Holography and the Polyakov action

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(Received 12 March 2002; published 18 June 2002)

In two-dimensional conformal field theory the generating functional for correlators of the stress-energy tensor is given by the nonlocal Polyakov action associated with the background geometry. We study this functional holographically by calculating the regularized on-shell action of asymptotically AdS gravity in three dimensions, associated with a specified (but arbitrary) boundary metric. This procedure is simplified by making use of the Chern-Simons formulation, and a corresponding first-order expansion of the bulk dreibein, rather than the metric expansion of Fefferman and Graham. The dependence of the resulting functional on local moduli of the boundary metric agrees precisely with the Polyakov action, in accord with the AdS/conformal field theory correspondence. We also verify the consistency of this result with regard to the nontrivial transformation properties of bulk solutions under Brown-Henneaux diffeomorphisms.

DOI: 10.1103/PhysRevD.65.126008 PACS number(s): 11.25.Hf, 04.62.+v

I. INTRODUCTION

The interplay between two-dimensional conformal field theories and classical three-dimensional gravity with a negative cosmological constant can be traced back to the identification, by Brown and Henneaux [1], of an infinite-dimensional symmetry acting on the space of gravitational solutions asymptotic to anti–de Sitter space (AdS_3). On the two-dimensional conformal boundary at infinity this symmetry reduces to two copies of the Virasoro algebra with a central charge

$$c=\frac{3l}{2G_3}, \tag{1}$$

where l is the AdS_3 scale (set to unity from here on), and G_3 is the three-dimensional Newton constant. This structure, now embedded within the general AdS/conformal field theory (CFT) correspondence [2–5], has come under considerable recent scrutiny. In particular, Strominger's observation [6] that a unitary CFT on the boundary, with the central charge (1), would have a density of states sufficiently large to account for the entropy of 3D Bañados-Teitelboim-Zanelli (BTZ) black holes [7,8] has stimulated further work on particular realizations of this system in string theory, such as configurations of fundamental strings and Neveu-Schwarz 5-branes (NS5-branes) wrapped on either T^4 or $K3$, in the hope of understanding the dual CFT [9] and consequently the microscopic origin of the black hole entropy. However, string theory on noncompact target spaces such as AdS_3 is still rather mysterious [10] (see, e.g., [11,12] for earlier work), and recent studies [13] (see also [14–21]) are only now leading in particular cases to a consistent picture of the perturbative spectrum.

In this context, it is interesting to explore how information about the dual CFT is encoded at the purely gravitational level, namely, in the space of solutions to asymptotically AdS 3D gravity. All such solutions are locally AdS, and starting from pure AdS_3 the classical phase space may be constructed in terms of orbits of the Brown-Henneaux mapping. Recall, for example, that a solution of the form

$$ds^2_{3D}=dr^2+e^{2r}dz d\bar{z}+\frac{12\pi}{c}T(z)dz^2+\cdots, \tag{2}$$

asymptotic to $\mathrm{AdS}_3$for large r, where r is the radial coordinate, admits infinitesimal Brown-Henneaux diffeomorphisms [1] [with parameter $\epsilon(z)$] as an asymptotic symmetry under which the metric remains form invariant, up to the shift

$$\delta T=\epsilon\partial T+2T\partial\epsilon-\frac{c}{24\pi}\partial^3\epsilon, \tag{3}$$

where $\partial=\partial/\partial z$. These mappings reduce on the boundary to infinitesimal local conformal transformations, and we see from Eq. (3) that in accord with the AdS/CFT correspondence we can identify the subleading components $T(z)$ [and $\overline{T(z)}$] of the metric (2) with the expectation value of chiral (and antichiral) components of the boundary stress-energy tensor [22–24,5,25–27]. Extending the consideration to *finite* Brown-Henneaux mappings relates solutions with a different topology, given suitable coordinate identifications. However, in practice it is convenient to consider orbits of fixed topology generated by Eq. (3) which are characterized by a given background topology (say pure AdS_3), perturbed by Brown-Henneaux ''gravitational waves'' (described by T) in Eq. (2).

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This picture of the classical phase space generalizes to solutions with a specified conformal structure $[g]$ at infinity, which then depend on a general representative metric g_{ij} in this conformal equivalence class [in Eq. (2) g_{ij} is simply the flat metric]. These bulk geometries may be obtained via the construction of Fefferman and Graham [28] (see also [29]), and take the asymptotic form [27,26,30,31]

$$ds^2_{3D}=dr^2+e^{2r}g_{ij}dx^idx^j + \frac{1}{2}\left(\mathcal{R}g_{ij}+\frac{24\pi}{c}\langle T_{ij}\rangle\right)dx^idx^j+\cdots, \tag{4}$$

where $\langle T_{ij}\rangle$, which is traceless, is unconstrained by the bulk Einstein equations [28]. This is consistent with its anomalous transformation under Brown-Henneaux diffeomorphisms [as in Eq. (3)], and $\langle T_{ij}\rangle$ can again be identified with the expectation value of the boundary stress-energy tensor [27,26,30,31]. Thus, in general, 3D bulk Einstein solutions can be reconstructed given two pieces of boundary data $\{g,\langle T\rangle\}$. In addition, one has holonomy data which describe the global properties[1] of the 3D geometry.

The possibility of turning on sources g_{ij} for the stress-energy tensor T_{ij} in the boundary CFT allows consideration of how the bulk dynamics reproduces the current sector of the CFT associated with correlators of the stress-energy tensor. This is the topic we will now focus on, and we aim in the present paper to tackle the local part of this problem, which may also be inverted as the question of how the bulk theory encodes (holographically) the local geometric moduli of the boundary CFT. In this context the basic quantity in the CFT is the generating functional $W[g]$ for correlators of the stress-energy tensor,

$$e^{iW[g_{ij}]}\equiv\left\langle \exp\left[i\int_\Sigma g^{ij}T_{ij}\right]\right\rangle_{CFT} \tag{5}$$

where $\{\Sigma,g_{ij}\}$ is the conformal boundary of the asymptotically AdS bulk geometry. This boundary coupling leads to a Weyl anomaly given by

$$\langle T^i_i\rangle_{CFT}=-\frac{c}{24\pi}\mathcal{R}, \tag{6}$$

where $\mathcal{R}$ is the scalar curvature associated with the metric g_{ij}, and c is the corresponding central charge. Using this, the anomalous Ward identity for Weyl transformations can be integrated [32], leading to the Polyakov action as the expression for the generating functional $W[g]$ in covariant form,

$$W_P[g]=\frac{c}{96\pi}\int\int\left[\mathcal{R}\frac{1}{\nabla^2}\mathcal{R}+\lambda^2\right], \tag{7}$$

where λ^2 is a cosmological constant.

