_0^1 dz[e^{i\bar{n}\cdot p(2z-\zeta)y}+e^{-i\bar{n}\cdot p(2z-\zeta)y}]\mathcal{F}^g_\zeta(z;t) + \frac{\bar{n}\cdot p}{2}h(\sigma',\sigma)\int_0^1 dz[e^{i\bar{n}\cdot p(2z-\zeta)y}+e^{-i\bar{n}\cdot p(2z-\zeta)y}]\mathcal{K}^g_\zeta(z;t). \tag{42}$$

In SCET the definition of the NFPDFs in terms of collinear fields is

$$\langle p'_n,\sigma'|\bar{\chi}^{(i)}_{n,\omega}\bar{n}\!\!\!/\chi^{(i)}_{n,\omega'}|p_n,\sigma\rangle = 2\,\delta(\omega_-+\bar{n}\cdot p\zeta)\int_0^1 dz\{e(\sigma',\sigma)[\,\delta(\omega_+-(2z-\zeta)\bar{n}\cdot p)\mathcal{F}^{(i)}_\zeta(z;t)-\delta(\omega_++(2z-\zeta)\bar{n}\cdot p) \times\bar{\mathcal{F}}^{(i)}_\zeta(z;t)]+h(\sigma',\sigma)[\,\delta(\omega_+-(2z-\zeta)\bar{n}\cdot p)\mathcal{K}^{(i)}_\zeta(z;t)-\delta(\omega_++(2z-\zeta)\bar{n}\cdot p)\bar{\mathcal{K}}^{(i)}_\zeta(z;t)]\}, \tag{43}$$

$$\langle p'_n,\sigma'|\mathrm{Tr}[B^{\mu}_{n,\omega}B^{n,\omega'}_{\mu}]|p_n,\sigma\rangle = -\frac{\bar{n}\cdot p}{2}\,\delta(\omega_-+\bar{n}\cdot p\zeta)\int_0^1 dz\{e(\sigma',\sigma)\,\delta(\omega_+-(2z-\zeta)\bar{n}\cdot p)\mathcal{F}^g_\zeta(z;t) + h(\sigma',\sigma)\,\delta(\omega_+-(2z-\zeta)\bar{n}\cdot p)\mathcal{K}^g_\zeta(z,t)\},$$

where the spinors in $e(\sigma',\sigma)$ and $h(\sigma',\sigma)$ are two component effective theory spinors, so that $u\to u_n$ where $\bar{n}\!\!\!/u_n=0$. Note that for $p'\to p$ both $\zeta\to0$ and $t\to0$. In this limit the NFPDFs reduce to the standard PDFs:

$$\lim_{p\to p'} \mathcal{F}_\zeta^{(i)} = f_{i/p}(z),$$
$$\lim_{p\to p'} \overline{\mathcal{F}}_\zeta^{(i)} = \overline{f}_{i/p}(z), \tag{44}$$
$$\lim_{p\to p'} \mathcal{F}_\zeta^{(g)} = z f_{g/p}(z).$$

D. Symmetries for collinear fields

In this section we discuss spin and discrete symmetry constraints on operators involving collinear fields.

The possible spin structures of currents with $\xi_{n,p}$ fields is restricted by the fact that they have only two components, $\not{n}\xi_{n,p}=0$. The four most general spin structures for currents with two collinear particles moving in the same or opposite directions are

$$\overline{\xi}_{n,p'}\Gamma_1\xi_{n,p} \quad \Gamma_1=\{\not{\bar n},\not{\bar n}\gamma_5,\not{\bar n}\gamma_\perp^\mu\},$$
$$\overline{\xi}_{\bar n,p'}\Gamma_2\xi_{n,p} \quad \Gamma_2=\{1,\gamma_5,\gamma_\perp^\mu\}. \tag{45}$$

Other choices for Γ_1 and Γ_2 either vanish between the fields or are related to those in Eq. (45). This result can be expressed in a compact way by the trace formulas

$$\overline{\xi}_{n,p'}\Gamma\xi_{n,p},$$
$$\Gamma=\frac{\not{\bar n}}{8}\mathrm{Tr}[\not{n}\Gamma]-\frac{\not{\bar n}\gamma_5}{8}\mathrm{Tr}[\not{n}\gamma_5\Gamma]-\frac{\not{\bar n}\gamma_\perp^\mu}{8}\mathrm{Tr}[\not{n}\gamma_\mu^\perp\Gamma],$$

and

$$\overline{\xi}_{\bar n,p'}\Gamma\xi_{n,p},$$
$$\Gamma=\frac{1}{8}\mathrm{Tr}[\not{n}\not{\bar n}\Gamma]+\frac{\gamma_5}{8}\mathrm{Tr}[\gamma_5\not{n}\not{\bar n}\Gamma]+\frac{\gamma_\perp^\mu}{8}\mathrm{Tr}[\gamma_\mu^\perp\not{n}\not{\bar n}\Gamma], \tag{46}$$

which reduce a general Γ to a linear combination of the terms in Eq. (45). For instance, it implies that $2i\overline{\xi}_n\sigma^{\mu\nu}\xi_n = n^\nu\overline{\xi}_n\not{\bar n}\gamma_\perp^\mu\xi_n - n^\mu\overline{\xi}_n\not{\bar n}\gamma_\perp^\nu\xi_n$, and $\overline{\xi}_{\bar n}\gamma_\perp^\mu\gamma_5\xi_n = i\epsilon_\perp^{\mu\nu}\overline{\xi}_{\bar n}\gamma_\nu^\perp\xi_n$ where $\epsilon_\perp^{\mu\nu}=\epsilon^{\mu\nu\alpha\beta}n_\beta\bar n_\alpha/2$. Furthermore, each of the two components of ξ_n and also $\xi_{\bar n}$ can be chosen to be eigenstates of their helicity operators, $h=\hat p\cdot\vec S$ with eigenvalues $\pm 1/2$. For these fields h is equivalent to the chiral rotation, $h=\gamma_5/2$. The structures in Eq. (45) split into two classes depending on whether they conserve or flip the helicity

$$\text{chiral even:}\quad \overline{\xi}_{n,p'}\{\not{\bar n},\not{\bar n}\gamma_5\}\xi_{n,p}, \quad \overline{\xi}_{\bar n,p'}\gamma_\perp^\mu\xi_{n,p},$$
$$\text{chiral odd:}\quad \overline{\xi}_{n,p'}\not{\bar n}\gamma_\perp^\mu\xi_{n,p}, \quad \overline{\xi}_{\bar n,p'}\{1,\gamma_5\}\xi_{n,p}. \tag{47}$$

Since gluon interactions in QCD preserve helicity, integrating out hard QCD fluctuations results in effective theory operators with the same helicity structure as the original operators at leading order in λ.

The presence of labels on the effective theory fields makes their transformation properties under the discrete symmetries C, P, and T slightly different from QCD. For example under P or T we have $n\leftrightarrow\bar n$ so these transformations relate collinear fields for different directions. Under charge conjugation, parity, and time-reversal the collinear fields transform as

$$C^{-1}\xi_{n,p}(x)C=-[\overline{\xi}_{n,-p}(x)\mathcal{C}]^T,$$
$$C^{-1}A_{n,p}^\mu(x)C=-[A_{n,p}^\mu(x)]^T,$$
$$P^{-1}\xi_{n,p}(x)P=\gamma_0\xi_{\bar n,\tilde p}(x_P),$$
$$P^{-1}A_{n,p}^\mu(x)P=g_{\mu\nu}A_{\bar n,\tilde p}^\nu(x_P),$$
$$T^{-1}\xi_{n,p}(x)T=\mathcal{T}\xi_{\bar n,\tilde p}(x_T),$$
$$T^{-1}A_{n,p}^\mu(x)T=g_{\mu\nu}A_{\bar n,\tilde p}^\nu(x_T), \tag{48}$$

where $\mathcal{C}^{-1}\gamma_\mu\mathcal{C}=-\gamma_\mu^T$ and $\mathcal{T}=\gamma^5\mathcal{C}$, while if $x^\mu=(x^+,x^-,x^\perp)$ and $p^\mu=(p^+,p^-,p^\perp)$ then $\tilde p^\mu\equiv(p^-,p^+,-p_\perp)$, $x_P^\mu\equiv(x^-,x^+,-x^\perp)$, and $x_T^\mu\equiv(-x^-,-x^+,x^\perp)$. The transformation properties of W_n can be worked out using Eq. (48), for instance $C^{-1}W_nC=[W_n^\dagger]^T$.

The collinear effective Lagrangian (10) is invariant under the transformations in Eq. (48) (adding the $\bar n\leftrightarrow n$ terms). These symmetries also constrain the form of nonperturbative matrix elements. As an example, for a meson which is an eigenstate of C one finds

$$\begin{aligned}\langle M_n|\overline{\xi}_{n,p}W_n\not{\bar n}\gamma_5\delta(\omega-\overline{\mathcal{P}}_+)W_n^\dagger\xi_{n,p'}|0\rangle&\\ =(-1)^C\langle M_n|(\mathcal{C}W_n^\dagger\xi_{n,p})^T\not{\bar n}\gamma_5\delta(\omega-\overline{\mathcal{P}}_+)(\overline{\xi}_{n,p'}W_n\mathcal{C})^T|0\rangle&\\ =(-1)^C\langle M_n|\overline{\xi}_{n,p'}W_n\not{\bar n}\gamma_5\delta(\omega+\overline{\mathcal{P}}_+)W_n^\dagger\xi_{n,p}|0\rangle.&\end{aligned} \tag{49}$$

For the isotriplet pion state $(-1)^C=+1$ so combining Eq. (49) with Eq. (34) gives

$$\begin{aligned}\langle\pi_{n,p}^a|\overline{\xi}_{n,p_1}W\Gamma_\pi^b\delta(\omega-\overline{\mathcal{P}}_+)W^\dagger\xi_{n,p_2}|0\rangle&\\ =-if_\pi\delta^{ab}\bar n\cdot p\int_0^1 dx\,\delta[-\omega-(2x-1)\bar n\cdot p]\phi_\pi(x)&\\ =-if_\pi\delta^{ab}\bar n\cdot p\int_0^1 dx\,\delta[\omega-(2x-1)\bar n\cdot p]\phi_\pi(1-x).&\end{aligned} \tag{50}$$

Together with Eq. (34), charge conjugation therefore implies that $\phi_\pi(1-x)=\phi_\pi(x)$.

III. EXCLUSIVE PROCESSES

A. π-γ form factor

The pion-photon form factor $F_{\pi\gamma}(Q^2)$ is perhaps the simplest setting for factorization since there is only one hadron in the external state. The form factor is measurable in single-

tagged two photon $e^-e^-\to e^-e^-\pi^0$ reactions. This process involves the scattering of a highly virtual photon and a quark–anti-quark constituent pair off an on-shell photon. The photon scatters the quark pair away from the incoming photon into a pion, so that $\gamma^*\gamma\to\pi^0$. The matrix element for this transition defines the π-γ form factor

$$\begin{aligned}&\langle\pi^0(p_\pi)|J_\mu(0)|\gamma(p_\gamma,\epsilon)\rangle\\&\quad=ie\epsilon^\nu\int d^4z e^{-ip_\gamma\cdot z}\langle\pi^0(p_\pi)|\mathrm{T}J_\mu(0)J_\nu(z)|0\rangle\\&\quad=-ieF_{\pi\gamma}(Q^2)\epsilon_{\mu\nu\rho\sigma}p_\pi^\nu\epsilon^\rho q^\sigma.\end{aligned}\tag{51}$$

Here $J_\mu=\bar\psi\hat{\mathcal{Q}}\gamma_\mu\psi$ is the full theory electromagnetic current with isodoublet field ψ and charge matrix $\hat{\mathcal{Q}}=\tau_3/2+\mathbf{1}/6$, and $-q^2=Q^2\gg\Lambda_{\rm QCD}^2$ where $q=p_\pi-p_\gamma$ is the virtual photon momentum. It has been shown that the form factor can be written as a one-dimensional convolution of a hard coefficient with the light-cone pion wave function [27]. Here we show how this factorization takes place in the SCET.

