Associativity anomaly in string field theory

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We give a detailed study of the associativity anomaly in open string field theory from the viewpoint of the split string and Moyal formalisms. The origin of the anomaly is reduced to the properties of the special infinite size matrices which relate the conventional open string to the split string variables, and is intimately related to midpoint issues. We discuss two steps to cope with the anomaly. We identify the field subspace that causes the anomaly which is related to the existence of closed string configurations, and indicate a decomposition of open or closed string sectors. We then propose a consistent cutoff method with a finite number of string modes that guarantees associativity at every step of any computation.

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I. INTRODUCTION

Recent developments in vacuum string field theory [1–18] are promising for a description of D-branes and closed strings from the viewpoint of open string field theory [19,20]. In particular, a simple picture of the stringy solitons emerges as noncommutative solitons of open string fields.

The algebraic structure of string field theory is greatly simplified by describing the open string in terms of two halves separated by a midpoint—the split string formalism [21,1–4]. By doing so, the open string field is regarded as an infinite dimensional matrix. Furthermore, by transforming to a Fourier space of the odd full string modes and using some special matrices that naturally emerged in the split string formalism (the T,R discussed below), Witten's star product is translated into the standard Moyal product involving the phase space of the even full string modes [4]. This establishes an explicit link between open string field theory and noncommutative geometry in a form which is familiar in old [22] and recent literature [23]. In this context, string field theory computations, including the construction of noncommutative solitons, become greatly simplified [24].

There are, however, some singularities in the split string formalism that require deeper understanding. In particular, in the description of D-branes some infinities and zeros are encountered [11,14]. So one must learn how to consistently extract finite quantities from infinite dimensional matrix calculations or Moyal-star computations that have singular behavior. Related phenomena were observed long ago [25–27], such as the breakdown of associativity of the star product for certain string field configurations. Such anomalies typically appear for string fields that correspond to closed string excitations, such as those that represent space-time diffeomorphisms.

The breakdown of associativity would have a huge influence on the very structure of open string field theory. For example, Witten's action would not enjoy a gauge symmetry in the presence of anomalies. It is therefore important to know precisely when and how such an anomaly occurs and how it can be treated.

The purpose of this paper is to present a systematic study of such anomalies. We will show that the associativity anomaly emerges from the very properties of the infinite dimensional matrices T,R that relate the open full string degrees of freedom to the split string degrees of freedom, thus clarifying the origin and the structure of the anomaly. Indeed, we will see that the Horowitz-Strominger anomaly is hidden in the matrices T,R themselves.

In order to tame such an anomaly, we will discuss two steps: (1) separation of the open or closed string sectors and (2) a consistent cutoff method.

In the first step, we study the structure of the Hilbert space for split strings more carefully. We find that the Hilbert space can be decomposed into two sectors. The first sector is the subspace in which associativity is maintained. We may regard it as the Hilbert space of open string fields. In the second sector associativity is explicitly broken. This subspace is characterized by the fact that under star products with singular fields the location of the midpoint shifts (contrary to the definition of the original star product). Thus, we show the simplified origin of the anomaly, with a direct relation to the Horowitz-Strominger anomaly, through its relation to the gauge variation of closed string degrees of freedom that are hidden in the open string formalism.

It is not clear how to precisely separate the open or closed sectors while maintaining the infinite number of string modes. Therefore, in the second step, we propose a consistent cutoff method using a finite number of string modes, and sending the number of modes to infinity at the end of computations. The essence of our cutoff method is to maintain all the crucial algebraic relations satisfied by the matrices T and R for any number of modes. This cutoff method is then valid in both the split string and Moyal formalisms. With a finite number of modes, associativity is maintained at all stages of any computation. When the number of modes is sent to infinity the origin of the anomaly emerges in the form of ∞/∞.

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The ambiguity in such quantities is seen to be the origin of the anomaly. With the consistent cutoff method the ambiguity is resolved and a unique value is obtained in the limit. With the cutoff method all quantities of the open string field theory (off-shell vertex, integration, etc.) are readily expressed in terms of a finite number of modes, and computations are carried out in a straightforward way without worrying about the associativity anomaly.

We expect that our consistent cutoff theory would also be quite useful in the numerical study of vacuum string field theory since it is a more reliable method as compared to the level truncation which has been used in the recent literature.

II. SPLIT STRING AND MOYAL FORMALISMS

We first recall the basic definitions of the split string and Moyal formalisms in order to set up the notation [1–4]. For ease of notation space-time indices and ghost degrees of freedom will be suppressed in most formulas.

In Witten's open string field theory, the three string vertex operator is defined by an overlap of the right half of the first open string with the left half of the second:

$$(\Psi_1\star\Psi_2)[z(\sigma)]\equiv\int \Psi_1[x(\sigma)]\Psi_2[y(\sigma)] \times\prod_{\pi/2\leqslant\sigma\leqslant\pi}\delta[x(\sigma)-y(\pi-\sigma)] \times dx(\sigma)dy(\pi-\sigma), \tag{1}$$

with the identification $z(\sigma)=x(\sigma)$ for $0\leqslant\sigma\leqslant\pi/2$ and $z(\sigma)=y(\sigma)$ for $\pi/2\leqslant\sigma\leqslant\pi$.

The mode expansion of the open string,

$$x(\sigma)=x_0+\sqrt{2}\sum_{n=1}^{\infty} x_n\cos(n\sigma), \tag{2}$$

is not the most convenient set of degrees of freedom to describe the star product since one cannot say whether x_n belongs to the left or right side of the string. In the operator formalism of the open string field theory, such description causes the Neumann coefficients appearing in the three string vertex operator to become rather complicated matrices. This obscures the understanding of the overall structure and leads to rather complex computations.

The Moyal formulation is obtained by performing a Fourier transform on the odd string modes. If the original string field written in terms of modes is $\psi(x_0,x_{2n},x_{2n-1})$, its Fourier image in the Moyal basis is $A(\bar{x},x_{2n},p_{2n})$ given as follows [4]:

$$A(\bar{x},x_{2n},p_{2n})=\det(2T)^{d/2}\left(\int dx^{\mu}_{2n-1}\right) \times e^{-(2i/\theta)\eta_{\mu\nu}\Sigma_{k,l=1}^{\infty}p^{\mu}_{2k}T_{2k,2l-1}x^{\nu}_{2l-1}} \times\psi(x_0,x_{2n},x_{2n-1}) \tag{3}$$

where d is the number of dimensions (26 plus 1 for the bosonized ghosts), θ is a parameter that has units of area in phase space, $T_{2n,2m-1}$ is a special infinite matrix intimately connected to split strings (see below) and $\bar{x}$ is the string midpoint which may be rewritten in terms of x_0,x_{2n} as $\bar{x}=x_0+\sqrt{2}\Sigma_{n=1}^{\infty}x_{2n}(-1)^{2n}$. Then the Witten star product becomes the Moyal star product in the phase space of each even mode except the midpoint:

$$(A\star B)(\bar{x},x_{2n},p_{2n})=e^{(3i/2)\bar{x}_{27}}A(\bar{x},x_{2n},p_{2n}) \times\exp\left[\frac{i\theta}{2}\eta_{\mu\nu}\sum_{n=1}^{\infty}\left(\frac{\overleftarrow{\partial}}{\partial x^{\mu}_{2n}}\frac{\overrightarrow{\partial}}{\partial p^{\nu}_{2n}} -\frac{\overleftarrow{\partial}}{\partial p^{\nu}_{2n}}\frac{\overrightarrow{\partial}}{\partial x^{\mu}_{2n}}\right)\right]B(\bar{x},x_{2n},p_{2n}). \tag{4}$$

Note that the product is local at the midpoint in all dimensions, and that there is a midpoint insertion $e^{i3\bar{x}^{27}/2}$ in the 27^{th} dimension which is the bosonized ghost coordinate. It is understood that the midpoint ghost insertion is present in all versions of the star product although it is not always explicitly indicated. For simplicity of notation we will continue this tradition of omitting the midpoint insertion in our formulas below unless we need to do an explicit computation. This reformulation of the star product greatly simplifies computations of interacting string fields as shown with many examples in [24].

