# QCD vacuum structure in strong magnetic fields

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(Received 30 April 2002; published 8 July 2002)

We study the response of the QCD vacuum to strong magnetic fields, using a potential model for the quark-antiquark interaction. We find that production of spin-polarized $u\bar{u}$ pairs is energetically favorable for fields $B > B_{\rm crit} \sim 10\ \ {\rm GeV}^2$. We contrast the resulting $u\bar{u}$ condensate with the quark condensate which is present at zero magnetic field, and we estimate the corresponding magnetization as a function of $B$.

DOI: 10.1103/PhysRevD.66.014004 PACS number(s): 12.38.−t, 12.39.Pn

## I. INTRODUCTION

Strong magnetic fields are interesting from several perspectives. From a theoretical point of view, an external magnetic field allows one to probe the vacuum structure and correlation functions of a quantum field theory. Strong magnetic fields are also of interest in astrophysics. There may be neutron stars with fields of up to $10^{14}$–$10^{15}$ G [1], while it has been suggested that much larger fields existed in the early universe [2].

With increasing magnetic field, the first place one might expect something interesting to happen is at the scale set by the electron mass:

$$B = m_e^2/\sqrt{\alpha} = 4.4\times 10^{13}\ \ {\rm G}. \tag{1}$$

At this scale an electron's Landau energy equals its rest energy. Magnetic fields of this strength have a significant effect on atomic and molecular physics, as reviewed in [3]. However, the structure of the QED vacuum does not change dramatically in fields of this magnitude. Corrections to the energy of a free electron in the lowest Landau level are small, proportional to the electron's anomalous magnetic moment, so it is not energetically favorable to produce $e^+e^-$ pairs. The binding energy of positronium is small and does not change this conclusion, so the QED vacuum is stable.[1]

The next place one might expect something interesting to happen is at the QCD scale,

$$B = \Lambda_{QCD}^2/\sqrt{\alpha} \approx 10^{19}\ \ {\rm G}. \tag{2}$$

This regime will be the main focus of this paper. We will argue that the perturbative QCD vacuum becomes unstable with respect to the formation of a quark-antiquark condensate. The basic physics is easy to understand: the strong magnetic field restricts quarks and antiquarks to move in one dimension, and the strongly attractive QCD potential then leads to the formation of a bound state with negative energy. We will argue that at sufficiently large magnetic fields the effective coupling becomes weak and perturbative QCD can be used.

The next interesting regime starts at the electroweak scale:

$$B = m_W^2/\sqrt{\alpha} \approx 10^{24}\ \ {\rm G}. \tag{3}$$

A field of this magnitude has been argued to drive electroweak symmetry restoration [5]. Finally, a grand unified theory with magnetic monopoles of mass $M_{\rm mon}$ could provide an absolute upper bound on a possible magnetic field. Extrapolation of semiclassical calculations [6] suggests that monopole pair production would become copious and short out any existing magnetic field when

$$B \sim \sqrt{\alpha} M_{\rm mon}^2 \sim 10^{52} \left( \frac{M_{\rm mon}}{10^{17}\ \ {\rm GeV}} \right)^2 {\rm G}. \tag{4}$$

Although the approximations underlying the semiclassical calculation break down before such fields are reached, this is probably an overestimate of the maximum possible field. Arguments similar to those leading to Eq. (3) indicate that there should be a local restoration of the grand unified theory (GUT) symmetry when $B \sim \alpha^{3/2} M_{\rm mon}^2$. (The resulting regions of symmetric vacuum can be viewed as condensed monopole and antimonopole cores.) With the unbroken symmetry enlarged to a simple non-Abelian group, magnetic flux is not conserved and a coherent long-range magnetic field can no longer be sustained.

There are several approaches one could adopt for studying QCD in a strong magnetic field. At the hadronic level, one can study the effect of a magnetic field on hadron spectra [7] and the nuclear equation of state [8], based on the large

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[1]This is discussed in more detail in [3]. Exponentially large magnetic fields, in contrast, have been shown to catalyze chiral symmetry breaking in QED [4].

