PHYSICAL REVIEW D, VOLUME 65, 126003

Cosmic evolution in a cyclic universe

Paul J. Steinhardt
Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08544

Neil Turok
DAMTP, CMS, Wilberforce Road, Cambridge CB3 0WA, United Kingdom

(Received 4 December 2001; published 24 May 2002)

Based on concepts drawn from the ekpyrotic scenario and M theory, we elaborate our recent proposal of a cyclic model of the universe. In this model, the universe undergoes an endless sequence of cosmic epochs which begin with the universe expanding from a ''big bang'' and end with the universe contracting to a ''big crunch.'' Matching from ''big crunch'' to ''big bang'' is performed according to the prescription recently proposed with Khoury, Ovrut and Seiberg. The expansion part of the cycle includes a period of radiation and matter domination followed by an extended period of cosmic acceleration at low energies. The cosmic acceleration is crucial in establishing the flat and vacuous initial conditions required for ekpyrosis and for removing the entropy, black holes, and other debris produced in the preceding cycle. By restoring the universe to the same vacuum state before each big crunch, the acceleration ensures that the cycle can repeat and that the cyclic solution is an attractor.

DOI: 10.1103/PhysRevD.65.126003 PACS number(s): 11.25.−w, 04.50.+h, 98.80.Cq

I. INTRODUCTION

In a recent paper [1], we introduced the possibility of a cyclic universe, a cosmology in which the universe undergoes a periodic sequence of expansion and contraction. Each cycle begins with a ''big bang'' and ends in a ''big crunch,'' only to emerge in a big bang once again. The expansion phase of each cycle includes a period of radiation, matter, and quintessence domination, the last phase of which corresponds to the current epoch of cosmic acceleration. The accelerated expansion phase dilutes by an exponential factor the entropy and the density of black holes and any other debris produced since the preceding big bang. The acceleration ultimately ends, and it is followed by a period of decelerating expansion and then contraction. At the transition from big crunch to big bang, matter and radiation are created, restoring the universe to the high density required for a new big bang phase.

Historically, cyclic models have been considered attractive because they avoid the issue of initial conditions [2]. Examples can be found in mythologies and philosophies dating back to the beginning of recorded history. Since the introduction of general relativity, though, various problems with the cyclic concept have emerged. In the 1930s, Tolman [3] discussed cyclic models consisting of a closed universe with a zero cosmological constant. He pointed out that entropy generated in one cycle would add to the entropy created in the next. Consequently, the maximal size of the universe, and the duration of a cycle, increases from bounce to bounce. Extrapolating backwards, the duration of the bounce converges to zero in a finite time. Consequently, the problem of initial conditions remains. In the 1960s, the singularity theorems of Hawking and Penrose showed that a big crunch necessarily leads to a cosmic singularity where general relativity becomes invalid. Without an available theory to replace general relativity, considerations of whether time and space could exist before the big bang were discouraged. ''Big bang'' became synonymous with the beginning of space-time. However, there is nothing in the Hawking-Penrose singularity theorems to suggest that cyclic behavior is forbidden in an improved theory of gravity, such as string theory and M theory, and some people have continued to speculate on this possibility [4,5]. In the 1990s, observations showed that the matter density is significantly less than the critical density and that the scale factor of the universe is accelerating [6]. Tolman's cyclic model based on a closed universe is therefore observationally ruled out.

Curiously, the same observations that eliminate Tolman's cyclic model fit perfectly the novel kind of cyclic model proposed here. In our proposal, the universe is flat, rather than closed. The transition from expansion to contraction is caused by introducing negative potential energy, rather than spatial curvature. Furthermore, the cyclic behavior depends in an essential way on having a period of accelerated expansion *after* the radiation and matter-dominated phases. During the accelerated expansion phase, the universe approaches a nearly vacuous state, restoring very nearly identical local conditions as existed in the previous cycle prior to the contraction phase. Globally, the total entropy in the universe grows from cycle to cycle, as Tolman suggested. However, the entropy density, which is all any real observer would actually see, has perfect cyclic behavior with entropy density being created at each bounce, and subsequently being diluted to negligible levels before the next bounce.

The linchpin of the cyclic picture is safe passage through the cosmic singularity, the transition from the big crunch to big bang. In recent work with Khoury, Ovrut and Seiberg, we have proposed that a smooth transition is possible in string theory [7,8]. In ordinary 4D general relativity, the big crunch is interpreted as the collapse and disappearance of four-dimensional space-time. Densities and curvatures diverge and there is no sign that a transition is possible. But in the theory considered here, what appears to be a big crunch in the 4D effective theory actually corresponds to the momentary collapse of an additional fifth dimension. As far as matter which couples to the higher dimensional metric is concerned, the three large spatial dimensions remain large and time continues smoothly. The temperature and density are finite as one approaches the crunch, and, furthermore, the geometry is flat just before and just after the bounce. In short, there is nothing to suggest that time comes to an end when the fifth spatial dimension collapses. Quite the contrary, the most natural possibility is that time continues smoothly. Efforts are currently under way to establish this conclusion rigorously in string theory [9]. The cyclic scenario considered here exploits this concept and is absolutely dependent on its validity. In the absence of a detailed theory of the transition from big crunch to big bang, we will parametrize the bounce in terms of simple matching conditions incorporating energy and momentum conservation.

The appeal of a cyclic model is that it provides a description of the history of the universe which applies arbitrarily far back into our past. The model presented here suggests novel answers to some of the most challenging issues in cosmology: How old is the universe—finite or infinite? How large is it? What was the physical cause of its homogeneity, isotropy and flatness? What was the origin of the energy density inhomogeneities that seeded cosmic structure formation and are visible on the cosmic microwave sky? What is the resolution of the cosmic singularity puzzle? Was there time, and an arrow of time, before the big bang? In addition, our scenario has a number of surprising implications for other major puzzles such as the value of the cosmological constant, the relative densities of different forms of matter, and even for supersymmetry breaking.

The cyclic model rests heavily on ideas developed as part of the recently proposed "ekpyrotic universe" [7,8]. The basic physical notion is that the collision between two brane worlds approaching one another along an extra dimension would have literally generated a hot big bang. Although the original ekpyrosis paper focused on collisions between bulk branes and boundary branes [7], here the more relevant example is where the boundary branes collide, the extra dimension disappears momentarily and the branes then bounce apart [8]. The ekpyrotic scenario introduced several important concepts that serve as building blocks for the cyclic scenario.

(i) Boundary branes approaching one another (beginning from rest) correspond to contraction in the effective 4D theoretic description [7].

(ii) Contraction produces a blueshift effect that converts gravitational energy into brane kinetic energy [7].

(iii) Collision converts some fraction of brane kinetic energy into matter and radiation that can fuel the big bang [7,12].

(iv) The collision and bouncing apart of boundary branes corresponds to the transition from a big crunch to a big bang [8].

A key element is added to obtain a cyclic universe. The ekpyrotic scenario assumes that there is only one collision after which the interbrane potential becomes zero (perhaps due to changes in the gauge degrees of freedom on the branes that zero out the force). The cyclic model assumes instead that the interbrane potential is the same before and after collision. After the branes bounce and fly apart, the interbrane potential ultimately causes them to draw together and collide again. To ensure cyclic behavior, we will show that the potential must vary from negative to positive values. (In the ekpyrotic examples, the potentials are zero or negative for all interbrane separations.) We propose that, at distances corresponding to the present-day separation between the branes, the interbrane potential energy density should be positive and correspond to the currently observed dark energy, providing roughly 70% of the critical density today. That is, the dark energy that is causing the cosmic acceleration of the universe today is, in this scenario, interbrane potential energy. The dark energy and its associated cosmic acceleration play an essential role in restoring the universe to a nearly vacuous state thereby allowing the cyclic solution to become an attractor. As the brane separation decreases, the interbrane potential becomes negative, as in the ekpyrotic scenario. As the branes approach one another, the scale factor of the universe, in the conventional Einstein description, changes from expansion to contraction. When the branes collide and bounce, matter and radiation are produced and there is a second reversal transforming contraction to expansion so a new cycle can begin.

The central element in the cyclic scenario is a four dimensional scalar field ϕ, parametrizing the inter-brane distance or equivalently the size of the fifth dimension. The brane separation goes to zero as ϕ tends to $-\infty$, and the maximum brane separation is attained at some finite value ϕ_{max}. For the most part our discussion will be framed completely within the four dimensional effective theory of gravity and matter coupled to the scalar field ϕ. This description is universal in the sense that many higher dimensional brane models converge to the same four dimensional effective description in the limit of small brane separation. We shall not need to tie ourselves to a particular realization of the brane world idea, such as heterotic M theory for the purposes of this discussion, although such an underlying description is certainly required, both for actually deriving the scalar potential we shall simply postulate and for the ultimate quantum consistency of the theory. The extra dimensional, and string theoretic interpretation is also crucial at the brane collision, where the effective four dimensional Einstein-frame description is singular and at which point we postulate a big-crunch–big-bang transition as outlined in Ref. [8]. Again, for the present discussion we simply parametrize the outcome of this transition in terms of the density of radiation produced on the branes, and the change in the kinetic energy of the scalar field, corresponding to a change in the contraction/expansion rate of the fifth dimension.

The scalar field ϕ plays a crucial role in the cyclic scenario, in regularizing the Einstein-frame singularity. Matter and radiation on the brane couple to the Einstein frame scale factor a times a function $\beta(\phi)$ with exponential behavior as $\phi\rightarrow-\infty$, such that the product is generically finite at the brane collision, even though $a=0$ and $\phi=-\infty$ there. For finite ϕ, the coupling of the matter and radiation to ϕ is more model-dependent. Models in which ϕ is massless at the current epoch, such as we describe in this paper, face a strong constraint due to the fact that ϕ can mediate a "fifth force," which is in general composition dependent and violates the equivalence principle. Again, without tying ourselves to a particular brane world scenario we shall consider models in which the coupling function $\beta(\phi)$ tends to a constant at current values of ϕ (large brane separations), and the corresponding fifth force is weak. An example of such a model is the Randall-Sundrum model with the nonrelativistic matter we are made of localized on the positive tension brane (see e.g. Ref. [10] for a recent discussion). In models where $\beta(\phi)$ does not tend to a constant at current values of ϕ, one must invoke some physical mechanism to give the ϕ field a small mass so that the fifth force is only short-ranged. This modification still allows for cyclic behavior, with an epoch of false vacuum domination followed by tunneling [11].