The split string formalism defines split string modes which are also convenient to describe string interactions. In terms of the continuous parameter σ, these are defined by explicitly splitting the left and right variables of the open string relative to a midpoint at $\sigma=\pi/2$:

$$l(\sigma)\equiv x(\sigma),\quad r(\sigma)\equiv x(\pi-\sigma)\quad \text{for}\quad 0\leqslant\sigma\leqslant\pi/2. \tag{5}$$

With these new variables, the star product can be written as an infinite matrix multiplication:

$$(\Psi_1\star\Psi_2)[l(\sigma),r(\sigma)] =\int\prod_{0\leqslant\sigma\leqslant\pi/2}dt(\sigma)\Psi_1[l(\sigma),t(\sigma)]\Psi_2[t(\sigma),r(\sigma)]. \tag{6}$$

This expression may be rewritten in terms of the split string modes discussed below. The open string variable $x(\sigma)$ has no *a priori* boundary condition at the midpoint. Therefore, a subtlety in identifying the split string modes is the boundary condition of the half-string variables $l(\sigma),r(\sigma)$ at the midpoint. Up to this point, two standard choices have been considered: the Dirichlet and Neumann boundary conditions [21,4]. While we do not exclude other possibilities, we will concentrate on these two choices in the following. Either case is compatible with the Moyal basis given above [4].

A. Dirichlet at the midpoint

We first examine the Dirichlet case $\bar{x}=x(\pi/2)=l(\pi/2)=r(\pi/2)$. Since we have Neumann boundary conditions at the other end of $l(\sigma)$ or $r(\sigma)$, we arrive at the mode expansion in terms of the odd cosines:

$$l(\sigma)=\bar{x}+\sqrt{2}\sum_{n=1}^{\infty} l_{2n-1}\cos(2n-1)\sigma,$$

$$r(\sigma)=\bar{x}+\sqrt{2}\sum_{n=1}^{\infty} r_{2n-1}\cos(2n-1)\sigma. \tag{7}$$

The Fourier coefficients are related to each other as

$$l_{2n-1}=\frac{2\sqrt{2}}{\pi}\int_0^{\pi/2}d\sigma(l(\sigma)-\bar{x})\cos(2n-1)\sigma$$
$$=\frac{2\sqrt{2}}{\pi}\int_0^{\pi/2}d\sigma(x(\sigma)-\bar{x})\cos(2n-1)\sigma$$

$$r_{2n-1}=\frac{2\sqrt{2}}{\pi}\int_0^{\pi/2}d\sigma(r(\sigma)-\bar{x})\cos(2n-1)\sigma$$
$$=\frac{2\sqrt{2}}{\pi}\int_0^{\pi/2}d\sigma(x(\pi-\sigma)-\bar{x})\cos(2n-1)\sigma$$

$$x_{n\neq 0}=\frac{\sqrt{2}}{\pi}\int_0^{\pi}d\sigma\, x(\sigma)\cos(n\sigma)$$
$$=\frac{\sqrt{2}}{\pi}\int_0^{\pi/2}d\sigma[l(\sigma)+(-1)^n r(\sigma)]\cos(n\sigma).$$

They imply

$$x_{2n-1}=\frac{1}{2}(l_{2n-1}-r_{2n-1}) \tag{8}$$

$$x_{2n\neq 0}=\frac{1}{2}\sum_{m=1}^{\infty} T_{2n,2m-1}(l_{2m-1}+r_{2m-1}) \tag{9}$$

$$x_0=\bar{x}+\frac{1}{2\sqrt{2}}\sum_{m=1}^{\infty} T_{0,2m-1}(l_{2m-1}+r_{2m-1}) \tag{10}$$

where

$$T_{2n,2m-1}=\frac{4}{\pi}\int_0^{\pi/2}d\sigma\cos((2n)\sigma)\cos((2m-1)\sigma)$$
$$=\frac{2(-1)^{m+n+1}}{\pi}\left(\frac{1}{2m-1+2n}+\frac{1}{2m-1-2n}\right). \tag{11}$$

This matrix T is directly related to the matrix X in [20,2] as follows:

$$X_{2m-1,2n}=-X_{2n,2m-1}=iT_{2n,2m-1}\quad (n>0), \tag{12}$$

$$X_{0,2m-1}=\frac{i}{\sqrt{2}}T_{0,2m-1}. \tag{13}$$

The inverse relations of Eqs. (8)–(10) are

$$l_{2m-1}=x_{2m-1}+\sum_{n=1}^{\infty} R_{2m-1,2n}x_{2n} \tag{14}$$

$$\bar{x}=x_0+\sqrt{2}\sum_{n=1}^{\infty}(-1)^n x_{2n} \tag{15}$$

$$r_{2m-1}=-x_{2m-1}+\sum_{n=1}^{\infty} R_{2m-1,2n}x_{2n} \tag{16}$$

where

$$R_{2m-1,2n}=\frac{4}{\pi}\int_0^{\pi/2}d\sigma\cos(2m-1)\sigma[\cos 2n\sigma-(-1)^n] \tag{17}$$

$$=\frac{4n(-1)^{n+m}}{\pi(2m-1)}\left(\frac{1}{2m-1+2n}-\frac{1}{2m-1-2n}\right). \tag{18}$$

Note that

$$R_{2m-1,2n}=T_{2n,2m-1}\frac{(2n)^2}{(2m-1)^2}$$
$$=T_{2n,2m-1}-(-1)^n T_{0,2m-1}. \tag{19}$$

It must be mentioned that $R_{2k-1,2m}$ is the inverse of $T_{2m,2n-1}$ on both sides

$$(RT)_{2m-1,2k-1}=\delta_{m,k},\quad (TR)_{2m,2k}=\delta_{m,k}. \tag{20}$$

From Eqs. (19) and (20) one obtains the relations

$$\sum_{n=1}^{\infty} T_{2n,2m-1}(2n)^2 T_{2n,2k-1}=(2m-1)^2\delta_{m,k} \tag{21}$$

$$\sum_{n=1}^{\infty} T_{2m,2n-1}\frac{1}{(2n-1)^2}T_{2k,2n-1}=\frac{1}{(2m)^2}\delta_{m,k} \tag{22}$$

$$\sum_{n=1}^{\infty} R_{2n-1,2m}(2n-1)^2 R_{2n-1,2k}=(2m)^2\delta_{m,k} \tag{23}$$

$$\sum_{n=1}^{\infty} R_{2m-1,2n}\frac{1}{(2n)^2}R_{2k-1,2n}=\frac{1}{(2m-1)^2}\delta_{m,k}. \tag{24}$$

These equations reflect the fact that the matrices T and R are transformations between two bases of the form $\cos(2n\sigma)$, $\cos((2n-1)\sigma)$ which diagonalize the Laplacian $-\partial_\sigma^2$ with two different boundary conditions.

B. Neumann at the midpoint

First we note the following properties of trigonometric functions when $0\leqslant\sigma\leqslant\pi$ for integers $m,n\geqslant 1$:

$$\cos((2n-1)\sigma) = \mathrm{sgn}\left(\frac{\pi}{2}-\sigma\right)\sum_{m=1}^{\infty}[\cos(2m\sigma)-(-1)^m]T_{2m,2n-1} \tag{25}$$

$$[\cos(2m\sigma)-(-1)^m] = \mathrm{sgn}\left(\frac{\pi}{2}-\sigma\right)\sum_{n=1}^{\infty}\cos((2n-1)\sigma)R_{2n-1,2m}. \tag{26}$$

Both sides of these equations satisfy Neumann boundary conditions at $\sigma=0$ and Dirichlet boundary conditions at $\sigma=\pi/2$, and both are equivalent complete sets of trigonometric functions for the range $0\leqslant\sigma\leqslant\pi/2$. In the previous section we made the choice of expanding $l(\sigma),r(\sigma)$ in terms of the odd modes. Now we see that we could also expand them in terms of the even modes as follows:

$$\begin{aligned} l(\sigma)&=\bar{x}+\sqrt{2}\sum_{m=1}^{\infty}l_{2m}[\cos(2m\sigma)-(-1)^m] \\ &=l_0+\sqrt{2}\sum_{m=1}^{\infty}l_{2m}\cos(2m\sigma) \end{aligned} \tag{27}$$

and similarly for $r(\sigma)$. The even modes l_{2m} are now associated with $\cos(2m\sigma)$ which is a complete set that satisfies Neumann boundary conditions at $\sigma=0,\pi/2$. Comparing to the expressions in the previous subsection, and using Eqs. (25),(26) we can find the relation between the odd modes (l_{2n-1},r_{2n-1}) and the even modes (l_{2n},r_{2n})

$$l_{2n-1}=\sum_{m=1}^{\infty}R_{2n-1,2m}l_{2m}, \quad l_{2m}=\sum_{n=1}^{\infty}T_{2m,2n-1}l_{2n-1} \tag{28}$$

$$r_{2n-1}=\sum_{m=1}^{\infty}R_{2n-1,2m}r_{2m}, \quad r_{2m}=\sum_{n=1}^{\infty}T_{2m,2n-1}r_{2n-1}. \tag{29}$$

