Closed-form summation of renormalization-group-accessible logarithmic contributions to semileptonic B decays and other perturbative processes

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For any perturbative series that is known to k-subleading orders of perturbation theory, we utilize the process-appropriate renormalization-group (RG) equation in order to obtain all-orders summation of series terms proportional to $\alpha^n \log^{n-k}(\mu^2)$ with $k=\{0,1,2,3\}$, corresponding to the summation to all orders of the leading and subsequent-three-subleading logarithmic contributions to the full perturbative series. These methods are applied to the perturbative series for semileptonic b decays in both $\overline{\text{MS}}$ and pole-mass schemes, and they result in RG-summed series for the decay rates which exhibit greatly reduced sensitivity to the renormalization scale μ. Such summation via RG methods of all logarithms accessible from known series terms is also applied to perturbative QCD series for vector- and scalar-current correlation functions, the perturbative static potential function, the (single-doublet standard-model) Higgs decay amplitude into two gluons, as well as the Higgs-mediated high-energy cross section for $W^+W^- \to ZZ$ scattering. The resulting RG-summed expressions are also found to be much less sensitive to the renormalization scale than the original series for these processes.

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I. INTRODUCTION

The renormalization group equation (RGE) has long proven useful as a means of improving and extending results obtained from perturbative quantum field theory. In addition to giving rise to scale-dependent running parameters (coupling constants and masses) and concomitant scale properties (e.g. asymptotic freedom), the RGE can also be utilized to determine scale-dependent portions of higher-order contributions to perturbative expressions. For example, if the two-loop contribution to a physical process has been determined via explicit computation of pertinent Feynman diagrams, the RGE then determines leading-log and next-to-leading contributions to *all* subsequent orders of perturbation theory. We denote such logarithms to be "RG accessible." In the present paper we demonstrate how closed-form summation of such RG-accessible logarithm contributions is obtained for a number of physical processes whose field-theoretical series are known to two or more nonleading orders of perturbation theory.

Consider a perturbative series of the form

$$S[x(\mu),L(\mu)] = \sum_{n=0}^{\infty} x^n S_n[xL] = \sum_{n=0}^{\infty}\sum_{k=0}^{n} T_{n,k} x^n L^k \tag{1.1}$$

occurring within a physical decay rate Γ or measurable cross section σ, where $x(\mu)$ is the running coupling constant [for QCD $x(\mu) \equiv \alpha_s(\mu)/\pi$] and where $L(\mu)$ is a logarithm regulated by the renormalization mass scale μ that may or may not also depend on a running mass:

$$L(\mu) \equiv \log\left(\frac{\mu^2}{m^2}\right). \tag{1.2}$$

If m is a running mass, then

$$\mu^2 \frac{dm}{d\mu^2} = m\gamma_m[x(\mu)] = -m[\gamma_0 x + \gamma_1 x^2 + \gamma_2 x^3 + \cdots]. \tag{1.3}$$

If m is a pole mass (or for scattering processes, a kinematic variable), then γ_m as defined by Eq. (1.3) is zero.

For example, in the modified minimal-subtraction scheme ($\overline{\text{MS}}$) expression for the semileptonic $b \to u l^- \overline{\nu_l}$ rate obtained from five active flavors, m is the running mass $m_b(\mu)$,

$$\Gamma = \frac{G_F^2 |V_{ub}|^2}{192\pi^3}[m_b(\mu)]^5 S[x(\mu),L(\mu)] \tag{1.4}$$

and the successive-order series coefficients within $S[x,L]$, as defined by Eq. (1.1), are [1]

$$T_{0,0}=1,\ T_{1,0}=4.25360,\ T_{1,1}=5,\ T_{2,0}=26.7848,$$
$$T_{2,1}=36.9902,\ T_{2,2}=17.2917. \tag{1.5}$$

The five active-flavor pole-mass expression for the same rate is obtained by replacing $m_b(\mu)$ with the renormalization-scale independent pole mass m_b^{pole} in Eqs. (1.4) and (1.2), as well as a concomitant alteration of the following series coefficients [1]:

$$T_{1,0}=-2.41307,\ \ T_{1,1}=0,\ \ T_{2,0}=-21.2955,$$
$$T_{2,1}=-4.62505,\ \ T_{2,2}=0. \tag{1.6}$$

Suppose for a given scattering or decay process that the series $S[x,L]$ is known to some order of perturbation theory:

$$S^{NL}=T_{0,0}+(T_{1,0}+T_{1,1}L)x \tag{1.7}$$

$$S^{NNL}=S^{NL}+(T_{2,0}+T_{2,1}L+T_{2,2}L^2)x^2 \tag{1.8}$$

$$S^{N^3L}=S^{NNL}+(T_{3,0}+T_{3,1}L+T_{3,2}L^2+T_{3,3}L^3)x^3 \tag{1.9}$$

$$S^{N^4L}=S^{N^3L}+(T_{4,0}+T_{4,1}L+T_{4,2}L^2+T_{4,3}L^3+T_{4,4}L^4)x^4. \tag{1.10}$$

These next-to-leading (NL) and higher-order expressions exhibit scale dependence as the magnitude of L increases. However, higher order polynomial coefficients of L can be determined via an appropriate RGE. For example, in $b\to ul^-\overline{\nu_l}$ the application of the RGE to the known [1] two-loop (NNL) $\overline{\text{MS}}$ expression for the rate is sufficient to determine the three-loop coefficients $T_{3,3}$, $T_{3,2}$, and $T_{3,1}$: for $n_f=5$, $T_{3,3}=50.914$, $T_{3,2}=178.76$, and $T_{3,1}=249.59$ [2]. This procedure is taken a step further in Ref. [3], in which the four loop coefficients $T_{4,4}$, $T_{4,3}$, and $T_{4,2}$ are determined via the RGE for this same process. Estimates for $T_{3,0}$ are also seen to determine $T_{4,1}$, yielding an S^{N^4L} expression characterized by only two unknown coefficients ($T_{3,0}$ and $T_{4,0}$) whose parameter space can be limited by the constraint [3] that successive orders of perturbation theory decrease in magnitude:

$$|S^{N^4L}-S^{N^3L}|\lesssim|S^{N^3L}-S^{NNL}|\lesssim|S^{NNL}-S^{NL}|. \tag{1.11}$$

In the present work, we wish to show how *all* RG-accessible logarithms may be summed if S is known to a given order. Specifically, we shall obtain explicit all-orders summations for the following four series, as defined by the intermediate expression in Eq. (1.1):

$$S_0[x(\mu)L(\mu)]\equiv T_{0,0}+T_{1,1}xL+T_{2,2}x^2L^2+T_{3,3}x^3L^3+\cdots$$
$$=\sum_{n=0}^{\infty}T_{n,n}x^nL^n \tag{1.12}$$

$$S_1[x(\mu)L(\mu)]\equiv T_{1,0}+T_{2,1}xL+T_{3,2}x^2L^2+\cdots$$
$$=\sum_{n=1}^{\infty}T_{n,n-1}(xL)^{n-1} \tag{1.13}$$

$$S_2[x(\mu)L(\mu)]\equiv T_{2,0}+T_{3,1}xL+T_{4,2}x^2L^2+\cdots$$
$$=\sum_{n=2}^{\infty}T_{n,n-2}(xL)^{n-2} \tag{1.14}$$

$$S_3[x(\mu)L(\mu)]\equiv T_{3,0}+T_{4,1}xL+T_{5,2}x^2L^2+\cdots$$
$$=\sum_{n=3}^{\infty}T_{n,n-3}(xL)^{n-3}. \tag{1.15}$$

The appropriate RGE $[\mu^2(\mathrm{d}/\mathrm{d}\mu^2)(\Gamma\ \text{or}\ \sigma)=0]$ is seen to determine all series coefficients of S_n in terms of its leading coefficient $T_{n,0}$, thereby facilitating the construction of *RG-summed* (RGΣ) perturbative expressions to any given order of perturbation theory:

$$S^{NL}_{RG\Sigma}=S_0[xL]+xS_1[xL] \tag{1.16}$$

$$S^{NNL}_{RG\Sigma}=S_0[xL]+xS_1[xL]+x^2S_2[xL] \tag{1.17}$$

$$S^{N^3L}_{RG\Sigma}=S_0[xL]+xS_1[xL]+x^2S_2[xL]+x^3S_3[xL]. \tag{1.18}$$

These RGΣ expressions are seen to exhibit reduced sensitivity to the renormalization scale μ even when the logarithms L are quite large. Compared with the truncated perturbative series, these resummed expressions more effectively implement the underlying idea behind the RGE, namely that the exact (all-orders) expression for any physical quantity is necessarily independent of the scale-parameter μ.

Although RGE determinations of higher-order terms have been known for some time to be of value in extracting divergent parts of bare parameters [4], the principle of incorporating *all* higher-order RG-accessible terms available to a given Feynman-diagram order of perturbation theory was, to the best of our knowledge, first articulated by Maxwell [5] as a method for eliminating unphysical renormalization-scale dependence. The all-orders summation of leading logarithms has been subsequently applied by Maxwell and Mirjalili [6] to moments of QCD leptoproduction structure functions and to NNL-order correlation functions. Such a summation of leading-logarithm contributions to all orders has also been explicitly performed by McKeon to extract one-loop RG functions from the effective actions of ϕ^4-field theory in four dimensions and ϕ^6-field theories in three dimensions [7]. In Sec. II of the present work, we extend McKeon's summation procedure to derive closed-form expressions for all-orders summations of leading (1.12), NL(1.13), NNL(1.14), and N^{3}L logarithms (1.15) by using the RGE appropriate to the perturbative series (1.1) within the QCD expression for the inclusive semileptonic B-decay rate. Such summations enable one to construct RGΣ perturbative expressions inclusive of up to three nonleading logarithmic contributions to all orders of the perturbative series (1.1).

In Sec. III, these results are applied to the $b\to ul^-\overline{\nu_l}$ rate computed to NNL order by van Ritbergen [1], later extended via Padé-approximant methods to a subsequent N^3L prediction [2]. The renormalization-scale dependence of the unsummed perturbative rate truncated to a given order is shown to be much greater than that of the RGΣ rates obtained from the same perturbative expression.

In Sec. IV, RGΣ expressions are obtained for the decays $b\to ul^-\overline{\nu_l}$ and $b\to cl^-\overline{\nu_l}$ in the "pole mass" scheme in which only the couplant $\alpha_s(\mu)/\pi$ exhibits renormalization-scale dependence. This scheme, already known to have difficulties for the $b\to u$ case [1], exhibits a rate which increases with the renormalization scale μ, making the identification of a "correct" or optimal value of μ problematical. However, RG summation is shown effectively to remove such μ dependence, leading to reliable order-by-order pole-mass-scheme predictions for the $b\to u$ semileptonic rate consistent with the $b\to u$ rate obtained from an $\overline{\rm MS}$ scheme inclusive of a running b-quark mass. RGΣ expressions are also obtained for the $b\to c$ semileptonic rate based upon its (approximately) known NNL series [8] and its Padé-estimated N^3L series in the pole-mass scheme [9].

The RGE appropriate for the perturbative series for semileptonic B decays in the pole mass scheme is also the appropriate RGE for the fermionic vector-current correlation function utilized to obtain QCD corrections to the cross-section ratio $\sigma(e^+e^-\to\text{hadrons})/\sigma(e^+e^-\to\mu^+\mu^-)$. In Sec. V we obtain RG-summation expressions for the QCD series embedded within the vector-current correlation function that include all higher-order logarithmic contributions that are accessible from the three fully known nonleading orders of perturbative corrections in the $\overline{\rm MS}$ scheme. We are thus able to compare directly the renormalization-scale dependence of the unsummed series S^{NL} (1.7), S^{NNL} (1.8), and S^{N^3L} (1.9) to their corresponding RGΣ expressions $S^{NL}_{RG\Sigma}$ (1.16), $S^{NNL}_{RG\Sigma}$ (1.17), and $S^{N^3L}_{RG\Sigma}$ (1.18). We find that the latter expressions provide a set of virtually scale-independent order-by-order perturbative predictions for the vector correlator.

In Sec. VI, we show how the use of process-appropriate RGE's can be used to obtain RGΣ perturbative expressions for a number of other processes. We obtain full RG summations for:

(1) the momentum-space series for perturbative contributions to the QCD static-potential function,

(2) the gluonic scalar-current correlation function characterising scalar gluonium states in QCD sum rules,

(3) the (standard-model-) Higgs-mediated cross section $W_L^+W_L^-\to Z_LZ_L$ at high energies, which is characterized by the two physical scale parameters s and M_H,

(4) the decay of a standard-model Higgs boson into two gluons [a process also characterized by two physical (pole-mass) scales (M_H and M_t) in addition to the renormalization scale μ], and

(5) the fermionic scalar-current correlation function that characterises both Higgs $\to b\bar{b}$ decays and scalar-meson-channel QCD sum rules.

We also discuss how RG summation of the two scalar-current correlators considered in Sec. VI removes much of the unphysical μ dependence of the unsummed series at low s that would otherwise percolate through QCD sum-rule integrals sensitive to the low-s region.

In Sec. VII we summarize our paper. We discuss not only the reduction of μ dependence via RG summation, but also the comparison of RGΣ results with those of unsummed series when minimal sensitivity or fastest apparent convergence criteria are used to extract an optimal value for the renormalization scale.

Finally, an alternative all-orders summation procedure to that of Sec. II is presented in the Appendix.

II. RG SUMMATION OF LOGARITHMS FOR SEMILEPTONIC *B* DECAYS

For semileptonic b decays, the μ-sensitive portion of the rate (1.4) must, as a physically measurable quantity, exhibit renormalization scale invariance:

$$\mu^2\frac{d}{d\mu^2}\{[m_b(\mu)]^5 S[x(\mu),L(\mu)]\}=0. \tag{2.1}$$

This constraint is easily seen to lead to the RGE

$$[1-2\gamma_m(x)]\frac{\partial S}{\partial L}+\beta(x)\frac{\partial S}{\partial x}+5\gamma_m S=0, \tag{2.2}$$

where

$$\beta(x)\equiv\mu^2\frac{d}{d\mu^2}x(\mu)=-(\beta_0x^2+\beta_1x^3+\beta_2x^4+\cdots), \tag{2.3}$$

$[x(\mu)\equiv\alpha_s(\mu)/\pi]$ and where the anomalous mass dimension is the series defined by Eq. (1.3). Substitution of the series expansion (1.1) into the RGE yields the following series equation:

$$\begin{aligned}0=&(1+2\gamma_0x+2\gamma_1x^2+2\gamma_2x^3+\cdots)\sum_{n=1}^{\infty}\sum_{k=1}^{n}T_{n,k}kx^nL^{k-1}\\&-(\beta_0x^2+\beta_1x^3+\beta_2x^4+\cdots)\sum_{n=1}^{\infty}\sum_{k=0}^{n}T_{n,k}nx^{n-1}L^k\\&-5(\gamma_0x+\gamma_1x^2+\gamma_2x^3+\cdots)\sum_{n=0}^{\infty}\sum_{k=0}^{n}T_{n,k}x^nL^k.\end{aligned} \tag{2.4}$$

A. Evaluation of S_0

To evaluate $S_0[xL]$, as defined by Eq. (1.12), we use Eq. (2.4) to extract the aggregate coefficient of x^nL^{n-1} and to obtain the recursion formula ($n\geqslant1$)

$$nT_{n,n}-[\beta_0(n-1)+5\gamma_0]T_{n-1,\,n-1}=0. \tag{2.5}$$

We multiply Eq. (2.5) by u^{n-1} and sum from $n=1$ to ∞ to obtain the differential equation,

$$(1-\beta_0u)\frac{dS_0[u]}{du}-5\gamma_0S_0[u]=0, \tag{2.6}$$

where $S_0[u]$ is given by Eq. (1.12) with xL replaced by u. The solution of Eq. (2.6) for the initial condition $S_0[0]=T_{0,0}$ is

$$S_0[u]=T_{0,0}(1-\beta_0u)^{-5\gamma_0/\beta_0}. \tag{2.7}$$

For the special case of pole-mass renormalization schemes $[\gamma_m[x]=0]$, $S_0=T_{0,0}=1$, corresponding to the complete absence of $x^n L^n$ terms from the perturbative series (1.1) when $n\geqslant 1$.