The outline of this paper is as follows. In Sec. II, we describe the requisite properties of the scalar field (interbrane) potential and present a brief tour through one complete cosmic cycle. In subsequent sections, we focus in technical detail on various stages of the cycle: the bounce (Sec. III), passing through the potential well after the big bang (Sec. IV), the radiation-, matter- and quintessence-dominated [15] epochs (Sec. V), the onset of the contraction phase and the generation of density perturbations (Sec. VI). In Sec. VII, we show that the cyclic solution is a stable attractor solution under classical and quantum fluctuations. In Sec. VIII, we discuss the implications for the fundamental questions of cosmology introduced above.

II. A BRIEF TOUR OF THE CYCLIC UNIVERSE

The various stages of a cyclic model can be characterized in terms of a scalar field ϕ which moves back and forth in an effective potential $V(\phi)$. In Sec. II A, we discuss the basic properties that $V(\phi)$ must have in order to allow cyclic solutions.

The stages of expansion and contraction can be described from two different points of view. First, one can choose fields and coordinates so that the full extra-dimensional theory is reduced to an effective four-dimensional theory with a conventional Einstein action. The key parameters are the scale factor a and the modulus scalar field ϕ that determines the distance between branes. In this picture, the terms "big bang" and "big crunch" seem well-merited. The scale factor collapses to zero at the big crunch, bounces, and grows again after the big bang. However, what is novel is the presence of the scalar field ϕ which runs to $-\infty$ at the bounce with diverging kinetic energy. The scalar field acts as a fifth force, modifying in an essential way the behavior of matter and energy at the big crunch. Namely, the temperature and matter density remain finite at the bounce because the usual blueshift effect during contraction is compensated by the fifth force effect due to ϕ. The arrangement seems rather magical if one is unaware that the 4D theory is derived from a higher dimensional picture in which this behavior has a clear geometrical interpretation. Nevertheless, for most of this paper we shall keep to the four dimensional Einstein description, switching to the higher dimensional picture only when necessary to understand the bounce, or to discuss global issues where matching one cycle to the next is important. The description of a single cycle from the 4D effective theory point-of-view is given in Sec. II B.

The same evolution appears to be quite different to observers on the visible brane who detect matter and radiation confined to three spatial dimensions. In this picture, depending on the details, the brane is either always, or nearly always expanding except for tiny jags near the big-crunch–big-bang transition when it contracts by a modest amount. The branes stretch at a rate that depends on which form of energy dominates the energy density of the universe. As the big crunch is approached, however, the expansion rate changes suddenly, and new matter and radiation is created (a brane has instantaneously collided into the visible brane and bounced from it). We describe some aspects of the visible brane viewpoint in Sec. III C.

This picture makes it clear that the big crunch does not correspond to the disappearance of all of space and the end of time but, rather, to the momentary disappearance of a fifth dimension. However, the behavior of gravity itself appears quite wild because it depends on the full bulk space-time, which is changing rapidly. One way of describing this picture is that one has mapped the conventional big bang singularity onto the mildest form of singularity possible, namely the disappearance of a single dimension for an instant of time. Nevertheless there are delicate issues involved, as are discussed in Ref. [8], such as the fact that the effective four dimensional Planck mass hits zero at the singularity, so that gravitational fluctuations can become large. There are suggestions in specific calculations [12] that physical quantities are nevertheless well behaved although a great deal more remains to be done to make the picture rigorous.

A. The effective potential for a cyclic universe

We will consider in this paper potentials $V(\phi)$ of the form shown in Fig. 1, with the following key features:

(i) The potential tends to zero rapidly as $\phi \to -\infty$. One natural possibility for the extra dimension parametrized by ϕ is the eleventh dimension of M theory. In this case the string coupling constant $g_s \propto e^{\gamma\phi}$, with some positive constant γ, and g_s vanishes as $\phi \to -\infty$. Nonperturbative potentials should vanish faster than any finite power of g_s, i.e., faster than an exponential in ϕ.

(ii) The potential is negative for intermediate ϕ, and rises with a region of large negative curvature, $V''/V \gg 1$ covering a range of ϕ of order unity in Planck mass units. This region is required for the production of scale invariant density perturbations, as proposed in Ref. [7] and detailed in Ref. [12]. Attractive exponential potentials of this type could be produced, for example, by the virtual exchange of massive particles between the boundary branes.

(iii) As ϕ increases, the potential rises to a shallow plateau, with $V''/V \ll 1$ and a *positive* height V_0 given by the present vacuum energy of the universe as inferred from cosmic acceleration and other astronomical evidence. The positive energy density is essential for having a cyclic solution since it produces a period of cosmic acceleration that restores the universe to a nearly vacuous state before the next bounce.

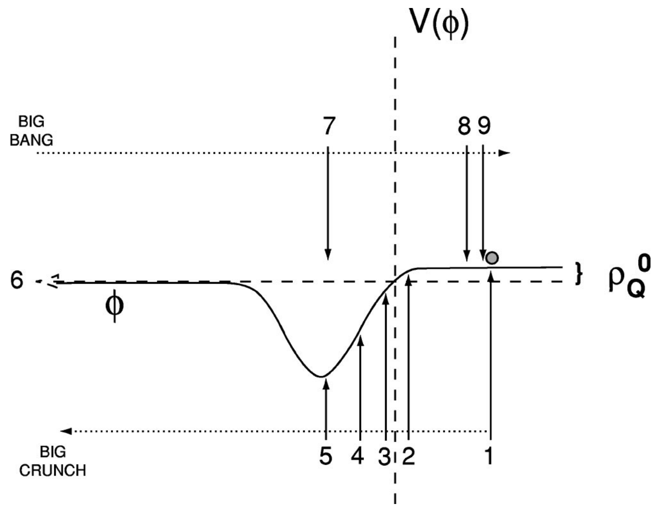

FIG. 1. The interbrane potential $V(\phi)$ versus ϕ, whose value $(-\infty<\phi<\phi_\infty)$ determines the distance between branes. The shaded circle represents the maximum positive value of ϕ during the cycle. The various stages are: (1) quintessence/potential domination and cosmic acceleration (duration $\geq$ trillion years); (2) ϕ kinetic energy becomes non-negligible, decelerated expansion begins (duration ~ 1 billion year); (3) $H=0$, contraction begins; (4) density fluctuations on observed scales created $[(t_0 t_R)^{1/2}\approx 1$ ms before big crunch]; (5) ϕ kinetic energy domination begins ($t_{min}\sim 10^{-30}$ s before big crunch); (6) bounce and reversal from big crunch to big bang; (7) end of ϕ kinetic energy domination, potential also contributes ($t_{min}\sim 10^{-30}$ s after big bang); (8) radiation dominated epoch begins ($t_R\sim 10^{-25}$ s after big bang); (9) matter domination epoch begins ($\sim 10^{10}$ s after big bang). As the potential begins to dominate and the universe returns to stage (1), the field turns around and rolls back towards $-\infty$.

The discussion here can be extended to potentials of a more general form. For example, it is not essential that the positive plateau persist to arbitrarily large ϕ since the cyclic solution only explores a finite range of $\phi>0$. Provided the condition $V''/V\ll 1$ is satisfied over that range, the universe undergoes cosmic acceleration when the field rolls down that portion of the potential. However, for simplicity, we will consider the example in Fig. 1.

An explicit model for $V(\phi)$ which is convenient for analysis is

$$V(\phi)=V_0(1-e^{-c\phi})F(\phi), \tag{1}$$

where, without loss of generality, we have shifted ϕ so that the zero of the potential occurs at $\phi=0$. The function $F(\phi)$ is introduced to represent the vanishing of nonperturbative effects described above: $F(\phi)$ turns off the potential rapidly as ϕ goes below ϕ_{min}, but it approaches one for $\phi>\phi_{min}$. For example, $F(\phi)$ might be proportional to e^{-1/g_s^2} or e^{-1/g_s}, where $g_s\propto e^{\gamma\phi}$ for $\gamma>0$. The constant V_0 is set roughly equal to the vacuum energy observed in today's Universe, of order 10^{-120} in Planck units. We do not attempt to explain this number. Various suggestions as to how a suitable small positive vacuum energy could arise have been made [13,14]. For large c, this potential has $V''/V\ll 1$ for $\phi\gtrsim 1$ and $V''/V\gg 1$ for $\phi_{min}<\phi<0$. These two regions account for cosmic acceleration and for ekpyrotic production of density perturbations, respectively [7,12]. In the latter region, the constant term is irrelevant and V may be approximated by $-V_0 e^{-c\phi}$ which may be studied using the scaling solution discussed in Sec. VI.

For an arbitrary scalar potential of the form sketched, i.e. rising with negative curvature towards a flat plateau, the scalar spectral index is given approximately by [12,16]

$$n_S\approx 1-4\left[1+\left(\frac{V}{V'}\right)^2-\frac{V''V}{(V')^2}\right], \tag{2}$$

to be evaluated when the modes on the length scales of interest are generated [stage (4) as described in Fig. 1]. For the exponential form here, Eq. (2) reduces to

$$n_S\approx 1-\frac{4}{c^2}. \tag{3}$$

Current observational limits from the cosmic microwave background and large scale structure data are safely satisfied for $c=10$, which we shall adopt as our canonical value.

The fact that the potential minimum is negative means that there are no strictly static solutions for ϕ except anti–de Sitter space. However, as we shall show, the generic behavior—indeed an attractor—is a dynamical "hovering" solution in which ϕ roams back and forth in cyclic fashion between the plateau and $-\infty$. The hovering solution is highly asymmetric in time. The field ϕ spends trillions of years or more on the plateau and mere instants traveling from the potential well to $-\infty$ and back. Gravity and the bounce provide transfers of gravitational to kinetic to matter-radiation density that keep the universe forever hovering around the anti–de Sitter minimum rather than being trapped in it.