Furthermore, by using the relation between the odd string modes $(l_{2n-1},\bar{x},r_{2n-1})$ and the full string modes (x_0,x_{2n},x_{2n-1}) in Eqs. (14)–(16) or by direct comparison to $x(\sigma)$, we derive the relation between the even split string modes and the full string modes:

$$l_{2m}=x_{2m}+\sum_{n=1}^{\infty}T_{2m,2n-1}x_{2n-1},$$

$$r_{2m}=x_{2m}-\sum_{n=1}^{\infty}T_{2m,2n-1}x_{2n-1}, \tag{30}$$

$$l_0=x_0+\frac{1}{\sqrt{2}}\sum_{n=1}^{\infty}T_{0,2n-1}x_{2n-1},$$

$$r_0=x_0-\frac{1}{\sqrt{2}}\sum_{n=1}^{\infty}T_{0,2n-1}x_{2n-1}. \tag{31}$$

The inverse relation is

$$x_0=\frac{l_0+r_0}{2}, \quad x_{2m}=\frac{l_{2m}+r_{2m}}{2},$$

$$x_{2m-1}=\sum_{n=1}^{\infty}R_{2m-1,2n}\frac{l_{2n}-r_{2n}}{2}. \tag{32}$$

Note that the matching condition at the midpoint $l(\pi/2)=r(\pi/2)=x(\pi/2)=\bar{x}$ is satisfied by the even modes. This is evident from the first expression in Eq. (27) and also by noting that l_0-r_0 automatically obeys the relation

$$l_0-r_0=\sqrt{2}\sum_{n=1}^{\infty}T_{0,2n-1}x_{2n-1}=-\sqrt{2}\sum_{n=1}^{\infty}(l_{2n}-r_{2n})(-1)^n \tag{33}$$

as seen by using Eqs. (31),(32) and inserting the relation $\bar{v}R=\bar{w}$ given below in Eq. (37). In working purely with even split string modes, Eq. (33) is a constraint on $(l_{0,},r_0,l_{2n},r_{2n})$ that must be imposed among those modes. However, an alternative strategy is to use the unconstrained modes $(\bar{x},l_{2n},r_{2n})$ as the independent modes instead of the constrained modes $(l_{0,},r_0,l_{2n},r_{2n})$. In this case, instead of Eq. (32), the center of mass x_0 is given in terms of the split string modes $(\bar{x},l_{2n},r_{2n})$ by

$$x_0=\bar{x}-\sqrt{2}\sum_{n=1}^{\infty}\frac{l_{2m}+r_{2m}}{2}(-1)^n, \tag{34}$$

while the expression for l_0-r_0 never enters and can take its allowed values in terms of $(l_{2n}-r_{2n})$ as seen in Eq. (33).

C. Relations among *T*, *R*, *v*, *w*

More relations among the special matrices T,R can be compactly written in matrix notation by defining the even and odd vectors w,v

$$w_{2m}=\sqrt{2}(-1)^{m+1},$$

$$v_{2n-1}=\frac{1}{\sqrt{2}}T_{0,2n-1}=\frac{2\sqrt{2}}{\pi}\frac{(-1)^{n+1}}{2n-1}, \tag{35}$$

and then noting the following identities among these matrices,

$$TR=1, \quad RT=1, \quad R=\bar{T}+v\bar{w}, \quad R=\kappa_o^{-2}\bar{T}\kappa_e^2, \tag{36}$$

$$v=\bar{T}w, \quad w=\bar{R}v, \quad \bar{R}R=1+w\bar{w}, \quad \bar{T}T=1-v\bar{v} \tag{37}$$

$$T\bar{T}=1,\quad Tv=0,\quad \bar{v}v=1, \tag{38}$$

where the bar on a symbol means transpose of the matrix. In Eq. (36) we have defined the odd and even diagonal matrices

$$\kappa_o=\mathrm{diag}(2n-1),\quad \kappa_e=\mathrm{diag}(2n), \tag{39}$$

to reproduce Eq. (19). We recall that the meaning of the eigenvalues of κ_o, κ_e are the frequencies of oscillation of the string modes.

As we see in the next section, these identities, while they come from the absolutely convergent sums, are not consistent with each other in the sense that they break associativity when some of these matrices occur in double sums. The culprits are the relations in Eq. (38) and the underlying reason is the infinite norm $\bar{w}w=\infty$. In the final section, we propose a finite size version of the matrices *T*, *R*, *w* and v to make all matrix relations consistent with associativity. Here we give a simple sketch of our idea. We suppose that we have a regularization scheme where a suitably redefined *w* has a finite norm ($\bar{w}w=$finite). Then there is a unique way to impose associativity consistent with the definitions of T,R,v,w as expressed in Eqs. (36),(37). Associativity forces us to modify the formulas in Eq. (38) to the unique form

$$T\bar{T}=1-\frac{w\bar{w}}{1+\bar{w}w},\quad Tv=\frac{w}{1+\bar{w}w},\quad \bar{v}v=\frac{\bar{w}w}{1+\bar{w}w}, \tag{40}$$

$$Rw=v(1+\bar{w}w),\quad R\bar{R}=1-v\bar{v}(1+\bar{w}w). \tag{41}$$

One derives them as, for example, $Tv=T(\bar{T}w)=T(R-v\bar{w})w=TRw-Tv\bar{w}w=w-(Tv)(\bar{w}w)$, which implies $Tv=w/(1+\bar{w}w)$. Of course, in the infinite norm limit of *w*, one reproduces Eq. (38). We will often come back to this issue in the text. The details of the cutoff procedure with finite rank matrices that preserve all the relations above are presented in Sec. V.

III. ASSOCIATIVITY ANOMALY

In this section, we explain the appearance of the associativity anomaly hidden in the split string formalism. The matrix algebras between *T*, *R*, *w*, v are defined by the absolutely convergent infinite sums as emphasized above. However the *double* sum appearing in the product of three elements can be only conditionally convergent and the two infinite sums in different order do not in general give the same answer, thus producing an anomaly. We will see that physically the anomaly appears as the subtleties at the midpoint.

We first show the most typical example. The matrices $T_{2n,2m-1}$ and $v_{2m-1}=(1/\sqrt{2})T_{0,2m-1}$ defined by Eq. (11) satisfy $v=\bar{T}w$, or

$$T_{0,2n-1}=-2\sum_{k=1}^{\infty}(-1)^k T_{2k,2n-1}. \tag{42}$$

Going back to the original definition in terms of the integrals of cosines as in Eq. (11), this equation is satisfied as follows:

$$\int_0^{\pi/2}\cos((2n-1)\sigma)d\sigma = -2\sum_{k=1}^{\infty}(-1)^k\int_0^{\pi/2}\cos((2n-1)\sigma)\cos(2k\sigma)d\sigma. \tag{43}$$

But this is rewritten as

$$\int_0^{\pi/2}\cos((2n-1)\sigma)\delta(\sigma-\pi/2)d\sigma=0, \tag{44}$$

where the periodic delta function is given by

$$\delta(\sigma-\pi/2)=\frac{1}{\pi}-\frac{2}{\pi}\sum_{k=1}^{\infty}(-1)^k\cos(2k\sigma) = \frac{1}{\pi}+\sqrt{2}\sum_{n=1}^{\infty}\frac{-w_{2n}}{\pi}\cos(2n\sigma). \tag{45}$$

Thus, through the delta function we see that $v=\bar{T}w$ is a relation involving the midpoint. Together with the identities $T\bar{T}=1$, $Tv=0$, $\bar{v}v=1$ given in Eq. (38), these matrices display an associativity anomaly as follows:

$$(T\bar{T})w=1\cdot w=w \quad \text{versus} \quad T(\bar{T}w)=Tv=0, \tag{46}$$

$$(\bar{v}\bar{T})w=0\cdot w=0 \quad \text{versus} \quad \bar{v}(\bar{T}w)=\bar{v}v=1. \tag{47}$$

Namely $(T\bar{T})w\neq T(\bar{T}w)$ and $(\overline{vT})w\neq\bar{v}(\bar{T}w)$. These examples clearly show the anomaly is intimately related to the midpoint.