In the Breit frame $q^\mu=Q(n^\mu-\bar n^\mu)/2$, the real photon's momentum is $p_\gamma^\mu=E\bar n^\mu\simeq Q\bar n^\mu/2$, and the pion is made up of collinear particles with momenta $\bar n\cdot p_i\simeq Q$. The particles exchanged between the two currents in Eq. (51) have hard momenta and can be integrated out. At leading order in λ the time ordered product of the two currents in Eq. (51) matches onto a single operator in the effective theory. For simplicity we restrict ourselves to the tensor and spin structures that are relevant when the meson is a pion,[4]

$$O=\frac{i}{Q}\epsilon_{\mu\nu}^\perp\ [\bar\xi_{n,p}W]\Gamma C(\bar{\mathcal{P}},\bar{\mathcal{P}}^\dagger,\mu)[W^\dagger\xi_{n,p'}],\tag{52}$$

where $\xi_{n,p}$ is an isodoublet collinear quark field, and $2\epsilon_{\mu\nu}^\perp=\epsilon_{\mu\nu\rho\beta}\bar n^\rho n^\beta$. O_1 is of dimension 2, just like the time-ordered product in Eq. (51), and a power of $1/Q$ is included to make $C(\mu,\bar{\mathcal{P}},\bar{\mathcal{P}}^\dagger)$ dimensionless. The time-ordered product in Eq. (51) is even under charge conjugation, so the operators in Eq. (52) must also be even. This implies $C_{\pi\gamma}(\mu,\bar{\mathcal{P}},\bar{\mathcal{P}}^\dagger)=C_{\pi\gamma}(\mu,-\bar{\mathcal{P}}^\dagger,-\bar{\mathcal{P}})$. The location of the W's in Eq. (52) is fixed by gauge invariance, and Γ contains the spin and flavor structure

$$\Gamma=(\bar n\!\!\!/\gamma_5)(3\sqrt{2}\hat{\mathcal{Q}}^2).\tag{53}$$

Since the off-shellness of the collinear particles in the pion is $p^2\sim\Lambda_{\rm QCD}^2$ we can also integrate out off-shell modes with $p^2\sim Q\Lambda_{\rm QCD}$ which come from soft-collinear interactions. For the collinear operators O_j, Eq. (17) implies that factors of the soft Wilson line S_n are induced. However, the location is such that $S_n^\dagger S_n=1$, so no coupling to soft gluons occurs at leading order. The coupling of the collinear fields to usoft gluons can be simplified with the field redefinitions

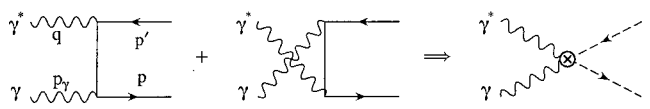

FIG. 1. Tree level matching onto O_j in the Breit frame. The graphs on the left include u and d quarks.

in Eq. (13). As discussed in Sec. II A, this moves all couplings into O_j, and using $Y_n^\dagger Y_n=1$ gives

$$O=\frac{i}{Q}\epsilon_{\mu\nu}^\perp[\bar\xi_{n,p}^{(0)}W^{(0)}]\Gamma C(\bar{\mathcal{P}},\bar{\mathcal{P}}^\dagger,\mu)[W^{(0)\dagger}\xi_{n,p'}^{(0)}].\tag{54}$$

Thus, usoft gluons also decouple.

In the Breit frame the pion momentum satisfies $p_\pi^\mu=E_\pi n^\mu+\mathcal{O}(\lambda)$, and comparing Eq. (51) with the SCET matrix element $i\langle\pi_{n,p_\pi}^0|O_{\pi\gamma}|0\rangle$, gives

$$\begin{aligned}\frac{Q^2}{2}F_{\pi\gamma}(Q^2)&=\frac{i}{Q}\langle\pi_n^0|\bar\xi_{n,p}^{(0)}W^{(0)}\Gamma C(\bar{\mathcal{P}},\bar{\mathcal{P}}^\dagger,\mu)\\&\quad\times W^{(0)\dagger}\xi_{n,p'}^{(0)}|0\rangle.\end{aligned}\tag{55}$$

Defining $\bar{\mathcal{P}}_\pm=\bar{\mathcal{P}}^\dagger\pm\bar{\mathcal{P}}$, the operator $\bar{\mathcal{P}}_-$ is related to $\bar{\mathcal{P}}$ acting from the outside on the fields. Using Eq. (6) it can therefore be set equal to the momentum label of the state, $\bar{\mathcal{P}}_-=\bar n\cdot p_\pi=Q$. Suppressing this dependence we write $C(\bar{\mathcal{P}},\bar{\mathcal{P}}^\dagger,\mu)=C_1(\bar{\mathcal{P}}_+,\mu)$ leaving

$$\begin{aligned}F_{\pi\gamma}(Q^2)&=\frac{2i}{Q^3}\langle\pi_n^0|\bar\xi_{n,p}^{(0)}W^{(0)}\Gamma C_1(\bar{\mathcal{P}}_+,\mu)W^{(0)\dagger}\xi_{n,p'}^{(0)}|0\rangle\\&=\frac{2i}{Q^3}\int d\omega C_1(\omega,\mu)\\&\quad\times\langle\pi_n^0|\bar\xi_{n,p}^{(0)}W^{(0)}\Gamma\,\delta(\omega-\bar{\mathcal{P}}_+)W^{(0)\dagger}\xi_{n,p'}^{(0)}|0\rangle.\end{aligned}\tag{56}$$

Using Eq. (34) the remaining matrix identity in Eq. (56) can be written in terms of the light-cone pion wave function

$$\begin{aligned}F_{\pi\gamma}(Q^2)&=\frac{2f_\pi}{Q^2}\int d\omega\int_0^1 dx\,\delta(\omega-(2x-1)2E_\pi)\\&\quad\times C_1(\omega,\mu)\phi_\pi(x,\mu)\\&=\frac{2f_\pi}{Q^2}\int_0^1 dx C_1((2x-1)Q,\mu)\phi_\pi(x,\mu).\end{aligned}\tag{57}$$

This is the final result and is valid to leading order in λ and all orders in α_s. From Eq. (50) charge conjugation implies that $\phi_\pi(x)=\phi_\pi(1-x)$ and $C_1(\omega)=C_1(-\omega)$. Equation (57) agrees with the Brodsky-Lepage [24] result that the form factor can be written as the convolution of a short distance function with the light-cone pion wave function. The SCET formalism gives a concise derivation of this result and

[4]Note that a pure glue operator would not have the same isospin as the pion state.

defines the short distance function in terms of the Wilson coefficient of an effective theory operator.

As an illustrative example consider the tree level matching onto C illustrated in Fig. 1. Since the location of the W's in O is fixed by gauge invariance, $C(\mu,\bar{\mathcal{P}},\bar{\mathcal{P}}^\dagger)$ can be determined by matching with $W=1$. Expanding the full theory graphs to leading order gives

$$i(\text{Fig. 1}) = \frac{ie}{2}\epsilon_{\mu\nu\rho\beta}\epsilon^{\nu}\bar{n}^{\rho}n^{\beta}\left(\frac{\bar{\hbar}}{2}\gamma_5\right) \times(\hat{\mathcal{Q}}^2)\left(\frac{1}{\bar{n}\cdot p}-\frac{1}{\bar{n}\cdot p'}\right), \tag{58}$$

where we have dropped isosinglet terms, contributions with opposite parity to the pion, as well as those proportional to $\bar{\hbar}\gamma_5$ since $\bar{\hbar}\xi_{n,p}=0$. Comparing Eq. (58) to Eq. (52) gives

$$C(\mu,\bar{\mathcal{P}},\bar{\mathcal{P}}^\dagger) = \frac{Q}{6\sqrt{2}}\left(\frac{1}{\bar{\mathcal{P}}^\dagger}-\frac{1}{\bar{\mathcal{P}}}\right)+\mathcal{O}(\alpha_s(Q)), \tag{59}$$

so that

$$C_1(\mu,\omega=(2x-1)Q) = \frac{1}{6\sqrt{2}}\left(\frac{1}{x}+\frac{1}{1-x}\right)+\mathcal{O}(\alpha_s(Q)). \tag{60}$$

This result is again in agreement with Ref. [24], and the order $\alpha_s(Q)$ corrections to this Wilson coefficient can be read off from the results in Refs. [28,29]. An identical analysis applies for operators with different spin structures such as the ones contributing to $\gamma^*\gamma\to\rho^0$.

B. The large Q^2 meson form factor

Another example of an exclusive process which can be treated in the effective theory is the classic case of the electromagnetic pion form factor at large space-like momentum transfer. For generality we consider in this section the electromagnetic form factors for arbitrary mesons (pseudoscalar P or vector V), defined as

$$\begin{aligned}
\langle P'(p')|J_\mu|P(p)\rangle &= F_P(Q^2)(p_\mu+p'_\mu),\\
\langle V'(p',\varepsilon')|J_\mu|P(p)\rangle &= G(Q^2)i\epsilon_{\mu\nu\alpha\beta}p^\nu p'^\alpha\varepsilon'^\beta,\\
\langle V'(p',\varepsilon')|J_\mu|V(p,\varepsilon)\rangle &= F_1(Q^2)(\varepsilon'^*\cdot\varepsilon)(p+p')_\mu\\
&\quad+F_2(Q^2)[(\varepsilon'^*\cdot p)\varepsilon_\mu\\
&\quad+(\varepsilon\cdot p')\varepsilon'^*_\mu],
\end{aligned} \tag{61}$$

where $q^2=-Q^2, q=p-p'$. For simplicity we suppress the dependence of the form factors on the isospin of the two mesons. We will restrict ourselves in the following to the case of hadrons made up only of u,d quarks. The electromagnetic current is defined as usual by $J_\mu=\bar{q}\hat{\mathcal{Q}}\gamma_\mu q$, with charge matrix $\hat{\mathcal{Q}}=\text{diag}(2/3,-1/3)$, which can be written in terms of the up and down quark charges as $\hat{\mathcal{Q}}=(Q_u-Q_d)\tau_3/2+(Q_u+Q_d)\mathbf{1}/2$.

We will be interested in the asymptotic form of the form factors in the region with $Q^2\gg m_P^2$, where it can be expanded in a power series in $1/Q^2$ [24]. It is convenient to work in the Breit frame, where the momentum transfer has the light-cone components $q=(q^+,q^-,q_\perp)=(Q,-Q,0)$. In this frame the meson momenta are $p=(Q,m_P^2/Q,0)$, $p'=(m_P^2/Q,Q,0)$, so the partons in the incoming/outgoing meson are collinear along the $\bar{n}_\mu/n_\mu$ direction.