From the bulk point of view, the AdS/CFT correspondence implies a relation between the generating functional $W[g]$ and the string partition function evaluated with conformal boundary data g_{ij}. In the classical gravitational regime, this relation takes the form[2] [2–5,27]

$$W[g]\sim I_{\rm reg}[G]=\lim_{\epsilon\to 0}(I_{\rm EH}[G_\epsilon]-I_{\rm ct}[g_\epsilon]), \tag{8}$$

where $I_{\rm EH}[G_\epsilon]$ is the bulk Einstein-Hilbert action (plus the appropriate boundary terms) evaluated on a solution $G_{ij}[g]$ of the form (4) with boundary conformal structure $[g]$, and an infrared regulator $r<1/\epsilon$ is used to subtract the bulk divergences with covariant counterterms $I_{\rm ct}$ [29,34] (for further work on holographic renormalization, see [25,27]).

The duality in Eq. (8) has already been sucessfully tested in the computation of Weyl anomalies of the boundary CFT via regularization of the bulk action. The calculation outlined in [4] and carried out explicitly by Henningson and Skenderis [29] makes use of diffeomorphism invariance in the bulk. Specifically, by looking at the logarithmically divergent terms in the Einstein-Hilbert action evaluated on a general asymptotically AdS solution [28], the variation of the action under Weyl transformations can be computed and the expected expressions for Weyl anomalies in various dimensions were obtained [29]. In particular, for $d=2$, the result (6) was reproduced with a central charge given by (1), consistent with the expected Weyl anomaly for a CFT realizing the Brown-Henneaux [1] asymptotic conformal symmetry.

The correspondence (8) is, however, considerably stronger than just a relation between the Weyl anomalies as it implies a direct equivalence between the generating functionals. In this paper, we will focus on the induced boundary effective action and its relation to the generating functional[3] $W[g]$ given in Eq. (7). An advantage of working within the $\mathrm{AdS}_3/\mathrm{CFT}_2$ framework is that the dependence of $W_P[g]$ [Eq. (7)] on the underlying moduli of the metric on Σ is well understood, and this can be contrasted with the corresponding dependence of the bulk action $I_{\rm reg}[g]$. Working within the Chern-Simons formalism, we will show that this moduli dependence is precisely that of the Polyakov action (7), a relation that has also been argued to hold by Skenderis and Solodukhin [25] using a different approach.

The appearance of the Polyakov functional through the relation (8) may be anticipated by observing that, from the point of view of the boundary CFT, the right hand side of Eq. (8) represents a particular covariant 3D "localization" of the generically nonlocal generating functional $W[g]$. To interpret this it is helpful to recall that the standard means of localizing the Polyakov action involves the introduction of a Liouville field ϕ. If the background metric on Σ is g_{ij}, then

[1]Such global data need to be specified in prescribing $\langle T\rangle$ since the stress-energy tensor undergoes Casimir-type shifts on changing the topology.

[2]In general, it may be necessary to specify topological data γ and to sum over inequivalent bulk topologies having the same boundary [33], although we will not need to consider the latter cases here.

[3]See [25,27,31] for other work on the induced effective action via AdS/CFT in various dimensions.

we consider a new background $e^{\phi}g_{ij}$ with constant curvature. The Liouville action (dropping the potential) for ϕ is then given by

$$W_L[g,\phi]=-\frac{c}{48\pi}\int_{\Sigma}d^2z\sqrt{-|g|}\left[\frac{1}{2}g^{ij}\nabla_i\phi\nabla_j\phi+\phi\mathcal{R}[g]\right], \tag{9}$$

which, on integrating out ϕ, reduces to the nonlocal Polyakov action (7) for g_{ij}. This procedure can be realized within the AdS/CFT correspondence if we consider the dynamics on a regulated boundary Σ_{r_0} at fixed radial coordinate $r=r_0$ in the asymptotic regime. We can regard the radial dependence as described by a field $r_0=r_0(\Sigma_{r_0})$. It follows from the form of the asymptotically AdS bulk metric (4) that, under local conformal transformations on Σ_{r_0}, r_0 transforms as a Liouville field. This identification of the bulk radial coordinate with a Liouville field [37] is simply the standard uv/ir correspondence [35] with radial shifts translated to scale transformations on the boundary. The corresponding behavior of regulating surfaces was discussed in some detail in [36] (see also [18]). This picture provides qualitative evidence for the appearance of the Polyakov action as the boundary generating functional. The main aim of this paper will be to verify this relation in detail.

Following the same theme, there is in the present context another aspect of the bulk "localization" which will be of interest. Recall that the Polyakov action evaluated in a light-cone gauge background was originally constructed [32] as a gravitational Weso-Zununo-Witten (WZW) model, and thus has a natural local representation in 3D. This is the appropriate background geometry on Σ in which to make explicit the dependence on "complex" structure moduli and we will find that there is a nice mapping between these chiral and antichiral moduli and the degrees of freedom entering via the two Chern-Simons fields which arise in the first-order formulation of the bulk dynamics. This in part motivates our approach to the calculation of the bulk action in Eq. (8) which makes use of the Chern-Simons formalism. The relevance of this formalism for realizing the holomorphic factorization associated with the generating functional of a CFT was also discussed recently in [31].

An important issue which arises in interpreting Eq. (8) is that fixing the boundary conformal structure $[g_{ij}]$ does not in general specify a unique bulk continuation [28], due to the anomalous transformation properties of $\langle T\rangle$, as discussed above. This concerns the first variation of Eq. (8),

$$\langle T\rangle_{\mathrm{CFT}}\sim\frac{1}{\sqrt{-|g|}}\frac{\delta I_{\mathrm{reg}}[g]}{\delta g}, \tag{10}$$

and we will show that bulk Brown-Henneaux transformations that shift $\langle T\rangle$ correctly maintain the correspondence in Eq. (10). Specifically, the ambiguities associated with bulk Brown-Henneaux diffeomorphisms enter, as expected, via local conformal transformations on the boundary, i.e. the moduli determine only the conformal class of the boundary metric and Brown-Henneaux diffeomorphisms amount to free-field shifts in the corresponding conformal mode.

In concluding this section, it is worth noting that, when using a conformal gauge for the boundary metric, the Polyakov generating functional we obtain reduces to that of Liouville theory, where the presence or otherwise of the potential term depends on the renormalization condition for the two-dimensional cosmological constant. However, it is important to realize that the Liouville theory found here should not be identified with that obtained by Coussaert, Henneaux, and van Driel (CHvD) [38] describing the asymptotic dynamics of gravity in AdS_3. More precisely, in [38] the Liouville dynamics describes gravitational perturbations propagating on a fixed 2D background metric g_{ij} (see [39,26,30] for a generalization of [38] to arbitrary backgrounds). Because of the lack of local degrees of freedom in 3D gravity, these perturbations actually encode the entire bulk dynamics (up to holonomies), for a given set of boundary conditions. In [38] only the constraint equations were solved, leading to dynamical fluctuations on the boundary. In contrast, in our analysis, the entire set of 3D equations are solved, for a given 2D background metric g_{ij}, thus obtaining the generating functional $W[g_{ij}]$ for stress tensor correlators. For a proper comparison the CHvD action should be put on shell, and the on-shell correspondence between bulk geometries and Liouville solutions has been discussed in [22,36].