Before we move on, let us point out what would happen to the double infinite sums if the infinite norm $\bar{w}w=\infty$ is not imposed in the single sums, as would be the case in any cutoff procedure. Then, instead of Eq. (38) we use Eq. (40). This gives

$$T\bar{T}w=w\frac{\bar{w}w}{1+\bar{w}w},\quad \overline{vT}w=\frac{\bar{w}w}{1+\bar{w}w} \tag{48}$$

independent of the order of the sums. The anomaly is circumvented if $\bar{w}w=\infty$ is imposed at the end of the computation since then there is a unique answer. After emphasizing the significance of the anomaly in terms of midpoint issues, we will propose a consistent cutoff procedure that will rely on this observation.

In the following, we show more specifically how the anomaly arises for the two choices of the midpoint boundary conditions considered in the previous section.

A. Dirichlet at the midpoint (odd modes)

We write the relation between the full string modes and the split string modes (8)–(10),(14)–(16) in matrix notation

$$\begin{pmatrix} x_0 \\ x^{(e)} \\ x^{(o)} \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{2}\bar{v} & \frac{1}{2}\bar{v} \\ 0 & \frac{1}{2}T & \frac{1}{2}T \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} \bar{x} \\ l^{(o)} \\ r^{(o)} \end{pmatrix} \equiv \mathcal{T}^{(o)} \begin{pmatrix} \bar{x} \\ l^{(o)} \\ r^{(o)} \end{pmatrix} \tag{49}$$

$$\begin{pmatrix} \bar{x} \\ l^{(o)} \\ r^{(o)} \end{pmatrix} = \begin{pmatrix} 1 & -\bar{w} & 0 \\ 0 & R & 1 \\ 0 & R & -1 \end{pmatrix} \begin{pmatrix} x_0 \\ x^{(e)} \\ x^{(o)} \end{pmatrix} \equiv \mathcal{R}^{(o)} \begin{pmatrix} x_0 \\ x^{(e)} \\ x^{(o)} \end{pmatrix} \tag{50}$$

where we use the notation $e=\text{even}$, $o=\text{odd}$ and the right-hand sides define the matrices $\mathcal{T}^{(o)}$ and $\mathcal{R}^{(o)}$. One may check $\mathcal{T}^{(o)}\mathcal{R}^{(o)}=\mathcal{R}^{(o)}\mathcal{T}^{(o)}=\underline{1}$ by using the formulas $TR=RT=1$, $\bar{v}R=\bar{w}$ and $\bar{v}=\bar{w}T$. A subtle point in this correspondence is that $\mathcal{T}^{(o)}$ has a state with zero eigenvalue given by $(\bar{x},l^{(o)},r^{(o)})\sim(-1,v,v)\equiv\mathcal{V}^{(o)}$

$$\mathcal{T}^{(o)}\mathcal{V}^{(o)} \equiv \begin{pmatrix} 1 & \frac{1}{2}\bar{v} & \frac{1}{2}\bar{v} \\ 0 & \frac{1}{2}T & \frac{1}{2}T \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} -1 \\ v \\ v \end{pmatrix} = \begin{pmatrix} -1+\bar{v}v \\ Tv \\ 0 \end{pmatrix} = 0, \tag{51}$$

which follows from $Tv=0$, $\bar{v}v=1$. Note that the eigenstate $\mathcal{V}^{(o)}$ has finite norm. These facts imply that associativity is broken explicitly as follows:

$$(\mathcal{R}^{(o)}\mathcal{T}^{(o)})\mathcal{V}^{(o)}=\mathcal{V}^{(o)}, \quad \text{versus} \quad \mathcal{R}^{(o)}(\mathcal{T}^{(o)}\mathcal{V}^{(o)})=0. \tag{52}$$

The interpretation of the eigenvector $\mathcal{V}^{(o)}$ is that the infinitesimal translation of the split string modes given by two translation parameters a^μ,b^μ

$$\delta\bar{x}_\mu=a_\mu, \quad \delta l^\mu_{2n-1}=b^\mu v_{2n-1}, \quad \delta r^\mu_{2n-1}=b^\mu v_{2n-1}, \tag{53}$$

does not generate any translation of the full string modes (x_{2n},x_{2n-1}) while x_0 is translated only by the sum $a^\mu+b^\mu$ but not the difference $a^\mu-b^\mu$

$$\delta x_0^\mu=a^\mu+b^\mu. \tag{54}$$

So, there is an extra zero mode in the split string formalism as compared to the full string formalism. In this sense, the correspondence between the split string modes and the full string modes does not seem to be one-to-one and either $\bar{x}$ or the variation of $l^{(o)},r^{(o)}$ along v appear to contain an extra zero mode. This redundancy gives the origin of the anomaly in this case. We will further clarify below the relation of this anomaly to the Horowitz-Strominger anomaly [25–27], and to the pure midpoint-ghost BRST operator recently suggested in the vacuum string field theory formalism [11].

As above, in a cutoff scheme, if the infinite norm $\bar{w}w=\infty$ is not imposed temporarily in the single sums, and we use Eq. (40) instead of Eq. (38), we get the temporarily nonzero result

$$\mathcal{T}^{(o)}\mathcal{V}^{(o)}=\begin{pmatrix} -1 \\ w \\ 0 \end{pmatrix} \frac{1}{1+\bar{w}w}. \tag{55}$$

Then $\mathcal{R}^{(o)}\mathcal{T}^{(o)}\mathcal{V}^{(o)}=\mathcal{V}^{(o)}$ follows without associativity anomalies in the double sums, provided the infinite norm $\bar{w}w=\infty$ is not imposed until the end of the computation.

B. Neumann at the midpoint (even modes)

The relations similar to Eqs. (49),(50) are

$$\begin{pmatrix} x_0 \\ x^{(e)} \\ x^{(o)} \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{2}\bar{w} & \frac{1}{2}\bar{w} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2}R & -\frac{1}{2}R \end{pmatrix} \begin{pmatrix} \bar{x} \\ l^{(e)} \\ r^{(e)} \end{pmatrix} \equiv \mathcal{R}^{(e)} \begin{pmatrix} \bar{x} \\ l^{(e)} \\ r^{(e)} \end{pmatrix} \tag{56}$$

$$\begin{pmatrix} \bar{x} \\ l^{(e)} \\ r^{(e)} \end{pmatrix} = \begin{pmatrix} 1 & -\bar{w} & 0 \\ 0 & 1 & T \\ 0 & 1 & -T \end{pmatrix} \begin{pmatrix} x_0 \\ x^{(e)} \\ x^{(o)} \end{pmatrix} \equiv \mathcal{T}^{(e)} \begin{pmatrix} x_0 \\ x^{(e)} \\ x^{(o)} \end{pmatrix}. \tag{57}$$

There is an eigenvector with zero eigenvalue when $(x_0,x^{(e)},x^{(o)})\sim(0,0,v)\equiv\mathcal{V}^{(e)}$

$$\mathcal{T}^{(e)}\mathcal{V}^{(e)} \equiv \begin{pmatrix} 1 & -\bar{w} & 0 \\ 0 & 1 & T \\ 0 & 1 & -T \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ Tv \\ -Tv \end{pmatrix} = 0 \tag{58}$$

which follows from the single sum in $Tv=0$. Again, we meet the associativity anomaly in the double sums

$$(\mathcal{R}^{(e)}\mathcal{T}^{(e)})\mathcal{V}^{(e)}=\mathcal{V}^{(e)}, \quad \text{versus} \quad \mathcal{R}^{(e)}(\mathcal{T}^{(e)}\mathcal{V}^{(e)})=0. \tag{59}$$

In this case, we have to be more careful since the zero eigenstate occurs on the full string side. That is, the translation of the full string mode $x^{(o)\mu}$ by $\epsilon^\mu v$ does not seem to induce any translation in the split string variables $(\bar{x},l^{(e)},r^{(e)})$

$$\begin{pmatrix}\delta\bar{x}\\ \delta l^{(e)}\\ \delta r^{(e)}\end{pmatrix}=\begin{pmatrix}1 & -\bar{w} & 0\\ 0 & 1 & T\\ 0 & 1 & -T\end{pmatrix}\begin{pmatrix}0\\0\\v\end{pmatrix}=\epsilon^\mu\begin{pmatrix}0\\ Tv\\ -Tv\end{pmatrix}=0. \tag{60}$$