B. Evaluation of S_1

To evaluate $S_1[u]$, as defined by Eq. (1.13) with $u=x(\mu)L(\mu)$ we first extract the aggregate coefficient of $x^n L^{n-2}$ from the RGE (2.4) for $n\geqslant 2$:

$$\begin{aligned}0=&(n-1)T_{n,n-1}+2\gamma_0(n-1)T_{n-1,n-1}\\&-\beta_0(n-1)T_{n-1,n-2}-\beta_1(n-2)T_{n-2,n-2}\\&-5\gamma_0 T_{n-1,n-2}-5\gamma_1 T_{n-2,n-2}.\end{aligned}\tag{2.8}$$

If one multiplies Eq. (2.8) by u^{n-2} and then sums from $n=2$ to infinity, one obtains the differential equation

$$\begin{aligned}(1-\beta_0 u)\frac{dS_1}{du}-(\beta_0+5\gamma_0)S_1[u]\\=5\gamma_1 S_0[u]+(\beta_1 u-2\gamma_0)\frac{dS_0}{du}.\end{aligned}\tag{2.9}$$

We find it convenient to reexpress this equation in terms of the variable

$$w\equiv 1-\beta_0 u\tag{2.10}$$

and the constant

$$A\equiv\frac{5\gamma_0}{\beta_0}.\tag{2.11}$$

We see from Eq. (2.7) that if $T_{0,0}=1$, then

$$S_0=w^{-A}\tag{2.12}$$

and find from Eq. (2.9) the following differential equation for S_1:

$$\frac{dS_1}{dw}+\frac{1+A}{w}S_1=Bw^{-A-1}+Cw^{-A-2}\tag{2.13}$$

where

$$B\equiv(A\beta_1-5\gamma_1)/\beta_0\tag{2.14}$$

$$C\equiv A(2\gamma_0-\beta_1/\beta_0).\tag{2.15}$$

For initial condition $S_1|_{u=0}=S_1|_{w=1}=T_{1,0}$, the solution to Eq. (2.13) is

$$S_1=Bw^{-A}+[T_{1,0}-B+C\log(w)]w^{-A-1}\tag{2.16}$$

with w, A, B and C respectively given by Eqs. (2.10), (2.11), (2.14) and (2.15).

C. Evaluation of S_2

The aggregate coefficient of $x^n L^{n-3}$ in Eq. (2.4) is $(n\geqslant 3)$

$$\begin{aligned}0=&(n-2)T_{n,n-2}+2\gamma_0(n-2)T_{n-1,n-2}\\&+2\gamma_1(n-2)T_{n-2,n-2}-\beta_0(n-1)T_{n-1,n-3}\\&-\beta_1(n-2)T_{n-2,n-3}-\beta_2(n-3)T_{n-3,n-3}\\&-5\gamma_0 T_{n-1,n-3}-5\gamma_1 T_{n-2,n-3}-5\gamma_2 T_{n-3,n-3}.\end{aligned}\tag{2.17}$$

If one multiplies Eq. (2.17) by u^{n-3} and sums from $n=3$ to infinity, one finds from the definitions

$$S_0[u]=1+\sum_{n=1}^{\infty}T_{n,n}u^n,\tag{2.18}$$

$$S_1[u]=\sum_{n=1}^{\infty}T_{n,n-1}u^{n-1},\tag{2.19}$$

$$S_2[u]=\sum_{n=2}^{\infty}T_{n,n-2}u^{n-2}\tag{2.20}$$

[following from Eqs. (1.12)–(1.14)] that

$$\begin{aligned}\frac{dS_2}{du}-&\frac{(2\beta_0+5\gamma_0)}{1-\beta_0 u}S_2\\&=\frac{(\beta_1 u-2\gamma_0)}{1-\beta_0 u}\frac{dS_1}{du}+\frac{(\beta_2 u-2\gamma_1)}{1-\beta_0 u}\frac{dS_0}{du}+\frac{(\beta_1+5\gamma_1)}{1-\beta_0 u}S_1\\&\quad+\frac{5\gamma_2}{1-\beta_0 u}S_0.\end{aligned}\tag{2.21}$$

If we incorporate the change of variable (2.10) in conjunction with the solutions (2.12) and (2.16) for S_0 and S_1, respectively, we find that

$$\begin{aligned}\frac{dS_2}{dw}+\frac{(2+A)}{w}S_2=&Dw^{-A-1}+Ew^{-A-2}+Fw^{-A-2}\log(w)\\&+Gw^{-A-3}+Hw^{-A-3}\log(w),\end{aligned}\tag{2.22}$$

where the constants $\{D,E,F,G,H\}$ are given by

$$D=[\beta_1 AB+\beta_2 A-(\beta_1+5\gamma_1)B-5\gamma_2]/\beta_0\tag{2.23}$$

$$\begin{aligned}E=&\left(2\gamma_0-\frac{\beta_1}{\beta_0}\right)AB+[(1+A)(T_{1,0}-B)-C]\frac{\beta_1}{\beta_0}\\&+\left(2\gamma_1-\frac{\beta_2}{\beta_0}\right)A+(B-T_{1,0})(\beta_1+5\gamma_1)/\beta_0\end{aligned}\tag{2.24}$$

$$F=(A\beta_1-5\gamma_1)C/\beta_0\tag{2.25}$$

$$G=[(1+A)(T_{1,0}-B)-C]\left(2\gamma_0-\frac{\beta_1}{\beta_0}\right)\tag{2.26}$$

$$H=\left(2\gamma_0-\frac{\beta_1}{\beta_0}\right)(1+A)C \tag{2.27}$$

with constants $\{A,B,C\}$ given by Eqs. (2.11), (2.14) and (2.15). The solution to the differential equation (2.21) with initial condition $S_2|_{u=0}=S_2|_{w=1}=T_{2,0}$ is

$$\begin{aligned}S_2=&\frac{D}{2}w^{-A}+(E-F)w^{-A-1}+Fw^{-A-1}\log(w)\\&+\left(T_{2,0}-\frac{D}{2}-E+F\right)w^{-A-2}+Gw^{-A-2}\log(w)\\&+\frac{H}{2}w^{-A-2}\log^2(w).\end{aligned} \tag{2.28}$$

D. Evaluation of S_3

The aggregate coefficient of x^nL^{n-4} in Eq. (2.4) is

$$\begin{aligned}0=&(n-3)[T_{n,n-3}+2\gamma_0T_{n-1,n-3}+2\gamma_1T_{n-2,n-3}\\&+2\gamma_2T_{n-3,n-3}]-\beta_0(n-1)T_{n-1,n-4}-\beta_1(n-2)\\&\times T_{n-2,n-4}-\beta_2(n-3)T_{n-3,n-4}-\beta_3(n-4)T_{n-4,n-4}\\&-5\gamma_0T_{n-1,n-4}-5\gamma_1T_{n-2,n-4}-5\gamma_2T_{n-3,n-4}\\&-5\gamma_3T_{n-4,n-4}.\end{aligned} \tag{2.29}$$

To evaluate the series

$$S_3[u]=\sum_{n=3}^{\infty}T_{n,n-3}u^{n-3}, \tag{2.30}$$

we multiply Eq. (2.29) by u^{n-4}, sum from $n=4$ to infinity, and, as before, make the Eq. (2.10) change of variable $w=1-\beta_0u$. We then find that

$$\begin{aligned}&\frac{dS_3}{dw}+\frac{3+A}{w}S_3\\&\quad=Kw^{-A-1}+Mw^{-A-2}+Nw^{-A-2}\log(w)+Pw^{-A-3}\\&\qquad+Qw^{-A-3}\log(w)+Rw^{-A-3}\log^2(w)+Uw^{-A-4}\\&\qquad+Vw^{-A-4}\log(w)+Yw^{-A-4}\log^2(w),\end{aligned} \tag{2.31}$$

by utilizing the explicit solutions (2.12), (2.16) and (2.28) for $\{S_0,S_1,S_2\}$, as defined by Eqs. (2.18), (2.19) and (2.20). The new constants within Eq. (2.31) are

$$\begin{aligned}K=&\frac{A}{\beta_0}(\beta_3+B\beta_2+D\beta_1/2)-\frac{1}{\beta_0}\Bigg[5\gamma_3+(5\gamma_2+\beta_2)B\\&+(5\gamma_1+2\beta_1)\frac{D}{2}\Bigg]\end{aligned} \tag{2.32}$$

$$\begin{aligned}M=&\left[\left(2\gamma_2-\frac{\beta_3}{\beta_0}\right)+\left(2\gamma_1-\frac{\beta_2}{\beta_0}\right)B+\left(2\gamma_0-\frac{\beta_1}{\beta_0}\right)\frac{D}{2}\right]A\\&+[(5\gamma_2+\beta_2)(B-T_{1,0})+(5\gamma_1+2\beta_1)(F-E)]\frac{1}{\beta_0}\\&+[(T_{1,0}-B)(1+A)-C]\frac{\beta_2}{\beta_0}+[E(1+A)\\&-F(2+A)]\frac{\beta_1}{\beta_0}\end{aligned} \tag{2.33}$$

$$N=\{(A\beta_2-5\gamma_2)C+[(A-1)\beta_1-5\gamma_1]F\}\frac{1}{\beta_0} \tag{2.34}$$

$$\begin{aligned}P=&\left(2\gamma_1-\frac{\beta_2}{\beta_0}\right)[(1+A)(T_{1,0}-B)-C]+\left(2\gamma_0-\frac{\beta_1}{\beta_0}\right)\\&\times[(1+A)E-(2+A)F]-\frac{(5\gamma_1+2\beta_1)}{\beta_0}\\&\times\left(T_{2,0}-\frac{D}{2}-E+F\right)-\frac{\beta_1}{\beta_0}\Bigg[G-(2+A)\\&\times\left(T_{2,0}-\frac{D}{2}-E+F\right)\Bigg]\end{aligned} \tag{2.35}$$

$$\begin{aligned}Q=&\left[\left(2\gamma_1-\frac{\beta_2}{\beta_0}\right)C+\left(2\gamma_0-\frac{\beta_1}{\beta_0}\right)F\right](1+A)\\&-[(5\gamma_1+2\beta_1)G+(H-(2+A)G)\beta_1]\frac{1}{\beta_0}\end{aligned} \tag{2.36}$$

$$R=(\beta_1A-5\gamma_1)\frac{H}{2\beta_0} \tag{2.37}$$

$$U=\left(2\gamma_0-\frac{\beta_1}{\beta_0}\right)\left[(2+A)\left(T_{2,0}-\frac{D}{2}-E+F\right)-G\right] \tag{2.38}$$

$$V=\left(2\gamma_0-\frac{\beta_1}{\beta_0}\right)[(2+A)G-H] \tag{2.39}$$

$$Y=\left(2\gamma_0-\frac{\beta_1}{\beta_0}\right)(2+A)\frac{H}{2}. \tag{2.40}$$

The constants $\{A,B,C,D,E,F,G,H\}$ within Eq. (2.31) are respectively given by Eqs. {(2.11), (2.14), (2.15), (2.23)–(2.27)}. The solution to Eq. (2.31), subject to the initial condition $S_3|_{w=1}=T_{3,0}$, is

$$S_3=\frac{K}{3}w^{-A}+\left(\frac{M}{2}-\frac{N}{4}\right)w^{-A-1}+\frac{N}{2}w^{-A-1}\log(w)$$
$$+(P-Q+2R)w^{-A-2}+(Q-2R)w^{-A-2}\log(w)$$
$$+Rw^{-A-2}\log^2(w)+\left(-\frac{K}{3}-\frac{M}{2}+\frac{N}{4}-P\right.$$
$$\left.+Q-2R+T_{3,0}\right)w^{-A-3}+Uw^{-A-3}\log(w)$$
$$+\frac{V}{2}w^{-A-3}\log^2(w)+\frac{Y}{3}w^{-A-3}\log^3(w), \tag{2.41}$$

where $w=1-\beta_0 u$ as in Eq. (2.10).

III. SEMILEPTONIC $b\rightarrow ul^-\overline{\nu_l}$ DECAY IN THE $\overline{\mathrm{MS}}$ SCHEME

In this section, we consider the semileptonic decay $b\rightarrow ul^-\overline{\nu_l}$ in the $\overline{\mathrm{MS}}$ scheme. The decay rate is given by Eq. (1.4) in terms of the series (1.1). The coefficients in this series are fully known to two loop order and are given by Eq. (1.5). The logarithms $L(\mu)$ within the series are characterized by a running b-quark mass, as given by Eqs. (1.2) and (1.3).

The coefficients (1.5) are listed for five active flavors, appropriate to analysis in an energy region containing $m_b(m_b)$. Consequently, the running b-quark mass $m_b(\mu)$ and the running couplant $x(\mu)=\alpha_s(\mu)/\pi$ should be characterized by $n_f=5$ values for the RG functions $\gamma_m[x]$ and $\beta[x]$:

$$\gamma_0=1,\ \ \gamma_1=\frac{253}{72},\ \ \gamma_2=7.41986,\ \ \gamma_3=11.0343 \tag{3.1}$$

$$\beta_0=\frac{23}{12},\ \ \beta_1=\frac{29}{12},\ \ \beta_2=\frac{9769}{3456},\ \ \beta_3=18.8522. \tag{3.2}$$

Given the computed values of $T_{1,0}$ and $T_{2,0}$ [1], it is straightforward to calculate the NL and NNL RG summations for the series S, as defined in Eqs. (1.16) and (1.17). The constants $\{A,B,C,\ldots,H\}$ that characterize the summations S_0, S_1 and S_2 are obtained via Eqs. (3.1), (3.2) and (1.5) from their definitions in Sec. II:

$$A=\frac{60}{23},\quad B=-\frac{18655}{3174},\quad C=\frac{1020}{529},\quad D=26.4461,$$
$$E=-58.8224,\quad F=-\frac{3171350}{279841},\quad G=25.5973,$$
$$H=\frac{1439220}{279841}. \tag{3.3}$$

Equations (2.12), (2.16) and (2.28) then lead to the following closed-form expressions for the summations S_0, S_1 and S_2:

$$S_0=\left[1-\frac{23}{12}x(\mu)L(\mu)\right]^{-60/23} \tag{3.4}$$

$$S_1=-\frac{18655}{3174}\left[1-\frac{23}{12}x(\mu)L(\mu)\right]^{-60/23}$$
$$+\left\{10.1310+\frac{1020}{529}\log\left[1-\frac{23}{12}x(\mu)L(\mu)\right]\right\}$$
$$\times\left[1-\frac{23}{12}x(\mu)L(\mu)\right]^{-83/23} \tag{3.5}$$

$$S_2=13.2231\left[1-\frac{23}{12}x(\mu)L(\mu)\right]^{-60/23}$$
$$-\left\{47.4897+\frac{3171350}{279841}\log\left[1-\frac{23}{12}x(\mu)L(\mu)\right]\right\}$$
$$\times\left[1-\frac{23}{12}x(\mu)L(\mu)\right]^{-83/23}$$
$$+\left\{61.0515+25.5973\log\left[1-\frac{23}{12}x(\mu)L(\mu)\right]\right.$$
$$\left.+\frac{719610}{279841}\log^2\left[1-\frac{23}{12}x(\mu)L(\mu)\right]\right\}$$
$$\times\left[1-\frac{23}{12}x(\mu)L(\mu)\right]^{-106/23}. \tag{3.6}$$

We first wish to compare the μ dependence of the 2-loop order expression

$$\frac{\Gamma^{NNL}}{\mathcal{K}}=[m_b(\mu)]^5[1+(4.25360+5L(\mu))x(\mu)+(26.7848$$
$$+36.9902L(\mu)+17.2917L^2(\mu))x^2(\mu)] \tag{3.7}$$

for the reduced rate ($\mathcal{K}\equiv G_F^2|V_{ub}|^2/192\pi^3$) to that of the corresponding RG-summed expression

$$\frac{\Gamma^{NNL}_{RG\Sigma}}{\mathcal{K}}=[m_b(\mu)]^5[S_0+S_1x(\mu)+S_2x^2(\mu)] \tag{3.8}$$

with S_0, S_1 and S_2 given by Eqs. (3.4), (3.5) and (3.6). To make this comparison, we evolve the running coupling and mass from initial values $x(4.17\ \mathrm{GeV})=0.0715492$ and $m_b(4.17\ \mathrm{GeV})=4.17$ GeV [2], where the former value arises from $n_f=5$ evolution of the running coupling from an assumed anchoring value $x(M_Z)=0.118000/\pi$ [10], and where the latter value is the $n_f=5$ central value in Ref. [11] for $m_b(m_b)$. Thus $x(\mu)$, $m_b(\mu)$, and $L(\mu)$ are fully determined via Eq. (1.2) and the RG equations (1.3) and (2.3), with γ_m- and β-function coefficients given by Eqs. (3.1) and (3.2).

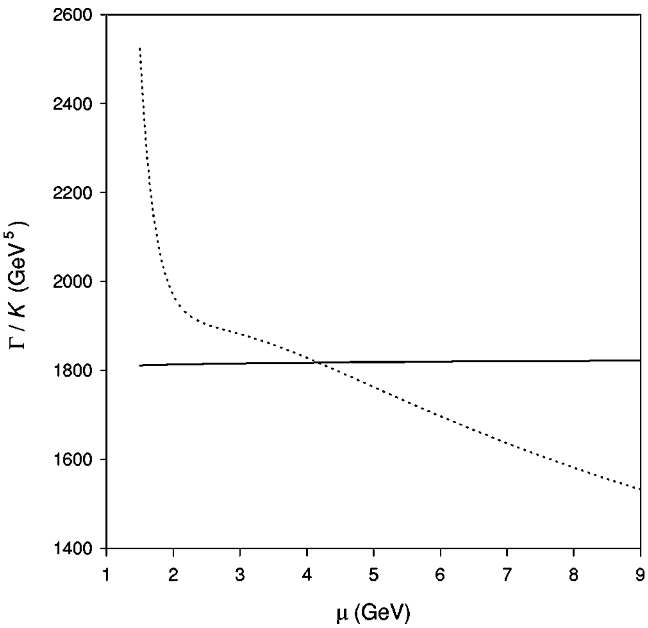

FIG. 1. Comparison of the next-to-next-to-leading- (NNL-) order unsummed (dotted line) and RG-summed (solid line) decay rates $\Gamma/\mathcal{K}$ for $b\to u l^- \overline{\nu_l}$ in the fully $\overline{\mathrm{MS}}$ scheme with five active flavors ($n_f=5$). The quantity $\mathcal{K}\equiv G_F^2|V_{ub}|^2/192\pi^3$.