The paper is organized as follows. In Sec. II we turn to the general parametrization of bulk metrics, introducing a first-order form for the expansion of Fefferman and Graham [28]. We then review in Sec. III how the generating functional $W_P[g]$ encodes the local moduli of the boundary metric in explicit form. In the present context we will restrict attention to simple boundary topologies, and the moduli are conveniently encoded in a chiral parametrization of the metric, the analogue of a Beltrami parametrization for Riemann surfaces. In Sec. IV we derive the dependence of $I_{\mathrm{reg}}[g]$ on the boundary moduli using a covariant Chern-Simons construction; the result is consistent with Eq. (7). We also discuss the action of Brown-Henneaux diffeomorphisms on the space of solutions, and the relation to the expectation value of the boundary stress tensor. Section VI contains some concluding remarks concerning global data.

II. CHERN-SIMONS FORMULATION AND A FIRST-ORDER EXPANSION

To implement the AdS/CFT prescription one needs to reconstruct a bulk Einstein metric with negative cosmological constant, given a representative $g_{ij}^{(0)}$ of the conformal structure at infinity. In the Chern-Simons formulation, this problem can be rephrased in first-order form, where the gauge freedom can be used to generate a solution in a straightforward manner. Before describing this, we recall some details of the conventional metric formalism.

Finding a bulk solution given a fixed boundary conformal structure $[g^{(0)}]$ is a nontrivial problem in general, although a particular existence theorem was obtained by Graham and

Lee [40] for the special case of boundary metrics sufficiently close to the standard one on the sphere. However, it was shown by Fefferman and Graham [28] (see also [41]) that an asymptotic expansion near infinity can be constructed starting from an arbitrary boundary metric. This expansion has the special form

$$ds^2=dr^2+e^{2r}g_{ij}(r,x)dx^idx^j, \tag{11}$$

with

$$g_{ij}(r,x)=g_{ij}^{(0)}+e^{-2r}g_{ij}^{(2)}+\cdots, \tag{12}$$

where r is a radial coordinate, related to that used in [28] by $e^{2r}\sim 1/\rho$. The Einstein equations in general determine almost all of the coefficients $g_{ij}^{(n)}$ as covariant functions of $g_{ij}^{(0)}$ and its derivatives. The coefficient $g_{ij}^{(0)}$, which is defined up to a Weyl rescaling, determines the boundary conformal structure and is identified with the boundary metric.

For the (2+1)D case of particular interest here, this structure is known to simplify with the expansion truncating at $O(e^{-4r})$ (for pure Einstein metrics which are all locally AdS). We can write [24,25]

$$ds^2=dr^2+(e^{2r}g_{ij}^{(0)}+g_{ij}^{(2)}+e^{-2r}g_{ik}^{(2)}g_j^{(2)k})dx^idx^j. \tag{13}$$

Given the boundary metric $g_{ij}^{(0)}$, the Einstein equations determine the trace of $g_{ij}^{(2)}$ automatically (which is enough to determine the boundary Weyl anomaly [29]), but do not specify the trace-free part of $g_{ij}^{(2)}$ [28,29]. This "ambiguity" is equivalent to the choice of a quadratic form[4] on the boundary [24], which as noted in the Introduction transforms anomalously under Brown-Henneaux diffeomorphisms. This is consistent with the corresponding transformation of the boundary stress tensor under local conformal mappings, and these quantities are identified via the AdS/CFT correspondence [27,26,30,31]. We will return to this issue in Sec. IV.

A Fefferman-Graham-type expansion can also be formulated in first-order form in terms of connections. Recall that a 3D Lorentzian geometry can be written in terms of two flat SL(2,$\mathfrak{R}$) gauge fields A and $\bar{A}$, i.e., the dreibein and spin connection are given by

$$e_\mu=(A_\mu-\bar{A}_\mu)/2, \quad w_\mu=(A_\mu+\bar{A}_\mu)/2, \tag{14}$$

where $A=A^aJ_a$, $\bar{A}=\bar{A}^aJ_a$. We use the SL(2,$\mathfrak{R}$) basis $\{J_+,J_-,J_3\}$ with $[J_+,J_-]=2J_3$, $[J_3,J_\pm]=\pm J_\pm$, and $\mathrm{Tr}(J_+J_-)=1$, $\mathrm{Tr}(J_3J_3)=1/2$.

The Einstein-Hilbert action is then equal to the difference of two Chern-Simons actions, supplemented by boundary terms which depend on the boundary conditions. Specifically, if we normalize the Einstein action as

$$I_{\mathrm{EH}}=\frac{1}{16\pi G}\int\sqrt{-|g|}(R-2\Lambda)+\text{boundary terms}, \tag{15}$$

with $\Lambda=-1$ (in units where $l=1$), then

$$\frac{4\pi}{k}I_{\mathrm{EH}}=I_{\mathrm{CS}}[A]-I_{\mathrm{CS}}[\bar{A}]+\text{boundary terms}, \tag{16}$$

where $k=1/4G$ is the level, and $I_{\mathrm{CS}}[A]$ is the Chern-Simons functional,

$$I_{\mathrm{CS}}[A]=\int_M AdA+\frac{2}{3}A^3, \tag{17}$$

with M the bulk 3D manifold. We will fix the required boundary terms once we have considered the asymptotic form of the on-shell connection in comparison with the Fefferman-Graham expansion in Eq. (13).

Recall that the space of solutions of three-dimensional Chern-Simons theory is the set of flat $SL(2,\mathfrak{R})$ connections $A=g^{-1}dg$ plus holonomies. We shall consider the pure AdS case with no holonomies, although we will comment on their inclusion in Sec. V. Note that the breakdown of gauge invariance at the boundary nonetheless prevents us from setting $g=1$.

The radial dependence of g is itself given by a gauge transformation, and an appropriate asymptotic radial coordinate can be introduced as follows [42,38]:

$$A=e^{-rJ_3}\alpha e^{rJ_3}+drJ_3, \tag{18}$$

$$\bar{A}=e^{rJ_3}\bar{\alpha}e^{-rJ_3}-drJ_3, \tag{19}$$

where α and $\bar{\alpha}$ are both SL(2,$\mathfrak{R}$)-valued flat connections defined on the surface at fixed[5] r.

Expanding α and $\bar{\alpha}$ in the basis J_+,J_-,J_3, we obtain the series expansion for the 3D forms A and $\bar{A}$,

$$A=(dr+\alpha^3)J_3+e^{-r}\alpha^+J_++e^r\alpha^-J_-, \tag{20}$$

$$\bar{A}=(-dr+\bar{\alpha}^3)J_3+e^r\bar{\alpha}^+J_++e^{-r}\bar{\alpha}^-J_-, \tag{21}$$

where we have used $e^{-rJ_3}J_\pm e^{rJ_3}=e^{\mp r}J_\pm$. This series is consistent with the structure of the Fefferman-Graham expansion (13), and the corresponding dreibein $e=(A-\bar{A})/2$ becomes

$$e=\left(dr+\frac{1}{2}(\alpha^3-\bar{\alpha}^3)\right)J_3+\frac{1}{2}e^r(\alpha^-J_--\bar{\alpha}^+J_+)$$
$$+\frac{1}{2}e^{-r}(\alpha^+J_+-\bar{\alpha}^-J_-). \tag{22}$$

In analogy with the metric treatment, we define the conformally induced boundary 2D zweibein as the leading term in the $r\rightarrow\infty$ limit,

[4]It has recently been emphasized that this ambiguity is equivalently understood in Euclidean signature as a choice of projective structure on the boundary [31].