The electromagnetic current in Eq. (61) is matched in the effective theory onto the most general combination of operators constructed from collinear fields which are compatible with collinear gauge invariance. Operators such as the dimension 3 current

$$[\bar{\xi}_{\bar{n}}W_{\bar{n}}]\Gamma C(\mu,\mathcal{P}^\dagger,\bar{\mathcal{P}})[W_n^\dagger\xi_n], \tag{62}$$

can contribute, but only overlap with the asymmetric meson states with one energetic collinear quark and one usoft or soft quark. Often this overlap is referred to as the tail of the wave function contribution or the Feynman mechanism of generating the form factor [30,31]. There are other operators with significant overlap with more symmetric meson states (where all the constituents are allowed to be energetic). The leading such operators have the form[5]

$$\frac{1}{Q^3}[\bar{\xi}_{n,p_1}W_n\Gamma W_n^\dagger\xi_{\bar{n},p_2}]C(\mu,\bar{\mathcal{P}},\bar{\mathcal{P}}^\dagger,\mathcal{P},\mathcal{P}^\dagger) \times[\bar{\xi}_{\bar{n},p_3}W_{\bar{n}}\Gamma' W_n^\dagger\xi_{n,p_4}], \tag{63}$$

with C a dimensionless Wilson coefficient. As usual, collinear gauge invariance is enforced by the location of the W's in Eqs. (62) and (63). There is some argument about the relative size of Eqs. (62) and (63) in the literature [30,31]. Often it is argued that the tail of the wave function is suppressed by an extra $\Lambda_{\rm QCD}^2/Q^2$ [30], in which case the operator in Eq. (63) dominates by two powers of Q. An analysis of the tail of wave function contributions has not yet been performed in the effective theory framework. Therefore, we choose to ignore the operator in Eq. (62), and below only analyze the operator in Eq. (63). We emphasize that we do not claim to have shown that this is justified by the effective theory power counting.

There are two different structures possible for the operator in Eq. (63), and we write the general matching for the electromagnetic current as

[5]There are also gluon operators that can contribute when one or more of the mesons is a neutral isosinglet, however for simplicity these are not discussed here.

$$J^\nu \to \frac{1}{Q^3}\int d\omega_j [C_0(\mu,\omega_j)\mathcal{J}_0^\nu(\omega_j,\mu) + C_8(\mu,\omega_j)\mathcal{J}_8^\nu(\mu,\omega_j)], \quad (64)$$

where $j=1,2,3,4$. The SCET currents are dimension-6 operators

$$\mathcal{J}_0^\nu = \bar{\chi}_{n,\omega_1}\Gamma\chi_{\bar{n},\omega_2}\bar{\chi}_{\bar{n},\omega_3}\Gamma'\chi_{n,\omega_4} - (\Gamma\leftrightarrow\Gamma', \omega_{1,2}\leftrightarrow -\omega_{4,3}),$$

$$\mathcal{J}_8^\nu = \bar{\chi}_{n,\omega_1}\Gamma T^a\chi_{\bar{n},\omega_2}\bar{\chi}_{\bar{n},\omega_3}\Gamma' T^a\chi_{n,\omega_4} - (\Gamma\leftrightarrow\Gamma', \omega_{1,2}\leftrightarrow -\omega_{4,3}), \quad (65)$$

where the χ fields are defined in Eq. (35). In terms of the charge matrix $\hat{\mathcal{Q}}$, the spin and flavor structure is

$$\Gamma\otimes\Gamma' = (n^\nu+\bar{n}^\nu)(\gamma_\alpha^\perp\hat{\mathcal{Q}}\otimes\gamma_\perp^\alpha\mathbf{1}). \quad (66)$$

The Wilson coefficients $C_{0,8}$ can be computed in a power series in $\alpha_s(Q)$. They are functions of μ, Q, and the ω_j which are the sum of momentum labels for gauge invariant products of collinear fields in the SCET currents.

The current operators in Eq. (65) are the most general allowed operators which are gauge invariant, transform the same way as J^μ under charge conjugation and satisfy current and helicity conservation. To see how these properties constrain the form of the allowed operators, we begin by noting that Eq. (45) implies that $\Gamma,\Gamma'=\{1,\gamma_5,\gamma_\perp^\mu\}$ are the most general allowed spin structures. For massless quarks the electromagnetic and QCD couplings preserve helicity, whereas $\bar{\xi}_n\{1,\gamma_5\}\xi_{\bar{n}}$ cause the helicity to flip. Thus, only the structure $\bar{\xi}_n\gamma_\perp^\mu\xi_{\bar{n}}$ is allowed. Current conservation $q^\nu\mathcal{J}_\nu=0$, together with $q^\nu=Q(\bar{n}^\nu-n^\nu)/2$ implies $\mathcal{J}_\nu\propto(n_\nu+\bar{n}_\nu)$. Under charge conjugation $J^\mu\to -J^\mu$ so the same must be true for the SCET currents. In the current operators, charge conjugation switches $\omega_1\leftrightarrow-\omega_4$, $\omega_2\leftrightarrow-\omega_3$, and $\Gamma\leftrightarrow\Gamma'$, as can be seen from Eq. (48). Thus, the second term in $\mathcal{J}_{0,8}^\nu$ is required to make these operators odd under charge conjugation. The operators $\mathcal{J}_{0,8}^\nu$ and the full electromagnetic current are invariant under a combined PT transformation. This requires that the Wilson coefficients are real.

The operators $\mathcal{J}_{0,8}$ are responsible for the $P_{\bar{n}}\to P_n$ transition, while the reverse transition $P_n\to P_{\bar{n}}$ is described by similar operators with $\bar{n}\leftrightarrow n$. Parity invariance requires the Wilson coefficients of these operators to be identical to $C_{0,8}(\omega_i)$. Demanding Hermiticity of the electromagnetic current in the effective theory then gives the relation $C_{0,8}(\omega_1,\omega_2,\omega_3,\omega_4)=C_{0,8}^*(\omega_2,\omega_1,\omega_4,\omega_3)$. Since the coefficients are real they must therefore satisfy $C_{0,8}(\omega_1,\omega_2,\omega_3,\omega_4)=C_{0,8}(\omega_2,\omega_1,\omega_4,\omega_3)$.

To compute the matrix elements in the effective theory, it is convenient to Fierz transform the four-quark operators in Eq. (65). This gives

$$C_0\mathcal{J}_0 + C_8\mathcal{J}_8 = (n^\nu+\bar{n}^\nu)\sum_{j=1}^4 C_j\mathcal{J}_j, \quad (67)$$

where

$$\mathcal{J}_j = [\bar{\chi}_{n,\omega_1}\Gamma_j\chi_{n,\omega_4}][\bar{\chi}_{\bar{n},\omega_3}\Gamma_j'\chi_{\bar{n},\omega_2}]. \quad (68)$$

The spin, flavor, and color structures are

$$\Gamma_1\otimes\Gamma_1' = -\frac{1}{4}(Q_u-Q_d)i\epsilon^{3bc}(\tau^b\otimes\tau^c)\times[\bar{\not{n}}\otimes\not{n}+\bar{\not{n}}\gamma_5\otimes\not{n}\gamma_5],$$

$$\Gamma_2\otimes\Gamma_2' = \frac{1}{4}[(Q_u+Q_d)(\mathbf{1}\otimes\mathbf{1}+\tau^a\otimes\tau^a) + (Q_u-Q_d)(\mathbf{1}\otimes\tau^3+\tau^3\otimes\mathbf{1})]\times[\bar{\not{n}}\otimes\not{n}+\bar{\not{n}}\gamma_5\otimes\not{n}\gamma^5], \quad (69)$$

while $\Gamma_{3,4}=T^a\Gamma_{1,2}$ and $\Gamma_{3,4}'=T^a\Gamma_{1,2}'$. The new Wilson coefficients are

$$C_1(\mu,\omega_j) = \left[\frac{1}{8}\left(1-\frac{1}{N_c^2}\right)C_8(\mu,\omega_j)+\frac{1}{4N_c}C_0(\mu,\omega_j)\right] + (\omega_{1,2}\leftrightarrow-\omega_{4,3}),$$

$$C_2(\mu,\omega_j) = \left[\frac{1}{8}\left(1-\frac{1}{N_c^2}\right)C_8(\mu,\omega_j)+\frac{1}{4N_c}C_0(\mu,\omega_j)\right] - (\omega_{1,2}\leftrightarrow-\omega_{4,3}), \quad (70)$$

with similar relations for $C_{3,4}$ which are also in terms of $C_{0,8}$.

A few general predictions follow from the form of the operators in Eq. (68).[6] For mesons with spin, only helicity conserving form factors appear, and furthermore no off-diagonal (e.g., $P\to V$) matrix elements are present at leading order in $1/Q^2$. These results agree with Ref. [27]. We also see that the form factors between arbitrary meson states are determined at leading power by only two hard coefficients, C_0 and C_8.

Now consider what factorization tells us about the matrix element of the operators in Eq. (68). For the decoupling of usoft and soft gluons we will follow Sec. II A. Integrating out off-shell modes with $p^2\sim Q\Lambda_{\rm QCD}$ induces soft Wilson lines S_n and $S_{\bar{n}}$, while the field redefinitions in Eq. (13) make all couplings to usoft gluons explicit in the operators. Together these give

[6]These predictions depend on the dominance of the operators in Eq. (63) over those in Eq. (62).

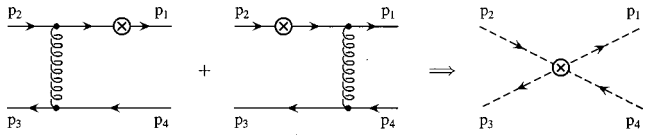

FIG. 2. Tree level matching onto $C_{0,8}$. The QCD graphs on the left plus the analogous graphs with the current on the bottom quark line are matched onto the collinear operator on the right.

$$\mathcal{J}_j^\nu = [\bar{\chi}^{(0)}_{n,\omega_1} Y_n^\dagger S_n^\dagger \Gamma_j S_n Y_n \chi^{(0)}_{n,\omega_4}] \times [\bar{\chi}^{(0)}_{n,\omega_3} Y_{\bar{n}}^\dagger S_{\bar{n}}^\dagger \Gamma_j' S_{\bar{n}} Y_{\bar{n}} \chi^{(0)}_{n,\omega_2}]. \tag{71}$$

Consider first the color singlet currents $j=1,2$. Here the Y's and S's all cancel using unitarity of the Wilson lines. Since the $A^\mu_{n,q}$ and $A^\mu_{\bar{n},q}$ gluons only interact with fields in the n and $\bar{n}$ directions respectively, collinear gluons are not exchanged between the n and $\bar{n}$ quark bilinears. Thus, the matrix element between states with particles moving in the n and $\bar{n}$ directions factors

$$\langle n|\mathcal{J}_{1,2}|\bar{n}\rangle = \langle n|\bar{\chi}^{(0)}_{n,\omega_1}\Gamma_{1,2}\chi^{(0)}_{n,\omega_4}|0\rangle\langle 0|\bar{\chi}^{(0)}_{n,\omega_3}\Gamma'_{1,2}\chi^{(0)}_{n,\omega_2}|\bar{n}\rangle. \tag{72}$$

Next consider the currents $\mathcal{J}_{3,4}$, which have color structure $T^a\otimes T^a$ in $\Gamma_j\otimes\Gamma_j'$. In this case the usoft and soft gluons do not cancel, but can all be moved into one quark bilinear using the color identity $Y_n^\dagger S_n^\dagger T^a S_n Y_n \otimes Y_{\bar{n}}^\dagger S_{\bar{n}}^\dagger T^a S_{\bar{n}} Y_{\bar{n}} = T^a \otimes Y_n^\dagger S_n^\dagger S_n Y_n T^a Y_n^\dagger S_n^\dagger S_{\bar{n}} Y_{\bar{n}}$. After this rearrangement it is clear that the (u)soft gluons and $A^\mu_{n,q}$ and $A^\mu_{\bar{n},q}$ gluons only interact with the fields in one of the quark bilinears. Thus, the matrix element $\langle n|\mathcal{J}_{3,4}|\bar{n}\rangle$ factors, similar to Eq. (72). For color singlet states, however, the matrix element of an octet operator vanishes identically since

$$\langle n|\bar{\chi}^{(0)}_{n,\omega_1} T^a \chi^{(0)}_{n,\omega_4}|0\rangle = 0. \tag{73}$$

Thus, the effective theory currents $\mathcal{J}_{3,4}$ do not contribute to the form factors at any order in perturbation theory.