B. The view from effective 4D theory

To set the context for the later sections, we present a brief tour through a single cycle, using the labels in Fig. 1 as the mileposts. Stage (1) represents the present epoch. The current value of the Hubble parameter is $H_0=(15\ \text{billion yr})^{-1}$. We are presently at the time when the scalar field is acting as a form of quintessence in which its potential energy has begun to dominate over matter and radiation. Depending on the specific details of the potential, the field ϕ may have already reached its maximal value (gray circle), turned back, and begun to evolve towards negative values. If not, it will do so in the near future. Because the slope of the potential is very small, ϕ rolls very slowly in the negative direction. As long as the potential energy dominates, the universe undergoes exceedingly slow cosmic acceleration (compared to inflationary expansion), roughly doubling in size every $H_0^{-1}=15$ billion years. If the acceleration lasts trillions of years or more (an easy constraint to satisfy), the entropy and black hole densities become negligibly small and the universe is nearly vacuous. The Einstein equations become

$$H^2 = \frac{8\pi G}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) \tag{4}$$

$$\frac{\ddot{a}}{a} = -\frac{8\pi G}{3}[\dot{\phi}^2 - V(\phi)] \tag{5}$$

where H is the Hubble parameter and G is Newton's constant. We will generally choose $8\pi G = 1$ except where otherwise noted. Accelerated expansion stops as $V(\phi)$ approaches zero and the scalar field kinetic energy becomes comparable to the potential energy, stage (2). The universe continues to expand and the kinetic energy of the scalar field continues to redshift as the potential drops below zero. A nearly scale invariant spectrum of fluctuations on large length scales (beyond our current Hubble horizon) begins to develop as the field rolls down the exponentially decreasing part of the potential. The evolution and perturbation equations are the same as in the ekpyrotic model [7,12]. Solving these equations, one finds that the decelerated expansion continues for a time H_0^{-1}/c, which is about one billion years [c is the parameter in $V(\phi)$, Eq. (1)]. At stage (3), the potential becomes sufficiently negative that the total scalar field energy density hits zero. According to Eq. (4), $H=0$ and the universe is momentarily static. From Eq. (5), $\ddot{a}<0$, so that a begins to contract. The universe continues to satisfy the ekpyrotic conditions for creating density perturbations. Stage (4), about one second before the big crunch, is the regime where fluctuations on the current Hubble horizon scale are generated. As the field continues to roll towards $-\infty$, the scale factor a contracts and the kinetic energy of the scalar field grows. That is, gravitational energy is converted to scalar field (brane) kinetic energy during this part of the cycle. Hence, the field races past the minimum of the potential at stage (5) and off to $-\infty$, with kinetic energy becoming increasingly dominant as the bounce approaches. The scalar field kinetic energy diverges as a tends to zero. At the bounce, stage (6), matter and radiation are generated, the scalar field gets a kick and increases speed as it reverses direction, and the universe is expanding. Through stage (7), the scalar kinetic energy density ($\propto 1/a^6$) dominates over the radiation ($\propto 1/a^4$) and the motion is almost exactly the time-reverse of the contraction phase between stage (5) and the big crunch. As the field rolls uphill, however, the small kick given the scalar field and, subsequently, the radiation become important, breaking the time-reversal symmetry. The universe becomes radiation dominated at stage (8), at say 10^{-25} s after the big bang. The motion of ϕ is rapidly damped so that it converges towards its maximal value and then very slowly creeps downhill. The damping continues during the matter dominated phase, which begins thousands of years later. The universe undergoes the standard big bang evolution for the next 15 billion years, growing structure from the perturbations created when the scalar field was rolling downhill at stage (4). Then, the scalar field potential energy begins to dominate and cosmic acceleration begins. Eventually, the scalar field rolls back across $\phi=0$. The energy density falls to zero and cosmic contraction begins. The scalar field rolls down the hill, density perturbations are generated and ϕ runs off to $-\infty$ for the next bounce. The evolution in terms of conventional variables is summarized in Fig. 2.

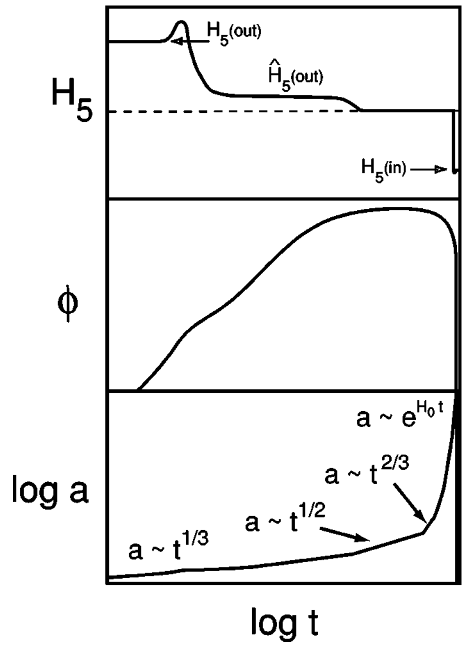

FIG. 2. Schematic plot of the scale factor $a(t)$, the modulus $\phi(t)$, and $H_5 \equiv \frac{2}{3} d(\exp(\sqrt{3/2}\phi))/dt$ for one cycle, where t is Einstein frame proper time. The scale factor starts out zero but expands as $t^{1/3}$, and the scalar field grows logarithmically with t, in the scalar kinetic energy dominated early regime. Then, when radiation begins to dominate we have $a \propto t^{1/2}$, and the scalar field motion is strongly damped. This is followed by the matter era, where $a \propto t^{2/3}$, and a potential dominated phase in which $a(t)$ increases exponentially, before a final collapse on a time scale H_0^{-1}, to $a=0$ once more. H_5 is proportional to the proper (five dimensional) speed of contraction of the fifth dimension. To obtain a cyclic solution, the magnitude of H_5 at the start of the big bang, $H_5(out)$, must be slightly larger than the value at the end of the big crunch, $H_5(in)$. This is the case if more radiation is generated on the negative tension brane (see the Appendix).

C. The view from the visible brane

Thus far, we have described the evolution in terms of the usual Einstein frame variables, a and ϕ. However, as emphasized in the next section, these variables are singular at the transition from big bang to big crunch, and they do not present an accurate picture of what an observer composed of matter confined to the brane would actually see. As a approaches zero, the density of matter and radiation scale as $1/[a\beta(\phi)]^3$ and $1/[a\beta(\phi)]^4$, respectively, where $\beta(\phi)$ is a function of ϕ which scales as $1/a$ as a tends to zero. Therefore the densities of matter and radiation on the branes are actually finite at $a=0$.

This scaling of the density with $a\beta(\phi)$ rather than a can be understood rather simply. First, the spatial volume element on the branes is that induced from five dimensions. When the brane separation is small, one can use the usual formula for Kaluza-Klein theory,

$$ds_5^2 = e^{-\sqrt{(2/3)}\phi} ds_4^2 + e^{2\sqrt{(2/3)}\phi} dy^2, \tag{6}$$

where ds_4^2 is the four dimensional line element, y is the fifth spatial coordinate which runs from zero to L, and L is a parameter with the dimensions of length. If we write the four dimensional line element in conformal time coordinates, as $ds_4^2 = a^2(-d\tau^2 + d\vec{x}^2)$, then since from the Friedmann equation we have $(a'/a)^2 = \frac{1}{6}(\phi')^2$, we see that a is proportional to $e^{\phi/\sqrt{6}}$ in the big crunch. Hence a three dimensional comoving volume element $d^3x a^3 e^{-\sqrt{(3/2)}\phi}$ remains finite as a tends to zero. Thus the density of massive particles tends to a constant. What about the density of radiation? First, recall the usual argument that the energy of a photon diverges at $a=0$. Consider a set of comoving massive particles in a space-time with metric $a^2\eta_{\mu\nu}$ where $\eta_{\mu\nu}$ is the Minkowski metric. The four velocities of the particles obey $u^\mu u^\nu g_{\mu\nu} = -1$. Hence, if they are comoving ($\vec{u}=\vec{0}$), then we must have $u^0 = a^{-1}$. Now a photon moving in such a space-time has a constant four-momentum, $p^\mu = E(1,\vec{n})$, with $\vec{n}^2 = 1$. The energy of the photon, as seen by the comoving particles, is $-u^\mu p^\nu g_{\mu\nu} = E/a$, which diverges as a tends to zero. However, in the present context, the metric to which the comoving particles couple is $e^{-\sqrt{(2/3)}\phi} a^2 \eta_{\mu\nu}$. Therefore, we have $u^0 = a^{-1} e^{\sqrt{(1/6)}\phi}$ and the energy of the detected photons is finite as a tends to zero. In other words, the effect of the scalar field approaching $-\infty$ is precisely such as to cancel the gravitational blueshift.

The second crucial use of the higher dimensional metric is in piecing together the global view of the space-time. If one only had the Einstein frame scale factor a, it would not be clear how to match to the next cycle, since $a=0$ at the bounce. But the scale factor on a brane, $a\beta(\phi)$, is nonzero at each bounce and may be so matched. In fact, in the examples studied in this paper, the scale factors a_0 and a_1 (which are the brane scale factors in the simplest models) both undergo a net exponential expansion within a cycle, and decrease for very brief periods—either just before the brane collision (for a_0) or just after it (for a_1). An observer on either brane would view the cosmology as one of almost uninterrupted expansion, with successive episodes of radiation, matter, and quintessence domination ending in a sudden release of matter and radiation.

Both matter and radiation are suddenly created by the impact of the other brane. The forewarning of this catastrophic event would be that as $\beta(\phi)$ started to rapidly change, one would see stronger and stronger violations of the equivalence principle (a "fifth force"), and the masses and couplings of all particles would change. In the case of M theory, the running of the string coupling to zero would presumably destroy all bound states such as nucleons and send all particle masses to zero.