 

anomalous magnetic moments of hadrons. Alternatively, one can take a diagrammatic approach and compute the quark condensate in a large magnetic field by resumming diagrams [9]. The effect of a magnetic field has also been investigated in the Nambu-Jona-Lasinio model [10–12] and in an instanton-inspired model for chiral symmetry breaking [11]. In contrast, we study the problem using the quark model. The advantages of this approach are simplicity and a clear physical picture of the QCD vacuum.

Throughout this paper we set $\hbar = c = 1$. The conversion factor is 1 $\mathrm{GeV}^2/(\hbar c)^{3/2} = 1.44\times 10^{19}$ G. In the Introduction we have used Gaussian units, but in the remainder of this paper we will exclusively use Heaviside-Lorentz units: $B_{\mathrm{Gaussian}} = \sqrt{4\pi} B_{\mathrm{Heaviside\text{-}Lorentz}}$ and $q_{\mathrm{Gaussian}} = q_{\mathrm{Heaviside\text{-}Lorentz}}/\sqrt{4\pi}$. Thus, for example, in the remainder of this paper the charge of an up quark is $q = \frac{2}{3}\cdot\sqrt{4\pi\alpha}$.

An outline of this paper is as follows. In Sec. II we study the behavior of a $q\bar{q}$ pair in a strong magnetic field, with the help of a potential model for the quark-antiquark interaction. In Section III we estimate the strength of the $q\bar{q}$ condensate in magnetic fields somewhat above the QCD scale. In Sec. IV we study the condensate in the regime of large fields, where perturbative QCD is applicable. Section V contains a summary and some concluding comments. For completeness, in the Appendix we compute the response of the QCD vacuum to weak magnetic fields by performing a pion loop calculation.

## II. MESONS IN A STRONG MAGNETIC FIELD

In a strong magnetic field quarks follow Landau orbits in the directions transverse to the magnetic field. These have a characteristic radius $R = 1/\sqrt{qB}$, so that for $B \gtrsim 1$ $\mathrm{GeV}^2$ the quarks can be localized in the two transverse directions to distances shorter than the QCD scale.

Moreover, there is no energy cost associated with this localization. Intuitively, this is because the quark kinetic energy is canceled when the magnetic moment of the quark lines up with the magnetic field. More precisely, the energy levels for a Dirac particle in a background magnetic field are

$$E(n,\sigma,p_z) = \pm\sqrt{|qB|(2n+\sigma+1)+p_z^2+m^2}. \tag{5}$$

Here $n = 0,1,2,\ldots$ labels the Landau levels, $\sigma = \pm 1$ specifies the spin orientation, and $p_z$ is the momentum in the $z$ direction. Thus in the lowest Landau level, with an appropriate spin orientation, the quark behaves just like a relativistic particle in $1+1$ dimensions.

One might expect that this localization enhances the attraction between a color-singlet quark and antiquark to the point where the energy of a $q\bar{q}$ state becomes negative. This would signal an instability with respect to the formation of a spin-polarized $q\bar{q}$ condensate.

To address this issue, we wish to estimate the energy of a $q\bar{q}$ state in a strong magnetic field. We do this by adopting a potential model for the $q\bar{q}$ interaction [13]. That is, we will take the Hamiltonian for a $q\bar{q}$ state to be given by the quasirelativistic expression

$$H = 2\sqrt{\mathbf{p}^2+m^2}+V(r). \tag{6}$$

A wide variety of potentials have been discussed in the literature; we will use the Cornell potential [14]

$$V(r) = Ar - \frac{\kappa}{r} + C. \tag{7}$$

We will be focusing on $u\bar{u}$ or $d\bar{d}$ states, and so use parameters [15]

$$\begin{aligned} A &= 0.203\ \mathrm{GeV}^2\\ \kappa &= 0.437\\ m &= 0.150\ \mathrm{GeV}\\ C &= -0.599\ \mathrm{GeV}\end{aligned} \tag{8}$$

chosen to fit the the spectrum of light mesons.