Equation (72) shows that for arbitrary meson states factorization occurs. It remains to show that the matrix elements in Eq. (72) are given by a two-dimensional convolution with the light-cone meson wave functions. To do this we consider the simple example of the $0^{-+}\to 0^{-+}$ form factor for the charged pion. It should be obvious that the same steps go through for other meson states.

The symmetry of the pion wave function $\phi_\pi(x)$ under charge conjugation ($x\to 1-x$) implies that only the $\mathcal{J}_1$ current contributes. Thus,

$$F_{\pi^\pm}(Q^2) = \frac{2}{Q^4}\int d\omega_j C_1(\mu,\omega_j) \times \langle \pi_n^\pm(p')|\bar{\chi}^{(0)}_{n,\omega_1}\Gamma_1\chi^{(0)}_{n,\omega_4}|0\rangle \times \langle 0|\bar{\chi}^{(0)}_{n,\omega_3}\Gamma_1'\chi^{(0)}_{n,\omega_2}|\pi_{\bar{n}}^\pm(p)\rangle. \tag{74}$$

The required matrix elements can be obtained from Eqs. (34) and (36) with $|\pi^\pm\rangle = \mp(\pi^1 \pm i\pi^2)/\sqrt{2}$. The momentum conserving delta functions fix $\omega_1-\omega_4 = \bar{n}\cdot p' = Q$ and $\omega_2-\omega_3 = \bar{n}\cdot p = Q$, while the $\omega=\omega_1+\omega_4$ and $\omega'=\omega_3+\omega_2$ integrations can be done with the delta functions. This leaves

$$F_{\pi^\pm} = \pm(Q_u - Q_d)\frac{f_\pi^2}{Q^2} \times \int_0^1 dx \int_0^1 dy\, T_1(x,y,\mu)\,\phi_\pi(x,\mu)\,\phi_\pi(y,\mu), \tag{75}$$

where $T_1(x,y)$ is defined in terms of $C_1(\omega_1,\omega_2,\omega_3,\omega_4)$ as

$$T_1(x,y) = C_1(xQ, yQ, (y-1)Q, (x-1)Q). \tag{76}$$

The coefficients $C_j(\mu,\omega_j)$, and therefore also $T_j(x,y)$, can be obtained at the scale $\mu=Q$ by a matching calculation, as illustrated in Fig. 2. For this purpose, it is sufficient to compute the matrix element of the currents with free collinear quarks. To lowest order in $\alpha_s(Q)$, only $C_8(\omega_j,\mu=Q)$ is nonvanishing

$$C_0(\omega_j,\mu=Q) = 0, \qquad C_8(\omega_j,\mu=Q) = 4\pi\alpha_s(Q)\frac{Q^2}{\omega_3\omega_4}. \tag{77}$$

This implies

$$T_1(x,y,\mu=Q) = \frac{4\pi\alpha_s(Q)}{9}\left[\frac{1}{xy} + \frac{1}{(1-x)(1-y)}\right]. \tag{78}$$

Using the asymptotic light-cone pion wave function $\phi_\pi(x) = 6x(1-x)$ we find agreement with Ref. [24],

$$F_{\pi^\pm}(Q^2) = \pm\frac{8\pi f_\pi^2\alpha_s(Q)}{9Q^2}\left[\int_0^1 dx\,\frac{\phi_\pi(x)}{x}\right]^2 \to \pm\frac{8\pi f_\pi^2\alpha_s(Q)}{Q^2}. \tag{79}$$

The order $\alpha_s^2(Q)$ corrections to Eq. (77) can be found in Refs. [32–34].

IV. INCLUSIVE PROCESSES

A. Deep inelastic scattering

DIS is a process which is both simple and rich in physics. As such it provides an ideal introduction to inclusive factorization in QCD, which we study from an effective field theory point of view in this section. The aim is to prove that to all orders in α_s and leading order in λ the DIS forward scattering amplitude can be written as an integral over hard coefficients times the parton distribution functions. This is done by matching onto local operators in SCET. The proper-

ties of SCET are used to show that matrix elements of the leading local operator can be written as a convolution of a hard coefficient with the parton distribution functions for the proton.

The first step is to understand the kinematics of the process. The hard scale $Q^2=-q^2$ is set by the invariant mass of the photon, and $x=Q^2/(2p\cdot q)$ is the Bjorken variable. In the Breit frame the momentum of the virtual photon is $q^\mu = Q(\bar{n}^\mu - n^\mu)/2$, and the incoming proton momentum is $p^\mu = n^\mu \bar{n}\cdot p/2 + \bar{n}^\mu m_p^2/(2\bar{n}\cdot p) \simeq n^\mu Q/(2x) + \bar{n}^\mu x m_p^2/(2Q)$ up to terms $\sim m_p^2/Q^2$, where m_p is the proton mass. By momentum conservation the final state momentum is $P_X^\mu = q^\mu + p^\mu$, which gives an invariant mass $P_X^2 = (Q^2/x)(1-x)+m_p^2$. Values $1-x \simeq \Lambda_{\rm QCD}/Q$ correspond to the end point region where the particles in X are collimated into a jet, while values $1-x \simeq \Lambda_{\rm QCD}^2/Q^2$ correspond to the resonance region. We will consider the standard OPE region where $1-x \gg \Lambda_{\rm QCD}/Q$ so that the final state has virtuality of order Q^2 and can be integrated out. In contrast, although the incoming proton has a large momentum component in the n^μ direction it has a small invariant mass $p^2=m_p^2\sim\Lambda_{\rm QCD}^2$, and therefore is described by collinear fields in the effective theory.

Consider the spin-averaged cross section for DIS which can be written as

$$d\sigma = \frac{d^3\mathbf{k}'}{2|\mathbf{k}'|(2\pi)^3}\frac{\pi e^4}{sQ^4} L^{\mu\nu}(k,k') W_{\mu\nu}(p,q), \qquad (80)$$

where k and k' are the incoming and outgoing lepton momenta with $q=k'-k$, $L^{\mu\nu}$ is the lepton tensor, and $s=(p+k)^2$. The hadronic tensor $W^{\mu\nu}$ can be related to the imaginary part of the DIS forward scattering amplitude:

$$W_{\mu\nu}(p,q) = \frac{1}{\pi}{\rm Im} T_{\mu\nu}(p,q),$$

$$T_{\mu\nu}(p,q) = \frac{1}{2}\sum_{\rm spin}\langle p|\hat{T}_{\mu\nu}(q)|p\rangle, \qquad (81)$$

$$\hat{T}_{\mu\nu}(q) = i\int d^4z e^{iq\cdot z}{\rm T}[J_\mu(z)J_\nu(0)],$$

where for an electromagnetic current J_μ we can write

$$T_{\mu\nu}(p,q) = \left(-g_{\mu\nu}+\frac{q_\mu q_\nu}{q^2}\right)T_1(x,Q^2) + \left(p_\mu+\frac{q_\mu}{2x}\right)\left(p_\nu+\frac{q_\nu}{2x}\right)T_2(x,Q^2). \qquad (82)$$

As explained above, the intermediate hadronic state has invariant mass $P_X^2\sim Q^2$. Therefore, one can perform an OPE and match $\hat{T}^{\mu\nu}(q)$ onto operators in SCET. All fields in the resulting operators are evaluated at the same residual space-time point, however, the presence of Wilson lines and label momenta make the operators nonlocal along a particular light cone direction. These nonlocal operators sum the infinite set of purely local operators of a given twist, however this is built into the formalism automatically. To match we write down the most general leading operator in SCET which contains collinear fields moving in the n^μ direction, and enforce the condition from current conservation $q^\mu\hat{T}_{\mu\nu}=0$. This leads to

$$\hat{T}^{\mu\nu} \to \frac{g_\perp^{\mu\nu}}{Q}\left(\sum_i O_1^{(i)} + \frac{O_1^g}{Q}\right) + \frac{(n^\mu+\bar{n}^\mu)(n^\nu+\bar{n}^\nu)}{Q}\left(\sum_i O_2^{(i)} + \frac{O_2^g}{Q}\right), \qquad (83)$$

where

$$O_j^{(i)} = \bar{\xi}_{n,p}^{(i)} W \frac{\bar{n}\!\!\!/}{2} C_j^{(i)}(\bar{\mathcal{P}}_+,\bar{\mathcal{P}}_-) W^\dagger \xi_{n,p}^{(i)},$$

$$O_j^g = \bar{n}_\mu \bar{n}_\nu {\rm tr}[W^\dagger (G_n)^{\mu\lambda} W C_j^g(\bar{\mathcal{P}}_+,\bar{\mathcal{P}}_-) W^\dagger (G_n)_\lambda^\nu W], \qquad (84)$$

where i labels the flavor of the fermions and $igG_n^{\mu\lambda} = [i\mathcal{D}_n^\mu + gA_{n,q}^\mu, i\mathcal{D}_n^\lambda + gA_{n,q'}^\lambda]$. The Wilson coefficients are dimensionless functions of $\bar{\mathcal{P}}_+$, $\bar{\mathcal{P}}_-$, Q, and μ. As in previous sections we can separate the hard coefficients from the long distance operators by introducing trivial convolutions. This gives

$$O_j^{(i)} = \int d\omega_1 d\omega_2 C_j^{(i)}(\omega_+,\omega_-)\left[\bar{\chi}_{n,\omega_1}^{(i)}\frac{\bar{n}\!\!\!/}{2}\chi_{n,\omega_2}^{(i)}\right],$$

$$O_j^g = -\int d\omega_1 d\omega_2 C_j^g(\omega_+,\omega_-){\rm tr}[B_{n,\omega_1}^\mu B_\mu^{n,\omega_2}], \qquad (85)$$

where $\omega_\pm = \omega_1 \pm \omega_2$, and $B_{n,\omega}^\mu \equiv \bar{n}_\nu (\mathcal{G}_{n,\omega})^{\nu\mu}$ with $(\mathcal{G}_{n,\omega})^{\mu\lambda}$ defined in Eq. (39). Next we factor the coupling of usoft gluons from the collinear fields using the field redefinitions in Eq. (13). The operator $O_j^{(i)}$ has the structure in Eq. (31) so the Y's cancel trivially, while for O_j^g we find

$$B_n^\mu = Y_n B_n^{\mu(0)} Y_n^\dagger, \qquad (86)$$

and the factors of Y cancel in the trace. It is easy to see that soft gluons also decouple using Eq. (17) or by noting that there is no nontrivial soft gauge invariant way of adding soft Wilson lines S_n to $O_j^{(i)}$ or O_j^g. Under charge conjugation the full theory electromagnetic current $J_\mu \to -J_\mu$ and therefore the operator $\hat{T}_{\mu\nu}\to\hat{T}_{\mu\nu}$. This implies relations for the effective theory Wilson coefficients since the operators $O_j^{(i)}$ must also respect this symmetry. Thus charge conjugation gives

$$\int d\omega_1 d\omega_2 C_j^{(i)}(\omega_+,\omega_-)\bar{\chi}_{n,\omega_1}^{(i)}\bar{n}\!\!\!/\chi_{n,\omega_2}^{(i)} \overset{\rm C}{=} -\int d\omega_1 d\omega_2 C_j^{(i)}(\omega_+,\omega_-)\bar{\chi}_{n,-\omega_2}^{(i)}\bar{n}\!\!\!/\chi_{n,-\omega_1}^{(i)} = \int d\omega_1 d\omega_2 [-C_j^{(i)}(-\omega_+,\omega_-)]\bar{\chi}_{n,\omega_1}^{(i)}\bar{n}\!\!\!/\chi_{n,\omega_2}^{(i)}. \qquad (87)$$

In the second line we changed variable $\omega_1 \to -\omega_2$ and $\omega_2 \to -\omega_1$ which takes $\omega_+ \to -\omega_+$ and $\omega_- \to \omega_-$. Thus, to all orders in perturbation theory $C_j^{(i)}(-\omega_+,\omega_-) = -C_j^{(i)}(\omega_+,\omega_-)$. This relates the Wilson coefficients for quarks and anti-quarks. Note that the above results are all independent of the collinear hadron on which DIS is performed.