[5]In general, to avoid singularities at $r=0$, and hence the introduction of other degrees of freedom, one must allow α and $\bar{\alpha}$ to depend on r near the horizon. We ignore this subtlety here as we are concerned with the asymptotics near $r\rightarrow\infty$.

$$e^{(0)}=\frac{1}{2}(\alpha^{-}J_{-}-\overline{\alpha}^{+}J_{+}), \tag{23}$$

where $\Sigma=\partial M$ is identified as the conformal boundary of the Poincaré patch, and not the global boundary of AdS_3. Note that this procedure yields a completely general zweibein on the surface Σ, which is determined by two independent one-forms from α and $\overline{\alpha}$. Note, however, that of the four components entering the zweibein, only three are (locally) independent as one may be decoupled in the line element by 2D Lorentz transformations.

At this point, it is clear that, since we wish to fix the conformal structure at the boundary, it is necessary to impose suitable (conformal) Dirichlet conditions on α^{-} and $\overline{\alpha}^{+}$. Actually, since we want to fix the conformal structure, but not a particular representative, we write the corresponding condition in terms of A and $\overline{A}$,

$$\delta(A-\overline{A})^{+}=0, \quad \delta(A-\overline{A})^{-}=0. \tag{24}$$

The remaining boundary condition on A^3 and $\overline{A}^3$ follows by comparing the metric determined by Eq. (22) with the block-diagonal metric of the Fefferman-Graham expansion (13). We see that the off-diagonal term $dr(\alpha_i^3-\overline{\alpha}_i^3)dx^i$ arising from Eq. (22) is canceled in the asymptotic regime on imposing the "Neumann" boundary condition,

$$\alpha^3-\overline{\alpha}^3=0. \tag{25}$$

We will see later that this condition is also sufficient to ensure that the boundary metric is torsion-free. These boundary conditions lead to a 3D geometry parametrized by the metric [following from Eq. (22)],

$$ds^2=dr^2-e^{2r}\alpha^{-}\overline{\alpha}^{+}+(\alpha^{-}\alpha^{+}+\overline{\alpha}^{+}\overline{\alpha}^{-}) -e^{-2r}\alpha^{+}\overline{\alpha}^{-}, \tag{26}$$

which we see is consistent with the truncated expansion in Eq. (13).

The appropriate action for imposing Dirichlet conditions on $(A-\overline{A})^{\pm}$ and a Neumann condition on $(A-\overline{A})^3$ is[6]

$$\frac{4\pi}{k}I=I[A]-I[\overline{A}]+\int_{\partial M}\Big(A^{+}\wedge\overline{A}^{-} +A^{-}\wedge\overline{A}^{+}-\frac{1}{2}A^3\wedge\overline{A}^3\Big). \tag{27}$$

The negative sign in front of the term $A^3\wedge\overline{A}^3$ is required to impose the Neumann condition on $A^3-\overline{A}^3$. In other words, the variation of the action (27) will have a term $\int_{\partial M}(A^3-\overline{A}^3)\delta(A^3+\overline{A}^3)$. We demand the action to be stationary with respect to arbitrary variations of $A^3+\overline{A}^3$ at the boundary. The condition $\alpha^3-\overline{\alpha}^3=0$ then follows as an equation of motion in the boundary theory.

It is important to note that this treatment has been manifestly covariant on the 2D surface Σ. In particular, we have not needed any coordinate choice to fix the boundary terms.

III. BOUNDARY MODULI AND THE POLYAKOV ACTION

Before turning to the analysis of the bulk action (27), we consider the expected form of the boundary generating functional (i.e., the Polyakov action) in suitable test geometries on Σ. The boundary metric (23) determined in the previous section is sufficiently general to allow an arbitrary dependence on local moduli.

To make this dependence manifest, we first recall that in 2D any metric is locally conformally flat, and thus can be represented in suitable coordinates as

$$ds^2=e^{\varphi'}dx d\overline{x}, \tag{28}$$

where $\varphi'(x,\overline{x})$ is the conformal mode. We are considering boundary geometries with Lorentzian signature, and so $\{x,\overline{x}\}$ are independent real light-cone coordinates, although the transition to Euclidean signature (for low genus) will be clear from the notation.

The parametrization (28), while simple, hides the dependence on the "complex" structure of the surface Σ. To make this manifest, we consider a quasiconformal mapping $\{x,\overline{x}\}\to\{z(x,\overline{x}),\overline{z}(x,\overline{x})\}$, where x and $\overline{x}$ satisfy the equations

$$\partial_{\overline{z}}x=\mu\partial_z x, \quad \partial_z\overline{x}=\overline{\mu}\partial_{\overline{z}}\overline{x}, \tag{29}$$

where μ and $\overline{\mu}$ are independent, real, and bounded ($|\mu|$ and $|\overline{\mu}|<1$) functions, and are the (Lorentzian) analogues of Beltrami parameters. The metric (28) then takes the form

$$ds^2=e^{\varphi}(dz+\mu d\overline{z})(d\overline{z}+\overline{\mu}dz). \tag{30}$$

We will restrict our attention to boundaries of cylindrical topology, and thus this parametrization is sufficient to describe an arbitrary metric on Σ given a fixed coordinate system $\{z,\overline{z}\}$. The metric (30) also makes explicit the dependence on a representative of the conformal class (φ), and on

[6]A comparison with the dreibein formulation may be useful here. The 3D Einstein-Hilbert action written in frame variables is $\int R_a\wedge e^a$ where $R^a=dw^a+(1/2)\epsilon^a_{bc}w^b\wedge w^c$. Varying this action one picks up the boundary term $\int\delta w_a\wedge e^a$ and thus either a Dirichlet condition on w^a or a Neumann condition on e^a is required. Conversely, one can write an action appropriate for Dirichlet conditions on e^a or Neumann conditions on w^a by adding the boundary term $-\int w_a\wedge e^a$. In the Chern-Simons formulation we consider the action $I_{\pm}=I_{\mathrm{CS}}[A]-I_{\mathrm{CS}}[\overline{A}]\pm\int A\wedge\overline{A}$ whose variation yields the boundary term $\int(A\pm\overline{A})\wedge\delta(A\mp\overline{A})$. The sign has to be chosen according to whether we want to fix the connection $A+\overline{A}$ or the dreibein, $A-\overline{A}$ and to whether these conditions are Dirichlet or Neumann. In our situation, we have a mixed case with Dirichlet conditions on $(A-\overline{A})^{\pm}$ and Neumann conditions on $(A-\overline{A})^3$. This leads directly to the action (27).

the deformations of the "complex" structure $(\mu,\bar{\mu})$ specifying the conformal class. These three moduli map to the three (independent) components of the boundary zweibein (23).

