PHYSICAL REVIEW D, VOLUME 65, 126005

Gauge-gravity duals with a holomorphic dilaton

Mariana Graña*
Department of Physics, University of California, Santa Barbara, California 93106

Joseph Polchinski†
Institute for Theoretical Physics, University of California, Santa Barbara, California 93106-4030

(Received 12 February 2002; published 5 June 2002)

We consider configurations of D7-branes and whole and fractional D3-branes with $\mathcal{N}=2$ supersymmetry. On the supergravity side these have a warp factor, three-form flux and a nonconstant dilaton. We discuss general type IIB solutions of this type and then obtain the specific solutions for the D7-D3 system. On the gauge side the D7-branes add matter in the fundamental representation of the D3-brane gauge theory. We find that the gauge and supergravity metrics on moduli space agree. However, in many cases the supergravity curvature is large even when the gauge theory is strongly coupled. In these cases we argue that the useful supergravity dual must be a type IIA configuration.

DOI: 10.1103/PhysRevD.65.126005 PACS number(s): 11.25.Mj

I. INTRODUCTION

The extension of Maldacena's gauge-gravity duality [1] to systems with less supersymmetry and richer matter content is an interesting one, both for understanding more general gauge theories and for application to the local geometries of warped compactifications [2]. A natural extension is to add D7-branes, as these contribute matter fields in the fundamental representation. Thus in this paper we consider $\mathcal{N}=2$ systems of D7-branes with whole and fractional D3-branes.

Gauge-gravity duality with many D7-branes has received little consideration. As far as we are aware, only Ref. [3] directly overlaps our work, with a discussion of D7-branes and whole D3-branes. Reference [4] allows for a position-dependent dilaton but requires that it be constant on an AdS_5 factor. There has also been substantial discussion of configurations of D7-branes and O7-planes such that the dilaton is everywhere constant, beginning with Refs. [5,3]. The nontrivial dilaton in the present case brings in new features and puzzles.

In Sec. II we review some of the special classes of type IIB supergravity solution that have played a role in gauge-gravity duality and string compactification, and develop the detailed form of the type IIB solutions with holomorphic τ.

In Sec. III we find solutions with D7-branes and whole and fractional D3-branes. The solutions are singular at long distance, but we conjecture that this can be thought of as a UV effect that decouples from the gauge dual. In the fractional D3 case the D7-branes are wrapped on the asymptotically locally euclidean ALE space $\mathbf{R}^4/\mathbf{Z}_2$. To fix the parameters in the solution we analyze the induced charges on the D7 world volume.

In Sec. IV we first determine the spectrum of the dual gauge theory and obtain its one-loop effective action. For D7-branes on $\mathbf{R}^4/\mathbf{Z}_2$ there are two choices of Chan-Paton action, just as for D3-branes on this space; we relate this to the induced D5 charge. We then find the one-loop metric on moduli space and show that it agrees with the action of a probe in the corresponding dual geometry.

In Sec. V we analyze the range of validity of the supergravity duals and find an unpleasant surprise: even when the gauge theory is strongly coupled, in many cases the supergravity curvature is large. This occurs, for example, in the simple and interesting case of the conformal theory of $SU(N)$ with $2N$ hypermultiplets. We argue that the correct supergravity dual is instead a type IIA configuration, whose study we leave for future work.

II. SOLUTIONS: GENERALITIES

A. Special type IIB solutions

Supersymmetric warped solutions of type IIB supergravity have recently played an extensive role in gauge-gravity duality and string compactification. The general solution of this type is not known. Early papers [6] obtained very restrictive results by use of the integrated Bianchi identity for the five-form flux. These restrictions need not hold when the transverse dimensions are noncompact, or when appropriate brane sources are included.[1]

Much recent work has involved two special cases, which can be characterized by the form of the ten-dimensional supersymmetry spinor ε. This can be decomposed

$$\varepsilon=\zeta\otimes\chi_1+\zeta^*\otimes\chi_2^*. \tag{2.1}$$

Here ζ is a four-dimensional chiral spinor, $\Gamma^4\zeta=\zeta$, and $\chi_{1,2}$ are six-dimensional chiral spinors, $\Gamma^6\chi_i=-\chi_i$. Each independent pair (χ_1,χ_2) gives rise to one $D=4$ supersymmetry. The two special cases are then

[1]Even without supersymmetry, the integrated Einstein equation implies that in a compact space without branes, warping is impossible in a Minkowski solution [7,8]. With appropriate brane sources, or in the noncompact case, warped solutions are possible; see Ref. [2] for a recent discussion.

*Email address: mariana@physics.ucsb.edu
†Email address: joep@itp.ucsb.edu

$$\text{type A(ndy):}\quad \chi_2 = e^{i\psi}\chi_1, \quad \psi \text{ real and constant;}$$
$$\text{Type B(ecker):}\quad \chi_1 = 0 \quad\text{or}\quad \chi_2 = 0. \tag{2.2}$$

The behavior of the spinor correlates with that of the complex three-form flux $G_{(3)}$. In type A solutions $G_{(3)}$ must have a constant phase. In type B solutions it must be imaginary self-dual; more specifically (see Sec. II B), it must be of type (2,1) and primitive with respect to the complex structure of the transverse space. Vanishing $G_{(3)}$ also gives type B solutions. Pure brane systems are of one or the other of these types: the D5-brane and NS5-brane are of type A, and the D3-brane and D7-brane are of type B.

The type A solutions are closely related to the warped heterotic solutions found by Strominger [9]. The type IIB form was discussed in Ref. [10]. The Maldacena-Nuñez solution [11] is a notable AdS/CFT (conformal field theory) example of this type.

The type B solutions are dual to M theory solutions found by Becker and Becker [12–14]. In the M theory form the corresponding restriction on the supersymmetry spinor is that it has definite eight-dimensional chirality. The explicit type IIB form was obtained in Refs. [15,16] for the special case of a constant dilaton. Such solutions have played an important role in gauge-gravity duality. The $\mathcal{N}=1$ fractional brane solution [17] is of this form, as well as its $\mathcal{N}=2$ generalization [18,19].

In general, the type B solutions allow a holomorphic dilaton. We find these solutions in Sec. II B. The various branes in our system—D7-branes and whole and fractional D3-branes—all preserve supersymmetries of type B. Moreover, the supersymmetries preserved by the different branes have a nontrivial intersection, which is the $\mathcal{N}=2$ of the whole system. Thus these solutions are the relevant ones.

Finally, we should note that there are interesting solutions which are of neither special form. A D3/D5 bound state will interpolate between type A at short distance and type B at long distance. Also, the $G_{(3)}$ flux corresponding to an $\mathcal{N}=1$ or $\mathcal{N}=2$ mass perturbation of the $\mathcal{N}=4$ gauge theory is of neither type, as one can see from the explicit expressions in Sec. III C of Ref. [20]. Full solutions are known only for a few special states in the mass-perturbed theory [21]. In Ref. [20] an approximate solution was found, whose supersymmetry was verified in Ref. [15]. This approximation is valid over most of parameter space, but it was emphasized that important physics occurs in regions where it breaks down.