Next we take the matrix element between proton states. Using the definitions of the nonperturbative matrix elements given in Sec. II C, and picking out the coefficients of the tensor structures we find that the delta functions in Eq. (38) set $\omega_+ = \pm 2Q\xi/x$ and $\omega_- = 0$. Since charge conjugation relates negative and positive values of ω_+, only coefficients, $C_j(\omega_+,0)$, with positive ω_+ are needed in the formulas for DIS. Therefore we define

$$H_j(z) \equiv C_j(2Qz,0,Q,\mu), \tag{88}$$

where here we have made the dependence on Q and μ explicit. Combining this with Eqs. (82), (83), (85), and (38), gives the final result

$$T_1(x,Q^2) = -\frac{1}{x}\int_0^1 d\xi \left\{ H_1^{(i)}\left(\frac{\xi}{x}\right)[f_{i/p}(\xi) + \overline{f}_{i/p}(\xi)] + \frac{\xi}{2x} H_1^g\left(\frac{\xi}{x}\right) f_{g/p}(\xi) \right\},$$
$$T_2(x,Q^2) = \frac{4x}{Q^2}\int_0^1 d\xi \left\{ \left[4H_2^{(i)}\left(\frac{\xi}{x}\right) - H_1^{(i)}\left(\frac{\xi}{x}\right) \right][f_{i/p}(\xi) + \overline{f}_{i/p}(\xi)] + \frac{\xi}{2x}\left[4H_2^g\left(\frac{\xi}{x}\right) - H_1^g\left(\frac{\xi}{x}\right) \right] f_{g/p}(\xi) \right\}, \tag{89}$$

where a sum over i is implicit. The hadronic tensor components $W_{1,2}(x,Q^2) = \mathrm{Im}T_{1,2}(x,Q^2)/\pi$ and therefore are determined by the imaginary part of the Wilson coefficients. The Wilson coefficients are dimensionless and therefore can only have $\alpha_s(Q)\ln(\mu/Q)$ dependence on Q. This reproduces the Bjorken scaling of the structure functions.

Finally, consider the tree level matching onto the Wilson coefficients shown in Fig. 3. From these graphs only the quark coefficient functions $C_j^{(i)}$ can be nonzero and we find

$$\mathrm{Im}H_1^{(i)}(z) = -Q_i^2 \pi \delta(z-1), \quad \mathrm{Im}H_2^{(i)} = 0, \tag{90}$$

where Q_i is the charge of parton i. The vanishing of $\mathrm{Im}H_2^{(i)}$ at tree level reproduces the Callan-Gross relation $W_1/W_2 = Q^2/(4x^2)$.

B. Drell-Yan, $p\overline{p} \to l^+ l^- X$

Next we will extend the DIS analysis to the Drell-Yan (DY) process $p\overline{p} \to l^+ l^- X$. Specifically we consider the Q^2 distribution, where Q^2 is the invariant mass of the lepton pair. The Drell-Yan process is more complicated than DIS because one has two hadrons in the initial state. In the center-of-mass frame the incoming proton and anti-proton move in opposite lightlike directions, and to prove factorization we use the fact that collinear modes in different lightlike directions can only couple to each other in external operators in SCET. We take the incoming proton to move in the n^μ direction and the incoming antiproton to move in the $\overline{n}^\mu$ direction. The hard scales in the DY process are Q^2 and the invariant mass of the colliding $p\overline{p}$ pair $s = (p+\overline{p})^2$. The lepton pair has an invariant mass Q^2, and the invariant mass of the final hadronic state is

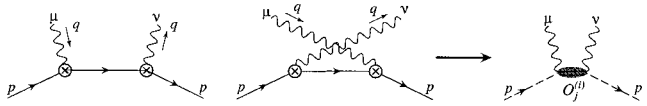

FIG. 3. Tree level matching onto the operator $O_j^{(i)}$ in DIS.

$$p_X^2 = Q^2\left(1 + \frac{1}{\tau} - \frac{1}{x_1} - \frac{1}{x_2}\right), \tag{91}$$

where

$$\tau = \frac{Q^2}{s}, \quad x_1 = \frac{Q^2}{2p\cdot q}, \quad x_2 = \frac{Q^2}{2\overline{p}\cdot q}. \tag{92}$$

We are interested in the kinematic region where $p_X^2 \sim Q^2$, which implies that both x_1 and x_2 are far away from 1. As τ approaches 1 the invariant mass becomes too small for the treatment given here to apply. However, the effective theory can be used to deal with this region as well. It is also possible to study the $q_\perp$ distribution, but this again requires a generalization of the discussion given below.

The spin averaged cross section for the Drell-Yan process is

$$d\sigma = \frac{32\pi^2\alpha^2}{Q^4 s} L_{\mu\nu} W^{\mu\nu} \frac{d^3k_1}{(2\pi)^3(2k_1^0)} \frac{d^3k_2}{(2\pi)^3(2k_2^0)}, \tag{93}$$

where

$$W^{\mu\nu} = \frac{1}{4}\sum_{\text{spins}}\sum_{X}(2\pi)^4\delta^{(4)}(p+\bar{p}-q-p_X) \times\langle p\bar{p}|J^\mu(0)|X\rangle\langle X|J^\nu(0)|p\bar{p}\rangle = \frac{1}{4}\sum_{\text{spins}}\int d^4x e^{-iq\cdot x}\langle p\bar{p}|J^\mu(x)J^\nu(0)|p\bar{p}\rangle. \tag{94}$$

The sum over spins refers to the initial hadron spins (the sum over final hadron spins is included in the sum over X). Integrating Eq. (93) over the emission angles of the final leptons one obtains

$$\frac{d\sigma}{dQ^2} = \frac{2\alpha^2}{3Q^2 s}\frac{1}{4}\sum_{\text{spins}}\langle p\bar{p}|\hat{W}|p\bar{p}\rangle, \tag{95}$$

where we have neglected the lepton masses and defined the operator

$$\hat{W}(\tau,Q^2) = -\int\frac{d^4q}{(2\pi)^3}\theta(q^0)\,\delta(q^2-Q^2)\int d^4x e^{-iq\cdot x}J^\mu(x)J_\mu(0). \tag{96}$$

As we discussed above in the region of phase space under consideration $p_X^2\sim Q^2$, so these hard fluctuations can be integrated out. Operationally this means we match $\hat{W}$ onto local operators in the effective theory. We would like to show that the minimal set of order λ^4 operators that contribute to the Drell-Yan process are

$$\hat{W}\to\frac{1}{Q^2}\int d\omega_i C_{qq}(\omega_i,Q)[\bar{\chi}^{(i)}_{n,\omega_1}\bar{\not{n}}\chi^{(i)}_{n,\omega_2}][\bar{\chi}^{(i)}_{n,\omega_3}\not{n}\chi^{(i)}_{n,\omega_4}] - \frac{1}{Q^3}\int d\omega_i C_{qg}(\omega_i,Q)[\bar{\chi}^{(i)}_{n,\omega_1}\bar{\not{n}}\chi^{(i)}_{n,\omega_2}]\mathrm{Tr}[\mathcal{B}^{\beta}_{\bar{n},\omega_3}\mathcal{B}^{\bar{n},\omega_4}_{\beta}] - \frac{1}{Q^3}\int d\omega_i C_{qg}(\omega_i,Q)\mathrm{Tr}[\mathcal{B}^{\nu}_{n,\omega_1}\mathcal{B}^{n,\omega_2}_{\nu}][\bar{\chi}^{(i)}_{n,\omega_3}\not{n}\chi^{(i)}_{n,\omega_4}] + \frac{1}{Q^4}\int d\omega_i C_{gg}(\omega_i,Q)\mathrm{Tr}[B^{\nu}_{n,\omega_1}B^{n,\omega_2}_{\nu}]\mathrm{Tr}[\mathcal{B}^{\beta}_{\bar{n},\omega_3}\mathcal{B}^{\bar{n},\omega_4}_{\beta}], \tag{97}$$

where the powers of Q are included to make the coefficients dimensionless. The operators displayed in Eq. (97) are just products of the operators that occurred in DIS, so for these terms the decoupling of soft and usoft gluons occurs in a straightforward manner. To show that the operators in Eq. (97) are the most general set needed we must show that all other operators that are order λ^4 either reduce to these or vanish between the matrix elements in Eq. (95). For instance, λ^4 operators also exist where a $B^{\nu}_{n,\omega}$ field is contracted with a $B^{\mu}_{\bar{n},\omega'}$ field, or the color structures of the operators in Eq. (97) could be arranged in a different way.

We now give a general argument for why we can always rewrite an arbitrary operator in the form of Eq. (97) or show that it does not contribute to the DY process. All operators relevant for the DY process contain four order λ collinear fields chosen from $\xi_{n,p}$, $\xi_{\bar{n},p}$, $B^{\mu}_{n,p}$, or $B^{\mu}_{\bar{n},p}$. Furthemore, two must move in direction n and two in the direction $\bar{n}$ (other possibilities end up vanishing by baryon number conservation or because they involve a set of fields between physical states that cannot possibly form a color singlet operator). For operators with 4 quark fields, Fierz transformations can always be made to arrange the fields such that those in the same direction sit in the same bilinear. Using as an example the operator with two collinear quarks in the n directions and two gluons in the $\bar{n}$ direction and leaving out the soft Wilson lines for the moment, the most general matrix element is

$$\langle p_n\bar{p}_{\bar{n}}|\bar{\chi}^{a,\alpha}_{n,\omega_1}\chi^{b,\beta}_{n,\omega_2}\mathcal{B}^{A,\mu}_{\bar{n},\omega_3}\mathcal{B}^{B,\nu}_{\bar{n},\omega_4}|p_n\bar{p}_{\bar{n}}\rangle\Delta^{ab;AB}_{\mu\nu;\alpha\beta}, \tag{98}$$

where a,b are quark colors, A,B are gluon colors, and α,β are spinor indices for the quarks. $\Delta^{ab;AB}_{\mu\nu;\alpha\beta}$ is some tensor that connects the indices in an arbitrary way. In the contraction of a,b and A,B there are two possible ways to make an overall color singlet, one where both the quarks and gluons are in a color singlet, and another where both the quarks and gluons are in a color octet. We will discuss both of these possibilities in turn.