We will find it useful to consider particular examples of the general boundary geometry (30). Recalling that conformal symmetry in 2D is generated by chiral and antichiral stress tensors $T(z)$ and $\overline{T(z)}$, it will be sufficient to consider the dependence on the conformal mode φ and one chiral parameter μ. These components of the metric are sources for the stress-energy tensors $T^{\varphi}(z,\bar{z})$ and $T(z)$, respectively. Note that the distinction here is tied up with the question of whether Eq. (29) has a global solution, or in other words whether or not ds^2 and the metric $d\tilde{s}^2=\exp(\varphi)dzd\bar{z}$ are in the same conformal class (see, e.g., [43]).

With this information in hand, we can determine the form of the Polyakov generating functional $W_P[\varphi,\mu,\bar{\mu}]$ which exhibits the explicit dependence on the moduli. The general decomposition of W_P for the metric (30) was obtained by Verlinde [44], but for simplicity we will restrict our attention to boundary geometries depending on either φ or μ.

A. Conformal gauge

Consider first the conformal gauge where, with a signature convention chosen for later convenience, the metric takes the form,

$$d\tilde{s}^2=-\exp(\varphi)dzd\bar{z}. \tag{31}$$

The corresponding zweibein (23) is given by

$$\begin{aligned} e^{(0)}&=\frac{1}{2}(\alpha^-J_- - \bar{\alpha}^+J_+)\\ &=\frac{1}{2}(e^{\phi}dzJ_- - e^{\bar{\phi}}d\bar{z}J_+), \end{aligned} \tag{32}$$

where $\varphi=\phi+\bar{\phi}$, and we see that in general the conformal mode will receive contributions from both gauge fields A and $\bar{A}$ entering the bulk action. In this background the Polyakov action reduces to that of Liouville theory,

$$W_P[\varphi]=-\frac{c}{96\pi}\int_{\Sigma}d^2z[\partial\varphi\bar{\partial}\varphi+\lambda^2e^{\varphi}], \tag{33}$$

where $\partial=\partial/\partial z$ and $\bar{\partial}=\partial/\partial\bar{z}$. There is clearly no curvature coupling as the reference background in this case (31) is flat.

B. Light-cone gauge

If instead we consider Polyakov's light-cone gauge, with

$$d\tilde{s}^2=-dzd\bar{z}-\mu d\bar{z}^2, \tag{34}$$

the corresponding zweibein (23) is given by

$$\begin{aligned} e^{(0)}&=\frac{1}{2}(\alpha^-J_- - \bar{\alpha}^+J_+)\\ &=\frac{1}{2}(dz+\mu d\bar{z})J_- - \frac{1}{2}d\bar{z}J_+, \end{aligned} \tag{35}$$

which makes the chiral structure quite manifest. We see that the nontrivial dependence on the "Beltrami" parameter μ enters only via A and not $\bar{A}$. In contrast, it is clear that in an antichiral gauge, the dependence on $\bar{\mu}$ would enter via $\bar{A}$. Recalling that μ and $\bar{\mu}$ act as sources for the chiral and antichiral stress tensors T and $\bar{T}$, which generate the two copies of the Virasoro algebra comprising the conformal group, this correspondence provides a simple map from "holomorphic" factorization on the boundary to the obvious factorization of the two Chern-Simons systems in the bulk [31].

This conclusion is perhaps not as obvious as it might seem due to the collapse of the bulk dynamics to the boundary, where the boundary terms couple A and $\bar{A}$. What we observe from Eqs. (32) and (35) is that as expected the conformal mode reflects a violation of this factorization, while the light-cone gauge is special in that it is preserved. We might anticipate that the simple factorization observed in the light-cone gauge is related to Polyakov's observation (see also [44]) of a SL(2,$\mathfrak{R}$) structure in 2D gravity. However, the fact that Eq. (35) still involves both gauge fields A and $\bar{A}$ makes the relation obscure,[7] and we will not explore this issue further here.

Returning to the generating functional, when evaluated in the background (35), one obtains the light-cone gauge Polyakov action [32] (see also [45])

$$W_P[\mu]=-\frac{c}{48\pi}\int_{\Sigma}d^2z\frac{\partial^2x}{\partial x}\partial\mu, \tag{36}$$

where we have ignored the (constant) potential term and $x=x(z,\bar{z})$ satisfies Eq. (29).

IV. THE HOLOGRAPHIC GENERATING FUNCTIONAL

We now return to the bulk action (27), and determine its dependence on the boundary moduli for comparison with Eqs. (33) and (36). The bulk equations of motion are simply $d\alpha+\alpha\wedge\alpha=0$ and similarly for $\bar{\alpha}$, and it is useful to write these equations explicitly in the SL(2,$\mathfrak{R}$) basis,

$$d\alpha^3+2\alpha^+\wedge\alpha^-=0,\quad d\bar{\alpha}^3+2\bar{\alpha}^+\wedge\bar{\alpha}^-=0, \tag{37}$$

$$d\alpha^- - \alpha^3\wedge\alpha^-=0,\quad d\bar{\alpha}^+ + \bar{\alpha}^3\wedge\bar{\alpha}^+=0, \tag{38}$$

$$d\alpha^+ + \alpha^3\wedge\alpha^+=0,\quad d\bar{\alpha}^- - \bar{\alpha}^3\wedge\bar{\alpha}^-=0. \tag{39}$$

[7]Note that the geometric data for Σ can be combined to form a *single* flat SL(2,$\mathfrak{R}$) connection $(A=-i\omega J_3+e^+J_+ +e^-J_-)$, where the constraint $dA+A\wedge A=0$ arises directly in Hamiltonian Chern-Simons theory.

The components of the zweibein α^- and $\bar{\alpha}^+$ appear in Eq. (38). In view of Eq. (25), the pullback of Eq. (38) to the boundary can be regarded as the torsion-free condition for the spin connection coefficient $w\equiv\alpha^3$, with associated two-form curvature $\mathcal{R}=dw$, and this observation will be useful in what follows.

The on-shell value of the Einstein-Hilbert action can be obtained straightforwardly by substituting Eqs. (20) and (21) in Eq. (27). We obtain (suppressing the $\wedge$ product)

$$\frac{4\pi}{k}I=-\frac{1}{3}\int_M \mathrm{tr}\{(\alpha)^3-(\bar{\alpha})^3+3[(\alpha)^2+(\bar{\alpha})^2]J_3dr\}$$
$$+\int_{\partial M}e^{2r}\alpha^-\bar{\alpha}^++e^{-2r}\alpha^+\bar{\alpha}^--\frac{1}{2}\alpha^3\bar{\alpha}^3. \tag{40}$$

As expected, this expression contains finite terms along with a quadratic (in e^r) and logarithmic divergence. Note that α and $\bar{\alpha}$ are still restricted by the flatness conditions (37),(38),(39).