B. Type B solutions

The solutions of type B could be obtained by duality [13,14] from those of Ref. [12], but we have found that it is generally simpler to work directly in type IIB variables. This section extends the results of Refs. [15,16], which were obtained for constant τ.

We first review the relevant results from type IIB supergravity [22]. The massless bosonic fields of the type IIB superstring theory consist of the dilaton Φ, the metric tensor G_{MN} and the antisymmetric 2-tensor B_{MN} in the Neveu-Schwarz–Neveu-Schwarz (NS-NS) sector, and the axion C, the two-form potential C_{MN}, and the four-form field C_{MNPQ} with self-dual five-form field strength in the Ramond-Ramond (RR) sector. Their fermionic superpartners are a complex Weyl gravitino ψ_M ($\Gamma^{10}\psi_M = -\psi_M$) and a complex Weyl dilatino λ ($\Gamma^{10}\lambda = \lambda$). The theory has $D=10$, $\mathcal{N}=2$ supersymmetry with a complex chiral supersymmetry parameter ε ($\Gamma^{10}\varepsilon = -\varepsilon$). The two scalars can be combined into a complex field $\tau = C + ie^{-\phi} \equiv \tau_1 + i\tau_2$ which parametrizes the $SL(2,\mathbf{R})/U(1)$ coset space.

We want to find backgrounds with four-dimensional Lorentz invariance that preserve some supersymmetry. Assuming that the background Fermi fields vanish, we have to find a combination of the bosonic fields such that the supersymmetry variation of the fermionic fields is zero. The dilatino and gravitino variations are [22]

$$\delta\lambda^* = -\frac{i}{\kappa}\gamma^M P_M^* \varepsilon + \frac{i}{4} G^* \varepsilon^*, \tag{2.3}$$

$$\delta\psi_M = \frac{1}{\kappa}\left(D_M - \frac{i}{2}Q_M\right)\varepsilon + \frac{i}{480}\gamma^{M_1\dots M_5}F_{M_1\dots M_5}\gamma_M\varepsilon - \frac{1}{16}\Gamma_M G\varepsilon^* - \frac{1}{8}G\Gamma_M\varepsilon^*. \tag{2.4}$$

Here $G = \frac{1}{6}G_{MNP}\gamma^{MNP}$, D_M is the covariant derivative with respect to the metric g_{MN}, and

$$P_M = f^2\partial_M B, \qquad Q_M = f^2\,\mathrm{Im}(B\partial_M B^*),$$
$$B = \frac{1+i\tau}{1-i\tau}, \qquad \tau = C + ie^{-\Phi}, \qquad f^{-2} = 1 - BB^*. \tag{2.5}$$

The field strengths are

$$G_{(3)} = f(F_{(3)} - BF_{(3)}^*), \qquad F_{(3)} = dA_{(2)},$$
$$F_{(5)} = dA_{(4)} - \frac{\kappa}{8}\mathrm{Im}(A_{(2)}\wedge F_{(3)}^*), \tag{2.6}$$

with $A_{(2)}$ complex and $A_{(4)}$ real.

We should note that the conventions used in supergravity are different from those usually used in string or brane actions, so for reference we give the relations. The complex potential is related to the NS-NS and RR potentials by

$$\kappa A_{(2)} = g(B_{(2)} + iC_{(2)}), \tag{2.7}$$

and the associated fluxes are related by

$$\kappa G_{(3)} = ige^{i\theta}\frac{F_{(3)\mathrm{s}} - \tau H_{(3)\mathrm{s}}}{\sqrt{\tau_2}}, \qquad e^{i\theta} = \left(\frac{1+i\tau^*}{1-i\tau}\right)^{1/2} \tag{2.8}$$

and

$$4\kappa F_{(5)} = gF_{(5)\mathrm{s}}. \tag{2.9}$$

The subscript "s" denotes the usual string quantities, e.g. the RR flux is $F_{(3)\mathrm{s}}=dC_{(2)}$, the NS-NS flux is $H_{(3)\mathrm{s}}=dB_{(2)}$, and the five-form flux is $F_{(5)\mathrm{s}}=dC_{(4)}+$Chern-Simons term. Define also

$$G_{(3)\mathrm{s}}=F_{(3)\mathrm{s}}-\tau H_{(3)\mathrm{s}}=-ie^{-i\theta}\frac{\kappa}{g}\sqrt{\tau_2}G_{(3)}. \tag{2.10}$$

Note that supergravity equations are usually written in terms of κ and string-brane equations in terms of g, but these are related

$$2\kappa^2=(2\pi)^7g^2\alpha'^4. \tag{2.11}$$

The general Einstein metric and five-form background with four-dimensional Poincaré invariance is

$$ds^2=Z^{-1/2}\eta_{\mu\nu}dx^\mu dx^\nu+Z^{1/2}\widetilde{ds}_6^2, \tag{2.12}$$

$$F_{0123m}=\partial_m h. \tag{2.13}$$

We use subindices $M,N,\ldots=0,\ldots,9$; $\mu,\nu=0,1,2,3$; and $m,n,\ldots=4,\ldots,9$. The warp factor Z, the potential $h\equiv C_{0123}$, and the dilaton axion τ depend only on the transverse x^m. The factor of $Z^{1/2}$ is included in the definition of the transverse metric for convenience.

For solutions of type B,

$$\varepsilon=\zeta\otimes\chi_1, \tag{2.14}$$

the terms proportional to ε and ε^* in the supersymmetry (SUSY) variations are linearly independent and so must vanish separately. Equivalently, the terms independent of $G_{(3)}$ and those containing $G_{(3)}$ must vanish separately. Let us start with the former.

First,

$$\delta\psi_\mu=\kappa^{-1}\partial_\mu\varepsilon-\frac{1}{8}\gamma_\mu\gamma^m(\kappa^{-1}\partial_m\ln Z-4Z\Gamma^4\partial_m h)\varepsilon. \tag{2.15}$$

The spin connection is calculated for tangent space axes $\hat{M}$ parallel to the Cartesian coordinate axes M. The Poincaré supersymmetries are independent of x^μ and and so the vanishing of $\delta\psi_\mu$ implies that

$$h=-\frac{1}{4\kappa Z}. \tag{2.16}$$

The variation of ψ_m now takes the form

$$\kappa\delta\psi_m=\left(\tilde{D}_m-\frac{i}{2}Q_m\right)\varepsilon+\frac{1}{8}\varepsilon\partial_m\ln Z, \tag{2.17}$$

where $\tilde{D}_m$ is the covariant derivative for $\widetilde{ds}_6^2$. Thus,

$$\tilde{\chi}_1=Z^{1/8}\chi_1 \tag{2.18}$$

is covariantly constant,

$$\left(\tilde{D}_m-\frac{i}{2}Q_m\right)\tilde{\chi}_1=0. \tag{2.19}$$

The connection $\tilde{D}_m$ is therefore in $U(3)$ and so $\widetilde{ds}_6^2$ is complex and Kähler. As in Calabi-Yau compactification, if the first Chern class of $\tilde{D}_m-(i/2)Q_m$ vanishes for a given metric, then there is a metric with the same Kähler class and complex structure such that a covariantly constant $\tilde{\chi}_1$ exists. We introduce complex coordinates z^i, where

$$\gamma^{\bar{\imath}}\chi_1=0; \tag{2.20}$$

acting on χ_1 with γ^i, γ^{ij}, and γ^{ijk} generate independent spinors. The final variation proportional to ε is that of the dilatino, whose vanishing implies

$$\gamma^M P_M^*\chi_1=\gamma^i P_{\bar{\imath}}^*\chi_1=0. \tag{2.21}$$

It follows that B, and so τ, is holomorphic.