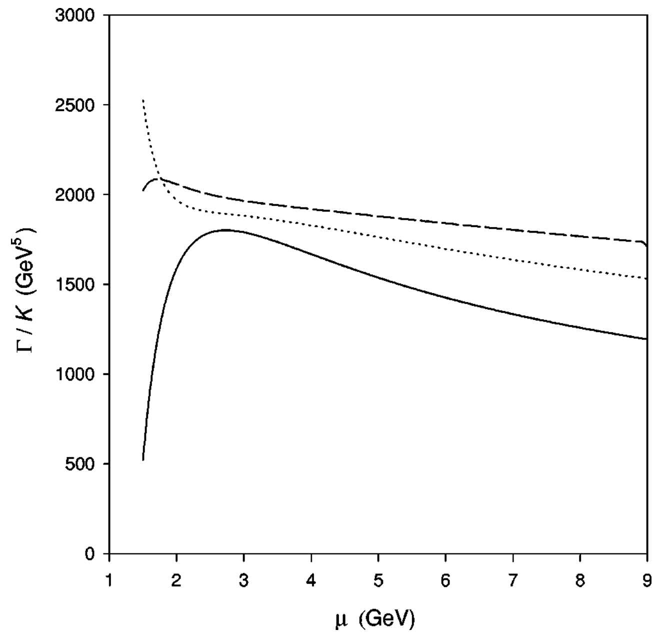

FIG. 2. Comparison of unsummed $b\to u l^- \overline{\nu_l}$ decay rates in the fully $\overline{\mathrm{MS}}$ scheme ($n_f=5$) truncated after NL order (solid line), NNL order (dotted line), and N^{3}L order (dashed line).

In Fig. 1, we use the $n_f=5$ evolution of $x(\mu)$ and $m_b(\mu)$, as described above, to compare Γ^{NNL} (3.7) to $\Gamma^{NNL}_{RG\Sigma}$ (3.8). It is clear from the figure that $\Gamma^{NNL}_{RG\Sigma}$ is almost perfectly flat. By contrast, the naive rate Γ^{NNL} is strikingly dependent on the renormalization scale μ, and does not exhibit any local extremum point of minimal sensitivity. Thus RG summation of leading, next-to-leading and next-to-next-to-leading logarithms is seen to remove the substantial theoretical uncertainty associated with the choice of μ from the (fully known) two-loop order $b\to u l^- \overline{\nu_l}$ rate.

It is useful to examine how the reduced $b\to u l^- \overline{\nu_l}$ rate develops in successive orders of perturbation theory. For example, the one-loop rates

$$\frac{\Gamma^{NL}}{\mathcal{K}}=[m_b(\mu)]^5[1+(4.25360+5L(\mu))x(\mu)] \quad (3.9)$$

$$\frac{\Gamma^{NL}_{RG\Sigma}}{\mathcal{K}}=[m_b(\mu)]^5[S_0+S_1x(\mu)] \quad (3.10)$$

can be compared to the corresponding higher precision results of Eqs. (3.7) and (3.8). Three-loop order (N^3L) reduced rates can be estimated through incorporation of an asymptotic Padé-approximant prediction of the three-loop coefficient $T_{3,0}=206$ [2]. The three-loop order expression for the reduced rate[1]

[1]Only the non-logarithmic three-loop coefficient 206 is estimated; the remaining three logarithmic coefficients in Eq. (3.11) are obtained via RG methods in Ref. [2].

$$\begin{aligned}\frac{\Gamma^{N^3L}}{\mathcal{K}}=&[m_b(\mu)]^5\{1+(4.25360+5L(\mu))x(\mu)\\&+(26.7848+36.9902L(\mu)+17.2917L^2(\mu))x^2(\mu)\\&+(206+249.592L(\mu)+178.755L^2(\mu)\\&+50.9144L^3(\mu))x^3(\mu)\}\end{aligned} \quad (3.11)$$

can then be compared to its RG-summation version

$$\frac{\Gamma^{N^3L}_{RG\Sigma}}{\mathcal{K}}=[m_b(\mu)]^5[S_0+S_1x(\mu)+S_2x^2(\mu)+S_3x^3(\mu)] \quad (3.12)$$

with S_0, S_1 and S_2 respectively given by Eqs. (3.4), (3.5) and (3.6). The RG-summation S_3 is obtained via Eq. (2.41). Given the estimate $T_{3,0}=206$, the known values (3.3) of $\{A,B,\ldots,H\}$ and values of $\{K,M,N,P,\ldots,Y\}$ defined via Eqs. (2.32)–(2.40),

$$\begin{aligned}&K=-14.3686,\quad M=146.729,\quad N=50.9925,\\&P=-317.085,\quad Q=-148.520,\\&R=-15.1138,\quad U=189.048,\quad V=83.3941,\\&Y=8.75961,\end{aligned} \quad (3.13)$$

we find that

$$S_3 = -\frac{4.7895}{\left(1-\frac{23}{12}xL\right)^{60/23}} + \frac{\left[60.617+25.496\log\left(1-\frac{23}{12}xL\right)\right]}{\left(1-\frac{23}{12}xL\right)^{83/23}} + \frac{\left[-198.79-118.29\log\left(1-\frac{23}{12}xL\right)-15.114\log^2\left(1-\frac{23}{12}xL\right)\right]}{\left(1-\frac{23}{12}xL\right)^{106/23}} + \frac{\left[\mathbf{348.96}+189.05\log\left(1-\frac{23}{12}xL\right)+41.697\log^2\left(1-\frac{23}{12}xL\right)+2.9199\log^3\left(1-\frac{23}{12}xL\right)\right]}{\left(1-\frac{23}{12}xL\right)^{129/23}}. \tag{3.14}$$

The bold-face number **348.96** is the only coefficient in the above expression dependent upon the asymptotic Padé-approximant estimate for $T_{3,0}$. We have included this estimate in order to demonstrate RG summation incorporating a three-loop diagrammatic contribution to $T_{3,0}$; when such a calculation is performed, the factor 348.96 in Eq. (3.14) should then be replaced by $T_{3,0}+142.96$.

We consider the μ dependence of three non-leading orders of perturbation theory first for the case in which logarithms are *not* summed to all orders. Figure 2 displays a comparison of the "unsummed" one-, two- and three-loop order reduced rates $\Gamma^{NL}/\mathcal{K}$, $\Gamma^{NNL}/\mathcal{K}$ and $\Gamma^{N^3L}/\mathcal{K}$, respectively given by Eqs. (3.9), (3.7) and (3.11). The μ dependence of all three orders is evident from the figure. Such μ dependence can be used to extract NL and N^3L values for Γ via the minimal-sensitivity criterion of Ref. [12]. Curiously, Γ^{NL} and Γ^{N^3L} are both seen to have comparable minimal-sensitivity extrema (1801 GeV^5 and 2085 GeV^5) at values of μ much less than $m_b(m_b)$. Γ^{NNL} exhibits some flattening between these extrema (≈ 1900 GeV^5) over the same range of μ, but with a continued negative slope. Indeed, one can employ fastest apparent convergence [13] to choose μ for Γ^{NNL} such that $|\Gamma^{NNL}(\mu)-\Gamma^{NL}(\mu)|$ is a minimum, and to choose μ for Γ^{N^3L} such that $|\Gamma^{N^3L}(\mu)-\Gamma^{NNL}(\mu)|=0$. As evident from Fig. 2, the former criterion leads to a value for μ (2.85 GeV) quite close to that value at which $\Gamma^{NL}(\mu)$ has an extremum (2.7 GeV), corresponding to $\Gamma^{NNL}/\mathcal{K}=1888$ GeV^5. The latter criterion indicates that Γ^{N^3L} should be evaluated at the point where Γ^{NNL} and Γ^{N^3L} cross, a point noted previously [2] to be virtually indistinguishable from the minimal-sensitivity extremum for $\Gamma^{N^3L}(\mu)$.

The point we wish to make here, however, is that all such values extracted for μ differ substantially from $m_b(m_b)$, in which case progressively large powers of large logarithms $L(\mu)\equiv\log[\mu^2/m_b^2(\mu)]$ enter the successive expressions (3.9), (3.7) and (3.11) for the NL, NNL and N^3L rate Γ. Moreover, in the absence of minimal-sensitivity or fastest-apparent-convergence criteria for extracting μ, even the N^3L rate exhibits a $\pm 8\%$ spread of values over the range $m_b/2 \lesssim \mu \lesssim 2m_b$.

RG summation eliminates renormalization scale dependence as a cause of theoretical uncertainty. In Fig. 3, we compare RG-summed versions of the reduced rate $\Gamma^{NL}_{RG\Sigma}$ (3.10), $\Gamma^{NNL}_{RG\Sigma}$ (3.8) and $\Gamma^{N^3L}_{RG\Sigma}$ (3.12). These three rates exhibit virtually no μ dependence whatsoever; rather, RG summation is seen to lead to clear order-by-order predictions of the rate that are insensitive to μ. We see from Fig. 3 that $\Gamma^{NL}_{RG\Sigma}/\mathcal{K}=1646\pm 2$ GeV^5, $\Gamma^{NNL}_{RG\Sigma}/\mathcal{K}=1816\pm 6$ GeV^5, and $\Gamma^{N^3L}_{RG\Sigma}/\mathcal{K}=1912\pm 4$ GeV^5 over the (more or less) physical range of μ considered in Fig. 3. Theoretical uncertainty in the calculated rate is now almost entirely attributable to truncation of the perturbation series to known contributions, an error which is seen to diminish as the order of known contributions increases.

It is important to realize, however, that these scale-independent predictions necessarily coincide with the $L(\mu)=0$ predictions of the unsummed rates (3.9), (3.7) and (3.11): Γ^{N^kL} and $\Gamma^{N^kL}_{RG\Sigma}$ equilibrate when $w=1$, i.e., when $L(\mu)=0$. Thus one can argue that the summation we have performed here of all RG-accessible logarithms supports the prescription of identifying as "physical" those perturbative results in which μ-sensitive logarithms are set equal to zero. We must nevertheless recognize the possibility that the μ-sensitivity of the unsummed rates, when exploited by minimal-sensitivity or fastest-apparent-convergence criteria, is capable of leading to more accurate order-by-order estimates of the true rate than corresponding scale-independent RGΣ rates. In comparing Figs. 2 and 3, it is noteworthy that the 1801 GeV^5 extremum of $\Gamma^{NL}/\mathcal{K}$, the *unsummed* one-loop rate, is quite close to $\Gamma^{NNL}_{RG\Sigma}/\mathcal{K}$, the RG-summed *two-loop* rate. Similarly, the 1888 GeV^5 fastest-apparent-convergence value of $\Gamma^{NNL}/\mathcal{K}$, the unsummed two-loop rate, is close to $\Gamma^{N^3L}_{RG\Sigma}$. This train of argument would suggest that the $\mathcal{O}(2085$ GeV$^5)$ extremum (or fastest-apparent-convergence value) of the unsummed rate $\Gamma^{N^3L}/\mathcal{K}$ may be a

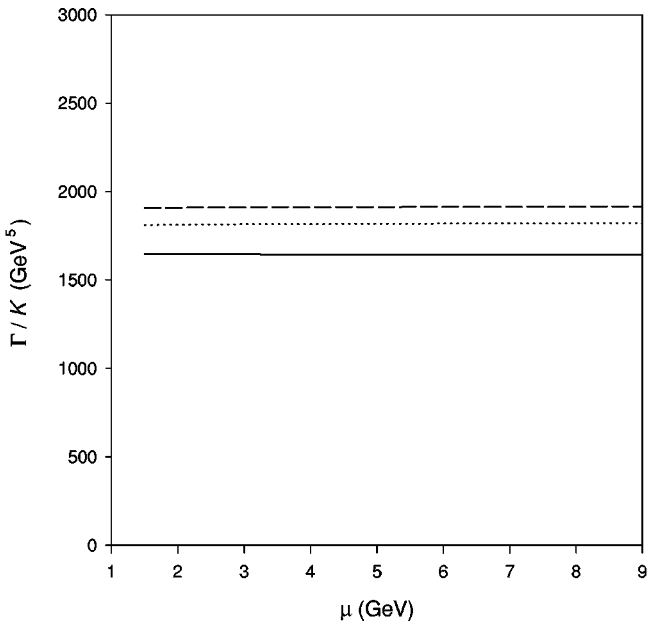

FIG. 3. Comparison of RG-summation expressions for the fully $\overline{\rm MS}$ $b \to u l^- \overline{\nu_l}$ decay rate ($n_f = 5$) obtained from the NL (solid line), NNL (dotted line), and N^{3}L (dashed line) perturbative series.

more accurate estimate of the *true* rate than $\Gamma_{RG\Sigma}^{N^3L}$. Such an argument, however, requires substantiation via explicit three- and four-loop order calculations, computations which are not yet available.

IV. APPLICATION TO SEMILEPTONIC *B* DECAYS IN THE POLE-MASS SCHEME

In the pole-mass renormalization scheme, the mass m appearing in logarithms (1.2) is independent of the renormalization mass scale μ. Thus the coefficients γ_k, as defined in Eq. (1.3) are all zero. The constants $\{A,B,C,D,E,F,H,K,M,N,Q,R,Y\}$, as defined in Sec. II, are all zero as well. The nonzero constants are

$$G = -T_{1,0}\frac{\beta_1}{\beta_0} \tag{4.1}$$

$$P = -T_{1,0}\left(\beta_2 - \frac{\beta_1^2}{\beta_0}\right)\frac{1}{\beta_0} \tag{4.2}$$

$$U = \left[-T_{1,0}\frac{\beta_1}{\beta_0} - 2T_{2,0}\right]\frac{\beta_1}{\beta_0} \tag{4.3}$$

$$V = 2T_{1,0}\frac{\beta_1^2}{\beta_0^2} \tag{4.4}$$

and the corresponding RGE summations appearing within $S_{RG\Sigma}^{NL}, S_{RG\Sigma}^{NNL}$, and $S_{RG\Sigma}^{N^3L}$ [Eqs. (1.16), (1.17) and (1.18)] are

$$S_0 = 1 \tag{4.5}$$

$$S_1[xL] = T_{1,0}/(1-\beta_0 xL) \tag{4.6}$$

$$S_2[xL] = T_{2,0}(1-\beta_0 xL)^{-2} + G(1-\beta_0 xL)^{-2} \times \log(1-\beta_0 xL) \tag{4.7}$$

$$S_3[xL] = P(1-\beta_0 xL)^{-2} + \Big[(T_{3,0}-P) + U\log(1-\beta_0 xL) + \frac{V}{2}\log^2(1-\beta_0 xL)\Big](1-\beta_0 xL)^{-3}. \tag{4.8}$$

In this section, we apply the above results toward the decay $b \to u l^- \overline{\nu_l}$ and $b \to c l^- \overline{\nu_l}$. The former rate is known fully to two-loop order in the pole mass scheme, though the result is argued to be of limited phenomenological utility [1]. The latter rate has been estimated within fairly narrow errors to two-loop order as well [8], and has been extended to a three-loop order estimated rate via asymptotic Padé-approximate methods [9].