A. Weyl anomaly

From the on-shell action (40), it is straightforward to recover the expression for the boundary Weyl anomaly which, following the discussion of [4,29], is given by the coefficient of the logarithmically divergent term, as all the other terms are Weyl invariant. The relevant term is $\int \mathrm{Tr}(\alpha^2+\bar{\alpha}^2)\wedge J_3dr=(1/2)\int dr\int_{\partial M}(\alpha^+\wedge\alpha^-+\bar{\alpha}^+\wedge\bar{\alpha}^-)$. Using the equations of motion (37) and the Neumann boundary condition (25) the coefficient reduces to $\int_{\partial M}dw=\int dr\int_{\partial M}\mathcal{R}$, yielding the Weyl anomaly

$$\langle T_i^i\rangle=-\frac{c}{24\pi}\mathcal{R}^{(0)}, \tag{41}$$

in agreement with the calculation in [29], where $\mathcal{R}^{(0)}$ is the boundary Ricci scalar, and we have used $c=6k$.

B. Boundary generating functional

The other divergent term (which is Weyl invariant) is $e^{2r}\int\alpha^-\wedge\bar{\alpha}^+$. This contribution is precisely the cosmological constant term in two dimensions because $\alpha^-\wedge\bar{\alpha}^+$ is equal to the determinant of the zweibein (23). The appearance of this divergence is consistent with the analysis of [34], and it can be associated with a divergent bare cosmological constant in the CFT. Therefore, we shall renormalize this term, canceling the divergence but retaining a finite cosmological constant λ^2. As in 2D quantum gravity, we find that the Liouville potential comes from this term.

The finite part of the action is therefore

$$\frac{4\pi}{k}I_{\mathrm{reg}}=-\int_M(\alpha^3\alpha^+\alpha^--\bar{\alpha}^3\bar{\alpha}^+\bar{\alpha}^-)$$
$$-\frac{1}{2}\int_{\partial M}(\alpha^3\bar{\alpha}^3+2\lambda^2\alpha^-\bar{\alpha}^+). \tag{42}$$

and, since $\alpha=g^{-1}dg$ and $\bar{\alpha}=\bar{g}^{-1}d\bar{g}$, it depends only on the boundary values of the fields. Note that in general the group elements g and $\bar{g}$ are not single valued.

Localizing the action (42) in general requires a parametrization of the holonomies of the gauge fields, thus specifying the topology in terms of the conjugacy classes of SL(2,$\Re$). We will return to this issue in Sec. V, but as mentioned earlier it will not be necessary to explicitly specify a particular class as we are interested in the dependence of I_{reg} on local quantities. It will be sufficient here to choose a suitable local patch of AdS$_3$, which is consistent with a choice of boundary coordinates.

1. Boundary conformal gauge

The simplest choice is one in which the metric may be written in conformally flat form. The appropriate boundary zweibein (32) is given by

$$e^{(0)}=\frac{1}{2}(e^{\phi}dxJ_--e^{\bar{\phi}}d\bar{x}J_+), \tag{43}$$

where the conformal mode of the metric is then $\varphi=\phi+\bar{\phi}$. Recalling the discussion of Sec. III, this coordinate choice hides the moduli in the conformal mode φ, as in Eq. (28). This coordinate system is convenient for deriving the effective action, and the dependence on moduli can then be made explicit by performing a quasiconformal transformation.

This choice of the boundary metric may be achieved by making use of the following Gauss decomposition of the SL(2,$\Re$) group elements:

$$g=e^{xJ_-}e^{\phi J_3}e^{y'J_+},\quad \bar{g}=e^{\bar{x}J_+}e^{-\bar{\phi}J_3}e^{\bar{y}'J_-}, \tag{44}$$

which yields the following expressions for the Chern-Simons currents:

$$\alpha^+=e^{-\phi}(-y^2dx+dy),\quad \bar{\alpha}^+=e^{\bar{\phi}}d\bar{x},$$
$$\alpha^-=e^{\phi}dx,\quad \bar{\alpha}^-=e^{-\bar{\phi}}(-\bar{y}^2d\bar{x}+d\bar{y}), \tag{45}$$
$$\alpha^3=-2ydx+d\phi,\quad \bar{\alpha}^3=2\bar{y}d\bar{x}-d\bar{\phi}.$$

Here we have made the convenient replacements $y'=e^{-\phi}y$ and $\bar{y}'=e^{-\bar{\phi}}\bar{y}$.

The action written in terms of these fields now takes a local form on Σ given by

$$\frac{4\pi}{k}I_{\mathrm{reg}}[\varphi,x,\bar{x};y,\bar{y}]=\frac{1}{2}\int 2\varphi(dx\wedge dy-d\bar{x}\wedge d\bar{y})$$
$$+(4y\bar{y}+\lambda^2e^{\varphi})dx\wedge d\bar{x} \tag{46}$$

with $\varphi=\phi+\bar{\phi}$ the conformal mode in the metric. Note that the mode $\phi-\bar{\phi}$ has decoupled, which is a consequence of the Neumann boundary condition (25) on the spin connection, and our neglect of holonomies (see, e.g., [23]).

The variables y and $\bar{y}$ are auxiliary fields which can be eliminated by their own equations of motion. Using the diffeomorphism invariance of Eq. (46) to choose x and $\bar{x}$ as coordinates, the action reads

$$I_{\rm reg}[\varphi,y,\bar{y}]=\frac{k}{8\pi}\int d^2x[2\varphi(\bar{\partial}y+\partial\bar{y})+4y\bar{y}+\lambda^2e^{\varphi}], \quad (47)$$

where the derivatives are with respect to x and $\bar{x}$. The fields y and $\bar{y}$ can be integrated out and (ignoring boundary terms) we obtain the Liouville action for the conformal mode φ:

$$I_{\rm reg}[\varphi]=-\frac{c}{96\pi}\int d^2x[\partial\varphi\bar{\partial}\varphi+\lambda^2e^{\varphi}], \quad (48)$$

where we have used $k=1/(4G)$ and the value of c given in Eq. (1). We see that this agrees with the evaluation of the Polyakov action in the background (43), as given in Sec. III. It is also straightforward to check that the constraint equations for $y=\partial\varphi/2$ and $\bar{y}=\bar{\partial}\varphi/2$ do imply $\alpha^{(3)}=\bar{\alpha}^{(3)}$, as required by the 3D variational principle. This result, combined with the fact that the remaining boundary conditions specify a fixed boundary zweibein, imply that the action (48) is indeed to be interpreted as a functional $I_{\rm reg}[\varphi]$ which generates correlation functions of the boundary stress tensor.

We can now verify the result for the Weyl anomaly (41), by considering the variation of the generating functional $I_{\rm reg}[\varphi]$ directly. We obtain

$$\langle T_i^i\rangle=-2e^{-\varphi}\frac{\delta I_{\rm reg}[\varphi]}{\delta\varphi}=-\frac{c}{24\pi}e^{-\varphi}\partial\bar{\partial}\varphi, \quad (49)$$

which is equivalent to Eq. (41) in the background (43), where $\mathcal{R}^{(0)}=e^{-\varphi}\partial\bar{\partial}\varphi$, and we have dropped the dependence on the cosmological constant.