In this case, the split string modes we have chosen do not seem to be enough to describe the open string degrees of freedom. However, let us analyze the zero mode (l_0-r_0) as given in Eq. (33). From the expression $l_0-r_0=2\bar{v}x^{(o)}$ we see that it certainly translates when the full string mode $x^{(o)\mu}$ is translated by $\epsilon^\mu v$, that is $\delta(l_0^\mu-r_0^\mu)=2\epsilon^\mu$ after using $\bar{v}v=1$. This shows that the infinite sum $\bar{w}(l^e-r^e)$ also must translate by the same amount even though the individual l^e,r^e did not seem to translate

$$\delta(l_0^\mu-r_0^\mu)=2\bar{v}\,\delta x^{(o)}=\bar{w}(\delta l^e-\delta r^e)=2\epsilon^\mu. \tag{61}$$

Thus, we see again that double infinite sums $\bar{w}Tv$ must be evaluated carefully as they are afflicted with the associativity anomaly. Once more, in a regularized theory, if we use $Tv=w(1+\bar{w}w)^{-1}$ as in Eq. (40) instead of the zero value in Eq. (38), then the correct result $\bar{w}(\delta l^e-\delta r^e)=2\epsilon^\mu$ is recovered by setting $\bar{w}w=\infty$ at the end of the calculation.

C. Relation to the Horowitz-Strominger anomaly

Actually the associativity anomaly which we encountered in this section is the split string version of the Horowitz-Strominger's anomaly in [25]. In that paper, the space-time translation generator is represented as the inner derivative of the open string fields. The generator is represented by the string field

$$P_L|\mathcal{I}\rangle \tag{62}$$

where P_L is the momentum density integrated over the left half of the open string. The string configuration described by P_L shifts the center of mass of the full string under commutation using the star product. This singular behavior gives rise to the Horowitz-Strominger anomaly.

More explicitly, in terms of the vertex operator, they observed that

$$(P_{1R}+P_{2L})|V_{1234}\rangle=0 \tag{63}$$

$$(\bar{x}_1-\bar{x}_3)|V_{1234}\rangle=0 \tag{64}$$

$$[P_{1R}+P_{2L},\bar{x}_1-\bar{x}_3]=-i/2. \tag{65}$$

The first equation represents the conservation of the momentum for the four string interaction. The second represents that the midpoint is fixed for the interaction. The third equation, however, says that P and $\bar{x}$ does not commute. Obviously, these equations are not consistent with each other if associativity is assumed

$$\bar{x}_1(P_{1R}|V_{1234}\rangle)-P_{1R}(\bar{x}_1|V_{1234}\rangle)\neq(\bar{x}_1P_{1R}-P_{1R}\bar{x}_1)|V_{1234}\rangle. \tag{66}$$

In the split string formalism, the product is defined by the path integral over the half string (1). The momentum conservation (63) is represented as the invariance of the constant shift of the integration variable $t(\sigma)$ on the right-hand side of Eq. (1). In this sense, $P_{L,R}$ operator should induce the infinitesimal translation of $l(\sigma),r(\sigma)$ by a constant. In the odd modding split string formalism, it is generated by the operator,

$$P_L^\mu=\sum_{n=1}^\infty v_{2n-1}\partial^\mu_{l_{2n-1}}, \quad P_R^\mu=\sum_{n=1}^\infty v_{2n-1}\partial^\mu_{r_{2n-1}}. \tag{67}$$

The sum generates exactly the type of the translation b_μ in Eq. (53) which caused the associativity anomaly in our case $(\delta l_\mu^{(o)}=b_\mu v=\delta r_\mu^{(o)})$. The associativity anomaly appears there because there is a redundancy in the split string description. We also noted in Eq. (54) that this translation causes a shift in the center of mass coordinate, as claimed by Horowitz and Strominger. We have therefore identified the Horowitz and Strominger anomaly with the anomaly in the very matrices R,T,v,w that occur naturally in the split string formulation.

In the Moyal formulation $-i\partial^\mu_{l_{2n-1}}\psi$ corresponds to left multiplication under the Moyal star product $\Sigma_m\bar{T}_{2n-1,2m}(p_{2m}^\mu\star A)$ and $i\partial^\mu_{r_{2n-1}}\psi$ corresponds to right multiplication[1] $\bar{T}_{2n-1,2m}(A\star p_{2m}^\mu)$. In particular the sum $(P_L^\mu+P_R^\mu)A(\bar{x},x_{2n},p_{2n})$ is given by the commutator $i\Sigma_{n,m}v_{2n-1}\bar{T}_{2n-1,2m}(p_{2m}^\mu\star A-A\star p_{2m}^\mu)$. Taking into account $Tv=0$, we see that the translation $(P_L^\mu+P_R^\mu)A(\bar{x},x_{2n},p_{2n})$ vanishes unless the string field A is such that the commutator $(p_{2m}^\mu\star A-A\star p_{2m}^\mu)$ behaves like w_{2m} (since the double sum wTv is ambiguous by the anomaly). Such a string field configuration must involve $\Sigma(-1)^n x_{2n}$ which is precisely related to the difference between the center of mass and midpoint $(x_0-\bar{x})$ as in Eq. (15). Hence Strominger's anomaly is closely connected to the associativity anomaly among the matrices R,T,v,w.

If we follow the discussion in this section, the anomaly would not exist if we exclude string fields that are nontrivial under the variation induced by P_L+P_R. If one takes such an

[1]These will be discussed in detail in a future paper [24].

approach the excluded string field configurations would live outside of the open string Hilbert space, and would belong to the closed string sector that are nontrivial under space-time diffeomorphisms generated by $P_L^\mu + P_R^\mu$ as advocated by Strominger.

IV. CONTROLLING THE ANOMALY

There are basically two natural ways to control the associativity anomaly. One method is to use projectors that separate the anomalous sector in the Hilbert space, thereby separating the open or closed string field sectors. This is along the lines of an old proposal by Strominger as described below. The other method is to consider a regularization which is by definition free from anomaly. In this section we consider the first strategy in the presence of an infinite number of modes. We first discuss a projection and its relation to old works. We then point out the relevance to midpoint issues that arise in recent proposals in the context of vacuum string field theory. In Sec. V we propose another way of controlling the anomaly through a new consistent regularization using a finite number of modes N, with the cutoff N to be sent to infinity at the end of the calculation. The essence of our regularization method is to maintain all the crucial relations satisfied by $R, T, v, w, \kappa_o, \kappa_e$ but with finite norm for a modified w as long as N is finite. The regularized theory automatically resolves the associativity anomaly.

A. Projecting out the anomalous sector

We start from the example which we first explained in the last section. We denote the mode space spanned by the basis $\cos(n\sigma)$ for $n=$odd (respectively $n=$even) as $\mathcal{H}_{odd}$ (respectively $\mathcal{H}_{even}$). The matrices T and R act on the mode spaces as

$$T:\ \mathcal{H}_{odd} \to \mathcal{H}_{even}, \quad R:\ \mathcal{H}_{even} \to \mathcal{H}_{odd}, \tag{68}$$

and they are the inverse of each other. We have discussed that the existence of an eigenvector v with zero eigenvalue implies the associativity anomaly as $v=(RT)v \neq R(Tv)=0$. From the mathematical viewpoint, such an anomaly should disappear in a sector with some restriction on the spaces $\mathcal{H}_{even,odd}$. Such a sector of string fields would presumably be identified with the open string sector.

One natural restriction is the limitation of the elements of $\mathcal{H}$ to square normalizable states. This restriction, however, is not enough to guarantee associativity as seen in the case of Eq. (51) that has a finite norm $\mathcal{V}^{(o)}$. Obviously the normalizable vector $v \in \mathcal{H}_{odd}$ breaks associativity. We therefore proceed to project it out from $\mathcal{H}_{odd}$ by using the projector,

$$P = 1 - v\bar{v} = \bar{T}T. \tag{69}$$

We limit $\mathcal{H}_{odd}$ by using this projector $\mathcal{H}'_{odd} = P\mathcal{H}_{odd}$ and redefine the operators in the surviving subspace

$$T' \equiv TP, \quad R' \equiv PR. \tag{70}$$

By using the identity $R = \bar{T} + v\bar{w}$ in Eq. (36), one may easily observe

$$R'T' = P, \quad T'R' = 1, \quad \text{and} \quad R' = \overline{T'}. \tag{71}$$

In a sense, T' and R' define the partial isometry between $\mathcal{H}_{even}$ and $\mathcal{H}'_{odd}$ and they become the transpose of each other in the restricted space.