III. THROUGH THE BOUNCE

To have repeating cycles, the universe must be able to pass smoothly from a big crunch to a big bang. Conventionally, the curvature and density singularity when the scale factor a approaches zero has been regarded as an impassable obstacle to the understanding of what came "before" the big bang. However, the brane world setup sheds new light on this problem. The key feature is that the apparent singularity in the effective four-dimensional description corresponds to a higher dimensional setup in which the four dimensional metric is completely nonsingular. When the extra dimension (or outer brane separation) shrinks to zero, there is no associated curvature singularity, and the density of matter on the branes remains finite. The most conservative assumption, based on the higher dimensional picture, is that the branes bounce from (or, equivalently, pass through) each other and time continues smoothly, with some conversion of brane kinetic energy to entropy. The separation of the two branes after the bounce corresponds to re-expansion in the four-dimensional effective theory.

How can this be reconciled with the singular four-dimensional description? The point explained in Ref. [8] is that the usual four-dimensional variables, the scale factor a and the scalar field ϕ, are a singular choice at $a=0$. Each is poorly behaved as the branes collide, but in the brane picture physical quantities depend on combinations of the two variables that remain well-behaved. These nonsingular variables may be treated as fundamental, and matching rules derived to parametrize the physics of inelastic brane collisions. If the system can, as conjectured in Ref. [8], be properly embedded within string theory, the matching conditions will be derivable from fundamental physics.

A. Nonsingular variables

The action for a scalar field coupled to gravity and a set of fluids ρ_i in a homogeneous, flat universe, with line element $ds^2 = a^2(\tau)(-N^2 d\tau^2 + d\vec{x}^2)$ is

$$\mathcal{S} = \int d^3x\, d\tau \left[N^{-1}\left(-3a'^2 + \frac{1}{2}a^2\phi'^2 \right) - N[(a\beta)^4 \Sigma_i \rho_i + a^4 V(\phi)] \right]. \tag{7}$$

We use τ to represent conformal time and primes to represent derivatives with respect to τ. N is the lapse function. The background solution for the scalar field is denoted $\phi(\tau)$, and $V(\phi)$ is the scalar potential.

The only unusual term in Eq. (7) is the coupling of the fluids ρ_i, which we treat as perfect fluids coupled only through gravity. The action for a perfect fluid coupled to gravity is just $-\int d^4x \sqrt{-g}\rho$, where the density ρ is regarded as a function of the coordinates of the fluid particles and the space-time metric [17]. For a homogeneous isotropic fluid, the equation of state $P(\rho)$ defines the functional dependence of ρ on the scale factor a, via energy-momentum conservation, $d\ln\rho/d\ln a = -3(1+w)$, with $w = P/\rho$. For example, for radiation, $\rho \propto a^{-4}$ and for matter $\rho \propto a^{-3}$.

We assume these fluids live on one of the branes, so that rather than coupling to the Einstein-frame scale factor a, the particles they are composed of couple to a conformally related scale factor $a\beta(\phi)$, being the scale factor on the appropriate brane. For simplicity we have only written the action for fluids on one of the branes, the action for fluids on the other brane being a xerox copy but with the appropriate $\beta(\phi)$.

The function $\beta(\phi)$ may generally be different for the two branes, and for different brane world setups. But as mentioned above there is an important universality at small separations corresponding to large negative ϕ. In this limit, which is relevant to the bounce, the bulk warp factor becomes irrelevant and one obtains $\beta \sim e^{-\phi/\sqrt{6}}$, the standard Kaluza-Klein result. This behavior ensures that $a\beta$ is finite at collision and so the matter and radiation densities are, as well.

The equations of motion for gravity, the matter and scalar field ϕ are straightforwardly derived by varying Eq. (7) with respect to a, N and ϕ, after which N may be set equal to unity. Expressed in terms of proper time t, the Einstein equations are

$$H^2 = \frac{8\pi G}{3}\left(\frac{1}{2}\dot{\phi}^2 + V + \beta^4\rho_R + \beta^4\rho_M\right), \tag{8}$$

$$\frac{\ddot{a}}{a} = -\frac{8\pi G}{3}\left(\dot{\phi}^2 - V + \beta^4\rho_R + \frac{1}{2}\beta^4\rho_M\right), \tag{9}$$

where a dot is a proper time derivative. As an example, we consider the case where there is radiation (ρ_R) and matter (ρ_M) on the visible brane only, which could in principle be either the positive or negative tension brane. Then the above equations are supplemented by the dynamical equation for the evolution of ϕ,

$$\ddot{\phi} + 3H\dot{\phi} = -V_{,\phi} - \beta_{,\phi}\beta^3\rho_M \tag{10}$$

and the continuity equation,

$$\hat{a}\frac{d\rho}{d\hat{a}} = a\frac{\partial\rho}{\partial a} + \frac{\beta}{\beta'}\frac{\partial\rho}{\partial\phi} = -3(\rho+p) \tag{11}$$

where $\hat{a} = a\beta(\phi)$ and p is the pressure of the fluid component with energy density ρ. Note that only the matter density contributes to the ϕ-equation, because, if $\rho_R \propto 1/(a\beta)^4$, the radiation term is actually just a constant times N in the action, contributing to the Friedmann constraint but not the dynamical equations of motion.

If $\beta(\phi)$ is sufficiently flat near the current value of ϕ, these couplings have modest effects in the late universe, and the successes of the standard cosmology are recovered. For example the total variation in ϕ since nucleosynthesis is very modest. In Planck units, this is of order $(t_r/t_N)^{1/2}$ where t_r is the time at the beginning of the radiation dominated epoch and nucleosynthesis begins at $t_N \sim 1$ sec. It is utterly negligible for values of t_r earlier than the electroweak era. However, the coupling of matter to ϕ produces other potentially measurable effects including a ‘‘fifth force’’ causing violations of the equivalence principle. Current constraints can be satisfied if $M_{Pl}(\ln\beta)_{,\phi} < 10^{-3}$ [18–20].

As the universe contracts towards the big crunch, $a \to 0$, the scalar field runs to $-\infty$ and the scalar potential becomes negligible. The universe becomes dominated by the scalar field kinetic energy density since it scales as a^{-6} whereas matter and radiation densities scale as a^{-3} and a^{-4} respectively (ignoring the β factor). As scalar kinetic domination occurs, the scale factor a begins to scale as $(-t)^{1/3}$, and the background scalar field diverges logarithmically in time. The energy density and Ricci scalar diverge as $(-t)^{-2}$, so that $t=0$ is a ‘‘big crunch’’ singularity.

As explained in Ref. [8], in the simplest treatment of brane world models there is only one scalar field modulus, the ‘‘radion,’’ which runs off to minus infinity as the scale factor a approaches zero. The singular variables, a and ϕ, can be replaced by the nonsingular variables:

$$a_0 = 2a\cosh((\phi-\phi_\infty)/\sqrt{6})$$

$$a_1 = -2a\sinh((\phi-\phi_\infty)/\sqrt{6}). \tag{12}$$

The kinetic terms in the action define the metric on moduli space. In terms of the old variables one has the line element $-3da^2 + \frac{1}{2}a^2 d\phi^2$, and $a=0$ is clearly a singular point in these coordinates. However, in the new coordinates in Eq. (12), the line element is $\frac{3}{4}(-da_0^2 + da_1^2)$, which is perfectly regular even when the Einstein frame scale factor $a = \frac{1}{2}\sqrt{a_0^2 - a_1^2}$ vanishes, on the ‘‘light-cone’’ $a_0 = a_1$. For branes in AdS, a_0 and a_1 are the scale factors on the positive and negative tension branes [7] so that $\beta = 2\cosh((\phi-\phi_\infty)/\sqrt{6})$ or $-2\sinh((\phi-\phi_\infty)/\sqrt{6})$ respectively for matter coupling to these branes.

Notice that the constant field shift ϕ_∞ is arbitrary. Its effect is a Lorentz boost on the (a_0, a_1) moduli space. In the Kaluza-Klein picture (6), a constant shift in ϕ can be removed by rescaling four dimensional space-time coordinates and redefining the length scale L of the extra dimensions. In the absence of matter which couples to ϕ, or of a potential $V(\phi)$, this shift is unobservable, a reflection of the global symmetry $\phi \to \phi + \text{const}$ of the $4D$ effective theory. However, this symmetry is broken by $V(\phi)$, and by matter couplings. In fact, the scale factor a_1 must be positive in order for it to be interpretable as a ‘‘brane scale factor,’’ and this requires that $\phi < \phi_\infty$.

We shall find it convenient to choose $\phi = 0$ to be the zero of the potential $V(\phi)$, and then to choose ϕ_∞ so that a_1 never vanishes for the solutions we are interested in. (In fact, since a_1 has a positive kinetic term [7], a suitable coupling to moduli fields will always guarantee that a_1 ‘‘bounces’’ away from zero [12]. In this paper, for simplicity we ignore this complication by picking ϕ_∞ large enough that no such ‘‘bounce’’ is necessary.)

Both a_0 and a_1 are ‘‘scale factors’’ since they transform like a under rescaling space-time coordinates. However, unlike a they tend to finite constants as a tends to zero, implying an alternative metrical description which is not singular at the ‘‘big crunch.’’ In the brane world models considered in Ref. [7], a_0 and a_1 actually represent the scale factors of the positive and negative tension branes respectively. Since there are no low energy configurations with $a_0 < a_1$, the ‘‘light cone’’ $a_0 = a_1$ is actually a boundary of moduli space and one requires a matching rule to determine what the trajectory

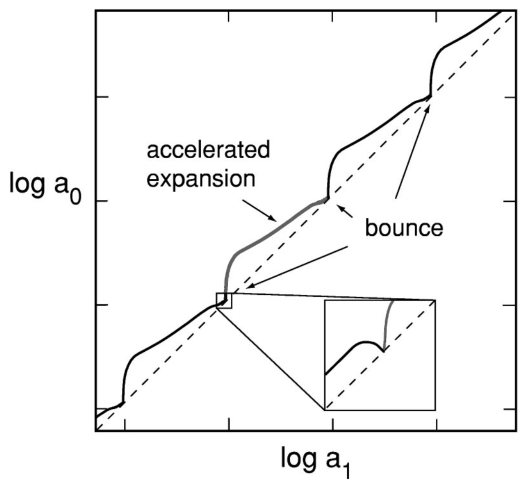

FIG. 3. Schematic plot of the a_0-a_1 plane showing a sequence of cycles of expansion and contraction (indicated by tick marks). The dashed line represents the ''light-cone'' $a_0 = a_1$ corresponding to a bounce ($a=0$). Each cycle includes a moduli kinetic energy, radiation, matter and quintessence dominated phase and lasts an exponentially large number of e-folds. The inset shows the trajectory near the big crunch and bounce. The potential energy $V(\phi)$ assumed takes the form shown in Fig. 1.

of the system does at that point. A natural matching rule is to suppose that at low energies and in the absence of potentials or matter, the branes simply pass through one another (or, equivalently, bounce) with the intervening bulk briefly disappearing and then reappearing after collision. This rule was detailed in Ref. [8], where simple models satisfying the string theory background equations to all orders in α' were given. In the Appendix we discuss the collision between boundary branes in terms of energy and momentum conservation, and the Israel matching conditions.