In the color singlet case, including the soft and ultrasoft Wilson lines is trivial, since using Eqs. (17), (19), and (A6) we see that they cancel due to unitarity/orthogonality of the various Wilson lines in the fundamental or adjoint representations. Thus, there are no soft, usoft, or collinear interactions that connect the n and the $\bar{n}$ fields. As in previous sections, this leads to a factorization of the matrix element in Eq. (98), namely

$$\langle p_n|\bar{\chi}^{(0)a,\alpha}_{n,\omega_1}\chi^{(0)b,\beta}_{n,\omega_2}|p_n\rangle\langle\bar{p}_{\bar{n}}|\mathcal{B}^{(0)A,\mu}_{\bar{n},\omega_3}\mathcal{B}^{(0)B,\nu}_{\bar{n},\omega_4}|\bar{p}_{\bar{n}}\rangle\Delta^{ab;AB}_{\mu\nu;\alpha\beta}. \tag{99}$$

Since the proton spins are summed over, we can write [with the help of Eq. (46)]

$$\langle p_n|\bar{\chi}^{(0)a,\alpha}_{n,\omega_1}\chi^{(0)b,\beta}_{n,\omega_2}|p_n\rangle\propto\delta^{ab}(\not{n})_{\alpha\beta}\langle p_n|\bar{\chi}^{(0)c,\gamma}_{n,\omega_1}\bar{\not{n}}\chi^{(0)c,\gamma}_{n,\omega_2}|p_n\rangle, \tag{100}$$

so that spin and color are summed over in the matrix element. Similarly the antiproton matrix element can be simplified to

$$\langle \overline{p_n}|\mathcal{B}^{(0)A,\mu}_{\overline{n},\omega_3}\mathcal{B}^{(0)B,\nu}_{\overline{n},\omega_4}|\overline{p_n}\rangle \propto \delta^{AB} g_\perp^{\mu\nu}\langle \overline{p_n}|\mathrm{Tr}[\mathcal{B}^{(0)\alpha}_{\overline{n},\omega_3}\mathcal{B}^{(0)\overline{n},\omega_4}_{\alpha}]|\overline{p_n}\rangle. \tag{101}$$

Here we used the fact that the matrix element is symmetric in μ and ν, and that only the perpendicular index μ of the field B^μ is order λ. Using Eqs. (100) and (101) the original matrix element in Eq. (98) can be written as

$$\begin{aligned}\langle p_n\overline{p_n}|\overline{\chi}^{a,\alpha}_{n,\omega_1}\chi^{b,\beta}_{n,\omega_2}\mathcal{B}^{A,\mu}_{\overline{n},\omega_3}\mathcal{B}^{B,\nu}_{\overline{n},\omega_4}|p_n\overline{p_n}\rangle\Delta^{ab;AB}_{\mu\nu;\alpha\beta}&\\ \propto \mathrm{Tr}[\Delta^\mu_\mu \overline{n}\!\!\!/]\langle\overline{p_n}|\mathrm{Tr}[\mathcal{B}^{(0)\nu}_{\overline{n},\omega_3}\mathcal{B}^{(0)\overline{n},\omega_4}_{\nu}]|\overline{p_n}\rangle&\\ \times\langle p_n|\overline{\chi}^{(0)}_{n,\omega_1}\frac{\overline{n}\!\!\!/}{2}\chi^{(0)}_{n,\omega_2}|p_n\rangle,&\end{aligned} \tag{102}$$

where the trace of Δ is over spin and color, and just gives an overall constant. The final result in Eq. (102) is identical to the matrix element of the second operator in Eq. (97).

If each of the n and $\overline{n}$ field bilinears involve color octet structures, then the soft and usoft Wilson lines do not cancel, since they do not commute with the SU(3) generators. However, one can use the color identity

$$Y_n^\dagger S_n^\dagger \mathcal{T}^x S_{\overline{n}} Y_{\overline{n}} \otimes Y_{\overline{n}}^\dagger S_{\overline{n}}^\dagger \mathcal{T}^x S_n Y_n = \mathcal{T}^x \otimes S_n Y_n Y_{\overline{n}}^\dagger S_{\overline{n}}^\dagger \mathcal{T}^x S_n Y_n Y_{\overline{n}}^\dagger S_{\overline{n}}^\dagger \tag{103}$$

where each $\mathcal{T}^x$, S, and Y factor is in the appropriate representation of the color group. Equation (103) moves all the soft and ultrasoft interactions between either the n or the $\overline{n}$ collinear fields. Thus, again the fields in one bilinear cannot be contracted with fields in the other bilinear and the matrix element factors. However this time the factored matrix element vanishes. For the example discussed above,

$$\langle p_n|\overline{\chi}^{(0)\alpha}_{n,\omega} T^C \chi^{(0)\beta}_{n,\omega'}|p_n\rangle = 0, \tag{104}$$

since a color octet operator vanishes between color singlet states. The same holds true for the matrix element of an octet gluon operator.

An identical proof of decoupling goes through for the case of 4 quarks, where we again either have two color singlet or two color octet n and $\overline{n}$ bilinears. With 4 gluon fields we can either have the two n and two $\overline{n}$ fields coupled as singlets, or coupled in the same higher representation (an **8**, {**10**, $\overline{\mathbf{10}}$}, or **27**). In the latter case the matrix element between color singlet states still vanishes so the proof for the 4 gluon operators also goes through in an identical way.

Thus we have shown that the matrix element of an operator with an arbitrary contraction of indices either vanishes or can be written in terms of a product of a matrix element which is related to a proton PDF and a matrix element which is related to an antiproton PDF as in the example in Eq. (102). This is the result we want. To see how the final formulas are derived, note that we can write the matrix element of Eq. (97) in the form of a convolution

$$\begin{aligned}\frac{1}{4}\sum_{\text{spins}}\langle p_n\overline{p_n}|\hat{W}|p_n\overline{p_n}\rangle = \sum_{a,b}\int d\omega_i C_{a,b}(\omega_i)&\\ \times\langle p_n|O_n^a(\omega_+,\omega_-)|p_n\rangle&\\ \times\langle\overline{p_n}|O_n^b(\omega'_+,\omega'_-)|\overline{p_n}\rangle,&\end{aligned} \tag{105}$$

where $\omega_\pm = \omega_1 \pm \omega_2$ and $\omega'_\pm = \omega_3 \pm \omega_4$. The operators here are the same as in DIS, with $a=(i)$ for the quark operator, and $a=g$ for the gluon operator

$$\begin{aligned}O_n^{(i)}(\omega_+,\omega_-) &= \frac{1}{Q}\left[\overline{\chi}^{(i)}_{n,\omega_1}\frac{\overline{n}\!\!\!/}{2}\chi^{(i)}_{n,\omega_2}\right],\\ O_n^g(\omega_+,\omega_-) &= -\frac{1}{Q^2}\mathrm{tr}[B^\mu_{n,\omega_1}B^{n,\omega_2}_\mu].\end{aligned} \tag{106}$$

Apart from the dependence on the labels, the Wilson coefficients in Eq. (105) can also depend on the renormalization point μ and the kinematic variable Q. Using Eq. (38) we see that the matrix elements in Eq. (106) set $\omega_- = \omega'_- = 0$ and $\omega_+ = 2\sqrt{s}z_1$, $\omega'_+ = 2\sqrt{s}z_2$ where z_1 and z_2 are the convolution variables. Since all kinematic variables aside from Q^2 are integrated over in Eq. (96) the only other variable that can appear in the Wilson coefficient is the center of mass energy which produces the l^+l^- pair, namely $4\hat{s} = \omega_+\omega'_+$. Thus, the Wilson coefficients only depend on $\omega_+\omega'_+ = 4sz_1z_2$. Defining new coefficients

$$H^{ab}(z_1z_2) = C^{ab}(\omega_+\omega'_+ = 4sz_1z_2, Q, \mu) \tag{107}$$

we can replace the matrix elements in Eq. (106) with parton distribution functions using Eq. (38) to obtain

$$\begin{aligned}\frac{1}{4}\sum_{\text{spins}}\langle p_n\overline{p_n}|\hat{W}|p_n\overline{p_n}\rangle = \frac{1}{\tau}\int_0^1 dz_1dz_2\Bigg\{&-H^{(i)(j)}(-z_1z_2)[f_{i/p}(z_1)\overline{f}_{j/\overline{p}}(z_2)+\overline{f}_{i/p}(z_1)f_{j/\overline{p}}(z_2)]\\ &+H^{(i)(j)}(z_1z_2)[f_{i/p}(z_1)f_{j/\overline{p}}(z_2)+\overline{f}_{i/p}(z_1)\overline{f}_{j/\overline{p}}(z_2)]+\frac{z_2}{2\sqrt{\tau}}H^{(i)g}(z_1z_2)f_{i/p}(z_1)f_{g/\overline{p}}(z_2)\\ &-\frac{z_2}{2\sqrt{\tau}}H^{(i)g}(-z_1z_2)\overline{f}_{i/p}(z_1)f_{g/\overline{p}}(z_2)+\frac{z_1}{2\sqrt{\tau}}H^{g(j)}(z_1z_2)f_{g/p}(z_1)f_{j/\overline{p}}(z_2)\\ &-\frac{z_1}{2\sqrt{\tau}}H^{g(j)}(-z_1z_2)f_{g/p}(z_1)\overline{f}_{j/\overline{p}}(z_2)+\frac{z_1z_2}{4\tau}H^{gg}(z_1z_2)f_{g/p}(z_1)f_{g/\overline{p}}(z_2)\Bigg\}.\end{aligned} \tag{108}$$

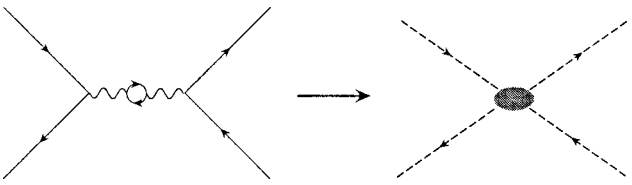

FIG. 4. Tree level matching onto the operators in the Drell-Yan process.

This is the final convolution formulas for the Drell-Yan process and is valid to all orders in α_s and leading order in the power expansion. At tree level the matching calculation shown in Fig. 4 yields zero for all the Wilson coefficients except

$$H^{(i)(i)}(-z_1 z_2) = -\frac{2\pi\tau}{3} Q_i^2 \delta(\tau - z_1 z_2), \qquad (109)$$

where Q_i is the charge of parton i. The coefficients $H^{(i)g}(\pm z_1 z_2)$ and $H^{g(j)}(\pm z_1 z_2)$ start at order $\alpha_s(Q)$, while $H^{(i)(i)}(z_1 z_2)$ and $H^{gg}(z_1 z_2)$ start at order $\alpha_s^2(Q)$.

C. Deeply virtual compton scattering, $\gamma^* p \to \gamma^{(*)} p'$

Next we examine deeply virtual Compton scattering (DVCS). To be more precise we examine the exclusive reaction $\gamma^* p \to \gamma^{(*)} p$, where the incoming photon is highly virtual, the final photon is either off-shell or real, and the incoming and outgoing protons have different momenta. The reason we have included this process in the inclusive section is that DVCS has the remarkable property that the nonperturbative physics is described by a so called non-forward parton distribution function (NFPDF). The NFPDF is a more general distribution function that reduces to the standard parton distribution functions (familiar from DIS) for some values of the momentum fraction, and behaves like a lightcone wave function (familiar from the pion examples) for other values. Deeply virtual Compton scattering was first studied in perturbative QCD in Refs. [36–38], and proofs of factorization to all orders in perturbation theory were later presented in Refs. [39,40]. In addition properties of NFPDFs were studied in Ref. [26]. Here we present a proof of factorization for DVCS based on SCET.

As with the previous proofs it is important to understand the kinematics of the process. We take the incoming proton and photon momenta to be p and q respectively, with $x = Q^2/(2p\cdot q)$ and $q^2 = -Q^2 \gg \Lambda_{\rm QCD}^2$. The outgoing proton and photon momentum are p' and q' respectively, with $0 \geqslant q'^2 \geqslant -Q^2$. It is convenient to define a parameter $\zeta \equiv 1 - \bar{n}\cdot p'/\bar{n}\cdot p$, which measures the change to the proton's large momentum. Working in the Breit frame and neglecting contributions that are $\ll \Lambda_{\rm QCD}^2/Q$ we have

	Label Momenta	Residual Momenta
$q^\mu =$	$\frac{Q}{2}(\bar{n}^\mu - n^\mu)$	$+0$
$p^\mu =$	$\frac{Q}{2x} n^\mu$	$+\frac{x}{2Q} m_p^2 (n^\mu + \bar{n}^\mu)$
$p'^\mu =$	$\frac{Q}{2x}(1-\zeta) n^\mu + p_\perp'^\mu$	$+\frac{x}{2Q}[m_p^2(1-\zeta) n^\mu + \{m_p^2(1+\zeta) - t\}\bar{n}^\mu]$
$q'^\mu =$	$\frac{Q}{2}\left(\frac{\zeta}{x} - 1\right) n^\mu + \frac{Q}{2}\bar{n}^\mu - p_\perp'^\mu$	$+\frac{x}{2Q}[\zeta m_p^2 n^\mu + (t - \zeta m_p^2)\bar{n}^\mu]$

(110)

Here the label momenta are order Q or $Q\lambda$, while the residual momenta are order $Q\lambda^2$ and depend on $p^2 = m_p^2$ and $t = (p'-p)^2 = (p_\perp'^2 - \zeta^2 m_p^2)/(1-\zeta)$, which are both $\sim \Lambda_{\rm QCD}^2$. The invariant mass of the intermediate hadronic state is $(p+q)^2 \approx Q^2(1-x)/x$ just like DIS, so for $1 - x \gg \Lambda_{\rm QCD}/Q$ the intermediate state can be integrated out.