One subtlety is that there is naively a vector w in $\mathcal{H}_{even}$ which causes a problem since $\overline{T'}w = P\bar{T}w = Pv = 0$ which seems to imply the existence of a problematic zero eigenvalue. However, we note that we restrict $\mathcal{H}$ to be square normalizable, and therefore the vector w does not belong to $\mathcal{H}_{even}$ in this sense.

A cost for using this prescription is that we lose some basic properties of T and R, Eqs. (21)–(24), after we project out the Hilbert spaces. In particular, κ_o should be replaced by a nondiagonal matrix $P\kappa_o P$. In fact, the relations (21)–(24) are quite singular since they imply that different sets of eigenvalues are related by unitary transformations (as observed in [14]).[2] In this sense, losing these identities after we properly define the space is natural. The failure of these identities is not desirable since this would create some problems in the construction of the Virasoro operators. Nevertheless one must also face the issue of anomalies that are in conflict with the basic gauge symmetry of the action. We will come back to this problem in our future work.

We can interpret our constraint $\mathcal{H}'_{odd}$ in terms of open strings. When we take the Dirichlet boundary condition at the midpoint (odd split string modes), we encountered a redundancy in the split string degree of freedom involving $(l^{(o)} + r^{(o)}) \propto v$ and $\bar{x}$. We note that $\bar{x}$ is physically essential to describe the vertex operator of the free boson which is the exponential of $x^\mu(\sigma)$. While it may be possible to remove the $\bar{x}$ variable, this reasoning suggests that it may not be a good idea to proceed in this direction. So we take the other option, namely projecting away the component of $(l^{(o)} + r^{(o)}) \propto v$ by applying the projector P. This prescription is obviously consistent with our analysis.

When we use the Neumann boundary condition at the midpoint, the split string variable is described by $\mathcal{H}_{even}$ and we do not need to consider the projector for this case.

Some years ago, Strominger [27] classified the inner derivation of the open string Hilbert space into three subclasses $\mathcal{O}, \mathcal{C}, \mathcal{I}$. The first one, $\mathcal{O}$, is the inner derivative with respect to the open string field in a narrow sense and the star product in this category is always associative. The second category, $\mathcal{C}$, describes the variation of the closed string background written in terms of the open string variable. He showed that the element belonging to this subspace breaks associativity. The associator for the closed string field then belongs to the third class $\mathcal{I}$ which is described by the midpoint insertion of

[2]We emphasize however that these are not really unitary transformations when the subtleties of the double sums are taken into account. Therefore, there really is no contradiction.

the primary field. The elements in $\mathcal{I}$ commute with all the elements of the inner derivative.

This scenario can be applied to our simple situation. The inner product of the open string is now represented by the commutators of the big matrices described by the split string variables or by commutators involving the Moyal star product. We have seen that associativity can be broken by the string field degree of freedom that generates $(P_L^\mu + P_R^\mu)$ that is related to the uniform translation of the open string (in the Moyal basis this is the string field $A = \sum_{n,m} p_{2m}^\mu T_{2m,2n-1} v_{2n-1}$ under commutation, as seen above). In Strominger's classification, this represents a single element in $\mathcal{C}$. We have seen only one element since we considered only the algebra of string fields linear in the modes x^μ, p^μ. For the nonlinear string configurations, the projection of the Hilbert space becomes more complicated and we get more and more elements which belong to $\mathcal{C}$. It is not easy to find a projection prescription to separate these configurations into open or closed sectors. Therefore, we will resort to the regularized theory given below which treats the issue of anomalies in a different manner.

B. Subtlety of the vertex operators

As we have seen, following Strominger's interpretation, the open string sector can be identified by imposing certain constraints. The constraints can be described in terms of the continuous variables $l(\sigma)$ and $r(\sigma)$ for which constant shifts are allowed only in the opposite directions $l(\sigma)+\varepsilon$ and $r(\sigma)-\varepsilon$. More precisely the allowed constant shifts are described by a kink at the midpoint and a translation of the midpoint as discussed in Eqs. (53),(54)

$$\delta x^\mu(\sigma) \propto b^\mu \left(1 + \mathrm{sgn}\left(\frac{\pi}{2} - \sigma\right)\right) \tag{72}$$

with a periodic sign function. This mode should be treated rather carefully.

This fact is relevant in recent developments in vacuum string field theory in two contexts namely (i) the open string coupling to the closed string vertex operator [12,11] and (ii) the ghost kinetic term as proposed by [11] namely fermionic ghost insertion at the midpoint.

In the first context, we recall that the proposed coupling of the open string to the closed string background is

$$\mathcal{O}_V = \int V(\pi/2)\Psi. \tag{73}$$

Here $V(\pi/2)$ is the midpoint insertion of the closed string vertex operator V acting on the open string field Ψ. The simplest choice for such a vertex operator is the closed string tachyon vertex $\exp(ik_\mu x^\mu(\pi/2))$. In the second context, the kinetic term of the VSFT was proposed as

$$S \sim \int \Psi \star (c(\pi/2)\Psi). \tag{74}$$

If we represent the ghost field in terms of the free boson field $\phi(z)$ [identified as a 27^{th} dimension $\phi(\sigma) = x_{27}(\sigma)$], then the $c(\pi/2)$ insertion is again written in the form of a vertex operator $\exp(i\phi(\pi/2))$.

In the following, we show that the midpoint insertion of the vertex operator discussed here can be precisely written in terms of the allowed kink configuration which we have just mentioned, and that a deviation from the midpoint kink is likely to create problems with associativity.

We take the fermionic ghost as the example. We consider its action at an arbitrary point σ_0. The ghost field $c^\pm(\sigma_0)$ acts on the string field in the 27^{th} direction by creating a kink at σ_0 (see Eq. (3.41) of [2])

$$c^\pm(\sigma_0)\Psi[\phi(\sigma)] = Ke^{i\epsilon(\sigma_0)(\pi/4)} e^{i\phi(\sigma_0)} \times \Psi[\phi(\sigma) \pm \pi\theta(\sigma_0 - \sigma)], \tag{75}$$

where the dependence on the other 26 dimensions is suppressed.

For generic $\sigma_0 \neq \pi/2$, the Fourier coefficients of the periodic shift $\theta(\sigma_0 - \sigma)$ are given by

$$\theta(\sigma_0 - \sigma) = \frac{\sigma_0}{\pi} + \sqrt{2}\sum_{n=1}^{\infty} \frac{\sqrt{2}\sin n\sigma_0}{n\pi}\cos(n\sigma). \tag{76}$$

The midpoint coordinate $\theta(\sigma_0 - \pi/2) = \overline{\theta}$ is

$$\overline{\theta} = \begin{cases} 0, & 0 \leq \sigma_0 < \pi/2, \\ 1, & \pi/2 < \sigma_0 \leq \pi. \end{cases} \tag{77}$$

An expansion of $\theta(\sigma_0 - \sigma)$ in terms of split string modes can be developed as in Secs. II A and II B (odd modes). For odd split string modes (Dirichlet at midpoint) the corresponding coefficients are given as follows:

$$\theta_{2n-1}^{(l)} = \frac{2\sqrt{2}}{\pi}\frac{\sin((2n-1)\sigma_0)}{2n-1}, \quad \theta_{2n-1}^{(r)} = 0, \quad \text{for} \quad 0 \leq \sigma_0 < \pi/2 \tag{78}$$

$$\theta_{2n-1}^{(r)} = -\frac{2\sqrt{2}}{\pi}\frac{\sin((2n-1)\sigma_0)}{2n-1}, \quad \theta_{2n-1}^{(l)} = 0, \quad \text{for} \quad \pi/2 < \sigma_0 \leq \pi/2. \tag{79}$$

An expansion in terms of even split string modes can also be easily obtained (Neumann at midpoint). We note that these odd split string mode expansion coefficients have nonvanishing inner product with v. For example the coefficients in Eq. (78) satisfy

$$\overline{v} \cdot \theta^{(l)} = 2\sigma_0/\pi. \tag{80}$$

This implies that the translation created by the kink $\theta(\sigma_0 - \sigma)$ has a mode along the vector $\mathcal{V}^{(o)}$ of Eq. (51). As we have already discussed, this is an extra mode which is at the very origin of the associativity anomaly.