A. Pole scheme semileptonic $b \to u l^- \overline{\nu_l}$ decay

The two-loop order $b \to u l^- \overline{\nu_l}$ rate in the pole mass scheme is given by substitution of known values of the coefficients $\{T_{1,0}, T_{1,1}, T_{2,0}, T_{2,1}, T_{2,2}\}$, as listed in Eq. (1.6), into the series (1.1) within the rate (1.4), with $m_b(\mu)$ replaced by m_b^{pole} as noted earlier. Although the reliability of the pole-mass scheme for this process is suspect because of the proximity of a renormalon pole [1], we have plotted this series for a range of $x(\mu) = \alpha_s(\mu)/\pi$ and a choice for m_b^{pole} that will facilitate comparison with phenomenology already obtained from the corresponding $\overline{\rm MS}$ process. We choose $n_f = 5$ active flavors in order to explore the μ dependence of the NNL rate in a region in which μ is considerably larger than m_b^{pole}. Corresponding results for four active flavors are easily obtainable as well. The evolution of $x(\mu) = \alpha_s(\mu)/\pi$ for five active flavors ultimately devolves from $\alpha_s(M_z) = 0.118$ and leads to the same $n_f = 5$ benchmark value $x(4.17\ \text{GeV}) = 0.071549$ as noted in Sec. III. Similarly we employ a value for $m_b^{pole} = 4.7659$ consistent to two-loop order with our use of the running mass value $m_b(m_b) = 4.17$ GeV, as obtained from the $n_f = 5$ relation between m_b^{pole} and $m_b(m_b)$ of Refs. [11,14]:

$$m_b^{pole} = 4.17\ \text{GeV}\left[1 + \frac{4}{3}x(4.17) + 9.27793x^2(4.17)\right]. \tag{4.9}$$

We then see from Fig. 4 that the μ-sensitive portion of the known two-loop rate in the pole mass scheme,

$$S^{NNL}(\mu) = 1 - 2.41307x(\mu) + \left[-21.2955 - 4.62505\log\left\{\left(\frac{\mu}{m_b^{pole}}\right)^2\right\}\right]x^2(\mu), \tag{4.10}$$

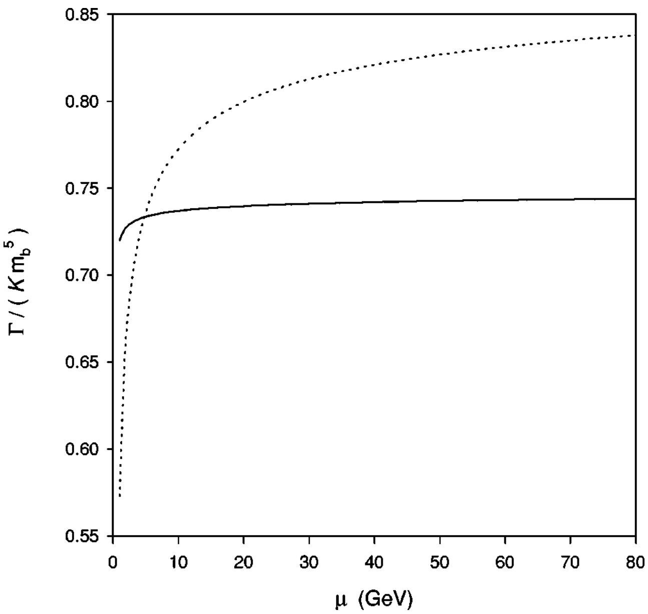

FIG. 4. Comparison of the large-μ behavior of the NNL unsummed (dotted line) and RG-summed (solid line) decay rates for $b \to u l^- \overline{\nu}_l$ within the RG-invariant pole-mass scheme with five active flavors.

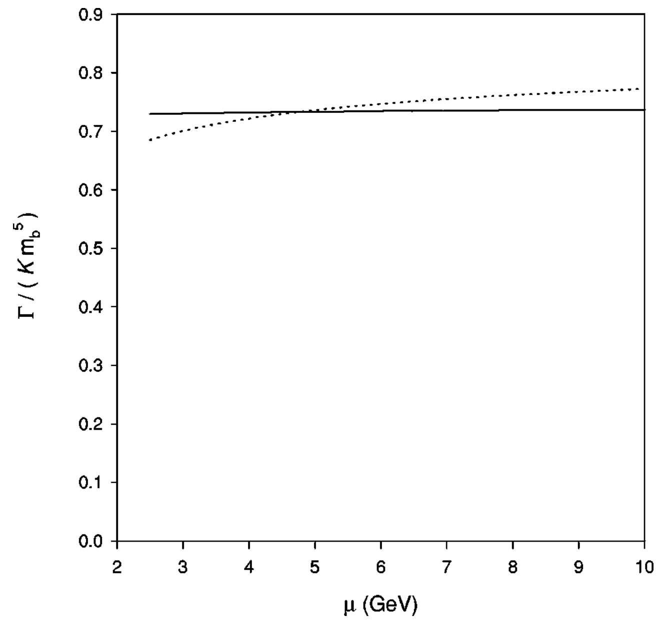

FIG. 5. Comparison of the NNL unsummed (dotted line) and RG-summed (solid line) decay rates for $b \to u l^- \overline{\nu}_l$ ($n_f = 5$) within the pole-mass scheme over the range $m_b^{pole}/2 \lesssim \mu \lesssim 2 m_b^{pole}$.

is indeed highly scale dependent. Specifically, we see that $S^{NNL}(\mu)$ increases monotonically with μ without exhibiting an extremum identifiable with a "physical" point of minimal sensitivity [12].

In Fig. 4 we have also plotted the RG-summed version of the 2-loop rate

$$S_{RG\Sigma}^{NNL} = 1 + x(\mu) S_1 \left[x(\mu) \log\left\{ \left(\frac{\mu}{m_b^{pole}} \right)^2 \right\} \right] + x^2(\mu) S_2 \left[x(\mu) \log\left\{ \left(\frac{\mu}{m_b^{pole}} \right)^2 \right\} \right] \qquad (4.11)$$

with $x(\mu)$ and m_b^{pole} as obtained above. The summations S_1 and S_2 are obtained via Eqs. (4.6) and (4.7) using the $n_f = 5$ pole-mass scheme values $T_{1,0} = -2.41307$, $T_{2,0} = -21.2955$ [1] and $n_f = 5$ QCD β-function coefficients $\beta_0 = 23/12$, $\beta_1 = 29/12$, and $\beta_2 = 9769/3456$. It is evident from the figure that renormalization scale dependence is considerably reduced by the summation of all orders of leading and next-to-leading logarithms in Eq. (3.11). The increase of $S_{RG\Sigma}^{NNL}$ with increasing μ is minimal compared to that of $S^{NNL}(\mu)$, the unsummed expression.

In Fig. 5, the comparison between $(m_b^{pole})^5 S^{NNL}$ and $(m_b^{pole})^5 S_{RG\Sigma}^{NNL}$ is exhibited over the physically relevant region $m_b^{pole}/2 \lesssim \mu \lesssim 2 m_b^{pole}$. The crossing point between the two curves necessarily occurs when $L(\mu) = 0$, corresponding to $\mu = m_b^{pole}$. Since $S_{RG\Sigma}^{NNL}$ is insensitive to μ, this crossover supports the expectation discussed in the previous section that the "physical" NNL rate is S^{NNL} with μ chosen to make all logarithms vanish. We would prefer, however, to argue that $S_{RG\Sigma}^{NNL}$ is an almost scale-independent formulation of the NNL rate, thereby obviating any need to define a physically appropriate value of μ to compute a meaningful two-loop order result. We also note that the asymptotic large-μ result we obtain for the "reduced rate"

$$\frac{\Gamma_{RG\Sigma}^{NNL}}{\mathcal{K}} \cong (m_b^{pole})^5 S_{RG\Sigma}^{NNL} \underset{\mu = M_W}{\longrightarrow} 1829 \ \text{GeV}^5 \qquad (4.12)$$

is surprising close to the 1817 GeV^5 $\overline{\text{MS}}$ two-loop order ("unsummed" NNL) estimate obtained at $\mu = m_b(\mu) = 4.17$ GeV, indicative of the utility of the pole-mass scheme when leading and next-to-leading logarithms are summed to all orders. In the absence of such summation the pole mass expression $S^{NNL}(\mu)$ spans values for the reduced rate between 1420 GeV^5 and 2060 GeV^5 as μ increases from 1 GeV to M_W, reflecting the problems with the pole-mass scheme already noted in Ref. [1]. By contrast, the RG-summed reduced rate varies only from 1774 GeV^5 to 1829 GeV^5 over the same region of μ.

B. Pole scheme semileptonic $b \to c l^- \overline{\nu}_l$ decay

The semileptonic decay of B into a charmed hadronic state is given by the following decay rate in the pole-mass renormalization scheme [8]:

$$\Gamma(b \to c l^- \overline{\nu}_l) = \frac{G_F^2 |V_{cb}|^2}{192\pi^3} F\left(\frac{m_c^2}{m_b^2} \right) m_b^5 S[x(\mu), L(\mu)]. \qquad (4.13)$$

In (4.13), m_b and m_c are μ-invariant pole masses, $L(\mu) \equiv \log(\mu^2/m_b m_c)$, $x(\mu) = \alpha_s(\mu)/\pi$, and $F(r)$ is the form factor

$$F(r)=1-8r-12r^2\log(r)+8r^3-r^4. \tag{4.14}$$

All sensitivity to the renormalization scale μ resides in the series $S[x,L]$, which may be expressed in the usual form

$$S[x,L]=1+(T_{1,0}+T_{1,1}L)x+(T_{2,0}+T_{2,1}L+T_{2,2}L^2)x^2 + (T_{3,0}+T_{3,1}L+T_{3,2}L^2+T_{3,3}L^3)x^3+\cdots. \tag{4.15}$$

For four active flavors, the perturbatively calculated coefficients of (4.15) are $T_{1,0}=-1.67$ and the (partially estimated) coefficient $T_{2,0}=-8.9(\pm 0.3)$ [8,15]. Except for $T_{3,0}$, the remaining coefficients in Eq. (4.15) are accessible from the RG equation

$$0=\left[\frac{\partial}{\partial L}+\beta(x)\frac{\partial}{\partial x}\right]S[x,L]. \tag{4.16}$$

These coefficients are $T_{n,n}=0$ $[n\geqslant 1]$, $T_{2,1}=T_{1,0}\beta_0=-3.479$, $T_{3,1}=2T_{2,0}\beta_0+T_{1,0}\beta_1=-42.4(\pm 1.3)$, and $T_{3,2}=T_{1,0}\beta_0^2=-7.25$ [9]. An asymptotic Padé-approximant estimate of $T_{3,0}=-50.1(\pm 2.6)$ has also been obtained in Ref. [9]. Consequently, one may list three orders for the μ-dependent portion of the $b\to cl^-\overline{\nu_l}$ rate:

$$S^{NL}[x(\mu),L(\mu)]=1-1.67x(\mu), \tag{4.17}$$

$$S^{NNL}[x(\mu),L(\mu)] =1-1.67x(\mu)+[-8.9-3.479L(\mu)]x^2(\mu), \tag{4.18}$$

$$S^{N^3L}[x(\mu),L(\mu)] =S^{NNL}[x(\mu),L(\mu)]+[-50.1-42.4L(\mu) -7.25L^2(\mu)]x^3(\mu), \tag{4.19}$$

with the caveat that NNL and N^{3}L expressions have increasing theoretical uncertainty arising from the (small) estimated error in $T_{2,0}$ and concomitant error in the estimation of $T_{3,0}$.

As before, we will compare the μ dependence of Eqs. (4.17), (4.18) and (4.19) to that of the corresponding RG-summed expressions

$$S_{RG\Sigma}^{NL}=1+x(\mu)S_1[x(\mu)L(\mu)] \tag{4.20}$$

$$S_{RG\Sigma}^{NNL}=1+x(\mu)S_1[x(\mu)L(\mu)]+x^2(\mu)S_2[x(\mu)L(\mu)] \tag{4.21}$$

$$S_{RG\Sigma}^{N^3L}=S_{RG\Sigma}^{NNL}+x^3(\mu)S_3[x(\mu)L(\mu)], \tag{4.22}$$

in order to illustrate how RG summation of higher logarithms affects the order-by-order renormalization-scale dependence of a perturbative series. Figure 6 displays a plot of the μ-sensitive portions of the decay rate considered to NL (4.17), NNL (4.18) and N^{3}L (4.19) orders. As in [8,9], m_c is assumed to be $m_b/3$. We have chosen the pole mass m_b to be 4.9 GeV consistent with phenomenological estimates [16].[2] The couplant $x(\mu)=\alpha_s(\mu)/\pi$ is chosen to devolve from $\alpha_s(m_\tau)=0.33$ [17] via four active flavors, where $\beta_0=25/12$, $\beta_1=77/24$, $\beta_2=21943/3456$ and $\beta_3=31.38745$. These choices permit careful attention to the 1.5 GeV$\lesssim\mu\lesssim m_b$ low-scale region anticipated to correspond to the physical rate, although we have chosen to extend the range of μ to $\sim 2m_b$ in Fig. 6.

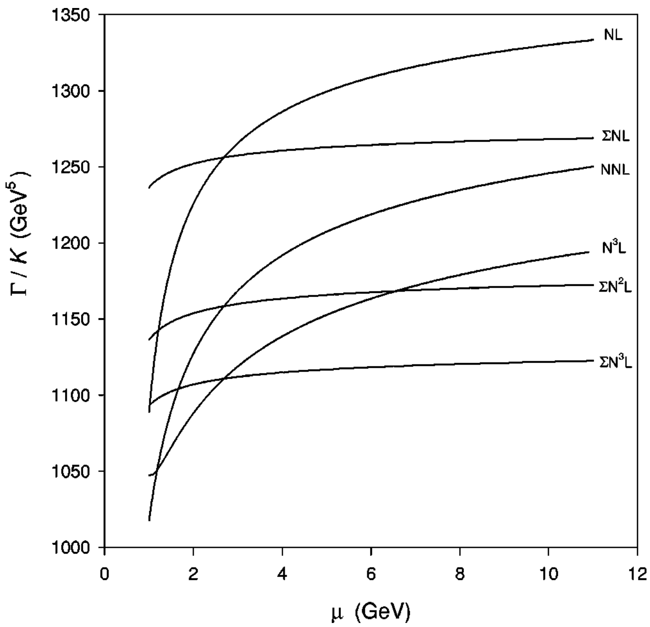

FIG. 6. Comparison of unsummed and RG-summed $b\to cl^-\overline{\nu_l}$ decay rates $\Gamma/\mathcal{K}$ $(n_f=4)$ in the pole-mass scheme, where $\mathcal{K}\equiv G_F^2|V_{cb}|^2/192\pi^3$. The curves representing the unsummed rates are labeled by NL, NNL and N^{3}L indicating the order at which they are truncated. Similiarly, the RG-summed curves are labeled by ΣNL, ΣN^{2}L and ΣN^{3}L.

Figure 6 demonstrates that the rate expressions appear to progressively flatten with the inclusion of higher order corrections, but that the residual scale dependence of each order remains comparable to the difference between successive orders. Figure 6 also displays rates proportional to the corresponding RG-summed expressions (4.20), (4.21) and (4.22) based upon the same phenomenological inputs. The expressions for S_1, S_2 and S_3 are given by Eqs. (4.6), (4.7) and (4.8). It is evident from Fig. 6 that the scale dependence of RG-summed expressions to a given order is dramatically reduced from the scale dependence of the corresponding nonsummed expressions.

Thus $S_{RG\Sigma}^{NL}$, $S_{RG\Sigma}^{NNL}$ and $S_{RG\Sigma}^{N^3L}$ are approximately scale-independent formulations of the one-, two- and three-loop perturbative series within the $b\to cl^-\overline{\nu_l}$ rate; once again the summation of progressively less-than-leading logarithms in the perturbative series is seen effectively to remove the choice of renormalization scale μ as a source of theoretical uncertainty to any given order of perturbation theory.

[2]Such an estimate is slightly larger than that based upon Eq. (4.9), as Eq. (4.9) is truncated after two-loop order.

V. THE VECTOR-CURRENT CORRELATION FUNCTION AND $R(s)$

The imaginary part of the $\overline{\mathrm{MS}}$ vector-current correlation function for massless quarks can be extracted from the Adler function [18]. This procedure is explicitly given in [19] and leads to an expression in the following form:

$$\frac{1}{\pi}\mathrm{Im}\Pi_v(s)=-\frac{4}{3}\sum_q Q_q^2 S[x(\mu),\log(\mu^2/s)], \quad (5.1)$$

where $x(\mu)=\alpha_s(\mu)/\pi$ and s is the kinematic variable p^2 [i.e., the square of the invariant mass in $e^+e^-\to$ hadrons]. The series $S[x,L]$ appearing in Eq. (5.1) is fully known to N^3L order:

$$S^{N^3L}[x,L]=1+x+(T_{2,0}+T_{2,1}L)x^2+(T_{3,0}+T_{3,1}L+T_{3,2}L^2)x^3. \quad (5.2)$$

The full series is, of course, identifiable with the generic series (1.1) [or Eq. (4.15)] provided $T_{0,0}=T_{1,0}=1$ and $T_{n,n}=0[n\geqslant 1]$. Values for the remaining vector correlation function constants in Eq. (5.2) are tabulated in Table I for three, four and five flavors. The coefficients $T_{2,0}$ and $T_{3,0}$ are obtained from the results of Ref. [18], and are well known from the standard expression for perturbative contributions to $R(s)$. The remaining coefficients

$$T_{2,1}=\beta_0, \quad T_{3,1}=2\beta_0 T_{2,0}+\beta_1, \quad T_{3,2}=\beta_0^2 \quad (5.3)$$

are easily determined from the renormalization scale invariance of the vector-current correlation function (5.1),

$$\mu^2\frac{d}{d\mu^2}S[x(\mu),\log(\mu^2/s)]=\left[\frac{\partial}{\partial L}+\beta(x)\frac{\partial}{\partial x}\right]S[x,L]=0. \quad (5.4)$$

This equation, of course, can be interpreted to reflect the imperviousness of the physical quantity

$$R(s)\equiv\frac{\sigma(e^+e^-\to\text{hadrons})}{\sigma(e^+e^-\to\mu^+\mu^-)}=-\frac{3}{4\pi}\mathrm{Im}\Pi_v(s), \quad (5.5)$$

TABLE I. Constants for determining the $\mathcal{O}(N^3L)$ RG summation of the N^3L-order vector-current correlation function.