In a strong magnetic field this three-dimensional model should reduce to an effective one-dimensional problem. However, at distances shorter than the magnetic length $R$ the problem again becomes three dimensional. We can take this into account by cutting off our one-dimensional potential at short distances. Thus we study the one-dimensional problem

$$H = 2\sqrt{p_z^2+m^2}+V(z)$$

$$V(z) = \begin{cases} AR - \dfrac{\kappa}{R} + C, & |z| < R,\\[2ex] Az - \dfrac{\kappa}{z} + C, & |z| > R. \end{cases} \tag{9}$$

By considering a Gaussian trial wave function, one can easily see that as $R \to 0$ the spectrum of this Hamiltonian is unbounded from below.

To estimate the energy levels of the Hamiltonian (9) we use a WKB approximation [16]. The classical turning points are at $z = \pm L$, where $V(L) = E - 2m$. The WKB quantization condition $\oint p_z dz = 2\pi(n+\frac{1}{2})$ becomes

$$\int_0^L dz\sqrt{[V(z)-E]^2-4m^2} = \pi\left(n+\frac{1}{2}\right),$$

$$n = 0,1,2,\ldots. \tag{10}$$

The resulting ground state energy is shown in Fig. 1. Note that the energy is negative for $qB \gtrsim 2$ $\mathrm{GeV}^2$. We expect the WKB approximation to give a reasonable estimate for the ground state energy in this regime.

The semiclassical turning point $L$ is shown in Fig. 2. Our reduction to one dimension only makes sense if $L$ is large compared to $R$. As can be seen in Fig. 3 this condition is reasonably well satisfied in the regime of interest. Note that $L$ decreases as the magnetic field gets bigger. This means that at sufficiently large magnetic fields the $q\bar{q}$ bound state is

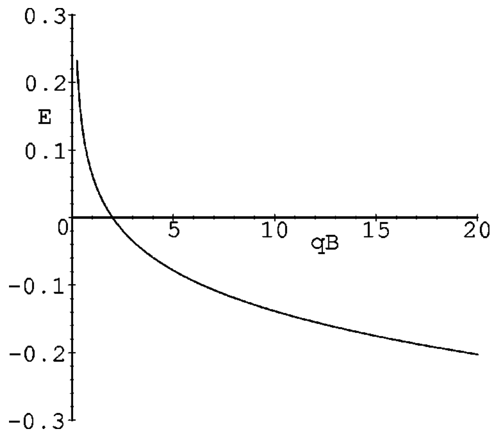


FIG. 1. Ground state energy as a function of $qB$. $E$ is in units of GeV, $qB$ is in units of $\mathrm{GeV}^2$.

driven into a short-distance regime where perturbative QCD can be applied. This regime is discussed in more detail in Sec. IV.

### III. MESON CONDENSATION

When the magnetic field is sufficiently large, $q\bar{q}$ pairs will start to condense. We first consider a single flavor, and discuss condensation of $u\bar{u}$ pairs. Condensation occurs when $qB \gtrsim 2\ \mathrm{GeV}^2$, which for a $u\bar{u}$ composite means

$$B > B_{\mathrm{crit}} \approx \frac{2\ \mathrm{GeV}^2}{\frac{2}{3}\cdot\sqrt{4\pi\alpha}} \approx 10\ \mathrm{GeV}^2. \tag{11}$$

The quark magnetic moments line up with the magnetic field, so $B$ is increased by the formation of the condensate. This would seem to make the vacuum unstable, but eventually the $u\bar{u}$ pairs will start to interact, and this effect presumably stabilizes the system.