Let us now consider the ghost very special point $\sigma_0 = \pi/2$. The kink now creates a translation proportional to the periodic function $\theta(\pi/2 - \sigma)$ which is given in terms of the periodic sign function by

$$\theta\left(\frac{\pi}{2}-\sigma\right)=\frac{1}{2}+\frac{1}{2}\mathrm{sgn}\left(\frac{\pi}{2}-\sigma\right). \tag{81}$$

This is an allowed translation as in Eq. (72). Indeed the mode expansion for this function is given in terms of the full string modes, odd split string modes, and even split string modes as follows:

$$\theta\left(\frac{\pi}{2}-\sigma\right)=\frac{1}{2}+\sqrt{2}\sum_{n=1}^{\infty}\frac{v_{2n-1}}{2}\cos((2n-1)\sigma) \tag{82}$$

$$=\begin{array}{lll} \text{left/right} & \text{odd split modes} & \text{even split modes} \\ \text{midpoint} & \frac{1}{2} & \frac{1}{2} \\ \text{left modes} & \frac{1}{2}v_{2n-1} & \dfrac{1}{2(1+\bar{w}w)}\,w_{2m} \\ \text{right modes} & -\frac{1}{2}v_{2n-1} & -\dfrac{1}{2(1+\bar{w}w)}\,w_{2m} \end{array} \tag{83}$$

In particular, the odd split string modes are given by the first column in Eq. (83)

$$\bar{\theta}=\frac{1}{2},\quad \theta^{(l)}_{2n-1}=-\theta^{(r)}_{2n-1}=\frac{1}{2}v_{2n-1}, \tag{84}$$

which shows again in detail that it is of the allowed type, as in Eqs. (53),(54). Thus, we see that except for the case, $\sigma_0=\pi/2$, the kink created by the vertex is anomalous. In this respect, the construction of the vertex operators is a very subtle problem.

A lesson that we can draw here may be the following. The guiding principle to determine the closed string coupling (73) or the kinetic term (74) was that they enjoyed an enhanced symmetry if inserted at the midpoint. From our point of view, the midpoint is the safest choice because associativity is preserved. However, the regularization offered to define them does not seem to enjoy the same properties.

V. CONSISTENT REGULARIZATION

The anomaly occurred because of the infinite norm of the vector w. To analyze the anomaly more carefully it is necessary to introduce a consistent regulator. This can be done by formulating a cutoff version of the theory using finite rank matrices which truncate the theory to a finite number of oscillator modes (x_{2n},p_{2n}) with $n=1,2,\ldots,N$. Such a cutoff is desirable more generally to regulate string field theory. It could also be useful for numerical estimates.

In this section we will denote the finite matrices with the same symbols as the infinite matrices. Thus, we have square $N\times N$ matrices T, R, κ_e,κ_o and N dimensional column matrices v,w. These finite matrices may depend on the cutoff N not only through their rank but also explicitly in the matrix elements. We will see that except for the general structure that we will explain, we seldom need the details of the N dependence, for which there is a certain amount of flexibility. In any case, the finite matrices become the infinite matrices we discussed before when $N\to\infty$.

To have a consistent theory, the finite rank matrices must obey the same relations among themselves that were obeyed by the infinite rank matrices. The list of all the relations that must be satisfied for arbitrary finite N are given in Eqs. (36)–(41) excluding the hasty infinite limit in Eq. (38). These relations are satisfied by the infinite matrices. Here we will present the general solution for $R,T,v,w,\kappa_o,\kappa_e$ that satisfies these relations at any N.

When $N\to\infty$ the sum $\bar{w}w$ diverges even though all components of the vector are finite. The associativity anomaly arises from this infinite norm. Associativity is restored by keeping track of this divergence when multiple sums are involved, and sending $N\to\infty$ only at the end of a calculation. In taking the limit any explicit N dependence in the matrix elements of the matrices T,R,κ_e,κ_o, v,w should be taken into account. However, the important infinity usually occurs in the form $\bar{w}w$, therefore keeping track of this expression is mainly what is needed in most cases to extract the unique values consistent with associativity.

First we give the general solution for R,T that satisfies all the relations except for $R=\kappa_o^{-2}\bar{T}\kappa_e^2$. This is given in terms of a general orthogonal matrix S as follows:

$$T=(1+w\bar{w})^{-1/2}S,\quad R=\bar{S}(1+w\bar{w})^{1/2}. \tag{85}$$

Expanding the square roots in a power series, one may write the matrices $(1+w\bar{w})^{\pm 1/2}$ in the form

$$(1+w\bar{w})^{\pm 1/2}=1+\frac{1}{\bar{w}w}((1+\bar{w}w)^{\pm 1/2}-1)w\bar{w}. \tag{86}$$

Thus, for any orthogonal matrix S that satisfies $S\bar{S}=1=\bar{S}S$, and any vector w, one constructs

$$v=\bar{T}w=\frac{\bar{S}w}{\sqrt{1+\bar{w}w}} \tag{87}$$

and then easily verifies that all the relations hold. Other than leaving S unspecified, this form of the solution for T,R,v is unique for a given w.

Next we turn to the condition $R=\kappa_o^{-2}\bar{T}\kappa_e^2$. By inserting the R,T given above, this relation takes the form

$$\kappa_o^2=\bar{T}\kappa_e^2T=\bar{S}(1+w\bar{w})^{-1/2}\kappa_e^2(1+w\bar{w})^{-1/2}S. \tag{88}$$

By a linear transformation of S and w, one may always go to a basis in which both spectrum matrices κ_o,κ_e are diagonal. Thus, without losing generality we assume diagonal κ_o,κ_e. In such a basis we see that the meaning of S in the above equation is that it is the orthogonal matrix that diagonalizes the symmetric matrix $(1+w\bar{w})^{-1/2}\kappa_e^2(1+w\bar{w})^{-1/2}$.

Let us now determine the spectrum κ_o by solving the eigenvalue condition for the matrix $(1+w\bar{w})^{-1/2}\kappa_e^2(1+w\bar{w})^{-1/2}$. We wish to solve the secular equation

$$\det((1+w\bar{w})^{-1/2}\kappa_e^2(1+w\bar{w})^{-1/2}-\lambda)=0. \tag{89}$$

The determinant can be computed as follows:

$$\det((1+w\bar{w})^{-1/2}[\kappa_e^2-\lambda(1+w\bar{w})](1+w\bar{w})^{-1/2}) \tag{90}$$

$$=\det((1+w\bar{w})^{-1/2})^2\det(\kappa_e^2-\lambda(1+w\bar{w})) \tag{91}$$

$$=\frac{\det(\kappa_e^2-\lambda)}{1+\bar{w}w}\det\left(1-\frac{\lambda}{\kappa_e^2-\lambda}w\bar{w}\right) \tag{92}$$

$$=\frac{\det(\kappa_e^2-\lambda)}{1+\bar{w}w}\left(1-\bar{w}\frac{\lambda}{\kappa_e^2-\lambda}w\right). \tag{93}$$

The only way to have a vanishing determinant is by the vanishing of the last factor, therefore the secular equation becomes

$$\frac{1}{\lambda}+\sum_{n=1}^{N}\frac{w_{2n}^2}{\lambda-\kappa_{2n}^2}=0. \tag{94}$$

This equation can be rewritten as an N^{th} order polynomial in λ, and therefore it has N roots for λ. The N roots correspond to the diagonal matrix κ_o. Therefore we have the following N relations among the $3N$ numbers $w_{2n},\kappa_{2n},\kappa_{2n-1}$

$$\frac{1}{\kappa_{2m-1}^2}+\sum_{n=1}^{N}\frac{w_{2n}^2}{\kappa_{2m-1}^2-\kappa_{2n}^2}=0 \tag{95}$$

for $m=1,2,\ldots,N$. Recall that the meaning of the eigenvalues $\kappa_{2n},\kappa_{2n-1}$ is that they represent the frequencies of oscillations of the string modes, while w_{2n} is related to the modes that determine the difference between the center of mass point and the midpoint $x_0-\bar{x}$. These N equations determine uniquely the w_{2n}^2 for arbitrary $\kappa_{2n},\kappa_{2n-1}$. The unique solution is

$$w_{2n}^2(N)=\sum_{m=1}^{N}(M^{-1})_{2n,2m-1}\frac{1}{\kappa_{2m-1}^2} \tag{96}$$

where $(M^{-1})_{2n,2m-1}$ is the inverse of the $N\times N$ matrix determined by the frequencies

$$M_{2m-1,2n}=\frac{-1}{\kappa_{2m-1}^2-\kappa_{2n}^2}.$$

We expect that in the large N limit $w_{2n}=\sqrt{2}(-1)^{n+1}$ when $\kappa_{2n}=2n$ and $\kappa_{2m-1}=2m-1$. Indeed it is easily verified that

$$\frac{1}{(2m-1)^2}+\sum_{n=1}^{\infty}\frac{2}{(2m-1)^2-(2n)^2}=0 \tag{97}$$

is satisfied for every integer m, showing that we have the expected solution in the large N limit.