The vanishing of the ε^* variations now implies

$$G\chi_1=G\chi_1^*=G\gamma^{\bar{\imath}}\chi_1^*=0. \tag{2.22}$$

Expanding these in term of the independent spinors gives

$$G_{ijk}=G^j_{ij}=G_{\bar{\imath}\bar{\jmath}\bar{k}}=G_{\bar{\imath}\bar{\jmath}k}=0. \tag{2.23}$$

In other words, $G_{(3)}$ is of type (2,1) and primitive, just as for a constant dilaton.

In addition the Bianchi identities must be satisfied. For the three-form flux these are simply

$$dF_{(3)}=dH_{(3)}=0. \tag{2.24}$$

These of course translate into more complicated identities for $G_{(3)}$ or $G_{(3)\mathrm{s}}$. The five-form flux Bianchi identity implies that

$$-\tilde{\nabla}^2 Z=(4\pi)^{1/2}\kappa\rho_3+\frac{\kappa^2}{12}G_{pqr}\widetilde{G^{pqr}}*. \tag{2.25}$$

III. SOLUTIONS WITH D7-BRANES

A. D7+D3-branes

As a warmup we consider D7-branes and D3-branes in a flat background, rederiving results obtained in Ref. [3]. The D3-branes are extended along the μ directions, and D7-branes along the noncompact μ directions as well as the 4567 directions.

From the discussion in the preceding section, we can take any solution without D3-branes ($Z=1$, implying $F_{(5)}=0$) and introduce D3-branes through a nontrivial Z. Thus we describe first the D7-branes [23]. We will use the complex coordinates

$$z^1=\frac{x^4+ix^5}{\sqrt{2}},\qquad z^2=\frac{x^6+ix^7}{\sqrt{2}},\qquad z=\frac{x^8+ix^9}{\sqrt{2}}. \tag{3.1}$$

The dilaton τ must be holomorphic, and in the given configuration it depends only on z. The transverse metric is of the form

$$\widetilde{ds}_6^2 = 2(dz^1 d\overline{z^1} + dz^2 d\overline{z^2} + e^{\psi(z,\bar{z})} dz\, d\bar{z}) \tag{3.2}$$

where ψ is to be determined in terms of the dilaton.

Now consider the supersymmetry of this solution. For arbitrary holomorphic $\tau(z)$, the covariant constancy condition (2.19) becomes

$$\begin{aligned}\left(\partial_i + \frac{1}{4}\tilde{\omega}_i^{ab}\Gamma_{ab} - \frac{1}{4}\partial_i \ln(1-BB^*)\right)\tilde{\chi}_1 = 0,\\ \left(\partial_{\bar{\imath}} + \frac{1}{4}\tilde{\omega}_{\bar{\imath}}^{ab}\Gamma_{ab} + \frac{1}{4}\partial_{\bar{\imath}} \ln(1-BB^*)\right)\tilde{\chi}_1 = 0,\end{aligned} \tag{3.3}$$

where $\tilde{\omega}_i^{ab}$ is the Christoffel connection for the tilded metric. For the metric (3.2) these become $\partial_1\tilde{\chi}_1 = \partial_2\tilde{\chi}_1 = \partial_{\bar{1}}\tilde{\chi}_1 = \partial_{\bar{2}}\tilde{\chi}_1 = 0$ and

$$\begin{aligned}\partial_z\tilde{\chi}_1 &= +\frac{1}{4}\tilde{\chi}_1\partial_z[\psi - \ln(1-BB^*)],\\ \partial_{\bar{z}}\tilde{\chi}_1 &= -\frac{1}{4}\tilde{\chi}_1\,\partial_{\bar{z}}[\psi - \ln(1-BB^*)].\end{aligned} \tag{3.4}$$

These are integrable provided

$$\psi - \ln(1-BB^*) = \gamma + \gamma^* \tag{3.5}$$

for arbitrary holomorphic $\gamma(z)$. Then $\tilde{\chi}_1 = e^{4(\gamma-\gamma^*)}\eta_0$, where η_0 is a constant spinor satisfying $\gamma^{\bar{\imath}}\eta_0 = 0$, and

$$e^{\psi} = (1-BB^*)e^{\gamma+\gamma^*}. \tag{3.6}$$

Noting that $1-BB^* = 4\tau_2/|1-i\tau|^2$, the holomorphic part is determined by modular invariance [23],

$$e^{\psi} = \tau_2|\eta(\tau)|^4\prod_{i=1}^{N_7}|z-z_i|^{-1/6}, \tag{3.7}$$

where η is the Dedekind eta function and z_i are the positions of the D7 branes. (To avoid clutter we have introduced dimensionless coordinates; to convert to the coordinates previously defined substitute $z \to z/r_0$ where r_0 is some fixed reference distance.)

For the purpose of the gauge-gravity duality we are interested in the local physics near N_7 D7-branes. In this limit

$$\tau = \frac{i}{g} + \frac{1}{2\pi i}\sum_{i=1}^{N_7}\ln(z-z_i), \tag{3.8}$$

with $z, z_i \ll 1$. The constant could be absorbed into the argument of the logarithm, but it is convenient to keep it explicit. When $z, z_i \ll 1$ then $\tau_2 \gg 1$ and the metric simplifies to

$$e^{\psi} = \tau_2 = \frac{1}{g} - \frac{1}{2\pi}\sum_{i=1}^{N_7}\ln|z-z_i|. \tag{3.9}$$

Notice that we are contemplating an arbitrarily large number of D7-branes. The local form (3.8), (3.9) becomes singular at $z \sim 1$, where τ_2 goes through zero. For $N_7 \leq 24$, this local solution can be extended to a nonsingular global solution (3.7). For $N_7 > 24$ there is no known nonsingular extension. Nevertheless, we will use this local solution and find that it gives sensible results at small z. It is an interesting question for the future, whether there is any physical realization of $N_7 > 24$ in string theory, and if not whether the use of this local solution in gauge-gravity duality is nonetheless justified.