Let us now comment on the character of the trajectory in the (a_0, a_1)-plane. The Friedmann constraint reads

$$a_0'^2 - a_1'^2 = \frac{4}{3}\left((a\beta)^4 \rho + \frac{1}{16}(a_0^2 - a_1^2)^2 V(\phi_0) \right). \quad (13)$$

If the energy density on the right hand side is positive, the trajectory is time-like. If the right hand side is zero (for example if the potential vanishes as $\phi_0 \to -\infty$ and if there is no matter or radiation), then the trajectory is light-like. If the right hand side is negative, the trajectory is space-like.

The trajectory for the cyclic solution in the $a_0 - a_1$ plane is shown in Fig. 3. The inset shows a blow-up of the behavior at the bounce in which the trajectory is light-like at contraction to the big crunch (the universe is empty) and time-like on expansion from the big bang (radiation is produced at the bounce). In these coordinates, the scale factor increases exponentially over each cycle, but the next cycle is simply a rescaled version of the cycle before. A local observer measures physical quantities such as the Hubble constant or the deceleration parameter, which entail ratios of the scale factor and its derivatives in which the normalization of the scale factor cancels out. Hence, to local observers, each cycle appears to be identical to the one before.

B. From big crunch to big bang

In this section we solve the equations of motion immediately before and after the bounce, and discuss how the incoming and outgoing states are connected. The nonsingular ''brane'' scale factors a_0 and a_1 provide the natural setting for this discussion, since neither vanishes at the bounce. As emphasized above, the Einstein frame scale factor a, and the scalar field ϕ are singular coordinates on field space at the bounce. Nevertheless, since our intuition is much better in the Einstein frame, we shall also give formulas for a and ϕ near the bounce. In subsequent sections, we shall frame the discussion almost entirely in terms of Einstein frame variables, for the most part using the nonsingular variables a_0 and a_1 solely as a ''bridge'' connecting the incoming big crunch to the outgoing big bang.

Before the bounce there is little radiation present since it has been exponentially diluted in the preceding quintessence-dominated accelerating phase. Furthermore, the potential $V(\phi)$ becomes negligible as ϕ runs off to minus infinity. The Friedmann constraint reads $(a'/a)^2 = \frac{1}{6}\phi'^2$, and the scalar field equation, $(a^2\phi')' = 0$, where primes denote conformal time derivatives. The general solution is

$$\phi = \sqrt{\frac{3}{2}} \ln[A H_5(in)\tau],$$

$$a = A e^{\phi/\sqrt{6}} = A\sqrt{A H_5(in)\tau},$$

$$a_0 = A[\lambda + \lambda^{-1} A H_5(in)\tau],$$

$$a_1 = A[\lambda - \lambda^{-1} A H_5(in)\tau], \quad (14)$$

where $\lambda \equiv e^{\phi_\infty/\sqrt{6}}$. We choose $\tau = 0$ to be the time when a vanishes so that $\tau < 0$ before collision. A is an integration constant which could be set to unity by rescaling space-time coordinates but it is convenient not to do so. The Hubble constants as defined in terms of the brane scale factors are a_0'/a_0^2 and a_1'/a_1^2 which at $\tau = 0$ take the values $+\lambda^{-3} H_5(in)$ and $-\lambda^{-3} H_5(in)$ respectively.

Re-expressing the scalar field as a function of proper time $t = \int a\, d\tau$, we obtain

$$\phi = \sqrt{\frac{2}{3}} \ln\left(\frac{3}{2} H_5(in) t \right). \quad (15)$$

The integration constant $H_5(in) < 0$ has a natural physical interpretation as a measure of the contraction rate of the extra dimension [see Eq. (6)]:

$$H_5 \equiv \frac{dL_5}{L\, dt_5} \equiv \frac{d(e^{\sqrt{(2/3)}\phi})}{dt_5} = \sqrt{\frac{2}{3}} \dot{\phi} e^{\sqrt{(3/2)}\phi}, \quad (16)$$

where $L_5 \equiv L e^{\sqrt{(2/3)}\phi}$ is the proper length of the extra dimension, L is a parameter with dimensions of length, and t_5 is the proper time in the five-dimensional metric,

$$dt_5 \equiv a e^{-\sqrt{(1/6)}\phi} d\tau = e^{-\sqrt{(1/6)}\phi} dt, \quad (17)$$

with t being FRW proper time. Notice that a shift ϕ_∞ can always be compensated for by a rescaling of L. As the extra dimension shrinks to zero, H_5 tends to a constant, $H_5(in)$.

If the extra dimension shrinks adiabatically and backreaction from particle production can be ignored, then the matching rule conjectured in Ref. [8] states that H_5 after the bounce should be given by $H_5(out)=-H_5(in)$. However, if radiation is produced, $H_5(out)$ takes a different value. If one is given the densities of radiation produced on both branes, then $H_5(out)$ may be inferred from energy and momentum conservation, and the Israel matching conditions, as we show in the Appendix.

Immediately after the bounce, scalar kinetic energy dominates and H_5 remains nearly constant, as shown in Fig. 2. The kinetic energy of the scalar field scales as a^{-6} and radiation scales as a^{-4}, so the former dominates at small a. It is convenient to rescale a so that it is unity at scalar kinetic energy-radiation equality, t_r, and denote the corresponding Hubble constant H_r. The Friedmann constraint in Eq. (13) then reads

$$(a')^2=\frac{1}{2}H_r^2(1+a^{-2}), \tag{18}$$

and the solution is

$$\phi=\sqrt{\frac{3}{2}}\ln\left(\frac{2^{5/3}\tau H_5^{2/3}(out)H_r^{1/3}}{(H_r\tau+2^{3/2})}\right),$$
$$a=\sqrt{\frac{1}{2}H_r^2\tau^2+\sqrt{2}H_r\tau}. \tag{19}$$

The brane scale factors are

$$a_0\equiv a(\lambda^{-1}e^{\phi/\sqrt{6}}+\lambda e^{-\phi/\sqrt{6}})$$
$$=A\left[\lambda\left(1+\frac{H_r\tau}{2^{3/2}}\right)+\lambda^{-1}2^{1/6}H_r^{1/3}H_5^{2/3}(out)\tau\right],$$
$$a_1\equiv a(-\lambda^{-1}e^{\phi/\sqrt{6}}+\lambda e^{-\phi/\sqrt{6}})$$
$$=A\left[\lambda\left(1+\frac{H_r\tau}{2^{3/2}}\right)-2^{1/6}\lambda^{-1}H_r^{1/3}H_5^{2/3}(out)\tau\right]. \tag{20}$$

Here the constant $A=2^{1/6}[H_r/H_5(out)]^{1/3}$ has been defined so that we match a_0 and a_1 to the incoming solution given in Eq. (14). As for the incoming solution, we can compute the Hubble constants on the two branes after collision. They are $\pm\lambda^{-3}H_5(out)+2^{-5/3}\lambda^{-1}H_r^{2/3}H_5^{1/3}$ on the positive and negative tension branes respectively.

For $H_r<2^{5/2}\lambda^{-3}H_5$, the case of relatively little radiation production, a_0 is expanding but a_1 is contracting immediately after collision, whereas for $H_r>2^{5/2}\lambda^{-3}H_5$, both brane scale factors expand after collision. We shall concentrate on the former case in this paper, in which we are near the adiabatic limit. If no scalar potential $V(\phi)$ were present, the scalar field would continue to obey the solution (19), converging to

$$\phi_C=\sqrt{\frac{2}{3}}\ln\left(2^{5/2}\frac{H_5(out)}{H_r}\right). \tag{21}$$

This value is actually larger than ϕ_∞ for $H_r<H_5\lambda^{-3}2^{5/2}$, the case of weak production of radiation. However, the presence of the potential $V(\phi)$ alters the expression (21) for the final resting value of the scalar field. As ϕ crosses the potential well traveling in the positive direction, H_5 is reduced to a renormalized value $\hat{H}_5(out)<H_5(out)$, so that the final resting value of the scalar field can be smaller than ϕ_∞. If this is the case, then a_1 never crosses zero, instead reversing to expansion shortly after radiation dominance. (In the calculations of Ref. [12], where we assumed the potential *vanished* after collision, this effect did not occur. Instead, we invoked a coupling of a_1 to a modulus field which caused it to bounce off $a_1=0$.)

If radiation dominance occurs well after ϕ has crossed the potential well, Eq. (21) provides a reasonable estimate for the final resting value, if we use the corrected value $\hat{H}_5(out)$. The dependence of Eq. (21) is simply understood: while the universe is kinetic energy dominated, a grows as $t^{1/3}$ and ϕ increases logarithmically with time. However, when the universe becomes radiation-dominated and $a\propto t^{1/2}$, Hubble damping increases and ϕ converges to the finite limit above.

IV. ACROSS THE WELL

Using the potential described in Sec. II A and, specifically, the example in Eq. (1), this section considers the motion of ϕ back and forth across the potential well. We will show that evolution converges to a stable attractor solution. Our main purpose, though, is to explore the asymmetry in the behavior before and after the bounce that is an essential component of the cyclic solution.

Over most of this region, V may be accurately approximated by $-V_0e^{-c\phi}$. For this pure exponential potential, there is a simple scaling solution [12]

$$a(t)=|t|^p,\quad V=-V_0e^{-c\phi}=-\frac{p(1-3p)}{t^2},\quad p=\frac{2}{c^2}, \tag{22}$$

which is an expanding or contracting universe solution according to whether t is positive or negative. (We choose $t=0$ to be the bounce.) From the expression for V, we see that ϕ varies logarithmically with $|t|$.