	$n_f=3$	$n_f=4$	$n_f=5$
β_0	9/4	25/12	23/12
β_1	4	77/24	29/12
β_2	3863/384	21943/3456	9769/3456
$T_{2,0}$	1.63982	1.52453	1.40924
$T_{2,1}$	9/4	25/12	23/12
$T_{3,0}$	−10.2839	−11.6856	−12.8046
$T_{3,1}$	11.3792	9.56054	7.81875
$T_{3,2}$	81/16	625/144	529/144
G	−16/9	−77/50	−29/23
P	$-\frac{3397}{2592}$	$-\frac{121687}{180000}$	$\frac{17521}{152352}$
U	−8.99096	−7.06715	−5.14353
V	512/81	5929/1250	1682/529

to changes in the choice of QCD renormalization-scale μ [20].

Equation (5.4) is just the RGE (2.2) with $\gamma_m(x)$ set equal to zero—precisely the same RG equation as applicable to the pole-mass scheme semileptonic b decay rates considered in the previous section. Consequently, the RG summation of the series $S[x,L]$ within Eq. (5.1) involves the *same* series summations S_0, S_1, S_2 and S_3 as those given by Eqs. (4.5), (4.6), (4.7) and (4.8). The (nonzero) constants G,P,U and V appearing in these equations are found in terms of β-function coefficients β_0,β_1,β_2 and series coefficients $T_{1,0}(=1)$, $T_{2,0}$ and $T_{3,0}$ via Eqs. (4.1)–(4.4). These constants are all tabulated in Table I, and are seen to fully determine the $\mathcal{O}(N^3L)$ RG-summed version of the series $S[x,L]$,

$$S_{RG\Sigma}^{N^3L}[x,L]=1+xS_1[xL]+x^2S_2[xL]+x^3S_3[xL], \quad (5.6)$$

where $x=x(\mu)$ and $L=\log(\mu^2/s)$. For example, if $n_f=5$, we see from Eqs. (4.6), (4.7) and (4.8) and the Table I entries for $G,P,U,V,T_{2,0}$ and $T_{3,0}$ that

$$S_1[xL]=\frac{1}{\left(1-\frac{23}{12}xL\right)} \quad (5.7)$$

$$S_2[xL]=\frac{1.49024-1.26087\log\left(1-\frac{23}{12}xL\right)}{\left(1-\frac{23}{12}xL\right)^2} \quad (5.8)$$

$$S_3[xL]=\frac{0.115003}{\left(1-\frac{23}{12}xL\right)^2}+\frac{\left[-12.9196-5.14353\log\left(1-\frac{23}{12}xL\right)+1.58979\log^2\left(1-\frac{23}{12}xL\right)\right]}{\left(1-\frac{23}{12}xL\right)^3}. \quad (5.9)$$

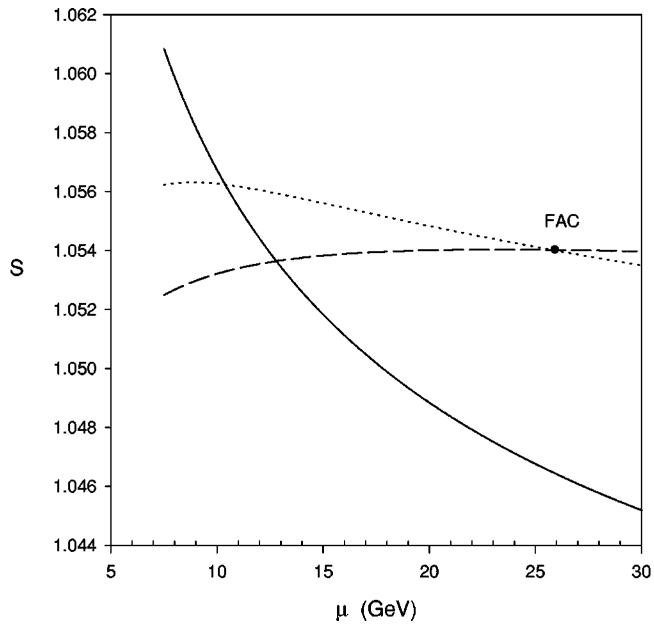

FIG. 7. Comparison of the following $n_f=5$ vector-current correlation-function series when $\sqrt{s}=15$ GeV: S^{NL} (solid line), the $\overline{\mathrm{MS}}$ perturbative series $S[x(\mu),\log(\mu^2/s)]$ truncated after NL-order contributions; S^{NNL} (dotted line), the same series truncated after NNL-order contributions; and S^{N^3L} (dashed line), the same series truncated after N^{3}L-order contributions. At the intersection of S^{NNL} with S^{N^3L}, the N^{3}L-order contribution to the perturbative series is zero, corresponding to the point of fastest apparent convergence (FAC).

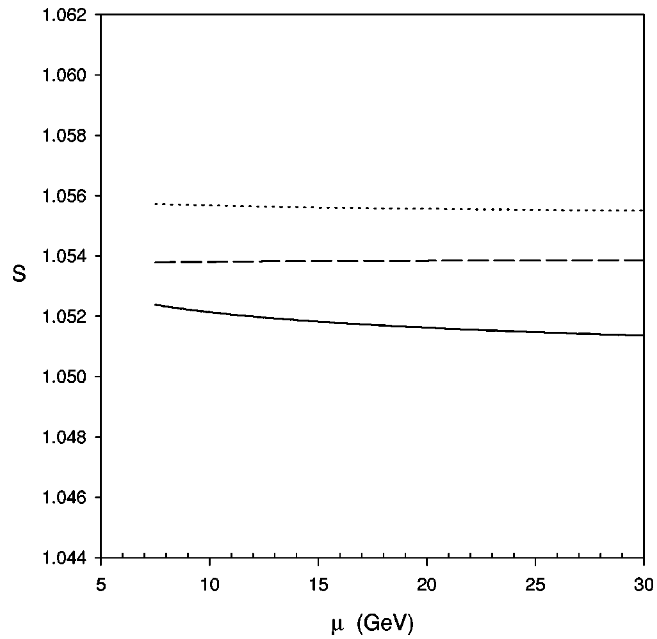

FIG. 8. Comparison of the following RG summations of the perturbative $\overline{\mathrm{MS}}$ series within the $n_f=5$ vector-current correlation function when $\sqrt{s}=15$ GeV: $S^{NL}_{RG\Sigma}$ (solid line), the summation based upon the perturbative series $S[x(\mu),\log(\mu^2/s)]$ truncated after NL-order contributions; $S^{NNL}_{RG\Sigma}$ (dotted line), the summation based upon the perturbative series truncated after NNL-order contributions; and $S^{N^3L}_{RG\Sigma}$ (dashed line), the summation based upon the perturbative series truncated after N^{3}L-order contributions.

In Figs. 7 and 8, we compare the μ dependence of the unsummed (5.2) and summed (5.6) expressions for $S[x(\mu),\log(\mu^2/s)]$, with the choice $s=(15\ \mathrm{GeV})^2$. The running coupling constant $x(\mu)$ is assumed to evolve via the $n_f=5$ (four-loop-order) β-function from an initial value $x(M_z)=0.11800/\pi$.[3] Although Fig. 7 does show a flattening of the unsummed expressions upon incorporation of successively higher orders of perturbation theory $[S^{NL},S^{NNL},S^{N^3L}]$, Fig. 8 demonstrates that the corresponding RG-summed expressions are order-by-order much less dependent on the renormalization scale μ. In particular, the full N^3L summed expression (5.6) exhibits virtually no dependence on μ, but is seen to maintain a constant value $S^{N^3L}_{RG\Sigma}=1.05372\pm0.00004$ over the entire $\sqrt{s}/2\leqslant\mu\leqslant2\sqrt{s}$ range of renormalization scale considered. By contrast the unsummed expression S^{N^3L} of (5.2) is seen to increase (modestly) over this same range from 1.0525 to 1.0540. The point marked FAC in Figure 7 is the intersection of the unsummed expressions for S^{NNL} and S^{N^3L}. This point is the particular choice of μ at which the $\mathcal{O}(x^3)$ contribution to (5.2) vanishes, i.e., the point of fastest apparent convergence (FAC). It is noteworthy that this FAC value for S^{N^3L} is quite close to the RG-summation value $S^{N^3L}_{RG\Sigma}$, a result anticipated by Maxwell [5]. In other words, if one were to use the FAC criterion to reduce the theoretical uncertainty arising from μ dependence in the original expression (5.2) for $S^{N^3L}(\mu)$ over the $\sqrt{s}/2\leqslant\mu\leqslant2\sqrt{s}$ range considered, the specific value one would obtain $[S^{N^3L}_{FAC}=S^{N^3L}(26\ \mathrm{GeV})=1.05402]$ corresponds very nearly to the RG-summation value extracted from Eq. (5.6) $[S^{N^3L}_{RG\Sigma}=1.05372\pm0.00004]$.

This behavior is not peculiar to the choice of s in Figs. 7 and 8. In Figs. 9 and 10, we consider S^{NNL}, S^{N^3L} and $S^{N^3L}_{RG\Sigma}$ for $\sqrt{s}=30$ GeV and $\sqrt{s}=45$ GeV. The FAC point in each figure occurs at the value of μ for which $S^{NNL}(\mu)=S^{N^3L}(\mu)$, as discussed above. In both figures it is evident that S^{N^3L} has substantially more variation with μ than $S^{N^3L}_{RG\Sigma}$, which is virtually independent of μ. Nevertheless, both figures also show that the FAC point of S^{N^3L} is very close to the $S^{N^3L}_{RG\Sigma}$-level value. For $\sqrt{s}=30$ GeV [Fig. 9] $S^{N^3L}_{FAC}=S^{N^3L}(\mu=52\ \mathrm{GeV})=1.04700$ and $S^{N^3L}_{RG\Sigma}=1.04689\pm0.00002$. For $\sqrt{s}=45$ GeV [Fig. 10], $S^{N^3L}_{FAC}=S^{N^3L}(\mu=77\ \mathrm{GeV})=1.04369$, and $S^{N^3L}_{RG\Sigma}=1.04360\pm0.00001$.

Similarly, the PMS points in all three figures, corresponding to maxima of $S^{N^3L}(\mu)$, are very near the FAC points and also quite close to the $S^{N^3L}_{RG\Sigma}$ level. In Fig. 7

[3]For purposes of comparing the μ dependence of S^{N^3L} and $S^{N^3L}_{RG\Sigma}$, we are assuming (as in Secs. III and IV) there to be no uncertainty in the value of $\alpha_s(\mu)$.

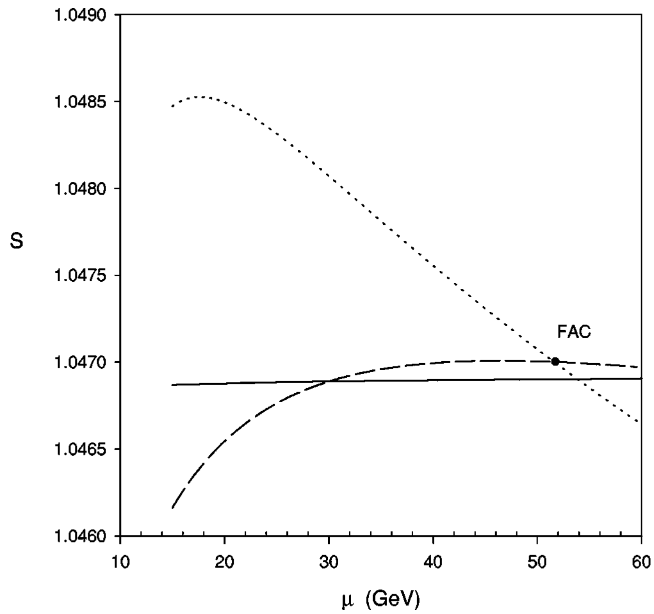

FIG. 9. Comparison of S^{NNL} (dotted line), S^{N^3L} (dashed line), and $S^{N^3L}_{RG\Sigma}$ (solid line) expressions for the series $S[x(\mu),\log(\mu^2/s)]$ within the $n_f=5$ vector-current correlation function when $\sqrt{s}=30$ GeV.

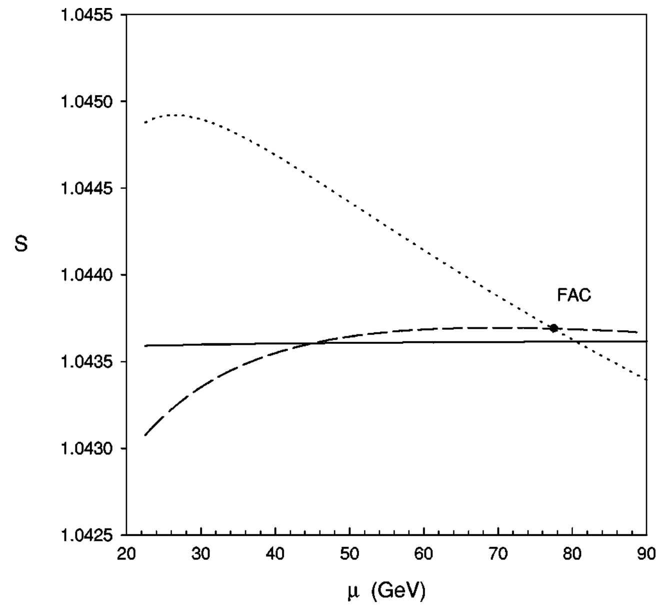

FIG. 10. Comparison of S^{NNL} (dotted line), S^{N^3L} (dashed line), and $S^{N^3L}_{RG\Sigma}$ (solid line) expressions for the series $S[x(\mu),\log(\mu^2/s)]$ within the $n_f=5$ vector-current correlation function when $\sqrt{s}=45$ GeV.

$[\sqrt{s}=15\ \text{GeV}]$, $S^{N^3L}_{PMS}=S^{N^3L}(23\ \text{GeV})=1.05403$; in Fig. 9 $[\sqrt{s}=30\ \text{GeV}]$, $S^{N^3L}_{PMS}=S^{N^3L}(46\ \text{GeV})=1.04701$; and in Fig. 10 $[\sqrt{s}=45\ \text{GeV}]$, $S^{N^3L}_{PMS}=S^{N^3L}(70\ \text{GeV})=1.04370$. Of course, equality between $S^{N^3L}(\mu)$ and $S^{N^3L}_{RG\Sigma}$ is necessarily exact when $L=0$, i.e., when μ^2 is chosen equal to s. This is indeed the prescription employed in the standard [10] prescription relating S^{N^3L} to $R(s)$ [18]:

$$R(s)=1+x(\sqrt{s})+T_{2,0}\ x^2(\sqrt{s})+T_{3,0}\ x^3(\sqrt{s}), \qquad (5.10)$$

where $T_{2,0}$ and $T_{3,0}$ are given in Table I for $n_f=\{3,4,5\}$. The point here, however, is that this prescription is justified *not* by the μ invariance of $S^{N^3L}[x(\mu),\log(\mu^2/s)]$, the truncated perturbative series, but by that of $S^{N^3L}_{RG\Sigma}$, the perturbative series incorporating the closed-form summation of all RG-accessible logarithms within higher-order terms. Moreover, the nearly μ-independent result $S^{N^3L}_{RG\Sigma}$ appears to be quite close to the result one would obtain from $S^{N^3L}[x(\mu),\log(\mu^2/s)]$ either by imposing FAC or PMS criteria to establish an optimal value of μ, as graphically evident from Figs. 9 and 10. That optimal value for μ, however, is *not* $\mu=\sqrt{s}$, but a substantially larger value of μ for each case considered.

These results are not peculiar to the choice $n_f=5$. Using the Table I entries for the parameters $T_{2,0}, T_{3,0}, G, P, U$ and V appearing in Eqs. (4.7) and (4.8), we find for $n_f=3$ that

$$S^{N^3L}_{RG\Sigma}[x,L]=1+\frac{x}{\left(1-\frac{9}{4}xL\right)}+\frac{x^2\left[1.63982-\frac{16}{9}\log\left(1-\frac{9}{4}xL\right)\right]+x^3\left(-\frac{3397}{2592}\right)}{\left(1-\frac{9}{4}xL\right)^2}$$
$$+\frac{x^3\left[-8.97333-8.99096\log\left(1-\frac{9}{4}xL\right)+\frac{256}{81}\log^2\left(1-\frac{9}{4}xL\right)\right]}{\left(1-\frac{9}{4}xL\right)^3} \qquad (5.11)$$

and for $n_f=4$ that

$$S_{RG\Sigma}^{N^3L}[x,L]=1+\frac{x}{\left(1-\frac{25}{12}xL\right)}+\frac{x^2\left[1.52453-\frac{77}{50}\log\left(1-\frac{25}{12}xL\right)\right]+x^3\left(-\frac{121687}{180000}\right)}{\left(1-\frac{25}{12}xL\right)^2}$$
$$+\frac{x^3\left[-11.0096-7.06715\log\left(1-\frac{25}{12}xL\right)+\frac{5929}{2500}\log^2\left(1-\frac{25}{12}xL\right)\right]}{\left(1-\frac{25}{12}xL\right)^3}. \tag{5.12}$$

In Figs. 11 and 12 we display the expression (5.12) for $S_{RG\Sigma}^{N^3L}$ together with (unsummed $n_f=4$ expressions for) S^{N^3L} and S^{NNL} for $\sqrt{s}=4$ GeV and $\sqrt{s}=8$ GeV. To generate these figures, we evolve $x(\mu)$ using an initial condition $x(4.17\ \text{GeV})=0.0716218$ [2] appropriate for $n_f=4$ and obtained from the threshold matching conditions [21] to the $n_f=5$ running couplant $x(4.17\ \text{GeV})=0.0715492$ evolved from $x(M_Z)=0.118/\pi$ [10]. As in Figs. 9 and 10 we see that $S_{PMS}^{N^3L}$ and $S_{FAC}^{N^3L}$ are both close to the very nearly constant $S_{RG\Sigma}^{N^3L}$ level, although the PMS and FAC values for μ are substantially larger than $\sqrt{s}$ in each case.