We will proceed in a manner analogous to the analysis for DIS. The amplitude (up to an overall momentum conserving δ-function) is given by a time ordered product of currents:

$$T_{\mu\nu}(p,q,q')=\langle p',\sigma'|\hat{T}_{\mu\nu}(q,q')|p,\sigma\rangle$$

$$\hat{T}_{\mu\nu}(q,q')=i\int d^4z e^{i(q+q')\cdot z/2}\times T[J_\mu(-z/2)J_\nu(z/2)]. \quad (111)$$

This time ordered product is contracted with a lepton tensor to obtain the amplitude. Now current conservation requires $q^\mu T_{\mu\nu}=q'^\nu T_{\mu\nu}=0$, however the DVCS $T_{\mu\nu}$ is not symmetric under $\mu\leftrightarrow\nu$. For electromagnetic currents J_μ we have

$$\begin{aligned}T_{\mu\nu}=&-\left(g_{\mu\nu}-\frac{q'_\mu q_\nu}{q\cdot q'}\right)T_1\\&+\left(p_\mu-\frac{q'_\mu p\cdot q}{q\cdot q'}\right)\left(p_\nu-\frac{q_\nu p\cdot q'}{q\cdot q'}\right)T_2\\&+l_\mu l'_\nu T_3+l_\mu\left(p_\nu-\frac{q_\nu p\cdot q'}{q\cdot q'}\right)T_4\\&+\left(p_\mu-\frac{q'_\mu p\cdot q}{q\cdot q'}\right)l_\nu T_5+\cdots,\end{aligned} \quad (112)$$

where the functions $T_i=T_i(x,\zeta,Q^2,t)$, and the vectors $l_\mu\equiv q'_\mu-q_\mu+p_\mu(q^2-q'^2+t)/(2p\cdot q)$ and $l'_\mu\equiv q_\mu-q'_\mu+p_\mu(q'^2-q^2+t)/(2p\cdot q')$ are defined so that $q\cdot l=q'\cdot l'=0$. In Eq. (112) and below the ellipses denote spin dependent terms. For simplicity we will show how factorization is achieved for the spin independent contributions shown in Eq. (112) with the understanding that it is no more difficult to also include the other terms.

It is convenient to define a parameter $0\leqslant\alpha\leqslant 1$, by $q'^2\equiv-\alpha Q^2$. The DIS hadronic time-ordered product is obtained in the limit $p'\to p$, where $\alpha\to 1$ and $\zeta\to 0$. From Eq. (110) we see that

$$\zeta=x(1-\alpha)+\mathcal{O}\left(\frac{t}{Q^2},\frac{m_p^2}{Q^2}\right), \quad (113)$$

so these parameters are not independent. Since the intermediate hadronic state has invariant mass $\mathcal{O}(Q^2)$ we can match $\hat{T}_{\mu\nu}$ onto operators in SCET. Requiring $q^\mu\hat{T}_{\mu\nu}=0$ and $q'^\nu\hat{T}_{\mu\nu}=0$ for the order Q label momenta leads to

$$\begin{aligned}\hat{T}_{\mu\nu}\to&\frac{g_\perp^{\mu\nu}}{Q}\left(O_1^{(i)}+\frac{O_1^g}{Q}\right)\\&+\frac{1}{Q}(n^\mu+\bar{n}^\mu)(\alpha n^\nu+\bar{n}^\nu)\left(O_2^{(i)}+\frac{O_2^g}{Q}\right)+\cdots,\end{aligned} \quad (114)$$

where the ellipses are spin dependent terms and the displayed operators are

$$O_j^{(i)}=\bar{\xi}^{(i)}_{n,p'}W\frac{\bar{n}\!\!\!/}{2}C_j^{(i)}(\bar{\mathcal{P}}_+,\bar{\mathcal{P}}_-)W^\dagger\xi^{(i)}_{n,p}, \quad (115)$$

$$O_j^g=\bar{n}_\mu\bar{n}_\nu\mathrm{tr}[W^\dagger(G_n)^{\mu\lambda}WC_j^g(\bar{\mathcal{P}}_+,\bar{\mathcal{P}}_-)W^\dagger(G_n)^\nu_\lambda W].$$

We have suppressed the dependence of the Wilson coefficients $C(\bar{\mathcal{P}}_+,\bar{\mathcal{P}}_-)$ on Q, α, and μ. The form of the operators in Eq. (115) looks the same as the DIS operators given in Eq. (84), however the operators here are more general because the Wilson coefficients depend on α. In the limit $\alpha\to 1$ the DVCS operators reduce to the DIS operators. However, since the field structure of the DVCS operators is identical to DIS several results follow immediately. For instance, the steps which factorize soft and usoft gluons and leave fields with superscript (0) are the same and are not repeated here:

$$O_j^{(i)}=\bar{\chi}^{(0)(i)}_{n,\omega}\frac{\bar{n}\!\!\!/}{2}C_j^{(i)}(\bar{\mathcal{P}}_+,\bar{\mathcal{P}}_-)\chi^{(0)(i)}_{n,\omega'}, \quad (116)$$

$$O_j^g=-\mathrm{tr}[B^{(0)\mu}_{n,\omega}C_j^g(\bar{\mathcal{P}}_+,\bar{\mathcal{P}}_-)(B^{(0)}_{n,\omega'})_\mu].$$

The restrictions on the DVCS Wilson coefficients from charge conjugation are the same as in Eq. (87), $C_j(\bar{\mathcal{P}}_+,\bar{\mathcal{P}}_-)=-C_j(-\bar{\mathcal{P}}_+,\bar{\mathcal{P}}_-)$, however because $p\neq p'$ this is not simply a relation between quark and anti-quark Wilson coefficients. The way in which DVCS is unique is that the matrix elements involve nucleon states with different momenta. This is what leads to results in terms of nonforward parton distribution functions.

The definition of the NFPDFs is given in Eq. (43), and can be used along with the relations above to obtain expressions for the T_i in terms of the NFPDFs. Before we give this result we note that the Wilson coefficients depend on the operators $\bar{\mathcal{P}}_\pm$ which become the variables $\omega_\pm$ after introducing trivial convolutions and the χ_{n,ω_i} fields in Eq. (35). The delta functions in Eq. (43) then set $\omega_-=-Q\zeta/x$ and $\omega_+=\pm Q(2\xi-\zeta)/x$, where ξ is the convolution variable. Note that $\zeta/x=1-\alpha$, and just like DIS it is the combination ξ/x which appears. Since charge conjugation relates the Wilson coefficients for $\omega_+>0$ and $\omega_+<0$ it is convenient to define

$$\mathcal{H}_j(\xi/x)\equiv C_j(Q(2\xi/x-1+\alpha),Q(\alpha-1),Q,\alpha,\mu), \quad (117)$$

where in the last three arguments we have made the dependence on Q, α, and μ explicit. Combining Eqs. (43), (116), and (117) then gives

$$T_1 = -\frac{e(\sigma',\sigma)}{2Q}\int_0^1 d\xi\left\{\mathcal{H}_1^{(i)}\left(\frac{\xi}{x}\right)[\mathcal{F}_\zeta^i(\xi;t)+\overline{\mathcal{F}}_\zeta^i(\xi;t)]+\frac{1}{2x}\mathcal{H}_1^g\left(\frac{\xi}{x}\right)\mathcal{F}_\zeta^g(\xi;t)\right\}+\cdots,$$

$$T_2 = \frac{x^2(1+\alpha)}{Q^3}e(\sigma',\sigma)\int_0^1 d\xi\left\{\left[2(1+\alpha)\mathcal{H}_2^{(i)}\left(\frac{\xi}{x}\right)-\mathcal{H}_1^{(i)}\left(\frac{\xi}{x}\right)\right][\mathcal{F}_\zeta^i(\xi;t)+\overline{\mathcal{F}}_\zeta^i(\xi;t)]\right.$$
$$\left.+\frac{1}{2x}\left[2(1+\alpha)\mathcal{H}_2^g\left(\frac{\xi}{x}\right)-\mathcal{H}_1^g\left(\frac{\xi}{x}\right)\right]\mathcal{F}_\zeta^g(\xi;t)\right\}+\cdots, \tag{118}$$

$$T_3=0,\quad T_4=0,\quad T_5=0,$$

which are the final convolution results valid to all orders in α_s and leading order in the power expansion. The structure functions $T_{3,4,5}$ vanish since the vectors $l^\mu = l'^\mu = 0$ at leading order in the power expansion. The terms with ellipses are for the spin flip terms and involve the NFPDF $\mathcal{K}$ defined in Eq. (43). The results for these terms have a similar form to those in Eq. (118).

Finally we match at tree level. The tree level diagram in QCD is the same as in Fig. 3 except the outgoing photon and proton have momenta q' and p' respectively. Only the quark Wilson coefficients are nonzero at tree level. We find

$$C_1^{(i)}(\omega_+,\omega_-,Q,\alpha) = e^2Q_i^2\left(\frac{2Q}{2Q+\omega_+-\omega_-}-\frac{2Q}{2Q-\omega_+-\omega_-}\right)$$
$$C_2^{(i)}(\omega_+,\omega_-,Q,\alpha)=0, \tag{119}$$

which gives

$$\mathcal{H}_1^{(i)}\left(\frac{\xi}{x}\right) = -e^2Q_i^2\left(\frac{1}{1-\xi/x}-\frac{1}{1+(\xi-\zeta)/x}\right)$$
$$\mathcal{H}_2^{(i)}\left(\frac{\xi}{x}\right)=0. \tag{120}$$

Since $\mathcal{H}_2^{(i)}=0$ at tree level, DVCS also obeys a Callan-Gross relation.

V. CONCLUSION

What we hope we have demonstrated here is the power of effective field techniques in the context of factorization for hard scattering processes. The explicit separation of modes and the implementation of gauge invariance for these modes greatly simplifies seemingly complex problems. What is normally accomplished by diagrammatic Ward identities and induction techniques now falls out as a consequence of the gauge symmetry of operators in a low energy soft-collinear effective theory.

As we have emphasized, the factorization formulas derived in this paper are not new. The purpose here was simply to extend the formalism introduced in [9–13] to cases with back-to-back collinear particles, and apply these ideas to more general processes than previously considered. Furthermore, the factorization proofs presented are perhaps simpler than those previously given (certainly they are more concise). We believe that within the confines of the SCET, more difficult, and unresolved problems can be addressed, such as power corrections in cases without an OPE, and proofs of factorization for more complex processes.

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APPENDIX: FACTORIZATION OF SOFT AND COLLINEAR n AND $\bar{n}$ MODES

This appendix discusses the simultaneous factorization of the soft $(\lambda,\lambda,\lambda)$ modes, n-collinear $(\lambda^2,1,\lambda)$ modes, and $\bar{n}$-collinear $(1,\lambda^2,\lambda)$ modes. These three classes of modes cannot interact with each other in a local manner and therefore do not couple through the SCET Lagrangian. However, they can couple in a gauge invariant way through external operators and currents. These interactions in currents are built up by integrating out off-shell fluctuations with $p^2 \gg (Q\lambda)^2$. For the special case of factorization of soft form n-collinear modes this was shown in detail in the Appendix of Ref. [12]. There it was shown that integrating out certain modes with off-shellness $p^2 \sim Q^2\lambda$ causes the Wilson lines W_n and S_n to appear in operators in a gauge invariant way. Here we will extend this approach to the factorization of modes for cases involving two classes of collinear particles. For simplicity we restrict ourselves to the case where the original operators involve only collinear quark or gluon fields. This type of factorization was used for the pion form factor example discussed in Sec. III B and the Drell-Yan process presented in Sec. IV B.