Because the $u\bar{u}$ pairs are color singlets, they will not interact strongly until their wavefunctions begin to overlap.

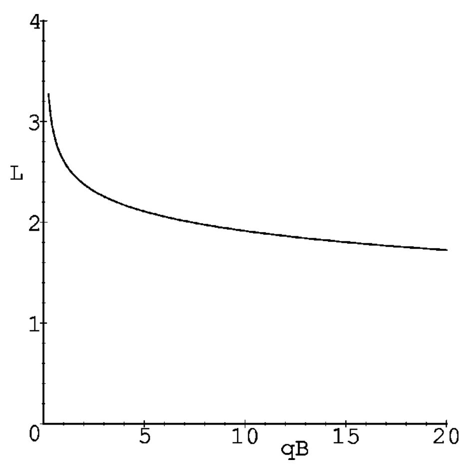


FIG. 2. Turning point $L$ as a function of $qB$. $L$ is in units of $\mathrm{GeV}^{-1}$, $qB$ is in units of $\mathrm{GeV}^2$.

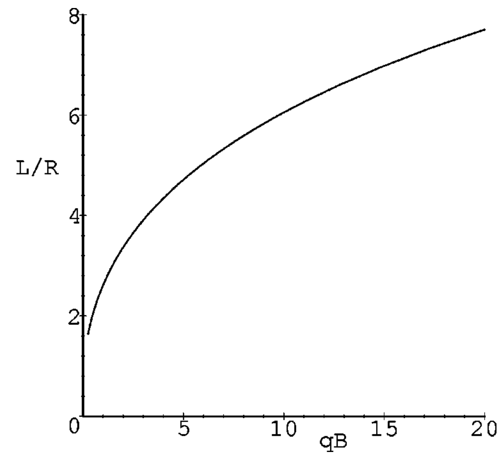


FIG. 3. $L/R$ as a function of $qB$ (measured in units of $\mathrm{GeV}^2$).

Hence, for $B > B_{\mathrm{crit}}$ pair production should proceed unimpeded until the density of $u\bar{u}$ pairs reaches a value of roughly

$$\rho = \frac{1}{\pi R^2 L} = \frac{qB}{\pi L}. \tag{12}$$

Once this density is attained, QCD interactions between pairs will tend to suppress further growth in the condensate. We will use Eq. (12) as our estimate for $\rho$, although the actual value that emerges from the interplay of magnetic and QCD effects will presumably have a somewhat more complicated dependence on $B$.

Treating the quark and antiquark as elementary Dirac fermions, the magnetic moment of a pair is $\mu = q/m$ and the magnetization is

$$M = \mu\rho = \frac{q^2 B}{m\pi L}. \tag{13}$$

To evaluate this, note that Fig. 2 shows that for $B \gtrsim B_{\mathrm{crit}}$ the length $L$ is slowly varying, with $L \approx 2\ \mathrm{GeV}^{-1}$. The question of what value to use for the mass is a bit more subtle. At zero magnetic field one uses constituent quark masses $m \approx 300$ MeV to estimate magnetic moments, although for extremely large magnetic fields, where $R$ is very small, a current quark mass may be more appropriate. Using the $u$-quark charge in Eq. (13) gives

$$M = 0.022\left(\frac{300\ \mathrm{MeV}}{m}\right)\left(\frac{2\ \mathrm{GeV}^{-1}}{L}\right)B. \tag{14}$$

Hence, we expect that in the regime $B \gtrsim B_{\mathrm{crit}}$ the magnetization will be small compared to $B$ so that we are justified in ignoring the back reaction of the magnetization on the strength of the condensate.