To conclude, the primary result of interest is that closed-form summation of all RG-accessible logarithms to any given order of perturbation theory leads to expressions (e.g. Fig. 8) that order-by-order are substantially less scale-dependent than the corresponding truncated series (Fig. 7). The scale independence of $S_{RG\Sigma}^{N^3L}$ supports the prescription of choosing $\mu=\sqrt{s}$ in the unsummed series $S^{N^3L}[\mu^2,\log(\mu^2/s)]$, since the summed and unsummed series coincide at this value of μ. However, the unsummed series S^{N^3L} still exhibits noticeable scale dependence. The use of FAC and PMS criteria to find an optimal value for μ for S^{N^3L} leads to values for this unsummed series which are quite close to its RG-summation value.

VI. OTHER PERTURBATIVE APPLICATIONS

A. Momentum space QCD static potential

The momentum space expression for the perturbative portion of the QCD static potential function

$$V(r)=\int\frac{d^3q}{(2\pi)^3}e^{i\vec{q}\cdot\vec{r}}\left(-\frac{16\pi^2}{3\vec{q}^2}\right)W\left[\frac{\alpha_s(\mu)}{\pi},\log\left(\frac{\mu^2}{\vec{q}^2}\right)\right] \tag{6.1}$$

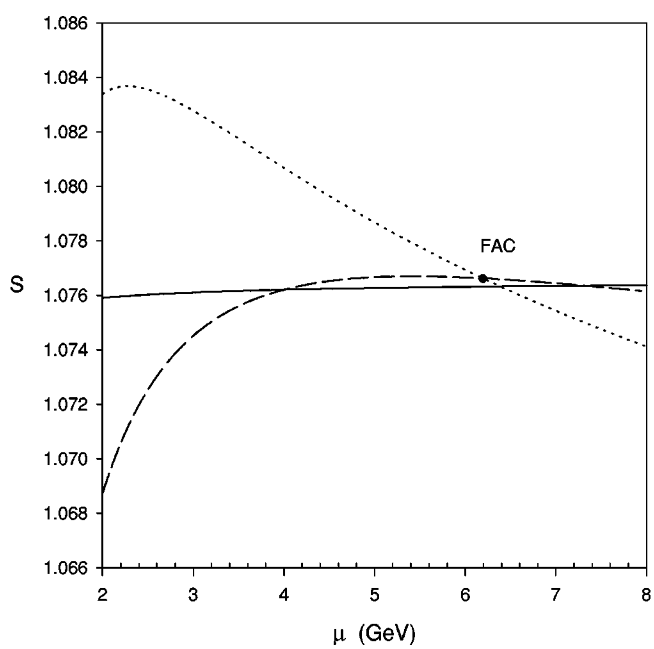

FIG. 11. Comparison of S^{NNL} (dotted line), S^{N^3L} (dashed line), and $S_{RG\Sigma}^{N^3L}$ (solid line) expressions for the series $S[x(\mu),\log(\mu^2/s)]$ within the $n_f=4$ vector-current correlation function when $\sqrt{s}=4$ GeV.

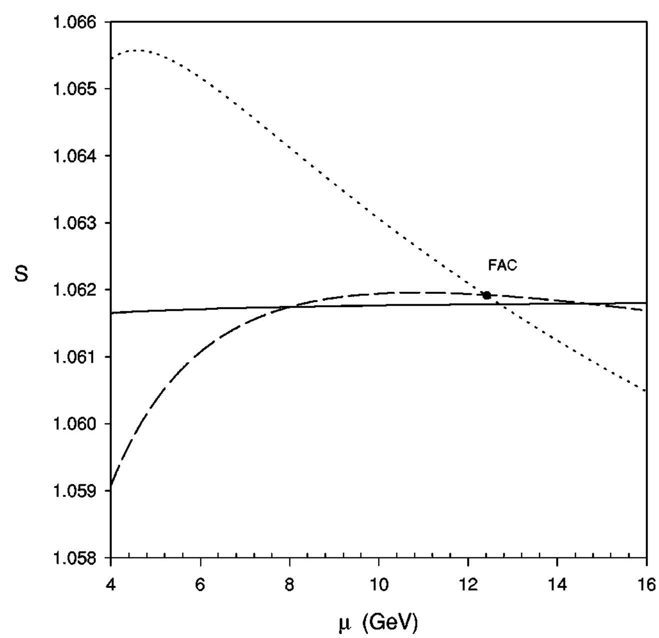

FIG. 12. Comparison of S^{NNL} (dotted line), S^{N^3L} (dashed line), and $S_{RG\Sigma}^{N^3L}$ (solid line) expressions for the series $S[x(\mu),\log(\mu^2/s)]$ within the $n_f=4$ vector-current correlation function when $\sqrt{s}=8$ GeV.

is given by the integrand series

$$W[x,L]=x+(T_{2,0}+T_{2,1}L)x^2 + (T_{3,0}+T_{3,1}L+T_{3,2}L^2)x^3+\cdots \tag{6.2}$$

where $x\equiv\alpha_s(\mu)/\pi$, $L=\log(\mu^2/\vec{q}^{\,2})$, and where the series coefficients within Eq. (6.2) are [22]

$$T_{2,0}=31/12-5n_f/18,\quad T_{2,1}=\beta_0=11/4-n_f/6,$$
$$T_{3,0}=28.5468-4.14714n_f+25n_f^2/324, \tag{6.3}$$
$$T_{3,1}=247/12-229n_f/72+5n_f^2/54,\quad T_{3,2}=\beta_0^2.$$

The function $W[x,L]$ is shown in [23] to satisfy the same RG equation (5.4) as the semileptonic $b\to u$ and $b\to c$ decay rate in the pole mass scheme. Consequently, the closed form summation of all RG-accessible logs is given by Eqs. (4.6)–(4.8), with the constants G,P,U and V as given by Eqs. (4.1)–(4.4). Since $T_{0,0}$, as defined by the generic series form (1.1), is zero for the series (6.2), the series $S_0=\Sigma_{n=0}^{\infty}T_{n,n}(xL)^n$ is trivially zero. Thus, the RG summation of $W[x,L]$ is

$$W_{RG\Sigma}=xS_1[xL]+x^2S_2[xL]+x^3S_3[xL]. \tag{6.4}$$

Noting that $T_{1,0}=1$ in Eq. (6.2), we find for $n_f=\{3,4,5\}$ that

$$W_{RG\Sigma}^{n_f=5}=\frac{x}{\left(1-\frac{23}{12}xL\right)}+\frac{x^2\left[\frac{43}{36}-\frac{29}{23}\log\left(1-\frac{23}{12}xL\right)\right]+x^3\left[\frac{17521}{152352}\right]}{\left(1-\frac{23}{12}xL\right)^2} + \frac{x^3\left[9.62511-\frac{43819}{9522}\log\left(1-\frac{23}{12}xL\right)+\frac{841}{529}\log^2\left(1-\frac{23}{12}xL\right)\right]}{\left(1-\frac{23}{12}xL\right)^3} \tag{6.5}$$

$$W_{RG\Sigma}^{n_f=4}=\frac{x}{\left(1-\frac{25}{12}xL\right)}+\frac{x^2\left[\frac{53}{36}-\frac{77}{50}\log\left(1-\frac{25}{12}xL\right)\right]-x^3\left[\frac{121687}{180000}\right]}{\left(1-\frac{25}{12}xL\right)^2} + \frac{x^3\left[13.8688-\frac{77693}{11250}\log\left(1-\frac{25}{12}xL\right)+\frac{5929}{2500}\log^2\left(1-\frac{25}{12}xL\right)\right]}{\left(1-\frac{25}{12}xL\right)^3} \tag{6.6}$$

$$W_{RG\Sigma}^{n_f=3}=\frac{x}{\left(1-\frac{9}{4}xL\right)}+\frac{x^2\left[\frac{7}{4}-\frac{16}{9}\log\left(1-\frac{9}{4}xL\right)\right]-x^3\left[\frac{3397}{2592}\right]}{\left(1-\frac{9}{4}xL\right)^2} + \frac{x^3\left[18.1104-\frac{760}{81}\log\left(1-\frac{9}{4}xL\right)+\frac{256}{81}\log^2\left(1-\frac{9}{4}xL\right)\right]}{\left(1-\frac{9}{4}xL\right)^3}. \tag{6.7}$$

In Fig. 13 we plot both the truncated series $W^{NNL}[x,L]$, consisting of the terms explicitly appearing in Eq. (6.2), as well as the corresponding RG-summed series (6.5) for the $n_f=5$ case with $|\vec{q}|=4$ GeV. As before, we determine $x(\mu)$ from Eq. (2.3) with the initial condition $\alpha_s(4.17\ \text{GeV})/\pi=0.071549$ devolving from $\alpha_s(M_z)=0.118000$. Although the truncated series varies only modestly $[0.0856\geqslant W^{NNL}\geqslant 0.0795]$ over the range $|\vec{q}|/2<\mu<2|\vec{q}|$, the RG-summed series is seen to exhibit less than 10% of the unsummed series' variation over this same range of μ $[0.08295\geqslant W_{RG\Sigma}\geqslant 0.08233]$.

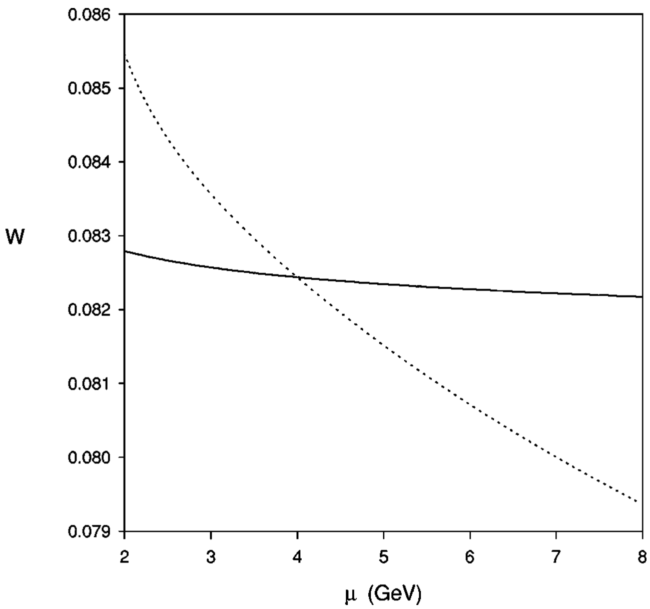

FIG. 13. The momentum-space $n_f=5$ static-potential function series W^{NNL} (dotted line), and the corresponding RG summation $W^{NNL}_{RG\Sigma}$ (solid line) with $|\vec{q}|=4$ GeV.

B. Gluonic scalar correlation function

The imaginary part of the correlator for gluonic scalar currents

$$j_G(y)=\frac{\beta(x(\mu))}{\pi x(\mu)\beta_0}G^a_{\mu\nu}(y)G^{\mu\nu,a}(y) \tag{6.8}$$

enters QCD sum rules pertinent to scalar glueball properties [24], and is given by $[s\equiv p^2, x(\mu)=\alpha_s(\mu)/\pi]$

$$\begin{aligned}\mathrm{Im}\Pi_G(s)&=\mathrm{Im}\left\{i\int d^4y e^{ip\cdot y}\langle 0|Tj_G(y)j_G(0)|0\rangle\right\}\\&=\frac{2x^2(\mu)s^2}{\pi^3}S[x(\mu),\log(\mu^2/s)]\end{aligned} \tag{6.9}$$

where

$$S[x,L]=1+\sum_{n=1}^{\infty}\sum_{m=0}^{n}T_{n,m}x^nL^m. \tag{6.10}$$

The leading coefficients $T_{n,m}$ within Eq. (6.10) can be extracted from a three-loop calculation by Chetyrkin, Kniehl and Steinhauser [19,25] and are tabulated in Table II.

The correlator $\mathrm{Im}\Pi_G(s)$ is RG invariant: $\mu^2 d\mathrm{Im}\Pi_G(s)/d\mu^2=0$. Consequently the series (6.10) can be shown to satisfy the RG equation

$$\left(\frac{\partial}{\partial L}+\beta(x)\frac{\partial}{\partial x}+\frac{2\beta(x)}{x}\right)S[x,L]=0. \tag{6.11}$$

TABLE II. NNL-order series coefficients within the gluonic scalar current correlation function.

	$n_f=2$	$n_f=3$	$n_f=4$	$n_f=5$	$n_f=6$
$T_{1,0}$	$\frac{6919}{348}$	$\frac{659}{36}$	$\frac{4999}{300}$	$\frac{4123}{276}$	$\frac{367}{28}$
$T_{1,1}$	$\frac{29}{6}$	$\frac{9}{2}$	$\frac{25}{6}$	$\frac{23}{6}$	$\frac{7}{2}$
$T_{2,0}$	246.434	197.515	150.210	104.499	60.3685
$T_{2,1}$	$\frac{7379}{48}$	$\frac{2105}{16}$	$\frac{1769}{16}$	$\frac{4355}{48}$	$\frac{1153}{16}$
$T_{2,2}$	$\frac{841}{48}$	$\frac{243}{16}$	$\frac{625}{48}$	$\frac{529}{48}$	$\frac{147}{16}$

Upon substituting Eq. (6.10) into (6.11), it is straightforward to show that the aggregate coefficients of x^nL^{n-1}, x^nL^{n-2} and x^nL^{n-3} respectively vanish provided

$$nT_{n,n}-\beta_0(n+1)T_{n-1,n-1}=0, \tag{6.12}$$

$$(n-1)T_{n,n-1}-\beta_0(n+1)T_{n-1,n-2}-\beta_1 nT_{n-2,n-2}=0, \tag{6.13}$$

$$\begin{aligned}(n-2)T_{n,n-2}&-\beta_0(n+1)T_{n-1,n-3}-\beta_1 nT_{n-2,n-3}\\&-\beta_2(n-1)T_{n-3,n-3}=0.\end{aligned} \tag{6.14}$$

We employ the definitions (2.18), (2.19) and (2.20) for S_0, S_1 and S_2. By multiplying Eq. (6.12) by u^{n-1}, Eq. (6.13) by u^{n-2} and Eq. (6.14) by u^{n-3} and summing from $n=1,2$ and 3, respectively, we obtain the following three linear differential equations

$$(1-\beta_0 u)\frac{dS_0}{du}-2\beta_0 S_0=0 \tag{6.15}$$

$$(1-\beta_0 u)\frac{dS_1}{du}-3\beta_0 S_1=\beta_1\left[u\frac{dS_0}{du}+2S_0\right], \tag{6.16}$$

$$\begin{aligned}(1-\beta_0 u)\frac{dS_2}{du}-4\beta_0 S_2=\beta_1\left[u\frac{dS_1}{du}+3S_1\right]\\+\beta_2\left[u\frac{dS_0}{du}+2S_0\right].\end{aligned} \tag{6.17}$$

Given the $u=0$ initial conditions $S_0=1, S_1=T_{1,0}, S_2=T_{2,0}$ and setting $u=xL$, we obtain the following RG-summed version of the series (6.10) to NNL order:

$$S_{RG\Sigma}^{NNL} = S_0[xL] + xS_1[xL] + x^2 S_2[xL]$$
$$= \frac{1}{(1-\beta_0 xL)^2} + \frac{x\left[T_{1,0} - \frac{2\beta_1}{\beta_0}\log(1-\beta_0 xL)\right] + x^2\left[\frac{2\beta_1^2}{\beta_0^2} - \frac{2\beta_2}{\beta_0}\right]}{(1-\beta_0 xL)^3}$$
$$+ x^2 \frac{\left[\left(T_{2,0} - \frac{2\beta_1^2}{\beta_0^2} + \frac{2\beta_2}{\beta_0}\right) - \left(\frac{3T_{1,0}\beta_1}{\beta_0} + \frac{2\beta_1^2}{\beta_0^2}\right)\log(1-\beta_0 xL) + (3\beta_1^2/\beta_0^2)\log^2(1-\beta_0 xL)\right]}{(1-\beta_0 xL)^4} \tag{6.18}$$

where $x = \alpha_s(\mu)/\pi$ and $L = \log(\mu^2/s)$, as before, and where the β_k are as defined in Eq. (2.3). In Fig. 14 we compare the μ dependence of $x^2(\mu)S_{RG\Sigma}^{NNL}$ to that of the corresponding truncated series

$$x^2(\mu)S^{NNL}[x,L] = x^2[1 + (T_{1,0} + T_{1,1}L)x + (T_{2,0} + T_{2,1}L + T_{2,2}L^2)x^2] \tag{6.19}$$

for the $n_f = 3$ case with $\sqrt{s} = 2$ GeV. The evolution of $x(\mu)$ is assumed to follow an $n_f = 3$ β function with the initial condition $\alpha_s(m_\tau)/\pi = 0.33/\pi$ [17]. As evident from the figure, the severe μ dependence of $x^2 S^{NNL}$ is considerably diminished by RG summation. The RG-summed expression [Eq. (6.18) multiplied by $x^2(\mu)$] falls from 0.056 to 0.041 as μ increases from 1 GeV to 4 GeV. By contrast, the unsummed expression (6.19) falls precipitously from 0.259 to 0.017, a factor of 15, over the same range of μ. This unphysical dependence on renormalization scale suggests that S^{NNL} [as in Eq. (6.19)] be replaced by Eq. (6.18) within sum rule approaches to the lowest-lying scalar gluonium state.