The D3-D7 solution is now obtained by including a nontrivial Z. This is determined by the Bianchi identity:

$$\begin{aligned}-2(e^{\psi}\partial_1\partial_{\bar{1}} + e^{\psi}\partial_2\partial_{\bar{2}} + \partial_3\partial_{\bar{3}})Z = (4\pi)^{1/2}\kappa e^{\psi}\rho_3,\\ e^{\psi}\rho_3 = \sum_{j=1}^{N}\delta^6(x^m - x_j^m).\end{aligned} \tag{3.10}$$

As is well known [24], this cannot be solved exactly. In Sec. IV we will discuss some approximate features.

Finally, let us ask for all supersymmetries of this solution. For a more general spinor ε', the necessary conditions are first the vanishing of $\delta\psi_\mu$, Eq. (2.15), which implies that $\Gamma^4\varepsilon' = 0$ or $\varepsilon' = \zeta\otimes\chi_1'$. The vanishing of $\delta\psi_m$, Eq. (2.17), then implies

$$\left(\tilde{D}_m - \frac{i}{2}Q_m\right)\varepsilon + \frac{1}{8}\varepsilon\,\partial_m \ln Z = 0, \tag{3.11}$$

so that $\chi_1' = Z^{-1/8}e^{4(\gamma-\gamma^*)}\eta$ for any constant spinor η. Finally, the vanishing of $\delta\lambda$ implies that $\gamma^{\bar{\imath}}P_i^*\eta = 0$, and so $\gamma^{\bar{3}}P_i^*\eta = 0$. This has two solutions of the given chirality, $\eta = \eta_0$ and $\eta = \gamma^{12}\eta_0$. We can characterize these as the two spinors having definite chiralities in the 4567 and 89 directions,

$$i^2\gamma^{4567}\eta = i\gamma^{89}\eta = \eta. \tag{3.12}$$

Thus, as expected, this background has four complex or eight real supersymmetries, i.e. $D=4$, $\mathcal{N}=2$.

As shown in Sec. II B, we can add three-form flux to the above solution provided that $G_{(3)}$, or equivalently $G_{(3)\mathrm{s}}$, is (2,1), primitive, and satisfies the appropriate Bianchi identity. The simplest solution of this form is

$$G_{(3)\mathrm{s}} = \tau_2 g(\overline{z^3})dz^1\wedge dz^2\wedge d\overline{z^3} \tag{3.13}$$

for any antiholomorphic $g(\overline{z^3})$. The primitivity and (2,1) properties are evident, and the Bianchi identity can readily be verified. For $g = (\overline{z^3})^k$, this scales as a dimension $7+k$ perturbation of the gauge theory [25], and so does not affect the infrared physics.

B. D7+fractional D3-branes

From the study of gauge-gravity duals without D7-branes, we know that interesting gauge theories are obtained by taking a $\mathbf{Z}_2$ orbifold and including fractional D3-branes on the fixed plane, corresponding to D5-branes wrapped on the collapsed 2-cycle. These duals are developed in Refs. [26]. The full $\mathcal{N}=2$ supergravity solutions are given in Refs. [18,19]. Our main focus will be to generalize these by the inclusion of D7-branes.

We begin with a $\mathbf{Z}_2$ orbifold of the D3-D7 solution. The $\mathbf{Z}_2$ reflects the 4567 directions, so that the space transverse to the D3-branes is $(\mathbf{R}^4/\mathbf{Z}_2)\times\mathbf{R}^2$ with the D7-branes filling the $\mathbf{R}^4/\mathbf{Z}_2$ ALE space and at a point in the $\mathbf{R}^2$ parametrized by z. This preserves $\mathcal{N}=2$ supersymmetry.

The D3-D7 solution survives in the orbifolded theory, provided that ρ_3 and therefore Z are invariant under the orbifolding, and this solution is our starting point. Then, as discussed in Sec. II B, we can add the three-form flux subject to the appropriate conditions. The new feature of the orbifolded theory is the existence of a zero-size two-sphere at the fixed point, which is associated with a harmonic two-form $\omega_{(2)}$ also localized at the fixed point. It is a standard property of ALE spaces that $\omega_{(2)}$ is (1,1) and primitive in the ALE space. Then, as in the case without D7-branes (our discussion and notation follow Ref. [19], except that the signs of $B_{(2)}$ and $C_{(2)}$ are reversed to agree with conventions used elsewhere), we take

$$B_{(2)}=2\pi\alpha'\,\theta_{\rm B}(z,\bar{z})\,\omega_{(2)},\quad C_{(2)}=2\pi\alpha'\,\theta_{\rm C}(z,\bar{z})\,\omega_{(2)}. \tag{3.14}$$

Conversely, at each fixed z,

$$\theta_{\rm B}=\frac{1}{2\pi\alpha'}\int_{S^2}B_2,\quad \theta_{\rm C}=\frac{1}{2\pi\alpha'}\int_{S^2}C_2. \tag{3.15}$$

The Bianchi identity and primitivity condition are automatic. The condition that the (1,2) part of $G_{(3)}$, or equivalently of $G_{(3)\rm s}$, vanish is then

$$\partial_{\bar{z}}\theta=0,\qquad \theta=\theta_{\rm C}-\tau\theta_{\rm B}. \tag{3.16}$$

We have used $d\omega_{(2)}=0$. Thus θ is any holomorphic function,

$$\theta=\text{holomorphic}. \tag{3.17}$$

Writing $\theta(z)=\theta_1+i\theta_2$, the real and imaginary parts of $\theta=\theta_{\rm C}-\tau\theta_{\rm B}$ imply that

$$\theta_{\rm B}=-\frac{\theta_2}{\tau_2},\qquad \theta_{\rm C}=\theta_1-\frac{\tau_1\theta_2}{\tau_2}. \tag{3.18}$$

The angles $\theta_{\rm B}$ and $\theta_{\rm C}$ are periodic with period 2π. A wrapped D5-brane couples magnetically to $\theta_{\rm C}$ and so θ has a branch cut

$$\theta\sim\pm 2i\ln(z-z_5). \tag{3.19}$$

Here z_5 is the D5-brane position, the upper/lower signs refer to D5/anti-D5, and the factor of 2 arises because the two-sphere has self-intersection number 2 (this is discussed more generally in Ref. [26]).