We now consider the effects of the other quark flavors. If there were no interaction between quarks of different flavors, extension of the above analysis to $d$ quarks would predict that a $d\bar{d}$ condensate forms at a critical field which is twice as large as for $u\bar{u}$ pairs, and with a magnetization that is one-quarter as large for a given $B$. However, the different condensates will interact with each other, so that an increase in the condensate of one flavor will tend to cause a compensating decrease in the other condensates. Because of the relatively large $u$ quark electric charge, a $u\bar{u}$ condensate is energetically favored over $d\bar{d}$ or $s\bar{s}$, while the heavier quarks are suppressed by their mass. Hence we expect the condensate to be dominated by $u\bar{u}$ pairs.

## IV. CONDENSATION IN THE PERTURBATIVE REGIME

When the magnetic field is far above the QCD scale the $q\bar{q}$ composite is driven into a short-distance regime where perturbative QCD can be applied. In this section we discuss the magnetization in this perturbative regime.

At short distances we should replace the Cornell potential of Eq. (7) with the potential from one-gluon exchange:

$$V(r)=-\frac{4}{3}\frac{\alpha_s(r)}{r}\equiv-\frac{A}{r\log(1/\Lambda r)}$$

$$A=\frac{8\pi}{3b_0}=\frac{8\pi}{11N_c-2N_f}. \tag{15}$$

The WKB quantization condition for this potential is still given by Eq. (10).

If one makes the approximation of neglecting the quark mass, the WKB integral can be evaluated analytically to obtain a relation between the turning point $L$ and the radius $R=1/\sqrt{qB}$:

$$\frac{A}{\log(1/L\Lambda)}+A\log\log\frac{1}{L\Lambda}=\frac{A}{\log(1/R\Lambda)}+A\log\log\frac{1}{R\Lambda}-\frac{\pi}{2}. \tag{16}$$

For $R\to 0$ this, together with Eq. (13), gives a magnetization

$$M=\frac{q^2B}{m\pi L}\approx\frac{q^2B\Lambda}{m\pi}\left(\frac{qB}{\Lambda^2}\right)^{(1/2)\exp(-\pi/2A)}. \tag{17}$$

To evaluate the exponent in the last factor we set $N_c=3$ and take $N_f$ to be the number of quark flavors that are lighter than the mass scale set by $B$. The dependence on $N_f$ is actually rather weak, with any value between 2 and 6 yielding an exponent of about 0.1.

Of course these calculations are only valid if the turning point $L$ is small enough to trust the potential of Eq. (15). This requires magnetic fields that are far larger than those we have considered so far. For example, $L<\frac{1}{3}\Lambda^{-1}$ for a $u\bar{u}$ composite requires $B>5\times10^{12}\Lambda^2$; although several orders of magnitude above the electroweak scale, this is still far below the upper limit of Eq. (4).

We expect that our approximations give the right qualitative behavior of the magnetization at strong fields. However, an accurate quantitative calculation calls for more sophisticated techniques than we have employed here. For one thing, the WKB estimate of Eq. (16) gets worse as the magnetic field increases, since the condition for the validity of WKB, $|\partial p_z^{-1}/\partial z|\ll 1$, is violated near $z=R$. (The left-hand side grows logarithmically as $R\to 0$.) A more serious concern is that as $R\to 0$ we should really treat the $q\bar{q}$ composite in a fully relativistic manner, e.g., by solving a Bethe-Salpeter equation.

Furthermore, at sufficiently high fields the nonlinearities become important enough that we must take into account the back reaction of the magnetization on the condensate. These nonlinearities can arise from several sources. First, there is the explicit nonlinearity in Eq. (17), which shows that $M/B$ includes a factor that grows as a small power of the magnetic field. Next, we expect the effective mass of the quarks to decrease as $B$ grows, also leading to an increase in $M/B$. A third possible source is the corrections to our estimate Eq. (12) for the density of condensed pairs; these should also give an increase in $M/B$ at stronger fields.

## V. CONCLUSIONS

In this paper we have used a quark model approach to study the behavior of QCD in the presence of a strong magnetic field. In the presence of such a field the quarks can be localized in the two transverse directions with no cost in energy. This enhances the quark-antiquark attraction to such an extent that the binding energy can compensate for the mass, thus making $u\bar{u}$ pair production energetically favorable if $B$ is greater than a critical value of about 10 GeV$^2$.