At the end of the expanding phase of the cyclic scenario, there is a period of accelerated expansion which makes the universe empty, homogeneous and flat, followed by ϕ rolling down the potential $V(\phi)$ into the well. After ϕ has rolled sufficiently and the scale factor has begun to contract [past stage (3) in Fig. 2], the universe accurately follows the above scaling solution down the well until ϕ encounters the potential minimum [stage (5) in Fig. 2].

Let us consider the behavior of ϕ under small shifts in the contracting phase. In the background scalar field equation and the Friedmann equation, we set $\phi=\phi_B+\delta\phi$ and $H=H_B+\delta H$, where ϕ_B and H_B are the background quantities given from Eqs. (22). To linear order in $\delta\phi$, one obtains

$$\delta\ddot{\phi}+\frac{1+3p}{t}\delta\dot{\phi}-\frac{1-3p}{t^2}\delta\phi=0, \tag{23}$$

with two linearly independent solutions, $\delta\phi\sim t^{-1}$ and t^{1-3p}, where $p\ll 1$. In the contracting phase, the former solution grows as t tends to zero. However, this solution is simply an infinitesimal shift in the time to the big crunch: $\delta\phi\propto\dot{\phi}$. Such a shift provides a solution to the Einstein-scalar equations because they are time translation invariant, but it is physically irrelevant since it can be removed by a redefinition of time. The second solution is a physical perturbation mode and it decays as t tends to zero. Hence, we find that the background solution is an attractor in the contracting phase.

We next consider the incoming and outgoing collision velocity, which we have parametrized as $H_5(in)$ and $H_5(out)$ in the previous section. Within the scaling solution (22), we can calculate the value of incoming velocity by treating the prefactor of the potential $F(\phi)$ in Eq. (1) as a Heaviside function which is unity for $\phi>\phi_{min}$ and zero for $\phi<\phi_{min}$, where ϕ_{min} is the value of ϕ at the minimum of the potential. We compute the velocity of the field as it approaches ϕ_{min} and use energy conservation at the jump in V to infer the velocity after ϕ_{min} is crossed. In the scaling solution, the total energy as ϕ approaches ϕ_{min} from the right is $\frac{1}{2}\dot{\phi}^2+V=3p^2/t^2$, and this must equal the total energy $\frac{1}{2}\dot{\phi}^2$ evaluated for ϕ just to the left of ϕ_{min}. Hence, we find that $\dot{\phi}=\sqrt{6}p/t=\sqrt{6pV_{min}/(1-3p)}$ at the minimum and, according to Eq. (16),

$$H_5(in)\approx-\frac{\sqrt{8}}{c}\frac{|V_{min}|^{1/2}e^{\sqrt{(3/2)}\phi_{min}}}{\sqrt{1-6c^{-2}}}. \tag{24}$$

At the bounce, this solution is matched to an expanding solution with

$$H_5(out)=-(1+\chi)H_5(in)>0, \tag{25}$$

where χ is a small parameter which arises because of the inelasticity of the collision.

In order to obtain cyclic behavior, we shall need χ to be positive or, equivalently, the outgoing velocity to exceed the incoming velocity. There are at least two effects that can cause χ to be positive. First, as we discuss in the Appendix, χ is generically positive if more radiation is generated on the negative tension brane than on the positive tension brane at collision. Secondly, χ can get a positive contribution from the coupling of $\beta(\phi)$ to the matter created on the branes by the collision; see Eq. (10). Both effects are equally good for our purposes. For the present discussion, we shall simply assume a small positive χ is given, and follow the evolution forward in time.

Since χ is small, the outgoing solution is very nearly the time reverse of the incoming solution as ϕ starts back across the potential well after the bounce: the scaling solution given in Eqs. (22), but with t positive. As time proceeds, however, the contribution of χ becomes increasingly significant. In the time-reversed scaling solution, H_5 tends to zero. For $\chi>0$, H_5 remains positive and ϕ overshoots the potential well. V_0 is exponentially smaller than the kinetic energy density at the bounce, so even a tiny fraction χ suffices to reach the plateau after crossing the potential well.

We can analyze this overshoot by treating χ as a perturbation and using the solution in Eq. (23) discussed above, $\delta\phi\sim t^{-1}$ and t^{1-3p}. The latter is a decaying mode in the contracting phase before the bounce but it grows in the expanding phase. One can straightforwardly compute the perturbation in δH_5 in this growing mode by matching at ϕ_{min} as before. One finds $\delta H_5=12\chi H_5^B/c^2$ where H_5^B is the background value, at the minimum. Beyond this point, δH_5 grows as $t^{\sqrt{6}/c}\propto e^{\sqrt{(3/2)}\phi}$, for large c, whereas in the background scaling solution H_5 decays with ϕ as $e^{[\sqrt{(3/2)}-c/2]\phi}$. When the perturbation is of order the background value, the trajectory departs from the scaling solution and the potential becomes irrelevant. The departure occurs when the scalar field has attained the value

$$\phi_{Dep}=\phi_{min}+\frac{2}{c}\ln\frac{c^2}{12\chi},\qquad |V|\lesssim\left(\frac{12\chi}{c^2}\right)^2|V_{min}|. \tag{26}$$

As ϕ passes beyond ϕ_{Dep} the kinetic energy overwhelms the negative potential and the field passes onto the plateau V_0 with H_5 nearly constant (see Fig. 2), and equal to

$$\hat{H}_5(out)\approx\chi\left(\frac{c^2}{12\chi}\right)^{\sqrt{6}/c}H_5(in), \tag{27}$$

until the radiation, matter and vacuum energy become significant and H_5 is then damped away to zero.

Before moving on to discuss these late stages, it is instructive to compare how rapidly ϕ travels over its range before and after the bounce. The time spent to the left of the potential well ($\phi<\phi_{min}$) is essentially identical in the incoming and outgoing stages for $\chi\ll 1$, namely

$$|t_{min}|\approx\frac{c}{3\sqrt{2|V_{min}|}}. \tag{28}$$

For the outgoing solution, when ϕ has left the scaling solution but before radiation domination, the definition Eq. (16) may be integrated to give the time since the big bang at each value of ϕ,

$$t(\phi)=\int\frac{d\phi}{\dot{\phi}}=\sqrt{\frac{2}{3}}\int d\phi\frac{e^{\sqrt{(3/2)}\phi}}{H_5(\phi)}\approx\frac{2}{3}\frac{e^{\sqrt{(3/2)}\phi}}{\hat{H}_5(out)}. \tag{29}$$

The time in Eq. (29) is a *microphysical* scale. The corresponding formula for the time before the big crunch is very different. In the scaling solution (22) one has, for large c,

$$t(\phi)=-\sqrt{\frac{2}{|V_{min}|}}\frac{e^{c(\phi-\phi_{min})/2}}{c}=-\frac{6e^{c(\phi-\phi_{min})/2}}{c^2}|t_{min}|. \tag{30}$$

The large exponential factor makes the time to the big crunch far longer than the time from the big bang, for each value of ϕ. This effect is due to the increase in H_5 after the bounce, which, in turn, is due to the positive value of χ.

V. THE RADIATION, MATTER AND QUINTESSENCE EPOCHS

As the scalar field passes beyond the potential well, it runs onto the positive plateau V_0. As mentioned in the last section, the value of $H_5(out)$ is nearly canceled in the passage across the potential well, and is reduced to $\hat{H}_5$ given in Eq. (27). Once radiation domination begins, the field quickly converges to the large t (Hubble-damped) limit of Eq. (19), namely

$$\phi_C=\sqrt{\frac{2}{3}}\ln[2^{5/2}\hat{H}_5(out)/H_r)], \tag{31}$$

where H_r is the Hubble radius at kinetic-radiation equality. The dependence is obvious: the asymptotic value of ϕ depends on the ratio of $\hat{H}_5(out)$ to H_r. Increasing $\hat{H}_5(out)$ pushes ϕ further, likewise lowering H_r delays radiation domination allowing the logarithmic growth of ϕ in the kinetic energy dominated phase to continue for longer.

As the kinetic energy redshifts away, the gently sloping potential gradually becomes important, in acting to slow and ultimately reverse ϕ's motion. The solution of the scalar field equation is, after expanding Eq. (19) for large τ, converting to proper time $t=\int a(\tau)d\tau$ and matching,

$$\dot{\phi}\approx\frac{\sqrt{3}H_r}{a^3(t)}-a^{-3}\int_0^t dt a^3 V_{,\phi}, \tag{32}$$

where as above we define $a(t)$ to be unity at kinetic-radiation equal density. During the radiation and matter eras, the first term scales as $t^{-3/2}$ and t^{-2} respectively. For a slowly varying field, $V_{,\phi}$ is nearly constant, and the potential gradient term in Eq. (32) scales linearly with t, so it eventually dominates.

When does ϕ turn around? We give a rough discussion here, ignoring factors of order unity. First, we use the fact that $V_0\sim t_0^{-2}$ where t_0 is the present age of the universe, and roughly we have $t_0\sim 10^5 t_m$, where t_m is the time of matter domination. As we shall see, ϕ may reach its maximal value ϕ_{max} and turn around during the radiation, matter or quintessence dominated epoch. All three possibilities are acceptable phenomenologically, although the case where turnaround occurs in the radiation epoch appears more likely. For example, ϕ_{max} is reached in the radiation era, if, from Eq. (32),

$$\frac{t_{max}}{t_m}\approx 10^4\left(\frac{t_r}{t_m}\right)^{1/5}\left(\frac{V}{V_{,\phi}}(\phi_C)\right)^{2/5}<1. \tag{33}$$

If the universe becomes radiation dominated at the grand unified theory (GUT) scale, $t_r\sim 10^{-25}$ s. Then, only if we fine-tune such that $V/V'>10^{10}$ does t_{max} exceed t_m. This corresponds to the case where we have 10^{10} e-foldings or more of cosmic acceleration at late times as ϕ rolls back, far more than required for the cyclic solution. The bound changes somewhat if the universe becomes radiation dominated as late as nucleosynthesis ($t_r\sim 1$ s). In that case, even if $V/V_{,\phi}(\phi_C)$ is not much greater than unity, the scalar field turns around in the matter era or later. For turnaround in the matter era, we require

$$3\times 10^{-4}\lesssim\left(\frac{t_r}{t_m}\right)^{1/6}\left(\frac{V}{V_{,\phi}}(\phi_C)\right)^{1/3}\lesssim 30. \tag{34}$$

Finally, if the field runs to very large ϕ_C, so that $V_{,\phi}/V(\phi_C)\approx ce^{-c\phi_C}$ is exponentially small, then ϕ only turns around in the quintessence-dominated era. For the example considered here, the natural range of parameters corresponds to turnaround occurring during the radiation-dominated epoch. Hence, by the present epoch, the field is rolling monotonically in the negative direction and slowly gaining in speed. Consequently, the ratio of the pressure to the energy density is increasing from its value at turnaround, $w=-1$, towards zero. Depending on the details of the scalar potential $V(\phi)$, it is conceivable that the increasing value of w could ultimately be observationally detectable.