The combination θ is invariant under the $SL(2,\mathbf{Z})$ monodromy of the D7-branes, but may still have branch cuts at the D7-branes arising from induced D5 charge. Consider then the Chern-Simons action for a D7-brane, whose relevant terms are

$$S_{\rm CS}=\mu_7\int_{M^4\times\rm ALE}\Big\{C_{(8)}+2\pi\alpha'\mathcal{F}_{(2)}\wedge C_{(6)} + \frac{1}{2}(2\pi\alpha')^2\mathcal{F}_{(2)}\wedge\mathcal{F}_{(2)}\wedge C_{(4)}\Big\} \tag{3.20}$$

(there is also a curvature term that will be discussed shortly). Here $2\pi\alpha'\mathcal{F}_{(2)}=2\pi\alpha' F_{(2)}-B_{(2)}$. Using the form (3.14) and defining $F_{(2)}=\Phi\,\omega_{(2)}$, this becomes

$$\begin{aligned} S_{\rm CS}&=\mu_7\int_{M^4\times\rm ALE}\Big\{C_{(8)}+2\pi\alpha'(\Phi-\theta_{\rm B})\,\omega_{(2)}\wedge C_{(6)} + \frac{1}{2}(2\pi\alpha')^2(\Phi-\theta_{\rm B})^2\omega_{(2)}\wedge\omega_{(2)}\wedge C_{(4)}\Big\}\\ &=\mu_7\int_{M^4\times\rm ALE}C_{(8)}+\frac{2\pi\alpha'\mu_7}{2}\int_{M^4\times S^2}(\Phi-\theta_{\rm B})C_{(6)} + \frac{(2\pi\alpha')^2\mu_7}{4}\int_{M^4}(\Phi-\theta_{\rm B})^2C_{(4)}. \end{aligned} \tag{3.21}$$

In going from the first line to the second we have used properties that follow from Poincaré duality, specifically

$$\int_{\rm ALE}\omega_{(2)}\wedge\alpha_{(2)}=\frac{1}{2}\int_{S^2}\alpha_{(2)},\quad \int_{S^2}\omega_{(2)}=1, \tag{3.22}$$

for any closed two-form $\alpha_{(2)}$. The $\frac{1}{2}$ again arises from the self-intersection number of the S^2.

Recall that $\mu_5=(2\pi)^2\alpha'\mu_7$ and $\mu_3=(2\pi)^4\alpha'^2\mu_7$ [27], and that in the orbifold theory $\theta_{\rm B}=\pi$ [28]. It follows from the coupling to $C_{(4)}$ that for $\Phi=0$, the induced D3 charge is $\frac{1}{16}$. However, we must also include the curvature terms in the Chern-Simons action [29]. These make a contribution $-\frac{1}{16}$, because on the space $\mathrm{K3}=T^4/\mathbf{Z}_2$ the total induced charge is -1. Thus the net induced D3 charge on the wrapped D7-brane with $\Phi=0$ is zero. Similarly the induced D5 charge is $-\frac{1}{4}$ times that of a wrapped D5. For $\Phi=2\pi$ the induced D3 charge is again zero and the induced D5 charge is $+\frac{1}{4}$.

These considerations suggest that

$$\theta \overset{?}{=} 2i\sum_{i=1}^{N_7}q_{5i}\ln(z-z_{7i})+2i\sum_{j=1}^{N_5}\ln(z-z_{5j}) -2i\sum_{k=1}^{N_{\bar{5}}}\ln(z-z_{\bar{5}k}). \tag{3.23}$$

Here i runs over D7-branes, j over D5-branes and k over anti-D5 branes. The induced charges q_{5i} are $\pm\frac{1}{4}$ from the above discussion. Recall also that

$$\tau = \frac{i}{g} + \frac{1}{2\pi i}\sum_{i=1}^{N_7}\ln(z - z_{7i}). \tag{3.24}$$

The form (3.23) is not quite correct, as we must include the explicit orbifold background $\theta_{\rm B} = \pi$, and so add $-\pi\tau$ to θ. The final result is

$$\theta = -\frac{i\pi}{g} + 2i\sum_{i=1}^{N_7}\left(q_{5i} + \frac{1}{4}\right)\ln(z - z_{7i}) + 2i\sum_{j=1}^{N_5}\ln(z - z_{5j}) - 2i\sum_{k=1}^{N_{\bar 5}}\ln(z - z_{\bar 5 k}). \tag{3.25}$$

This is correct in any configuration in which all D5 charges cancel locally, as it then just gives the orbifold background $\theta_{\rm B} = \pi$. It can then be verified for other configurations by moving the D5- and D7-branes around. Note that in the final result (3.25) the shifts in $\theta_{\rm C}$ around the D7-branes are properly quantized (multiples of 2π), whereas that did not hold for Eq. (3.23).

The D7-brane–fractional-D3-brane system has the same $\mathcal{N}=2$ supersymmetry as the D7-D3 system. The orbifolding preserves supersymmetries (3.12) of positive 4567 chirality. The fractional brane flux is manifestly (2,1) with respect to the complex structure defined by the spinor η_0. It is also (2,1) with respect to the complex structure defined by $\gamma^{12}\eta_0$: this is obtained by replacing $z^i \leftrightarrow \overline{z^i}$ in the ALE directions, so $\omega_{(2)}$ remains (1,1).

The three-form flux now acts as an additional effective D3-brane source for the warp factor Z. The Bianchi identity (3.10) becomes

$$-2(e^{\psi}\partial_1\partial_{\bar 1} + e^{\psi}\partial_2\partial_{\bar 2} + \partial_3\partial_{\bar 3})Z = (4\pi)^{1/2}\kappa e^{\psi}\rho_3 + \frac{1}{2}(2\pi\alpha' g)^2\delta_{\rm FP}|D_z\theta|^2, \tag{3.26}$$

where $D_z\theta = \partial_z\theta_{\rm C} - \tau\partial_z\theta_{\rm B}$. We have used the fact that $\omega_{pq}\widetilde{\omega^{pq}} = \delta_{\rm FP}$ is a δ function at the fixed point of the ALE space.

1. Probe actions

Regarding this as a four-dimensional system, the D7-brane positions are fixed while the D5-brane coordinates parametrize a moduli space of N_5 complex dimensions. The metric on this space can be computed both on the supergravity side and on the gauge theory side. In this section we find the supergravity metric, and in the next we will compare it with the gauge theory metric.

To that end we consider the action for a probe D5-brane at the fixed plane, whose moduli space has one complex dimension. The relevant terms in the probe brane action are

$$S/\mu_5 = -\int d^6\xi\, e^{-\Phi}[-\det(G + 2\pi\alpha'\mathcal{F}_{(2)})]^{1/2} + \int C_{(6)} + \int 2\pi\alpha'\mathcal{F}_{(2)}\wedge C_{(4)}, \tag{3.27}$$

where the metric G in the Dirac-Born-Infeld (DBI) action is in the string frame. The determinant splits into $\det G_{\|}\det(G_{ab} + 2\pi\alpha'\mathcal{F}_{ab})$, where $\|$ denotes the 0123 directions and a,b label the directions in the 2-cycle. If the probe is slowly moving with velocity v in the complex plane, then

$$e^{-\Phi}(-\det G_{\|})^{1/2} = g^{-1}Z^{-1}(1 - |v|^2 e^{\psi}Z)^{1/2} \approx g^{-1}Z^{-1} - \frac{1}{2}g^{-1}e^{\psi}|v|^2. \tag{3.28}$$