In the language of field theory, this pair production corresponds to the formation of a chiral symmetry breaking $u\bar{u}$ condensate. Of course, even in the absence of a magnetic field, nonperturbative QCD dynamics produce a nonzero quark condensate that breaks chiral symmetry. However, the zero-field and the high field condensates differ in some significant aspects.

At zero field the condensate is Lorentz invariant. In particular, $\langle\bar{q}\sigma^{\mu\nu}q\rangle=0$. By contrast, the quark pairs produced by a critical magnetic field are polarized along the direction of the magnetic field. For a field directed along the $z$ direction, this corresponds to a condensate with[2] $\langle\bar{q}q\rangle\approx\langle\bar{q}\sigma^{12}q\rangle$.

The flavor properties of the two condensates are also quite different. The zero temperature, zero field condensate is, to a good approximation, flavor SU(3) symmetric, with $\langle\bar{u}u\rangle\approx\langle\bar{d}d\rangle\approx\langle\bar{s}s\rangle$. This is not the case in the presence of a supercritical magnetic field, since the production of $u\bar{u}$ pairs is energetically favored over that of $d\bar{d}$ and $s\bar{s}$ pairs.

Finally, the zero field and high field condensates differ in magnitude. The former is of the order of $\Lambda_{\mathrm{QCD}}^3\sim(.25\ \mathrm{GeV})^3$. This should be compared with our estimate, Eq. (12), for the density of quark pairs. This density increases faster than linearly with $B$, but even at the critical field for $u\bar{u}$ production we have $\rho\sim(0.7\ \mathrm{GeV})^3$.

We would like to understand the transition between the

[2]The possibility of a $\langle\bar{q}\sigma^{\mu\nu}q\rangle$ condensate was raised in [11].

zero field and high field regimes. As $B$ is increased from zero, its initial effect is to gradually polarize the QCD chiral condensate. For weak fields, the relevant degrees of freedom are the Goldstone modes of the condensate. These can be studied by using a low-energy chiral effective Lagrangian. This leads to a pion-loop calculation, which we review in the Appendix, that gives a magnetization

$$M \sim \begin{cases} \dfrac{7e^4B^3}{1440\pi^2 m_\pi^4} & \text{for } |eB| \ll m_\pi^2, \\ \dfrac{e^2B}{48\pi^2}\log\dfrac{eB}{m_\pi^2} & \text{for } |eB| \gg m_\pi^2. \end{cases} \tag{18}$$

Using a similar approach and working in the chiral limit, Shushpanov and Smilga [9] find that the overall magnitude of the quark condensate is enhanced by a factor

$$\frac{\Sigma(B)}{\Sigma(0)} = 1 + \frac{eB \ln 2}{16\pi^2 F_\pi^2} + \dots . \tag{19}$$

As the field increases, higher order terms in the chiral Lagrangian become important. In any case, the chiral Lagrangian must be abandoned in favor of a description in terms of quarks for $eB \gtrsim (4\pi F_\pi)^2 \approx 1\ \text{GeV}^2$. It would be desirable to understand this transition region between the weak and strong field regimes.

Within the strong field regime, our quark model calculations give an estimate for the magnetization, given in Eqs. (14) and (17). One would like a clearer understanding of how the interplay of electromagnetic and QCD effects determines the density of quark pairs, thus leading to an improvement upon these estimates. The development of improved techniques for performing precise calculations in this regime is clearly needed. Finally, the relation between our methods and the more field-theoretic approaches to strong field chiral symmetry breaking of [9] should be better understood.

## ACKNOWLEDGMENTS

We are grateful to Al Mueller, D. T. Son and Mal Ruderman for valuable discussions. This work is supported in part by the U.S. DOE under contract DE-FG02-92ER40699 and in part by KOSEF 1998 Interdisciplinary Research Grant 98-07-02-07-02-5 (K.L.).