The basic idea is to first match onto a Lagrangian with couplings between on-shell and off-shell modes that give all order λ^0 diagrams. The off-shell modes [with $p^2 \gg (Q\lambda)^2$] can then be integrated out, so that all operators are expressed entirely in terms of the on-shell degrees of freedom. In Table I, a summary is given of the three types of off-shell momenta that are induced by adding soft, n-collinear, and $\bar{n}$-collinear momenta. For each type auxiliary quark and gluon fields are defined, and for convenience momentum labels are sup-

TABLE I. Summary of the on-shell modes discussed in Sec. II A, and the auxiliary fields introduced to represent the off-shell fluctuations that are integrated out in this appendix.

Type		Momenta$(+,-,\perp)$	Fields	Wilson lines
on-shell	collinear-n	$p_1^\mu \sim (\lambda^2,1,\lambda)$	ξ_n, A_n^μ	W_n
	collinear-$\bar{n}$	$p_2^\mu \sim (1,\lambda^2,\lambda)$	$\xi_{\bar{n}}$, $A_{\bar{n}}^\mu$	$W_{\bar{n}}$
	soft	$q^\mu \sim (\lambda,\lambda,\lambda)$	q_s, A_s^μ	S_n, $S_{\bar{n}}$
	usoft	$k^\mu \sim (\lambda^2,\lambda^2,\lambda^2)$	q_{us}, A_{us}^μ	Y_n, $Y_{\bar{n}}$
off-shell	$p=p_1+p_2$	$p^\mu \sim (1,1,\lambda)$	ψ_Q, A_Q^μ	X_n, $X_{\bar{n}}$
	$p=p_1+q$	$p^\mu \sim (\lambda,1,\lambda)$	ψ_n^L, $A_n^{X\mu}$	W_n^X, S_n^X
	$p=p_2+q$	$p^\mu \sim (1,\lambda,\lambda)$	$\psi_{\bar{n}}^L$, $A_{\bar{n}}^{X\mu}$	$W_{\bar{n}}^X$, $S_{\bar{n}}^X$

pressed in this section. For example, the ψ_Q quarks are created by the interaction of an n-collinear quark with an $\bar{n}$-collinear gluon, whereas the ψ_n^L quarks are created when a collinear quark ξ_n is knocked off-shell by a soft gluon. For the field ψ_Q we write $\psi_Q = \psi_n^Q + \psi_{\bar{n}}^Q$, where $\psi_n^Q = \frac{1}{4}\not{n}\not{\bar{n}}\psi_Q$ and $\psi_{\bar{n}}^Q = \frac{1}{4}\not{\bar{n}}\not{n}\psi_Q$. Then we have $\not{n}\psi_n^Q = \not{\bar{n}}\psi_{\bar{n}}^Q = 0$ and $\not{n}\psi_n^L = \not{\bar{n}}\psi_{\bar{n}}^L = 0$. Various Wilson lines are also required and are listed in the table. It is convenient to define a generic Wilson line $L[a,A]$ along direction a with field A by the solution of

$$(a\cdot\mathcal{P} + g a\cdot A) L[a,A] = 0. \tag{A1}$$

With this notation the on-shell Wilson lines are $W_n = L[\bar{n},A_n]$, $W_{\bar{n}} = L[n,A_{\bar{n}}]$, $S_n = L[n,A_s]$, and $S_{\bar{n}} = L[\bar{n},A_s]$. (Recall that the subscripts on W and S mean different things.) The Wilson lines involving off-shell fields that we will require are

$$\begin{aligned} X_n &= L[\bar{n},A_Q + A_n^X + A_n], \quad & X_{\bar{n}} &= L[n,A_Q + A_{\bar{n}}^X + A_{\bar{n}}], \\ W_n^X &= L[\bar{n},A_n^X + A_n], & W_{\bar{n}}^X &= L[n,A_{\bar{n}}^X + A_{\bar{n}}], \\ S_n^X &= L[n,A_n^X + A_s], & S_{\bar{n}}^X &= L[\bar{n},A_{\bar{n}}^X + A_s]. \end{aligned} \tag{A2}$$

Below we discuss the results which allow us to integrate out off-shell fluctuations. The structure of the auxiliary Lagrangians and construction of their solutions are very similar to the case presented in Ref. [12], to which we refer for a more detailed presentation.

From Table I we see that adding n and $\bar{n}$-collinear momenta gives $p^2 \sim Q^2$, whereas adding soft and collinear momenta gives $p^2 \sim Q^2\lambda$. Loops that are dominated by off-shell momenta only modify Wilson coefficients and not the infrared physics. Therefore, to determine the structure of SCET fields in an operator it is sufficient to integrate out the off-shell fields at tree level. For convenience we can integrate out the fluctuations starting with those with the largest off-shellness. Recall that we only wish to consider off-shell propagators connected to external operators. A subtlety for quarks is that distinct auxiliary fields are needed for the incoming and outgoing off-shell propagators. However, the solution for the outgoing field turns out to be the conjugate of the incoming field, so to avoid a proliferation of notation we simply denote the outgoing terms in the Lagrangian by $+$H.c., and present a solution for the incoming fields. Finally, note that for the gluon field A_Q the fields A_n, $A_{\bar{n}}$, A_n^X, $A_{\bar{n}}^X$, and A_s appear as background fields while for the fields A_n^X and $A_{\bar{n}}^X$ it is A_n, $A_{\bar{n}}$, and A_s that appear as background fields.

The terms in the auxiliary Lagrangian involving the $p^2 \sim Q^2$ fields are

$$\begin{aligned} \mathcal{L}_{\rm aux}^Q = {} & \bar{\psi}_n^Q g n\cdot(A_Q + A_{\bar{n}}^X + A_{\bar{n}}) \frac{\not{\bar{n}}}{2}(\psi_n^L + \xi_n) + \bar{\psi}_n^Q [n\cdot\mathcal{P} + g n\cdot(A_Q + A_{\bar{n}}^X + A_{\bar{n}})] \frac{\not{\bar{n}}}{2}\psi_n^Q \\ & + (n\leftrightarrow\bar{n}) + \text{H.c.} + \frac{1}{2g^2}\mathrm{tr}\{[iD_Q^\mu + gA_Q^\mu, iD_Q^\nu + gA_Q^\nu]^2\} + \frac{1}{\alpha_Q}\mathrm{tr}\{[iD_Q^\mu, A_{Q\mu}]^2\}, \end{aligned} \tag{A3}$$

where $iD_Q^\mu = \frac{1}{2}n^\mu[\bar{\mathcal{P}} + g\bar{n}\cdot(A_n^X + A_n)] + \frac{1}{2}\bar{n}^\mu[\mathcal{P} + g n\cdot(A_{\bar{n}}^X + A_{\bar{n}})]$. The solutions of the equations of motion for these modes are

$$\psi_n^Q = (X_{\bar{n}} - 1)(\psi_n^L + \xi_n), \quad \psi_{\bar{n}}^Q = (X_n - 1)(\psi_{\bar{n}}^L + \xi_{\bar{n}}), \quad X_n^\dagger X_n = W_n^X W_{\bar{n}}^{X\dagger}. \tag{A4}$$

(In addition to the last equation a constraint on the components $n\cdot A_Q$ and $\bar{n}\cdot A_Q$ also comes from the gauge fixing term, but will not be needed.) The terms in the auxiliary Lagrangian involving the $p^2 \sim Q^2\lambda$ fields are [12]

$$\mathcal{L}^X_{\rm aux} = \overline{\psi}^L_n g n\cdot(A^X_n + A_s)\frac{\overline{\not n}}{2}\xi_n + \overline{\psi}^L_n[n\cdot\mathcal{P} + g n\cdot(A^X_n + A_s)]\frac{\overline{\not n}}{2}\psi^L_n + (n\leftrightarrow\overline{n}) + \text{H.c.}$$
$$+ \frac{1}{2g^2}\mathrm{tr}\{[iD^\mu_{nX} + gA^{X\mu}_n, iD^\nu_{nX} + gA^{X\nu}_n]^2\} + \frac{1}{\alpha_n}\mathrm{tr}\{[iD^\mu_{nX}, A^X_{n\mu}]^2\} + (n\leftrightarrow\overline{n}), \tag{A5}$$

where $iD^\mu_{nX} = \frac{1}{2}n^\mu[\overline{\mathcal{P}} + g\overline{n}\cdot A_n] + \frac{1}{2}\overline{n}^\mu[n\cdot\mathcal{P} + g n\cdot A_s]$. The solutions for these modes are

$$\psi^L_n = (S^X_n - 1)\xi_n, \quad S^{X\dagger}_n W^X_n = W_n S^\dagger_n,$$
$$\psi^L_{\overline{n}} = (S^X_{\overline{n}} - 1)\xi_{\overline{n}}, \quad S^{X\dagger}_{\overline{n}} W^X_{\overline{n}} = W_{\overline{n}} S^\dagger_{\overline{n}}. \tag{A6}$$

Together Eqs. (A4) and (A6) can be used at leading order to eliminate the fields representing off-shell fluctuations with $p^2 \gg (Q\lambda)^2$. For collinear quarks this leads to the rules in Eq. (17). Note that we did not need to use the gauge fixing term to resolve the ambiguity in the implicit solution for the $\overline{n}\cdot A$ and $n\cdot A$ auxillary fields.

As an illustration of these results, we discuss the soft-collinear factorization for the production of a $q_n\overline{q}_{\overline{n}}$ pair with a large invariant mass Q^2. This process is mediated in the full theory by the electromagnetic current $J = \overline{\psi}\Gamma\psi$ (Γ a color singlet). This current will match onto a current in SCET that is built entirely out of on-shell fields. Using the results in this appendix this current can be systematically derived. To start, the quark field in J matches onto ξ_n plus all possible fields which the auxiliary Lagrangian can create starting from ξ_n, so

$$J \to (\overline{\xi}_n + \overline{\psi}^L_n + \overline{\psi}^Q_n)\Gamma(\xi_{\overline{n}} + \psi^L_{\overline{n}} + \psi^Q_{\overline{n}}). \tag{A7}$$

Integrating out the $p^2 \sim Q^2$ fluctuations with Eq. (A4) and inserting a hard Wilson coefficient C which depends on label operators turns Eq. (A7) into

$$(\overline{\xi}_n + \overline{\psi}^L_n)X^\dagger_n C\Gamma X_n(\xi_{\overline{n}} + \psi^L_{\overline{n}})$$
$$= (\overline{\xi}_n + \overline{\psi}^L_n)W^X_n C\Gamma W^{X\dagger}_n(\xi_{\overline{n}} + \psi^L_{\overline{n}}). \tag{A8}$$

To construct the first operator we used the equations of motion for ψ^Q_n and $\psi^Q_{\overline{n}}$, and in the second operator we used the equation of motion identity for the gluons in X_n and $X_{\overline{n}}$. In a similar fashion we can now integrate out the $p^2 \sim Q^2\lambda$ fluctuations with Eq. (A6) to give

$$\overline{\xi}_n S^{X\dagger}_n W^X_n C\Gamma W^{X\dagger}_n S^X_n \xi_{\overline{n}} = \overline{\xi}_n W_n S^\dagger_n C\Gamma S_{\overline{n}} W^\dagger_{\overline{n}}\xi_{\overline{n}}. \tag{A9}$$

The operator on the right is the final result used in Eq. (24), and is soft, collinear, and usoft gauge invariant. It should be obvious from this example how the equations of motion in Eqs. (A4) and (A6) can be used to determine the factorized form of a general leading order operator.

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