We have used $G_{\rm Einstein} = g^{1/2}e^{-\Phi/2}G_{\rm string}$, as well as the form (2.12) and (3.2) for the Einstein metric. For the other determinant, $\det(G_{ab} + 2\pi\alpha'\mathcal{F}_{ab}) = \det(2\pi\alpha'\mathcal{F}_{ab})$, since the 2-cycle is in the limit of zero area. Slightly enlarging this collapsing cycle so that it is a small two-sphere, we get

$$\int_{S^2}\det(2\pi\alpha'\mathcal{F}_{ab})^{1/2} = 2\pi\alpha'|2\pi n - \theta_B|, \tag{3.29}$$

where $\int_{S^2}F_{ab} = 2\pi n$ is the quantized D-brane gauge flux. Combining Eqs. (3.28) and (3.29), the DBI Lagrangian density in the noncompact dimensions becomes

$$\mathcal{L}_{\rm DBI} = -2\pi\alpha'\frac{\mu_5}{g}|2\pi n - \theta_B|\left(Z^{-1} - \frac{1}{2}e^{\psi}|v|^2\right). \tag{3.30}$$

The potential $C_{(6)}$ is obtained from the seven-form field strength

$$dC_{(6)} = -e^{\Phi}*(F_{(3)\rm s} - CH_{(3)\rm s}) + C_{(4)}\wedge H_{(3)\rm s}. \tag{3.31}$$

The exterior derivative of the right-hand side vanishes by the type IIB supergravity equations; in fact, this consistency condition determines the form of the Chern-Simons terms here. Inserting the type B form for the metric and $C_{(4)}$, the right-hand side is proportional to $\mathrm{Re}(\tilde{*}_6 G_{(3)\rm s} - iG_{(3)\rm s})$, where $\tilde{*}_6$ denotes the dual in the transverse directions using the metric $\widetilde{ds}^2$. This combination vanishes as a consequence of the supersymmetry conditions, so for all type B solutions the coupling to $C_{(6)}$ is at most a constant in the action, which can be ignored.

The Chern-Simons term, for a type B background, gives the Lagrangian density

$$\mathcal{L}_{\rm CS} = 2\pi\alpha'\frac{\mu_5}{gZ}(2\pi n - \theta_B). \tag{3.32}$$

As long as the induced D3 charge

$$q_3 = n - \frac{1}{2\pi}\theta_B \tag{3.33}$$

is positive, this cancels the potential from the DBI action. The final result for the DBI Lagrangian density is

$$\mathcal{L}_{\rm DBI}=\frac{1}{2}T_3 q_3 e^{\psi}|v|^2 \tag{3.34}$$

with $T_3=4\pi^2\alpha'\mu_5/g$ being the D3-brane tension.

This action has a simple interpretation: the inertial mass comes entirely from the induced D3-brane charge, with an additional factor of e^{ψ} from the effect of the D7-branes on the metric. For anti-D5-branes the same result holds with q_3 replaced by $n+(1/2\pi)\theta_B$. Recall that these branes are the correct degrees of freedom on the moduli space only for

$$0\leqslant q_3\leqslant 1. \tag{3.35}$$

Where the induced D3 charge becomes negative the 5-branes would no longer be Bogomol'nyi-Prasad-Sommerfield (BPS) type. The moduli space of 5-brane coordinates thus does not continue on into such a region but rather is joined onto the internal moduli space of the enhanced symmetry region defined by the curve where q_3 vanishes [30]. When the magnitude of the induced D3 charge exceeds unity, the 5-brane acquires additional moduli and can separate into elementary constituents in the range (3.35) [19].

Carrying out a similar expansion for the gauge field action yields an additional term

$$\mathcal{L}_{\rm DBI}=T_3 q_3\left\{\frac{1}{2}e^{\psi}|v|^2-\frac{1}{4}(2\pi\alpha')^2 e^{-\Phi}F_{\mu\nu}F^{\mu\nu}\right\}. \tag{3.36}$$

Finally, noting that $e^{\psi}=e^{-\Phi}=\tau_2$ in our solution, this becomes

$$\mathcal{L}_{\rm DBI}=\left(n\tau_2\pm\frac{\theta_2}{2\pi}\right)\left(\frac{1}{2}T_3|v|^2-\frac{1}{8\pi}F_{\mu\nu}F^{\mu\nu}\right), \tag{3.37}$$

where the plus (minus) corresponds to a D5 (anti-D5) brane. The kinetic and gauge terms have the same coefficient, implying that z is the $\mathcal{N}=2$ special coordinate. More generally, as in the F theory solution (3.7), $e^{\psi}/e^{-\Phi}$ is the modulus of a holomorphic function and the special coordinate is then a holomorphic function of z. For future reference we define $\tau_{2,\rm eff}$ to be the coefficient of $-(1/8\pi)F_{\mu\nu}F^{\mu\nu}$, hence

$$\tau_{2,\rm eff}=n\tau_2\pm\frac{\theta_2}{2\pi}. \tag{3.38}$$

IV. GAUGE THEORY DUALS

A. The D7-brane–fractional-D3-brane spectrum

The gauge theory dual to our supergravity solution is obtained from the open-string spectrum for D3- and D7-branes on the orbifold [31,32]. The $\mathbf{Z}_2$ reflection R acts on the D3 and D7 Chan-Paton degrees of freedom via matrices, which in a diagonal basis will be of the form

$$\gamma_{R3}=\begin{pmatrix} I_{N_{3^+}} & 0\\ 0 & -I_{N_{3^-}}\end{pmatrix},\qquad \gamma_{R7}=\begin{pmatrix} I_{N_{7^+}} & 0\\ 0 & -I_{N_{7^-}}\end{pmatrix}, \tag{4.1}$$

where I_N is the $N\times N$ identity matrix. The interpretation of γ_{R3} is well known [33]. This basis represents half D3-branes trapped on the fixed plane. Geometrically, the positive eigenvalues correspond to wrapped D5-branes on the collapsed S^2, and the negative eigenvalues to wrapped anti-D5-branes. Thus,

$$N_{3^+}=N_5,\quad N_{3^-}=N_5^-. \tag{4.2}$$

Each D5 carries one-half unit of D3 charge in the orbifold theory, so the D3 charge is one-half the number of D3 Chan-Paton indices (this is evident in a basis in which each D3-brane has an image); thus $Q_3=\frac{1}{2}N_3=\frac{1}{2}(N_{3^+}+N_{3^-})$.

We must similarly deduce the meaning of γ_{R7}. There is a natural guess, since we have seen in Sec. III B that the D7-brane has two ground states, with D5-charges $\pm\frac{1}{4}$. Indeed, one can argue for this connection as follows.[2] The reflection R relates opposite points on a given D7-brane, so γ_{R7} represents a phase under a closed motion on $\mathbf{R}^4/\mathbf{Z}_2$. This phase is a D7 Wilson line around the fixed point and so should arise from a localized flux, which is just the degree of freedom distinguishing the two D7 states. To be precise, a disk bounded by the given closed motion intersects the collapsed S^2 once, so the integral of the flux on this disk is one-half of its integral on the collapsed S^2, giving a phase difference of π between the two states.

In fact, the induced charge has already been calculated in Ref. [34], in T-dual 5-9 form, where the last line of Eq. (3.30) shows that the induced charge carried by the D9-brane is $-\frac{1}{4}$ of that carried by the D5-brane. So just as for D3-branes, the D7 Chan-Paton eigenvalue is related to the brane's D5 charge, though with a different proportionality: Chan-Paton eigenvalue ± 1 corresponds to charge $\mp\frac{1}{4}$.