At finite N, we have the freedom to choose freely $2N$ numbers $\kappa_{2n}(N),\kappa_{2n-1}(N)$ and determine the N numbers $w_{2n}^2(N)$ from Eq. (96). Symmetry considerations may dictate a particular pattern for the N dependence of $\kappa_{2n}(N),\kappa_{2n-1}(N)$ at some stage of our investigation. For now, as an example, suppose we make the choice $\kappa_{2n}=2n$ and $\kappa_{2n}=2n-1$ just like at infinite N, and then determine $w_{2n}^2(N)$ as a function of N. The solutions can be obtained numerically and their dependence on N can be studied. Also, the unique diagonalizing matrix S can be obtained numerically and its dependence on N can be studied. Some examples for $N=2,3,10$ are given below; as follows from Eq. (96), they show consistency with $w_{2n}=\sqrt{2}(-1)^{n+1}$ as N gets larger

$$w_{2n}^2|_{N=2}=\begin{pmatrix}\frac{1}{3} & \frac{1}{15}\\ -\frac{1}{5} & \frac{1}{7}\end{pmatrix}^{-1}\begin{pmatrix}1\\ \frac{1}{9}\end{pmatrix}=\begin{pmatrix}\frac{20}{9}\\ \frac{35}{9}\end{pmatrix}$$

$$=\begin{pmatrix}2.222\\ 3.888\end{pmatrix} \tag{98}$$

$$w_{2n}^2|_{N=3}=\begin{pmatrix} \frac{1}{3} & \frac{1}{15} & \frac{1}{35} \\ -\frac{1}{5} & \frac{1}{7} & \frac{1}{27} \\ -\frac{1}{21} & -\frac{1}{9} & \frac{1}{11} \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ \frac{1}{9} \\ \frac{1}{25} \end{pmatrix}$$

$$=\begin{pmatrix} \frac{21}{10} \\ \frac{63}{25} \\ \frac{231}{50} \end{pmatrix} = \begin{pmatrix} 2.1 \\ 2.52 \\ 4.62 \end{pmatrix}. \quad (99)$$

For $N=10$, we get the 10 numbers

$$w_{2n}^2=(2.009,2.039,2.091,2.171,2.290,2.465,2.734,$$
$$3.190,4.141,8.076)$$

which approach $w_{2n}^2=2$ except for the last few components. For faster convergence one may take advantage of the freedom in the choices we can make freely in the N dependence of $\kappa_{2n}(N),\kappa_{2n-1}(N)$. However, we will not exercise this choice until we determine whether some symmetry considerations dictate a specific N dependence.

There are some relations among the $w_{2n}^2(N),\kappa_{2n}(N)$, $\kappa_{2n-1}(N)$ which can be read off directly from Eq. (94) without knowing explicitly the $\kappa_{2n}(N),\kappa_{2n-1}(N)$. We first rewrite Eq. (94) by multiplying it with $\lambda\Pi_{n=1}^{N}(\lambda-\kappa_{2n}^2)$ and expressing the secular determinant in terms of its eigenvalues as in the right-hand side below:

$$\prod_{n=1}^{N}(\lambda-\kappa_{2n}^2)+\sum_{n=1}^{N}\bar{w}_{2n}w_{2n}\lambda\prod_{i(\neq n)}(\lambda-\kappa_{2i}^2)$$
$$=(1+\bar{w}w)\prod_{n=1}^{N}(\lambda-\kappa_{2n-1}^2). \quad (100)$$

The overall coefficient on the right-hand side is determined by comparing the highest power of λ on both sides. The same relation follows from Eq. (88) after subtracting λ from both sides and computing the determinant. By comparing the coefficients of various powers of λ on both sides we can derive many relations. In particular, by comparing the coefficient of the zeroth power one finds

$$\det\left(\frac{\kappa_e^2}{\kappa_o^2}\right)=\prod_{n=1}^{N}\frac{\kappa_{2n}^2}{\kappa_{2n-1}^2}=1+\bar{w}w. \quad (101)$$

The first power in λ gives

$$\sum_{i=1}^{N}\frac{1+w_{2i}^2}{\kappa_{2i}^2}=\sum_{i=1}^{N}\frac{1}{\kappa_{2i-1}^2} \quad (102)$$

and the kth power in λ yields

$$\sum_{i_1\neq i_2\cdots\neq i_k=1}^{N}\frac{1+w_{2i_1}^2+w_{2i_2}^2+\cdots+w_{2i_k}^2}{\kappa_{2i_1}^2\kappa_{2i_2}^2\cdots\kappa_{2i_k}^2}$$
$$=\sum_{i_1\neq i_2\cdots\neq i_k=1}^{N}\frac{1}{\kappa_{2i_1-1}^2\kappa_{2i_2-1}^2\cdots\kappa_{2i_k-1}^2}. \quad (103)$$

In particular, the ratio $\det(\kappa_e^2/\kappa_o^2)=1+\bar{w}w$ which was computed from the zeroth power in λ has a universal form in terms of w independent of the specific choice of κ_e,κ_o. Note that in the infinite N limit $1+\bar{w}w$ is the periodic delta function $\delta(\sigma-\pi/2)$ with vanishing argument, as seen from Eq. (45)

$$1+\bar{w}w=\pi\delta(0). \quad (104)$$

This relation also implies that the determinant of T, $\bar{T}$ can be computed from Eqs. (40),(88):

$$\det(\bar{T}T)=\det(\kappa_o^2/\kappa_e^2)=(1+\bar{w}w)^{-1}. \quad (105)$$

By replacing this in Eqs. (36),(85) we learn

$$(\det R)^{-1}=\det T=\det\left(\frac{\kappa_o}{\kappa_e}\right)=(1+\bar{w}w)^{-1/2}. \quad (106)$$

If we use the choice $w_{2n}=\sqrt{2}(-1)^{n+1}$ at large N, the right-hand side vanishes as $O(1/\sqrt{N})$ in the large N limit.

Having satisfied the crucial relations for $R,T,v,w,\kappa_o,\kappa_e$ at finite N, we may next easily represent some cutoff versions of distributions such as the theta or delta functions given in Eqs. (45),(82),(83) in which v,w appear as fundamental entities.

The above are examples of expressions and relations that could not be uniquely determined without a consistent cutoff. These results are useful in explicit computations that will be presented in [24].

VI. CONCLUSION AND OUTLOOK

In this paper, we presented fundamental aspects of the associativity anomaly in Witten's string field theory, and indicated that it arises from the matrices that map the open string variable to the split string variable. We argued that this anomaly does not come from the peculiarity of using the split string variables but can be related to the midpoint issues noticed in the literature in other contexts. We proposed some prescriptions to deal with the anomaly. One was based on an attempt to separate open or closed string modes in the presence of an infinite number of modes, the other was based on a systematic cutoff version of the theory.

We have to emphasize that our arguments are not yet complete. For example, in the first prescription, although we may identify some anomaly causing singular string fields that are linear, the vertex operators V_N [20] and other string field configurations are expected to contain nonlinear singular fields associated with closed strings. In this sense, it is not completely clear how to generalize the open or closed separation systematically to all string fields.

On the other hand, in the regularization with a finite number of modes, we do not have such a difficulty. However, we pointed out that there remains some arbitrariness in fixing the spectrum (matrices κ_e and κ_o) where we have not yet found a principle to determine them. A related issue is how to represent conformal symmetry with a finite number of modes. One idea which we have not explored yet is to use quantum groups, which may lead to a determination of κ_e and κ_o. While our prescription is successful in providing a systematic regulator, it is clear that we need additional insights to complete our proposal.

After providing a regularization as in this paper, the Moyal approach [4] gives a very simple framework to calculate various quantities in string field theory. In a forthcoming paper, we will discuss explicit computations, including a discussion of the Virasoro generators in terms of Moyal variables (in the $N\rightarrow\infty$ limit), and calculations of off-shell n-point amplitudes.

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