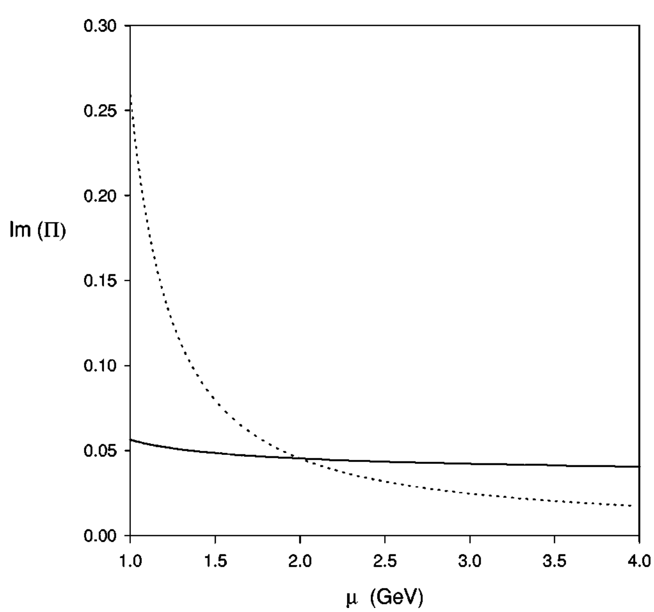

FIG. 14. The imaginary part of the gluonic scalar-current correlation function (6.9), as obtained from $x^2(\mu)S^{NNL}$ (dotted line) and from $x^2(\mu)S_{RG\Sigma}^{NNL}$ (solid line) with $n_f = 3$ and $\sqrt{s} = 2$ GeV.

C. Cross section $\sigma(W_L^+ W_L^- \to Z_L Z_L)$

The scattering of two longitudinal W's into two longitudinal Z's is mediated by the Higgs particle of standard-model electroweak physics. Assuming a single Higgs particle (devolving from the single doublet responsible for electroweak symmetry breaking), one finds the cross section for this process at very high energies ($s \gg M_H^2$) to be

$$\sigma(s) = \frac{8\pi^2}{9s} g^2(\mu) S[g(\mu), \log(\mu^2/s)], \tag{6.20}$$

where $g(\mu) = 6\lambda_{\overline{\rm MS}}(\mu)/16\pi^2$, the quartic scalar couplant of the single-doublet standard model,[4] and where the series S is [26]

$$S[g,L] = 1 + \sum_{n=1}^{\infty}\sum_{m=0}^{n} T_{n,m} g^n L^m \tag{6.21}$$

with $g = g(\mu)$ and $L = \log(\mu^2/s)$. The constants $T_{n,m}$ are fully known to NNL order [26]

$$T_{1,0} = -10.0, \;\; T_{1,1} = -4, \;\; T_{2,0} = 93.553 + \frac{2}{3}\log(s/M_H^2),$$
$$T_{2,1} = 68.667, \; T_{2,2} = 12. \tag{6.22}$$

The RG invariance of the physical cross section σ implies that the series S satisfies the RG equation

$$\left(\frac{\partial}{\partial L} + \beta(g)\frac{\partial}{\partial g} + \frac{2\beta(g)}{g}\right) S[g,L] = 0. \tag{6.23}$$

[4] $\lambda_{\overline{\rm MS}}(\mu)$ is perturbatively related to its on-mass-shell value $G_F M_H^2/\sqrt{2}$, as discussed in [26].

This is the same RG equation as Eq. (6.11) characterizing the gluonic scalar correlator, but with the β function appropriate for the single-Higgs-doublet $\overline{\mathrm{MS}}$ quartic scalar couplant $g(\mu)$ [27]:

$$\mu^2 \frac{dg}{d\mu^2} = \beta(g) \equiv -\beta_0 g^2 - \beta_1 g^3 - \beta_2 g^4 \ldots, \quad (6.24)$$

$$\beta_0 = -2, \;\; \beta_1 = \frac{13}{3}, \;\; \beta_2 = -27.803.$$

Consequently the RG-summed version of S is given by Eq. (6.18):

$$S_{RG\Sigma} = \frac{1}{(1+2gL)^2} + \frac{g\left[-10+\frac{13}{3}\log(1+2gL)\right] - 18.4141 g^2}{(1+2gL)^3} + \frac{g^2\left[111.967 + \frac{2}{3}\log\left(\frac{s}{M_H^2}\right) - \frac{1339}{18}\log(1+2gL) + \frac{169}{12}\log^2(1+2gL)\right]}{(1+2gL)^4}. \quad (6.25)$$

D. Higgs decay $H \to gg$

Higher-order expressions for the decay of a Higgs boson into two gluons have been obtained and studied both outside [28] and within [25,29,30] the context of a standard-model single-doublet Higgs field. In the limit $M_H^2 \ll 4M_t^2$, $M_b = 0$ the latter decay rate is of the form

$$\Gamma = [\sqrt{2} G_F M_H^3 / 72\pi] \times x^2(\mu) S[x(\mu), \log(\mu^2/m_t^2(\mu)), \log(M_H^2/M_t^2)]. \quad (6.26)$$

Capitalized masses (M_H, M_t) denote RG-invariant pole masses, whereas $m_t(\mu)$ is the running t-quark mass. The series S within Eq. (6.26) is of the generic form (1.1) with $L \equiv \log(\mu^2/m_t^2(\mu))$, but the coefficients $T_{n,m}$ are now dependent upon the RG-invariant logarithm $T \equiv \log(M_H^2/M_t^2)$. Using six active flavors to accommodate the running of $m_t(\mu)$, one can extract from Ref. [25] the following two subleading orders of series coefficients within S [29]:

$$T_{0,0} = 1, \;\; T_{1,0}(T) = \frac{215}{12} - \frac{23}{6}T, \;\; T_{1,1} = \frac{7}{2},$$

$$T_{2,0}(T) = 146.8912 - \frac{4903}{48}T + \frac{529}{48}T^2, \quad (6.27)$$

$$T_{2,1}(T) = \frac{1445}{16} - \frac{161}{8}T, \quad T_{2,2} = \frac{147}{16}.$$

RG invariance of the physical decay rate $(\mu^2 d\Gamma/d\mu^2 = 0)$ implies the following RG equation for the series $S[x,L,T]$ within Eq. (6.26):

$$\left[(1-2\gamma_m(x))\frac{\partial}{\partial L} + \beta(x)\frac{\partial}{\partial x} + \frac{2\beta(x)}{x}\right] S[x,L,T] = 0. \quad (6.28)$$

The $n_f = 6$ values for the $\overline{\mathrm{MS}}$ β and γ_m functions are

$$\beta(x) = -(\beta_0 x^2 + \beta_1 x^3 + \beta_2 x^4 + \cdots),$$

$$\beta_0 = 7/4, \quad \beta_1 = 13/8, \quad \beta_2 = -65/128$$

$$\gamma_m(x) = -(\gamma_0 x + \gamma_1 x^2 + \gamma_2 x^3 + \cdots),$$

$$\gamma_0 = 1, \gamma_1 = 27/8, \quad \gamma_2 = 4.83866. \quad (6.29)$$

Upon substituting the series S, as described above, into Eq. (6.28), one finds that the net coefficients of $x^n L^{n-1}, x^n L^{n-2}$ and $x^n L^{n-3}$ on the left-hand side of Eq. (6.28) vanish provided the following recursion relations are respectively upheld:

$$n T_{n,n} - \beta_0 (n+1) T_{n-1,n-1} = 0 \quad (6.30)$$

$$(n-1)T_{n,n-1} + 2\gamma_0(n-1)T_{n-1,n-1} - \beta_0(n+1)T_{n-1,n-2} - \beta_1 n T_{n-2,n-2} = 0 \quad (6.31)$$

$$0 = (n-2)T_{n,n-2} + 2\gamma_0(n-2)T_{n-1,n-2} + 2\gamma_1(n-2)T_{n-2,n-2} - \beta_0(n+1)T_{n-1,n-3} - \beta_1 n T_{n-2,n-3} - \beta_2(n-1)T_{n-3,n-3}. \quad (6.32)$$

By multiplying Eq. (6.30) by u^{n-1}, Eq. (6.31) by u^{n-2} and Eq. (6.32) by u^{n-3}, and by summing each equation from $n = 1,2$ and 3, respectively, we obtain the following linear differential equations for the summations S_0 (2.18), S_1 (2.19) and S_2 (2.20):

$$(1-\beta_0 u)\frac{dS_0}{du}-2\beta_0 S_0=0 \tag{6.33}$$

$$(1-\beta_0 u)\frac{dS_1}{du}-3\beta_0 S_1=\beta_1\left(u\frac{dS_0}{du}+2S_0\right)-2\gamma_0\frac{dS_0}{du} \tag{6.34}$$

$$(1-\beta_0 u)\frac{dS_2}{du}-4\beta_0 S_2=\beta_1\left(u\frac{dS_1}{du}+3S_1\right)-2\gamma_0\frac{dS_1}{du}+\beta_2\left(u\frac{dS_0}{du}+2S_0\right)-2\gamma_1\frac{dS_0}{du}. \tag{6.35}$$

Given the $u=0$ initial conditions $S_0=1, S_1=T_{1,0}(T), S_2=T_{2,0}(T)$, one can solve for $S_0[u], S_1[u]$ and $S_2[u]$. As before, all-orders summation of the RG-accessible logarithms within the series $S[x,L,T]$ is now possible, given the explicit form of $T_{1,0}(T)$ and $T_{2,0}(T)$ in Eq. (6.27) and the explicit β- and γ-function coefficients in Eq. (6.29). We thus find that

$$S_{RG\Sigma}^{NNL}=S_0[xL]+xS_1[xL]+x^2S_2[xL]=\frac{1}{\left(1-\frac{7}{4}xL\right)^2}+\frac{x\left[\frac{215}{12}-\frac{23}{6}T+\frac{15}{7}\log\left(1-\frac{7}{4}xL\right)\right]+\frac{9479}{784}x^2}{\left(1-\frac{7}{4}xL\right)^3}$$
$$+\frac{x^2\left[134.801-\frac{4903}{48}T+\frac{529}{48}T^2+\left(\frac{21675}{392}-\frac{345}{28}T\right)\log\left(1-\frac{7}{4}xL\right)+\frac{675}{196}\log^2\left(1-\frac{7}{4}xL\right)\right]}{\left(1-\frac{7}{4}xL\right)^4}. \tag{6.36}$$

E. Fermionic scalar correlation function

The imaginary part of the RG-invariant correlator for the fermionic scalar current

$$j_s(y)=m\overline{\Psi}(y)\Psi(y) \tag{6.37}$$

is

$$\mathrm{Im}\Pi(s)=\mathrm{Im}\left[i\int d^4y e^{ip\cdot y}\langle 0|Tj_s(y)j_s(0)|0\rangle\right]$$
$$=\frac{3s}{8\pi}m^2(\mu)S[x(\mu),\log(\mu^2/s)], \tag{6.38}$$

where the series $S[x,L]$ is of the form (1.1) and has been fully calculated to N^3L order [31]. For $n_f=\{3,4,5\}$ the series coefficients $T_{n,m}$ are tabulated for $(n,m)\leqslant 3$ in Table III. This correlation function is relevant both for QCD sum-rule analyses of scalar mesons, a topic of past and present interest [32], and for the decay of a single-doublet standard-model Higgs boson into a $b\overline{b}$ pair [29,31]. RG invariance of the correlator $[\mu^2 d\mathrm{Im}\Pi(s)/d\mu^2=0]$ implies the following RG equation for the series $S[x,L]$ within Eq. (6.38) [19]:

$$\left[\frac{\partial}{\partial L}+\beta(x)\frac{\partial}{\partial x}+2\gamma_m(x)\right]S[x,L]=0, \tag{6.39}$$

where $L=\log(\mu^2/s)$, $x=\alpha_s(\mu)/\pi$, $\beta(x)$ is the β-function series (2.3) and $\gamma_m(x)$ is the γ-function series (1.3). Substitution of Eq. (1.1) into Eq. (6.39) leads to the following recursion formulas for the elimination of terms proportional to x^nL^{n-1}, x^nL^{n-2}, x^nL^{n-3}, and x^nL^{n-4}:

$$0=nT_{n,n}-\beta_0(n-1)T_{n-1,n-1}-2\gamma_0 T_{n-1,n-1} \tag{6.40}$$

TABLE III. N^3L-order series coefficients within the fermionic scalar current correlation function, as calculated in [31]. Also listed are the four-loop β-function [33] and γ-function [34] coefficients β_3 and γ_3 required for the evaluation of the series S_3.

	$n_f=3$	$n_f=4$	$n_f=5$
$T_{0,0}$	1	1	1
$T_{1,0}$	17/3	17/3	17/3
$T_{1,1}$	2	2	2
$T_{2,0}$	31.8640	30.5054	29.1467
$T_{2,1}$	95/3	274/9	263/9
$T_{2,2}$	17/4	49/12	47/12
$T_{3,0}$	89.1564	65.1980	41.7576
$T_{3,1}$	297.596	267.589	238.381
$T_{3,2}$	229/2	22547/216	10225/108
$T_{3,3}$	221/24	1813/216	1645/216
β_3	47.2280	31.3874	18.8522
γ_3	44.2628	27.3028	11.0343

$$0=(n-1)T_{n,n-1}-\beta_0(n-1)T_{n-1,n-2}-\beta_1(n-2)\times T_{n-2,n-2}-2\gamma_0 T_{n-1,n-2}-2\gamma_1 T_{n-2,n-2} \quad (6.41)$$

$$0=(n-2)T_{n,n-2}-\beta_0(n-1)T_{n-1,n-3}-\beta_1(n-2)\times T_{n-2,n-3}-\beta_2(n-3)T_{n-3,n-3}-2\gamma_0 T_{n-1,n-3}-2\gamma_1 T_{n-2,n-3}-2\gamma_2 T_{n-3,n-3} \quad (6.42)$$

$$0=(n-3)T_{n,n-3}-\beta_0(n-1)T_{n-1,n-4}-\beta_1(n-2)\times T_{n-2,n-4}-\beta_2(n-3)T_{n-3,n-4}-\beta_3(n-4)\times T_{n-4,n-4}-2\gamma_0 T_{n-1,n-4}-2\gamma_1 T_{n-2,n-4}-2\gamma_2 T_{n-3,n-4}-2\gamma_3 T_{n-4,n-4}. \quad (6.43)$$

We follow the usual procedure of

(1) multiplying Eq. (6.40) by u^{n-1} and summing from $n=1$ to ∞,

(2) multiplying Eq. (6.41) by u^{n-2} and summing from $n=2$ to ∞,

(3) multiplying Eq. (6.42) by u^{n-3} and summing from $n=3$ to ∞, and

(4) multiplying Eq. (6.43) by u^{n-4} and summing from $n=4$ to ∞.