The dynamical fields in $D=4$ are obtained from the 3-3, 3-7 and 7-3 strings. The massless 3-3 spectrum is well known to be a $U(N_{3^+})\times U(N_{3^-})$ gauge theory with two $(N_{3^+},\overline{N}_{3^-})\oplus(\overline{N}_{3^+},N_{3^-})$ hypermultiplets [35,26]. The action of the orbifold on the 3-7 strings is

$$R|\psi,i,j\rangle=\gamma_{R3,ii'}\gamma_{R7,jj'}|R\psi,i',j'\rangle, \tag{4.3}$$

where ψ is the oscillator state and i and j are the D3 and D7 Chan-Paton indices. In the Ramond sector, the fermionic zero modes on the 3-7 strings come from the 23- and 89-planes, so that the massless fermionic states are labeled by the corresponding helicities $|s_1,s_4\rangle$ and the Gliozzi-Scherk-Olive (GSO) projection sets $s_1=-s_4$. The reflection in the 4567 directions has no action on this state, so the orbifold projection amounts to $\gamma_{R3,ii}\gamma_{R7,jj}=1$. Thus the 3-7 strings

[2]We thank M. Douglas for suggesting this.

contribute N_{7^+} Weyl fermions of each chirality (from $s_4 = \pm\frac{1}{2}$) in the fundamental $(N_{3^+},1)$ and N_{7^-} Weyl fermions of each chirality in the $(1,N_{3^-})$. The 7-3 strings contribute the antiparticles of these. In the NS sector, states are labeled by the 4567 helicities $|s_2,s_3\rangle$ and the GSO projection sets $s_2 = s_3$. Supersymmetry requires bosonic partners for the fermions in the spectrum, so R must act trivially on the oscillator part of these bosonic states. This is so if R is defined as $e^{i\pi(s_2-s_3)}$: this is the condition where the orbifold and D7-D3 supersymmetries are compatible.

In summary, the massless spectrum is an $\mathcal{N}=2$ gauge theory with

$$\begin{aligned} &\text{vector multiplets:} \quad U(N_{3^+})\times U(N_{3^-})\text{adjoint},\\ &\text{two hypermultiplets:} \quad (N_{3^+},\overline{N}_{3^-})\oplus(\overline{N}_{3^+},N_{3^-}),\\ &N_{7^+}\text{hypermultiplets:} \quad (N_{3^+},1)\oplus(\overline{N}_{3^+},1),\\ &N_{7^-}\text{hypermultiplets:} \quad (1,N_{3^-})\oplus(1,\overline{N}_{3^-}). \end{aligned} \tag{4.4}$$

Again, the superscripts $\pm$ refer to the action of the $\mathbf{Z}_2$ on the D3 and D7 Chan-Paton factors.

B. The metric on moduli space

The Coulomb branch of moduli space is defined classically by the eigenvalues of the vector multiplet scalars ϕ and $\tilde{\phi}$,

$$\phi = \text{diag}(a_1,\ldots,a_{N_{3^+}}), \quad \tilde{\phi} = \text{diag}(\tilde{a}_1,\ldots,\tilde{a}_{N_{3^-}}). \tag{4.5}$$

These are related to the positions of the fractional branes by

$$2\pi\alpha' a_i = z_{5i}, \quad 2\pi\alpha'\tilde{a}_i = z_{\overline{5i}}. \tag{4.6}$$

Define similarly for the D7-brane positions

$$2\pi\alpha' b_i = z_{7^+i}, \quad 2\pi\alpha'\tilde{b}_i = z_{7^-i}. \tag{4.7}$$

The moduli space metric is obtained from the $\mathcal{N}=2$ prepotential $\mathcal{F}$, whose perturbative form is

$$\mathcal{F} = \mathcal{F}_{\text{classical}} + \mathcal{F}_{\text{one loop}}, \tag{4.8}$$

$$\mathcal{F}_{\text{classical}} = \frac{2\pi i}{g_+^2}\sum_{i=1}^{N_{3^+}} a_i^2 + \frac{2\pi i}{g_-^2}\sum_{i=1}^{N_{3^-}} b_i^2,$$

$$\mathcal{F}_{\text{one loop}} = \frac{i}{8\pi}\left\{\sum_{i,j=1}^{N_{3^+}}(a_i-a_j)^2\ln\frac{(a_i-a_j)^2}{\mu^2} + \sum_{i,j=0}^{N_{3^-}}(\tilde{a}_i-\tilde{a}_j)^2\ln\frac{(\tilde{a}_i-\tilde{a}_j)^2}{\mu^2} - \sum_{\text{hypers}} m^2\ln\frac{m^2}{\mu^2}\right\}.$$

Here $g_\pm^2$ are the two classical gauge couplings, each equal to $8\pi g$ in the classical limit. The masses of the hypermultiplets in Eq. (4.4) are, respectively,

$$a_i - \tilde{a}_j, \quad a_i - b_j, \quad \tilde{a}_i - \tilde{b}_j. \tag{4.9}$$

The effective value of τ_2, normalized as in Eq. (3.38), is $\text{Im}(\mathcal{F}'')$, and represents the inverse of the effective coupling-squared for the $U(N_{3^+})$ factor. To obtain the effective action for a D5-brane probe, increase the rank of the gauge group by one, adding in the field a_0 and extending the ranges in the sums. Then

$$\begin{aligned} \tau_{2,\text{eff}} &= \text{Im}\left(\frac{\partial^2\mathcal{F}}{\partial a_0^2}\right)\\ &= \frac{1}{2g} + \frac{1}{2\pi}\sum_{i=1}^{N_{3^+}}\ln\left|\frac{(a_0-a_i)^2}{\mu^2}\right| - \frac{1}{2\pi}\sum_{i=1}^{N_{3^-}}\ln\left|\frac{(a_0-\tilde{a}_i)^2}{\mu^2}\right|\\ &\quad - \frac{1}{4\pi}\sum_{i=1}^{N_{7^+}}\ln\left|\frac{(a_0-b_i)^2}{\mu^2}\right|. \end{aligned} \tag{4.10}$$

(An uninteresting numerical constant has been absorbed into the definition of μ.)

The moduli space is divided into regions which are separated by enhançon curves [30,19]. Within each such region the supergravity calculation is supposed to match the appropriate perturbative description [36], with a nonperturbative rearrangement of degrees of freedom when an enhançon is crossed. The perturbative orbifold corresponds to the range $0<\theta_B/2\pi<1$, where Eqs. (3.35) and (3.33) imply that the D5-brane probe corresponds to $n=1$. Then Eq. (3.38) gives

$$\tau_{2,\text{eff}} = \tau_2 + \frac{1}{2\pi}\theta_2. \tag{4.11}$$

Inserting the results (3.24),(3.25) for τ and θ, one finds that the metrics do agree, where we identify $\mu = r_0/2\pi\alpha'$ [r_0 is the reference scale introduced below Eq. (3.7)]. The metric for an anti-D5 probe also agrees.