2. Boundary light-cone gauge

To exhibit the dependence on the chiral moduli μ, we consider the alternate boundary zweibein (35)

$$e^{(0)}=\frac{1}{2}(dz+\mu d\bar{z})J_- - \frac{1}{2}d\bar{z}J_+, \quad (50)$$

which leads to the light-cone gauge metric (34). The decomposition of the group element for this case is conveniently achieved by noting that Eq. (50) may be obtained from the zweibein (43) in conformal gauge by the quasiconformal mapping

$$\phi\to-\ln\partial x, \quad \bar{\phi}\to 0, \quad (51)$$

where x satisfies the Beltrami-type equation (29), while $\bar{x}=\bar{z}$, and ∂ now denotes $\partial/\partial z$.

The potential term is unchanged under this mapping, so we concentrate on the kinetic part. Evaluating Eq. (46) in this background, we find

$$I_{\rm reg}[x;y,\bar{y}]=\frac{k}{8\pi}\int d^2z[2\ln\partial x(\partial x\bar{\partial}y-\bar{\partial}x\partial y+\partial\bar{y}) + 4y\bar{y}\partial x], \quad (52)$$

and, again ignoring boundary terms, we can integrate out y and $\bar{y}$. Making use of Eq. (29) we find $2y=-\partial^2x/(\partial x)^2$ and $2\bar{y}=-\partial\mu$, leading to

$$I_{\rm reg}[\mu]=-\frac{c}{48\pi}\int_\Sigma d^2z\frac{\partial^2 x}{\partial x}\partial\mu, \quad (53)$$

which we recognize as the light-cone gauge Polyakov action $W_P[\mu]$ given in Eq. (36).

One can generalize this approach to a more general background, but these examples are sufficient for us to conclude that the dependence on the boundary moduli is correctly encoded in the bulk action, consistent with the AdS/CFT correspondence. However, as mentioned in Sec. III, specifying the boundary metric does not uniquely determine the bulk geometry due to the possibility for Brown-Henneaux transformations. We will now explore the consequences of this for the boundary generating functional.

C. Brown-Henneaux diffeomorphisms and stress tensor expectation values

The consistency of the results of the preceding section hides the ambiguity of the Fefferman-Graham expansion regarding the trace-free part of the first subleading term in the bulk metric. This ambiguity, equivalent to specifying the expectation value of the boundary stress tensor, is associated with the asymptotic symmetry of the space of bulk solutions under Brown-Henneaux diffeomorphisms [1], and the ambiguity can be rephrased as specifying the position of a particular solution along a Brown-Henneaux orbit.

The finite generalization of the mapping (3) corresponds to

$$T(z)\to(\partial f)^2T(f)-\frac{c}{24\pi}\{f,z\}, \quad (54)$$

where $f(z)$ represents a local diffeomorphism of the circle in the case that we start with a boundary of cylindrical topology, and $\{f,z\}$ is the Schwarzian derivative

$$\{f,z\}\equiv\frac{\partial^3 f}{\partial f}-\frac{3}{2}\left(\frac{\partial^2 f}{\partial f}\right)^2. \quad (55)$$

It is the Schwarzian term in the anomalous mapping (54) which leads to a finite charge [1] characterizing the position of a given solution on a Brown-Henneaux orbit. We will now explore the consequences of this mapping for the results of the previous section.

1. Brown-Henneaux diffeomorphisms acting on the currents

The Brown-Henneaux diffeomorphisms have a simple and direct action on the Chern-Simons currents. Let us go back to the expression for the currents (45) and impose the conditions $\alpha^3=\overline{\alpha}^3$ (which are equivalent to the equations of motion for y and $\overline{y}$). In conformal coordinates $x=z$ and $\overline{x}=\overline{z}$, these conditions read $y=(1/2)\partial\varphi$ and $\overline{y}=(1/2)\overline{\partial}\varphi$ and the currents take the form

$$\alpha^+=\tfrac{1}{2}e^{-\phi}[T^\varphi dz+\mathcal{R}d\overline{z}],$$
$$\alpha^-=e^{\phi}dz, \tag{56}$$
$$\alpha^3=-\partial\overline{\phi}dz+\overline{\partial}\phi d\overline{z}.$$

where $\mathcal{R}=\partial\overline{\partial}\varphi$ is the scalar coefficient of the two-form curvature of the background (43), and

$$T^\varphi=\frac{24\pi}{c}\langle T\rangle=\partial^2\varphi-\frac{1}{2}(\partial\varphi)^2 \tag{57}$$

can be identified with the Liouville stress-energy tensor of the conformal mode φ. Note that similar formulas arise in the ''antiholomorphic'' sector.

Now consider the following local conformal transformations:

$$z\to f(z),\quad e^{\phi}\to e^{\phi}(\partial f)^{-1},$$
$$\overline{z}\to\overline{f}(\overline{z}),\quad e^{\overline{\phi}}\to e^{\overline{\phi}}(\overline{\partial f})^{-1}. \tag{58}$$

We observe that the conformal factor φ then transforms appropriately as a Liouville field $e^{\varphi}\to e^{\varphi}(\partial f)^{-1}(\overline{\partial f})^{-1}$.

Both α^- and α^3 are trivially invariant under these transformations. In order to analyze the transformation of α^+ we note that the combination $e^{-\phi}dz$ has conformal dimension $(-2,0)$ while $e^{-\phi}d\overline{z}$ has conformal dimension $(-1,-1)$. Since T^φ has conformal dimension (2,0) and $\partial\overline{\partial}\phi$ has conformal dimension (1,1), the component α^+ transforms anomalously via the Schwarzian derivative. This is consistent with the discussion in Sec. I on recognizing that α^+ enters the subleading term in the metric and encodes the boundary stress-energy tensor via T^φ. This can be made more explicit by considering the form of the bulk metric.

2. *The general bulk metric and stress tensor expectation values*

We can insert the expressions for the currents into Eq. (26) to obtain the corresponding form of the bulk metric. We find,

$$ds^2_{3D}=dr^2-e^{2r+\varphi}dxd\overline{x}+\tfrac{1}{2}T^\varphi dx^2+\tfrac{1}{2}\overline{T}^\varphi d\overline{x}^2+\mathcal{R}dxd\overline{x}$$
$$-\tfrac{1}{2}e^{-2r-\varphi}(T^\varphi dx+\mathcal{R}d\overline{x})(\overline{T}^\varphi d\overline{x}+\mathcal{R}dx), \tag{59}$$

where, from the subleading terms, we identify the general form of the expansion given in Eq. (4). Indeed, the metric (59) is the general solution associated with a conformally flat boundary geometry, as determined recently in [46,31]. However, the mapping (58) implies that the metric (59), in which there is a direct correspondence between the subleading terms and the Liouville stress-energy tensor for the representative boundary metric, is not completely general. The fact that T^φ transforms anomalously is conveniently encoded in Eq. (59) by writing it in a new coordinate system $\{x,\overline{x}\}\to\{z=f(x),\overline{z}=\overline{f}(\overline{x})\}$,