## APPENDIX: WEAK-FIELD RESULTS

The response of the QCD vacuum to a weak magnetic field can be computed by using a chiral effective Lagrangian [9]. One can integrate out the matter fields to obtain an effective electromagnetic action $S_{\text{eff}}(\mathbf{B})$. At leading order the matter contribution to this effective action, $S_{\text{eff}}^{\text{matter}}(\mathbf{B})$, arises from a single pion loop. Thus, we consider a complex scalar field coupled to electromagnetism with action

$$S = \int d^4x \left[ -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} - |(\partial_\mu - ieA_\mu)\phi|^2 - m^2|\phi|^2 \right]. \tag{A1}$$

Schwinger's classic calculation [17] for a uniform magnetic field then gives

$$S_{\text{eff}}^{\text{matter}} = i \log\det(-\mathcal{D}_\mu \mathcal{D}^\mu + m^2) \tag{A2}$$

$$= -i \int_{\epsilon^2}^{\infty} \frac{ds}{s} \operatorname{Tr} e^{-is(-\mathcal{D}_\mu\mathcal{D}^\mu + m^2)} \tag{A3}$$

$$= \frac{1}{16\pi^2} \int d^4x \int_{\epsilon^2}^{\infty} \frac{ds}{s^3} e^{-sm^2} \frac{esB}{\sinh esB}. \tag{A4}$$

Expanding this in powers of $B$, one finds both a quartic and a logarithmic divergence. The former is a contribution to the vacuum energy, while the latter is the one pion loop contribution to the renormalization of the electric charge. Subtracting these divergences gives the renormalized matter effective action

$$S_{\text{eff}}^{\text{matter}} = \frac{1}{16\pi^2} \int d^4x \times \int_0^\infty \frac{ds}{s^3} e^{-sm^2} \left( \frac{esB}{\sinh esB} - 1 + \frac{1}{6} e^2 s^2 B^2 \right) \tag{A5}$$

where $e$ is the renormalized electric charge.

The magnetization $\mathbf{M} = \mathbf{B} - \mathbf{H}$ is given by

$$\mathbf{M} = \mathbf{B} + \frac{\delta S_{\text{eff}}}{\delta \mathbf{B}} = \frac{\delta S_{\text{eff}}^{\text{matter}}}{\delta \mathbf{B}}. \tag{A6}$$

Substituting the result from Eq. (A5) then leads to

$$M = \frac{\partial \mathcal{L}_{\text{eff}}^{\text{matter}}}{\partial B} \sim \begin{cases} \dfrac{e^2B}{16\pi^2}\left( \dfrac{7e^2B^2}{90m^4} + \mathcal{O}((eB/m^2)^4) \right) & \text{for } |eB| \ll m^2, \\ \dfrac{e^2B}{48\pi^2}\log\dfrac{eB}{m^2} & \text{for } |eB| \gg m^2. \end{cases} \tag{A7}$$

Since $M > 0$, the QCD vacuum is paramagnetic.

The effects of higher order terms in the chiral Lagrangian have been studied [18]. These become large for $eB \sim (4\pi F_\pi)^2 \approx 1\ \text{GeV}^2$, at which point the chiral Lagrangian approximation breaks down. For such fields the effective photon coupling

$$\frac{1}{e_{\text{eff}}^2} = \frac{1}{e^2}\left[ 1 - \frac{\partial^2 \mathcal{L}_{\text{eff}}^{\text{matter}}}{\partial B^2} \right] \tag{A8}$$

$$\approx \frac{1}{e^2} - \frac{1}{48\pi^2}\log\frac{eB}{m^2} \quad \text{for } |eB| \gg m^2 \tag{A9}$$

is still small, thus justifying our neglect of the quantum fluctuations in the electromagnetic field.

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