V. DISCUSSION

Now let us consider the conditions under which the supergravity solution gives a good description of the theory. We first summarize the solution

$$ds^2 = Z^{-1/2}\eta_{\mu\nu}dx^\mu dx^\nu + Z^{1/2}\widetilde{ds}_6^2,$$

$$\widetilde{ds}_6^2 = 2(dz^1 d\overline{z^1} + dz^2 d\overline{z^2} + \tau_2 dz d\overline{z}),$$

$$\tau = \frac{i}{g} + \frac{1}{2\pi i}\sum_{i=1}^{N_7} \ln(z - z_{7i}),$$

$$-2(\partial_1\partial_{\bar{1}} + \partial_2\partial_{\bar{2}} + \tau_2^{-1}\partial_3\partial_{\bar{3}})Z$$

$$= (4\pi)^{1/2}\kappa\rho_3 + \frac{1}{2\tau_2}(2\pi\alpha' g)^2\delta_{\mathrm{FP}}|D_z\theta|^2,$$

$$\theta = -\frac{i\pi}{g} + i\sum_{i=1}^{N_{7^-}} \ln(z - z_{7^-i}) + 2i\sum_{j=1}^{N_5} \ln(z - z_{5j})$$

$$-2i\sum_{k=1}^{N_{\bar{5}}} \ln(z - z_{\bar{5}k}). \tag{5.1}$$

As discussed in Sec. II A, there is a singularity in the dilaton at large radius, when $\ln r \sim 2\pi/gN_7$. In order that the unknown physics of the singularity decouple we then need $r \ll e^{2\pi/gN_7}$. It will be convenient to make the slightly stronger assumption that

$$r \ll 1. \tag{5.2}$$

This inequality also implies that $\tau_2^{-1} \ll 1$ so that the theory is weakly coupled.

In addition for a good supergravity dual the string metric must have curvature whose inertial components are small in string units [37],

$$\alpha' \mathcal{R}_{\mathrm{s}} \ll 1. \tag{5.3}$$

A typical term in the curvature is of order

$$\mathcal{R}_{\mathrm{s}} \sim G_{\mathrm{s}}^{z\bar{z}}|\partial_z \ln Z|^2 = \left(\frac{g}{\tau_2 Z}\right)^{1/2}|\partial_z \ln Z|^2. \tag{5.4}$$

Although we cannot find the warp factor Z exactly, we can estimate it. For convenience we take all branes to be at the origin. The arguments of the logarithms are small and so the logarithms are slowly varying. One thus obtains a good estimate of Z at any position by treating τ and θ as constants. The unwarped metric is then flat, and Z satisfies an ordinary Laplace equation, so that the warp factor reduces to that for a D3-brane system. Thus,

$$Z \approx \frac{R^4}{\tilde{r}^4}, \quad \tilde{r}^2 = r_1^2 + r_2^2 + \tau_2 r^2, \quad R^4 = 4\pi g Q_3, \tag{5.5}$$

where Q_3 is the total 3-brane charge

$$Q_3 = N_{3^+}\left(1 - \frac{\theta_B}{2\pi}\right) + N_{3^-}\frac{\theta_B}{2\pi}. \tag{5.6}$$

Evaluating this on the plane $z_1 = z_2 = 0$, where it is greatest, we find

$$\alpha' \mathcal{R}_{\mathrm{s}} \sim \sqrt{\tau_2/Q_3}. \tag{5.7}$$

This has a simple interpretation: it is the condition for the usual AdS/CFT duality to be valid [1], substituting the running values of Q_3 and τ. Thus we need

$$Q_3 \gg \tau_2 \Rightarrow r \gg e^{-2\pi Q_3/N_7}. \tag{5.8}$$

In particular, the supergravity dual is good over a range of scales only if

$$Q_3 \gg N_7. \tag{5.9}$$

This result is unfortunate. For example, an interesting case is to take N D3$^+$-branes and $2N$ D7$^+$-branes all at the origin: this gives a conformal $\mathcal{N}=2$ $SU(N)$ theory with fundamental matter. The solution for this case is

$$\tau = \frac{i}{g} - \frac{iN}{\pi}\ln z, \quad \theta = -\frac{i\pi}{g} + 2iN\ln z, \tag{5.10}$$

giving

$$\tau_{2,\mathrm{eff}} = \tau_2 + \frac{\theta_2}{2\pi} = \frac{1}{2g}, \quad \frac{\theta_{\mathrm{B}}}{2\pi} = \frac{\pi - 2gN\ln r}{2\pi - 2gN\ln r}. \tag{5.11}$$

The effective coupling is constant, as is already assured by the general agreement between the moduli space metrics as calculated in supergravity and the gauge theory. Note that we have here a conformal gauge theory even though the dilaton is nontrivial. Unfortunately $Q_3 \lesssim N_7$ and so we have not found a good dual. We will return to this issue shortly.

The condition (5.8) is satisfied over a wide range of scales in the case of N D3$^+$-branes and $N_7 \ll N$ D7$^+$-branes. This gives a $\mathcal{N}=2$ $SU(N)$ theory with a small amount of matter. However, there is a large negative β function so the coupling quickly becomes strong, and so there is interesting dynamics only over a small range of scales with an enhançon [30] in the IR.

The one case where we obtain a useful dual is $N_{3^+} \sim N_{3^-} \gg N_{7^\pm}$. This gives an approximately conformal $SU(N_{3^+}) \times SU(N_{3^-})$ theory with bifundamental matter plus a small amount of fundamental matter.

This raises the interesting question: what *is* the dual to the conformal $\mathcal{N}=2$ $SU(N)$ theory with fundamental matter, when the 't Hooft parameter gN is large? Let us see why our dual (5.10) fails. At $r \ll 1$, τ_2 quickly becomes large, so the underlying string theory is *weakly* coupled. The reason that the gauge dynamics remains strongly coupled is that at the same time θ_{B} rapidly approaches the value 2π at which the ALE space becomes singular. One could also try to obtain a dual description by starting with the dual to a product theory with bifundamental matter [26] and taking one gauge coupling to zero while holding the other fixed. The limit is again zero string coupling on a singular ALE space.

The problem of understanding weakly coupled string theory on a singular ALE space is a familiar one, and it has been argued that the correct effective description is obtained by T-duality on one of the angular directions of the ALE space, giving a IIA configuration with parallel NS5-branes [38]. With the inclusion of D3- and D7-branes on the IIB side one obtains D4- and D6-branes on the IIA side. Such configurations have of course been extensively considered

[39], and their *T*-duality to the IIB configurations discussed [40]. However, thus far they have been applied only to the moduli space dynamics. To obtain a complete dual to the large-*N* gauge theory one needs the full supergravity solution on the IIA side. There has been recent progress in this area [41], and we hope to return to this point in future work.

ACKNOWLEDGMENTS

We would like to thank I. Bena, O. de Wolfe, M. Douglas, A. Fayyazuddin, and D. Smith for discussions and communications. This work was supported by National Science Foundation grants PHY99-07949 and PHY97-22022.

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