$$ds^2_{3\,\mathrm{D}}=dr^2-e^{2r+\varphi}dzd\overline{z}+\tfrac{1}{2}(T^\varphi+\tilde{T})dz^2+\tfrac{1}{2}(\overline{T}^\varphi+\tilde{\overline{T}})d\overline{z}^2$$
$$+\mathcal{R}dzd\overline{z}-\tfrac{1}{2}e^{-2r-\varphi}[(T^\varphi+\tilde{T})dz+\mathcal{R}d\overline{z}]$$
$$\times[(\overline{T}^\varphi+\tilde{\overline{T}})d\overline{z}+\mathcal{R}dz], \tag{60}$$

where $\partial=\partial/\partial z$, and $\tilde{T}=\tilde{T}(z)$ is an arbitrary chiral function given by $\tilde{T}(z)=-c\{x,z\}/24\pi$. This representation makes explicit the anomalous shift (54) of the subleading terms under Brown-Henneaux diffeomorphisms, in accordance with the mapping $T(z)=\langle T_{\mathrm{CFT}}\rangle$ to the dual CFT [22,24,5,26,27].

In order to understand how the generating functional is consistent with this shift [via Eq. (10)], it is convenient to consider a particular example. Specifically, setting φ to zero in Eq. (60), the metric reduces to (dropping the tildes)

$$ds^2_{3\,\mathrm{D}}=dr^2-(e^{2r}+e^{-2r}T\overline{T})dzd\overline{z}+\tfrac{1}{2}Tdz^2+\tfrac{1}{2}\overline{T}d\overline{z}^2, \tag{61}$$

which is equivalent to the general solution with a flat boundary geometry obtained in [24]. In interpreting Eq. (10) for such a background it is important to recall that the boundary metric is specified only up to a Weyl rescaling. Thus, the actual boundary metric in Eq. (61) is any geometry in the same conformal class as the flat one. From Eq. (60), we see that the ambiguity reflects additive ''free-field'' shifts in the conformal mode φ. This explains why the generating functional is actually nonvanishing for Eq. (61) and leads on variation, via Eq. (10), to nonzero expectation values for $\langle T_{\mathrm{CFT}}\rangle$ and $\langle\overline{T}_{\mathrm{CFT}}\rangle$. Following [22], this interpretation of the allowed free-field shifts in φ may also be understood in terms of the connection between the oscillator modes in an expansion of the Liouville field φ, and the bulk ''gravitational wave'' perturbations specified by T and $\overline{T}$.

In the discussion above, we concentrated for simplicity on the boundary conformal gauge. If we instead consider boundary geometries in the conformal class of the light-cone metric (34), then the situation is similar with the complication that local conformal transformations on the boundary amount to ''chiral'' free-field shifts in the conformal mode. The stress-energy tensor $T(x)=c\{x,z\}/24\pi$ given by the variation of Eq. (53), then shifts according to Eq. (54) via the Schwarzian, which is now chiral with respect to the light-cone structure determined by μ (see, e.g., [44]).

V. GLOBAL ISSUES AND CONCLUDING REMARKS

Making use of the Chern-Simons formalism, we have shown that the dependence of the regularized bulk gravitational action on local moduli of the boundary metric is precisely as one would expect for the Polyakov action, consistent with the AdS/CFT correspondence. In this final section we comment on some of the important features that were ignored in the analysis of Sec. IV, specifically, global data associated with the holonomy of the gauge fields.

One of the main motivations for this work was to understand the manifestation of the data specifying the bulk solutions in the boundary effective action. In 3D, we can roughly characterize these bulk data in terms of Brown-Henneaux "gravitational waves" at the boundary, and global data conveniently described by holonomies of the Chern-Simons gauge field. The latter can alternatively be thought of as the modes associated with "large" Brown-Henneaux diffeomorphisms which change the topology. The construction of the boundary effective action in Sec. IV, in contrast to earlier analyses [38,46,31], which made use of the WZW model at intermediate stages,[8] should allow a more direct analysis of the dependence of the effective action on the global bulk data (see also [48,49]). We plan to report on this elsewhere. However, some comments are in order regarding the need for this extension of the effective action in the analysis of Sec. IV.

Recall that, in general, the action (42) can only be localized once a suitable choice of the holonomies $\mathrm{tr}(\exp\oint A)$ and $\mathrm{tr}(\exp\oint\bar{A})$ is made, thus specifying the topology. Within $SL(2,\mathfrak{R})$, these fall into three conjugacy classes: elliptic holonomies, conjugate to rotations, correspond to conical singularities; hyperbolic holonomies, conjugate to dilations, correspond to nonextreme back holes; while extreme black holes are associated with parabolic holonomies, conjugate to translations [8]. Including these global data will be important in extending the considerations both of this paper and of [38], to bulk and boundary geometries of more complicated topology.

As emphasized in [22], the classification of bulk conjugacy classes is mirrored in the classical solutions of Liouville theory, or in other words the uniformization of the boundary Σ. Thus one suspects that the holonomies map to zero modes of the Liouville field, and it would be interesting to understand how this applies in the context of the generating functional studied here, where the Liouville field in question can be associated with the bulk radial scale rather than a dynamical field as in [38]. In practice, this picture is complicated because in generic cases there is no global Liouville field corresponding to the bulk solution [23,39,30]. A straightforward example is given by a spinning black hole. For nonzero angular momentum, the corresponding Liouville solution can only be specified locally. It is interesting to note in this regard that the existence of global solutions for the Liouville field is closely connected with the existence of Killing spinors [30]. The absence of a global Liouville solution for the spinning black hole then correlates with the absence of Killing spinors for the generic BTZ solution [50]. This suggests that the holonomies of bulk solutions will be an important ingredient in mapping out the zero-mode structure of the Liouville field, and consequently the global structure of the geometry experienced by the dual CFT.

[8]As noted earlier, the work of Coussaert *et al.* [38] proceeded along different lines. Nonetheless, we can consider using our formulation to obtain the analogue of the CHvD action [38] by solving the constraints, with appropriate falloff conditions for the fields at infinity. In doing this, an important distinction is that we need to identify their Liouville field ϕ with the Bäcklund transform of the field φ appearing here (provided we choose $\lambda^2=0$ as the renormalization condition). That is, $\partial\varphi=-\partial\phi+e^{(\phi-\varphi)/2}$, $\bar{\partial}\varphi=\bar{\partial}\phi-e^{(\phi+\varphi)/2}$ [47]. This is due to the use of the Polyakov-Wiegmann identity in [38] which performs the appropriate canonical transformation, and this mapping then explains the form of the bulk metric which can be determined in the approach of [38].

ACKNOWLEDGMENTS

M.B. thanks M. Henneaux for a useful conversation, and A.R. thanks the Departamento de Física, P. Universidad Católica de Chile, where part of this work was completed, for warm hospitality. This work was supported in part by FONDECYT (Chile) Grants No. 1000744, No. 7000744, and No. 3000026.

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