Once the field has turned around and started to roll back towards the potential well, the second term in Eq. (32) dominates. For our scenario to be viable, we require there to be a substantial epoch of vacuum energy domination (inflation) before the next big crunch. The number of e-foldings N_e of inflation is given by the usual slow-roll formula,

$$N_e=\int d\phi\frac{V}{V_{,\phi}}\approx\frac{e^{c\phi_C}}{c^2}, \tag{35}$$

for our model potential. For example, if we demand that the number of baryons per Hubble radius be diluted to below unity before the next contraction, which is certainly over-kill in guaranteeing that the cyclic solution is an attractor, we set $e^{3N_e}\gtrsim 10^{80}$, or $N_e\gtrsim 60$. This is easily fulfilled if ϕ_C is of order unity in Planck units.

From the formulas given above we can also calculate the maximal value ϕ_C in the cyclic solution: for large c and for $t_r\gg\chi^{-1}t_{min}$, it is

$$\phi_C-\phi_{min}\approx\sqrt{\frac{2}{3}}\ln\left(\chi\frac{t_r}{t_{min}}\right), \tag{36}$$

where we used $H_r^{-1}\sim t_r$, the beginning of the radiation-dominated epoch. From Eq. (36) we obtain

$$\frac{t_r}{t_{min}}\sim\frac{1}{\chi}\left(\frac{c^2N_e|V_{min}|}{V_0}\right)^{\sqrt{3/2c^2}}. \tag{37}$$

This equation provides a lower bound on t_r. The extreme case is to take $|V_{min}|\sim 1$. Then using $V_0\sim 10^{-120}, c\sim 10, N_e\sim 60$, we find $t_r\sim 10^{-25}$ s. In this case the maximum temperature of the universe is $\sim 10^{10}$ GeV. This is not very different from what one finds in simple inflationary models.

As ϕ rolls down the hill, one can check that ϕ leaves the slow-roll regime when $e^{-c\phi}$ exceeds $3/c^2$. At this point the constant term V_0 in the potential becomes irrelevant and one can use the scaling solution for ϕ, all the way to the potential minimum. This is also the point at which density perturbations start to be generated via the ekpyrotic mechanism, while the Einstein frame scale factor a is still expanding. The universe continues to expand slowly, but with a slowly decreasing Hubble constant, and finally enters contraction when the density in the scalar field reaches zero, at a negative value of the potential energy. The ensuing contracting phase is accurately described by the scaling solution (22), in which $a \sim (-t)^p$ and $\dot{\phi} = \sqrt{2p}/t$, with $t<0$, and $t=0$ being the time of the next big crunch. From the formulas (12) one finds $\dot{a}_1 = (pa_1 - \sqrt{p/3}a_0)/t$, which is greater than zero for $p<\frac{1}{3}$, since a_0 is greater than a_1. Thus even when a is undergoing slow contraction, in the scaling era, the effect of the motion of ϕ is enough to make a_1 expand throughout this phase. Matter residing on this brane would see continuous expansion all the way to the big crunch. The same argument shows that a_1 actually undergoes a small amount of contraction in the very much shorter scaling epoch of the expanding phase.

VI. GENERATION OF DENSITY PERTURBATIONS

In the cyclic scenario, the period of exponential expansion occurring late in each cycle plays a key role in diluting the densities of matter, radiation and black holes to negligible levels, suppressing long wavelength perturbations and establishing a "clean slate," namely a flat vacuous universe in which all fields are in their quantum mechanical ground state. As the scalar field rolls down the potential in Eq. (1), entering the scaling solution in Eq. (22), the ekpyrotic mechanism for the generation of fluctuations derived in Refs. [1] and [12] sets in and a scale invariant spectrum of adiabatic perturbations is thereby developed. Quantum fluctuations of the usual inflationary sort are also developed in the slow-roll quintessence epoch, but these are: (a) negligible in amplitude because V_0 is tiny; and (b) only excited on scales of order t_0 and above in the contracting phase. These scales are shrunk only as $(-t)^p$ in the contracting, scaling solution, but then expanded as $t^{1/3}$, $t^{1/2}$ and $t^{2/3}$ in the kinetic dominated, radiation and matter eras in the big bang phase, which also lasts for a time of order t_0. Therefore, the modes amplified during inflation are exponentially larger in wavelength than the Hubble radius scale in the next cycle by the time of quintessence domination, which is the present epoch.

Let us concentrate on the fluctuations produced via the ekpyrotic mechanism [12]. Expanding the inhomogeneous fluctuations in the scalar field $\delta\phi(t,\vec{x}) = \Sigma_{\vec{k}} \delta\phi_{\vec{k}}(\tau) e^{i\vec{k}\cdot\vec{x}}$, we remove the damping term by setting $\delta\phi_{\vec{k}} = a^{-1}\chi_{\vec{k}}$, to obtain

$$\chi_{\vec{k}}'' = -k^2 \chi_{\vec{k}} + \left(\frac{a''}{a} - V_{,\phi\phi} a^2 \right) \chi_{\vec{k}} \equiv -(k^2 - k_F^2)\chi_{\vec{k}}, \quad (38)$$

where primes denote conformal time derivatives and we have defined k_F, the comoving "freeze-out" wave number. Modes with $k>k_F$ oscillate with fixed amplitude, whereas those with $k<k_F$ are amplified. In the regime of interest k_F grows monotonically so that shorter and shorter wavelengths progressively freeze out as the big crunch is approached. The physical scale at which modes freeze out is given by

$$\lambda_F = \left[\frac{a''}{a^3} - V_{,\phi\phi} \right]^{-1/2} = \left[\frac{2}{3} V - \frac{1}{6}\dot{\phi}^2 - V_{,\phi\phi} \right]^{-1/2}. \quad (39)$$

As usual we adopt units where $8\pi G = 1$, and denote proper time derivative with a dot. In the era of quintessence domination when V dominates over $V_{,\phi\phi}$, the freeze-out scale λ_F is nearly constant, and comoving wavelengths are exponentially stretched beyond it. As $V_{,\phi\phi}$ begins to dominate, however, Hubble damping becomes irrelevant, and the system approaches the scaling solution given in Eq. (22), in which $V_{,\phi\phi} \approx -2/t^2$, where t is the proper time to the big crunch. The freeze-out scale drops linearly with time to zero, as the scale factor is falling, like $(-t)^{1/3}$. Therefore progressively shorter and shorter wavelength modes are frozen out and amplified, with waves of physical wavelength t_F being frozen out at a time t_F.

An exponentially large band of comoving wavelengths is amplified and frozen in as ϕ rolls from $\phi=0$ down towards ϕ_{min}. Modes with all physical wavelengths from the microphysical scale t_{min}, which could be not much larger than the Planck length, to the macroscopic scale t_0/c which is of order a tenth the present Hubble radius, acquire scale invariant perturbations. Once the perturbations are generated, their wavelength scales as $(t_{min}/t_F)^p$ in the collapsing phase. Then as ϕ crosses the potential well and races off to minus infinity, the Einstein frame physical wavelength goes to zero. But this is not the relevant quantity to track, since we match the variables a_0 and a_1 and therefore should match the physical wavelengths as measured by these scale factors. In the kinetic dominated phase, a_0 and a_1 are nearly constant, so in effect the physical wavelength of the modes is matched when ϕ crosses ϕ_{min}, in the contracting and expanding phases. Furthermore, the contracting and expanding solutions are nearly time-reverses of one another, until the time t_{Dep} computed above when the expanding solution deviates from scaling. Therefore one is effectively matching at t_{Dep}, from which one sees that the time t_F at which perturbations on the current Hubble radius scale t_0 were generated, is given by

$$|t_F| \left(\frac{|t_{Dep}|}{|t_F|} \right)^p \approx t_0 \left(\frac{t_m}{t_0} \right)^{2/3} \left(\frac{t_r}{t_m} \right)^{1/2} \left(\frac{t_{Dep}}{t_r} \right)^{1/3}, \quad (40)$$

where the bracketed factors are: (a) the contraction of the scale factor in the scaling solution, between the time t_F at which the perturbations were generated and the time t_{Dep} at which the expanding solution departs from scaling; (b) the scaling back of the present comoving Hubble radius scale to the time of matter-domination t_m; (c) the scaling back to the time of radiation-domination t_r; and (d) the scaling back to the time t_{Dep} using $H_5\sim$ const, corresponding to kinetic domination in the expanding solution.

From Eq. (30), it follows that perturbations on the scale of the present Hubble radius were generated at a field value

$$\phi_F \approx \phi_{min} + \frac{2}{c}\ln\left[\frac{c^2}{6}\left(\frac{c^2}{12\chi}\right)^{1/3}\frac{t_0^{1/2}t_r^{1/6}}{t_{min}^{2/3}}\right]. \qquad (41)$$

Comparing with Eq. (36) for the resting value of the field ϕ_C, and the expression

$$\phi_{min} \approx -\frac{2}{c}\ln(ct_0/t_{min}), \qquad (42)$$

for the field value at the potential minimum, which follows from Eq. (28), one finds that

$$\phi_{Gen} - \phi_{min} \approx -\frac{1}{2}\phi_{min} + \frac{1}{\sqrt{6}c}(\phi_C - \phi_{min}), \qquad (43)$$

where the first term dominates. In other words, the fluctuations we see today were generated at a field value approximately halfway between the zero and the minimum of $V(\phi)$.