Using the definitions (2.18), (2.19), (2.20) and (2.30) for $\{S_0,S_1,S_2,S_3\}$, we then obtain the following four linear differential equations for these summations:

$$(1-\beta_0 u)\frac{dS_0}{du}-2\gamma_0 S_0=0,\quad S_0[0]=1 \quad (6.44)$$

$$(1-\beta_0 u)\frac{dS_1}{du}-(\beta_0+2\gamma_0)S_1=\beta_1 u\frac{dS_0}{du}+2\gamma_1 S_0,\quad S_1[0]=T_{1,0} \quad (6.45)$$

$$(1-\beta_0 u)\frac{dS_2}{du}-(2\beta_0+2\gamma_0)S_2=\beta_1 u\frac{dS_1}{du}+\beta_2 u\frac{dS_0}{du}+(\beta_1+2\gamma_1)S_1+2\gamma_2 S_0,\quad S_2[0]=T_{2,0} \quad (6.46)$$

$$(1-\beta_0 u)\frac{dS_3}{du}-(3\beta_0+2\gamma_0)S_3=\beta_1 u\frac{dS_2}{du}+\beta_2 u\frac{dS_1}{du}+\beta_3 u\frac{dS_0}{du}+(2\beta_1+2\gamma_1)S_2+(\beta_2+2\gamma_2)S_1+2\gamma_3 S_0,\quad S_3[0]=T_{3,0}. \quad (6.47)$$

The solutions to these equations are

$$S_0[u]=(1-\beta_0 u)^{-A} \quad (6.48)$$

$$S_1[u]=C_1(1-\beta_0 u)^{-A}+\frac{[T_{1,0}-C_1+C_2\log(1-\beta_0 u)]}{(1-\beta_0 u)^{A+1}} \quad (6.49)$$

$$S_2[u]=\frac{D_1}{2}(1-\beta_0 u)^{-A}+\frac{[D_2-D_3+D_3\log(1-\beta_0 u)]}{(1-\beta_0 u)^{A+1}}+\frac{\left[T_{2,0}-\frac{D_1}{2}-D_2+D_3+D_4\log(1-\beta_0 u)+\frac{D_5}{2}\log^2(1-\beta_0 u)\right]}{(1-\beta_0 u)^{A+2}} \quad (6.50)$$

$$S_3[u]=\frac{F_1}{3}(1-\beta_0 u)^{-A}+\frac{\left[\frac{F_2}{2}-\frac{F_3}{4}+\frac{F_3}{2}\log(1-\beta_0 u)\right]}{(1-\beta_0 u)^{A+1}}+\frac{F_4-F_5+2F_6+(F_5-2F_6)\log(1-\beta_0 u)+F_6\log^2(1-\beta_0 u)]}{(1-\beta_0 u)^{A+2}}+\frac{\left[T_{3,0}-\frac{F_1}{3}-\frac{F_2}{2}+\frac{F_3}{4}-F_4+F_5-2F_6+F_7\log(1-\beta_0 u)+\frac{F_8}{2}\log^2(1-\beta_0 u)+\frac{F_9}{3}\log^3(1-\beta_0 u)\right]}{(1-\beta_0 u)^{A+3}}, \quad (6.51)$$

where

$$A=2\gamma_0\beta_0 \quad (6.52)$$

$$C_1=2\beta_1\gamma_0/\beta_0^2-2\gamma_1/\beta_0 \quad (6.53)$$

$$C_2=-2\beta_1\gamma_0/\beta_0^2 \quad (6.54)$$

$$D_1=[C_1(A\beta_1-\beta_1-2\gamma_1)+A\beta_2-2\gamma_2]/\beta_0 \quad (6.55)$$

$$D_2 = [C_1(2\gamma_1 - 2A\beta_1) - C_2\beta_1 - A\beta_2 + T_{1,0}(A\beta_1 - 2\gamma_1)]/\beta_0 \tag{6.56}$$

$$D_3 = C_2(A\beta_1 - 2\gamma_1)/\beta_0 \tag{6.57}$$

$$D_4 = \beta_1[C_2 - (A+1)(T_{1,0} - C_1)]/\beta_0 \tag{6.58}$$

$$D_5 = -C_2\beta_1(A+1)/\beta_0 \tag{6.59}$$

$$F_1 = [C_1(A\beta_2 - 2\gamma_2 - \beta_2) + D_1(A\beta_1/2 - \gamma_1 - \beta_1) + A\beta_3 - 2\gamma_3]/\beta_0 \tag{6.60}$$

$$F_2 = \frac{1}{\beta_0}\left[D_3(2\gamma_1 - A\beta_1) + D_2(A\beta_1 - \beta_1 - 2\gamma_1) - \frac{D_1}{2}A\beta_1 - C_2\beta_2 + C_1(2\gamma_2 - 2A\beta_2) + T_{1,0}(A\beta_2 - 2\gamma_2) - A\beta_3\right] \tag{6.61}$$

$$F_3 = [D_3(A\beta_1 - \beta_1 - 2\gamma_1) + C_2(A\beta_2 - 2\gamma_2)]/\beta_0 \tag{6.62}$$

$$F_4 = \frac{1}{\beta_0}[(T_{2,0} - D_1/2 - D_2 + D_3)(A\beta_1 - 2\gamma_1) - D_4\beta_1 + D_3(A+2)\beta_1 - D_2(A+1)\beta_1 + C_2\beta_2 + C_1(A+1)\beta_2 - T_{1,0}(A+1)\beta_2] \tag{6.63}$$

$$F_5 = [D_4(A\beta_1 - 2\gamma_1) - D_3(A+1)\beta_1 - D_5\beta_1 - C_2(A+1)\beta_2]/\beta_0 \tag{6.64}$$

$$F_6 = D_5(A\beta_1/2 - \gamma_1)/\beta_0 \tag{6.65}$$

$$F_7 = \beta_1[D_4 - (A+2)(T_{2,0} - D_1/2 - D_2 + D_3)]/\beta_0 \tag{6.66}$$

$$F_8 = \beta_1[D_5 - D_4(A+2)]/\beta_0 \tag{6.67}$$

$$F_9 = -\beta_1 D_5(A+2)/(2\beta_0). \tag{6.68}$$

The $\mathcal{O}(N^3L)$ RG summation of the series $S[x(\mu),\log(\mu^2/s)]$ appearing in the correlator (6.38) is then found to be

$$S_{RG\Sigma}^{N^3L} = S_0\left[x(\mu)\log\left(\frac{\mu^2}{s}\right)\right] + x(\mu)S_1\left[x(\mu)\log\left(\frac{\mu^2}{s}\right)\right] + x^2(\mu)S_2\left[x(\mu)\log\left(\frac{\mu^2}{s}\right)\right] + x^3(\mu)S_3\left[x(\mu)\log\left(\frac{\mu^2}{s}\right)\right] \tag{6.69}$$

where the RG summations S_0, S_1, S_2 and S_3 are given by Eqs. (6.48)–(6.51) with $u = x(\mu)\log(\mu^2/s)$.

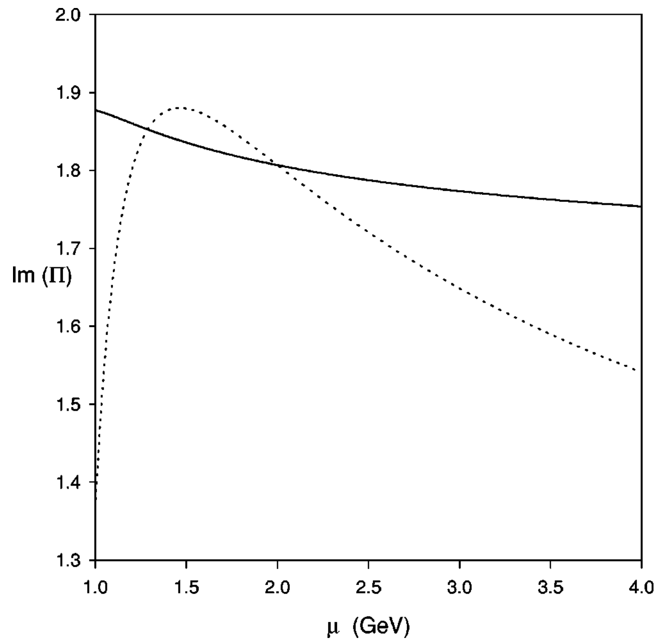

FIG. 15. The imaginary part of the fermionic scalar-current correlation function (6.38), as obtained from $m^2(\mu)S^{N^3L}$ (dotted line) and from $m^2(\mu)S_{RG\Sigma}^{N^3L}$ (solid line) with $n_f = 3$ and $\sqrt{s} = 2$ GeV.

In Fig. 15 we compare the μ dependence of the RG-summed scalar fermionic-current correlator,

$$\mathrm{Im}\Pi_{RG\Sigma}^{N^3L} \sim m^2(\mu)S_{RG\Sigma}^{N^3L} \tag{6.70}$$

to that of the correlator (6.38) when truncated after its fully known $\mathcal{O}(x^3)$ contributions,

$$\mathrm{Im}\Pi^{N^3L} \sim m^2(\mu)\sum_{n=0}^{3}\sum_{m=0}^{n} T_{n,m}x^n(\mu)\left[\log\left(\frac{\mu^2}{s}\right)\right]^m, \tag{6.71}$$

over the range $\sqrt{s}/2 \leqslant \mu \leqslant 2\sqrt{s}$ with $\sqrt{s} = 2$ GeV. The coefficients $T_{n,m}$ appearing in Eq. (6.71) are given in Table III. We choose to work in the $\sqrt{s} = 2$ GeV, $n_f = 3$ regime appropriate for QCD sum rule applications, where the couplant $x(\mu)$ is large. The evolution of $x(\mu)$ is assumed to proceed via the $n_f = 3$ four-loop β function with initial condition $x(m_\tau) = 0.33/\pi$ [17]. The running mass $m(\mu)$ is normalized to 1 GeV at $\mu = m_\tau$ to facilitate comparison of Eqs. (6.70) to (6.71). In Fig. 15, the unsummed correlator is seen to achieve a sharp maximum near 1.5 GeV, followed by a precipitous fall as μ approaches 1 GeV from above. By contrast, the RG-summed correlator exhibits a much flatter profile, falling from 1.88 GeV^2 to 1.75 GeV^2 as μ increases from 1 to 4 GeV. As in Sec. VI B for the gluonic scalar-current correlator, these results indicate that even N^3L-order expressions for the perturbative contribution to QCD correlation functions exhibit substantial μ dependence. Such dependence, which we have shown to be largely eliminated via the RG-summation process, would otherwise percolate through QCD sum-rule integrals as spurious sensitivity to the

Borel parameter on the theory side. The incorporation of RG-summed correlators within QCD sum-rule extractions of lowest-lying scalar resonances is currently under investigation.

VII. SUMMARY

In this paper we have explicitly summed all RG-accessible logarithms within a number of perturbative processes known to at least two nonleading orders, a procedure originally advocated by Maxwell [5]. As anticipated, we have found the dependence on the renormalization scale μ in every case examined to be considerably diminished over that of the original series' known terms.

In semileptonic $b \to u$ decays in the fully $\overline{\mathrm{MS}}$ scheme (Sec. III), we observe the intriguing possibility that PMS-FAC criteria for the unsummed series truncated to a given order may anticipate the RG-summed series for the next order of perturbation theory.[5] This behavior, however, is not evident in the other processes we consider. For the vector correlation function (Sec. V), PMS-FAC criteria for the unsummed series truncated to a given order coincide closely with the RG-summed series for that same order, but do not anticipate the level of the next-order RG summation. In the pole-mass scheme version of semileptonic B decays (Sec. IV), PMS and FAC criteria do not appear applicable to the unsummed series, which monotonically increase with the renormalization scale μ. Indeed, one of the virtues of RG summation is the sensible scale-independent results it provides for inclusive semileptonic B decays in the pole mass scheme, a scheme whose unsummed expressions for $b \to u$ are already known to be problematic [1].

The μ independence of RG summation, particularly for the vector-current correlation function (Sec. V), is seen to justify the prescription of zeroing all logarithms by setting the renormalization scale to μ^2 equal to the kinematic variable s. This prescription necessarily equates the unsummed and RG-summed series, and, since the RG-summed series is virtually independent of μ, the zeroing of logarithms in the unsummed series equilibrates it to the flat RG-summation level we obtain.

In Sec. VI, RG summation is also applied to the perturbative contributions to the momentum-space QCD static potential, the decay rate of a standard-model Higgs boson to two gluons, the Higgs-mediated cross section $\sigma(WW \to ZZ)$, and to two scalar-current correlation functions. Examination of these last two quantities in the low-s region appropriate for QCD sum rules suggests the utility of RG summation in reducing the unphysical scale dependence of the perturbative QCD contributions to the field-theoretical side of sum rules in these channels.

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APPENDIX: AN ALTERNATIVE CLOSED-FORM SUMMATION PROCEDURE

The body of our paper has addressed the evaluation of $S_n[xL] = \Sigma_{k=0}^{\infty} T_{n+k,k}(xL)^k$, where the full perturbative series is $S(x,L) = \Sigma_{n=0}^{\infty} S_n[xL]x^n$. It is, however, also possible to group the terms within $S(x,L)$, as defined by Eq. (1.1), such that the dependence of each series term on x and L fully factorizes:

$$S(x,L) = \sum_{n=0}^{\infty} R_n(x) L^n, \tag{A1}$$

$$R_n(x) = \sum_{k=n}^{\infty} T_{k,n} x^k. \tag{A2}$$

Let us suppose, for example, that the series $S(x,L)$ satisfies the RGE (2.2) appropriate to semileptonic b decays. By substituting Eq. (A1) into Eq. (2.2), we find that

$$R_{n+1}(x) = \frac{1}{n+1}\left[\frac{1}{2\gamma(x)-1}\right]\left[\beta(x)\frac{d}{dx} + 5\gamma(x)\right] R_n(x), \tag{A3}$$

where we have relabeled the anomalous mass dimension $\gamma_m(x) \to \gamma(x)$ to avoid any misinterpretation of the label m as a subscript. If

$$R_n(x) \equiv \exp\left(-\int^x \frac{5\gamma(x')}{\beta(x')} dx'\right) P_n(x), \tag{A4}$$

then the recursion relation (A3) implies that

$$P_{n+1}(x) = \frac{1}{n+1}\left(\frac{\beta(x)}{2\gamma(x)-1}\right)\frac{d}{dx} P_n(x). \tag{A5}$$

If one defines x to be implicitly a function of y via the equation

$$\frac{d}{dy} x(y) = \frac{\beta(x)}{2\gamma(x)-1}, \tag{A6}$$

then Eq. (A5) simplifies to

$$P_{n+1}(x(y)) = \frac{1}{n+1}\frac{d}{dy} P_n(x(y)), \tag{A7}$$

in which case

[5]Although this result incorporates a Padé estimate for an RG-inaccessible $\mathcal{O}(x^3)$ coefficient, this estimate occurs in both the unsummed and RG-summed expression.

$$\sum_{n=0}^{\infty} P_n(x(y))L^n = \sum_{n=0}^{\infty} \left(\frac{1}{n!}L^n \frac{d^n}{dy^n}\right) P_0(x(y)) = P_0(x(y+L)). \tag{A8}$$

This last result implies via Eq. (A4) that the series $S(x,L)$ is fully determined by knowledge of the log-free summation R_0, i.e. that

$$S(x,L) = \exp\left[\int_{x(y)}^{x(y+L)} \frac{5\gamma(x')}{\beta(x')} dx'\right] R_0(x(y+L)) \tag{A9}$$

where $x(y)$ is defined implicitly by the constraint

$$y = \int^{x} \frac{2\gamma(x')-1}{\beta(x')} dx' \tag{A10}$$

obtained by integrating Eq. (A6).

Using lowest-order expressions $\beta(x) = -\beta_0 x^2$ and $\gamma(x) = -\gamma_0 x$, we find from Eq. (A10) that

$$y = \frac{1}{\beta_0}\left(-\frac{1}{x} + 2\gamma_0 \log(x)\right) + K. \tag{A11}$$

If we set

$$x = \frac{1}{2\gamma_0 W}, \quad K = \frac{2\gamma_0}{\beta_0}\log(2\gamma_0) \tag{A12}$$

we find from Eq. (A11) that

$$W e^{W} = \exp\left(-\frac{\beta_0 y}{2\gamma_0}\right). \tag{A13}$$

Equation (A13) is the defining relationship for the Lambert W-function $W[\exp(-\beta_0 y/2\gamma_0)]$, as discussed in Ref. [35]. Since

$$x(y) = \frac{1}{2\gamma_0 W[\exp(-\beta_0 y/2\gamma_0)]} \tag{A14}$$

in the approximation $\beta(x) \cong -\beta_0 x^2$, $\gamma(x) \cong -\gamma_0 x$, we then find from Eq. (A9) that

$$S(x,L) = \left[\frac{W[\exp(-\beta_0 y/2\gamma_0)]}{W[\exp(-\beta_0(y+L)/2\gamma_0)]}\right]^{5\gamma_0/\beta_0} \times R_0\left(\frac{1}{2\gamma_0 W[\exp(-\beta_0(y+L)/2\gamma_0)]}\right), \tag{A15}$$

where

$$R_0(x) = \sum_{k=0}^{\infty} T_{k,0} x^k = S(x,0) \tag{A16}$$

and where

$$y = -\frac{1}{\beta_0 x} + \frac{2\gamma_0}{\beta_0}\log(2\gamma_0 x). \tag{A17}$$

For the RGE (4.16), corresponding to Eq. (2.2) with $\gamma_m(x)$ chosen to be zero, the solution (A9) still applies provided $x(y)$ is defined implicitly via the constraint

$$y = -\int^{x} \frac{dx'}{\beta(x')}. \tag{A18}$$

In the approximation $\beta(x) = -\beta_0 x^2$, one can choose

$$x(y) \equiv -\frac{1}{\beta_0 y} \tag{A19}$$

in which case

$$S(x,L) = R_0\left(\frac{x}{1-\beta_0 L x}\right), \tag{A20}$$

where the function R_0 is as defined via Eq. (A16).

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