VII. CYCLIC SOLUTIONS AND CYCLIC ATTRACTORS

We have shown that a cyclic universe solution exists provided we are allowed to pass through the Einstein-frame singularity according to the matching conditions elaborated in Sec. IV, Eqs. (24) and (25). Specifically, we assumed that $H_5(out) = -(1+\chi)H_5(in)$ where χ is a non-negative constant, corresponding to branes whose relative speed after collision is greater than or equal to the relative speed before collision. Our argument showed that, for each $\chi \geqslant 0$, there is a unique value of $H_5(out)$ that is perfectly cyclic. In the Appendix, we show that an increase in velocity is perfectly compatible with energy and momentum conservation in a collision between a positive and negative tension brane, provided a greater density of radiation is generated on the negative tension brane. [A similar outcome can occur through the coupling of ϕ to the matter density, as discussed below Eq. (25), but we will only discuss the first effect for the purpose of simplicity.]

In this section, we wish to show that, under reasonable assumptions, the cyclic solution is a stable attractor, typically with a large basin of attraction. Without the attractor property, the cyclic model would seem fine-tuned and unstable. One could imagine that there would still be brane collisions and periods of contraction and expansion, but there would be no regularity or long-term predictability to the trajectories. If this were the case, fundamental physics would lose its power to explain the masses and couplings of elementary particles. The masses and couplings depend on ϕ and other moduli fields. If there were no attractor solution, the precise trajectory of ϕ through cosmic history would depend on initial conditions and could not be derived from fundamental physics alone. In our proposal, the nature of the attractor solution depends on microphysics at the bounce which is computable, in principle, from fundamental theory. Hence, masses and couplings of particles change during the course of the cycle, but fundamental theory retains predictive power in determining the way they change and, specifically, their values at the current epoch.

The essential feature for attractor behavior is the extended period of accelerated expansion that damps the motion of ϕ. Let us consider how this works. Assuming χ is fixed by microphysics, there is a value $\overline{H}_5(out)$ which corresponds to the cyclic solution. Now, suppose the value of $H_5(out)$ exceeds $\overline{H}_5(out)$. This means that the outgoing velocity exceeds the cyclic value and ϕ runs out farther on the plateau than in the cyclic case. Once the field stops, turns around, and quintessence-domination begins, the field is critically damped. By the time $V(\phi)$ falls to zero, the transient behavior of ϕ which depends on the initial value of H_5 has damped away exponentially so that the field accurately tracks the slow-roll solution. Following the solution forwards, $H_5(in)$ at the next bounce is then exponentially close to what it would have been for the cyclic solution. By erasing memory of the initial conditions, the acceleration insures that $H_5(out)$ after the next bounce is very nearly $\overline{H}_5(out)$.

How many *e*-foldings of accelerated expansion are actually required to make the cyclic solution an attractor? If there is no epoch of accelerated expansion, perturbations will grow each cycle, becoming self-gravitating and nonlinear so that no attractor will occur. A minimal requirement for obtaining an attractor is that linear density perturbations grown during the matter era should be damped away during the subsequent exponential expansion. This requires at least $\ln(10^5)\sim 10$ *e*-foldings of exponential expansion. Equally, diluting the number density of baryons below one per Hubble volume is certainly over-kill in terms of ensuring an attractor, and this requires of order 60 *e*-foldings. In fact, as we discussed above, obtaining a far larger number of *e*-foldings is perfectly possible.

To discuss the nature of the attractor solution, it is helpful to plot the trajectories of the system in the phase space given by the (H_5,ϕ)-plane, shown in Fig. 4–6. Recall that H_5 is proportional to $\dot{\phi}$; see Eq. (16). Figure 4 illustrates the cyclic trajectory for the case where no radiation is generated at the bounce $(\chi=0)$ and the cycle is exactly time-symmetric.

The phase space plot must always satisfy three properties. First, for a flat universe, the Friedmann constraint equation $H^2 = \frac{1}{3}\rho$ implies that the energy density $\rho = [\frac{1}{2}\dot{\phi}^2 + V(\phi)]$ must be positive. Without negative space curvature, the system is simply not allowed to explore negative energies. We show the classically excluded region as the shaded area in the figures. In Fig. 4 where there is no radiation and $V(\phi)\to 0$ as $\phi\to 0$, the excluded region extends along $H_5=0$ out to $\phi\to -\infty$. In Figs. 5 and 6 the shape of the excluded region is modified due to the presence of radiation. For example, the gray region pinches off on the left hand side for some finite value of ϕ. However, the effect is negligible for the trajectories considered in our discussion and so we show the same excluded region in the figures as in the case of no radiation.

The second property is that phase space trajectories are double-valued on the (H_5,ϕ)-plane. Given the scalar field

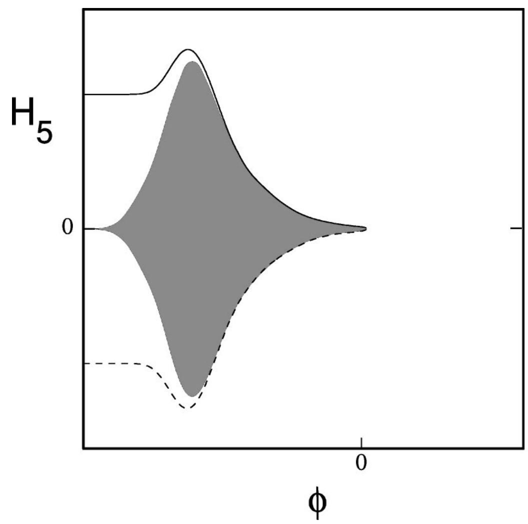

FIG. 4. The cyclic trajectory in the (H_5,ϕ)-plane for the case where no matter and radiation are produced at the bounce $(\chi=0)$. The gray region, which corresponds to negative energy density, is forbidden. The solid (dashed) line represents the trajectory during an expanding (contracting) phase. Expansion turns to contraction and vice versa when the trajectory hits the zero energy surface (the rightmost tip of the gray region in this case).

and H_5, one may have either a contracting or an expanding universe. We represent expanding trajectories as solid lines and the contracting trajectories as dashed lines. Two expanding trajectories are not allowed to cross, and neither are two contracting trajectories for the usual reasons that hold for particle trajectories on phase space. However, an expanding trajectory may certainly intersect a contracting trajectory.

The final rule is that there are only two ways an expanding trajectory can turn into a contracting trajectory. If reversal occurs at finite ϕ it can only happen if the trajectory hits

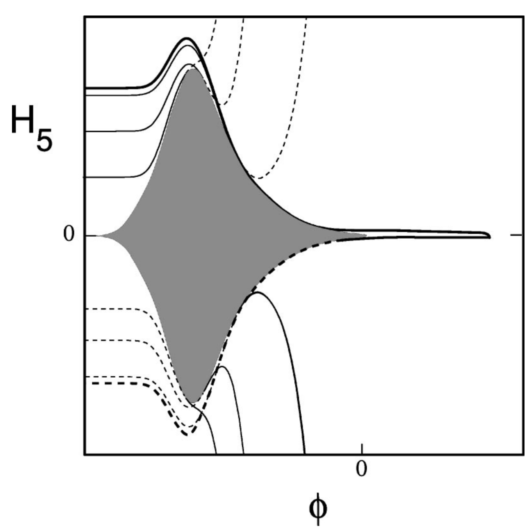

FIG. 5. Trajectories in the (H_5,ϕ)-plane for the case where there is radiation. The solid (dashed) curves represent the trajectory during an expanding (contracting) phase. The thin lines illustrate undershoot solutions and the heavy line represents an overshoot solution.

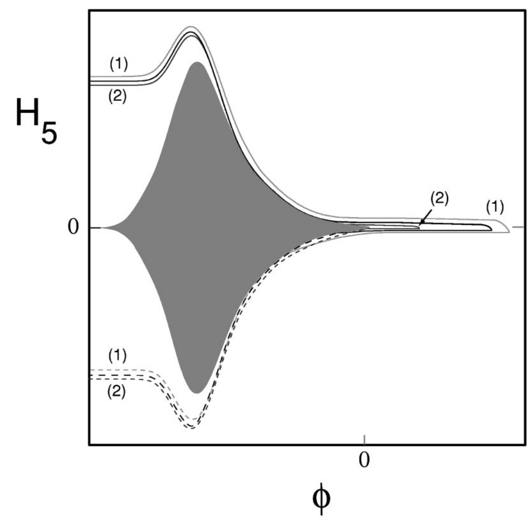

FIG. 6. Trajectories in the (H_5,ϕ)-plane showing the attractor nature of the cyclic solution (the middle trajectory). Path (1) is an overshoot solution that begins with slightly greater velocity (H_5) than the attractor, bounces off the gray zero-energy surface, and then has a contracting trajectory whose value of H_5 is smaller in magnitude. Path (2) is an undershoot solution which begins with slightly less velocity in the expanding phase than the cyclic value and ends in a contracting phase with H_5 having a slightly greater magnitude. Following the next bounce, therefore, overshoot turns into undershoot and vice versa. In either case, the deviation from the attractor value shrinks.

the forbidden zero density region (shaded), since ρ has to vanish if H is to pass smoothly through zero. The shaded region is analogous to the ‘‘egg’’ region described by Brustein and Veneziano [21]. The second way in which contraction can turn into expansion is if the system runs off to $\phi=-\infty$. Then, the ‘‘bounce’’ described in Sec. III and Ref. [8] occurs.

The trajectory shown in Fig. 4 is a cyclic solution (albeit not a very interesting one) in which no matter-radiation is produced at the bounce and the value of $H_5(out)$ is precisely equal to $H_5(in)$. The field rolls out in the expanding phase (solid line emanating from the upper left side of the figure) to the value where $V(\phi)=0$ and stops (the rightmost tip of the gray region). The total energy density is momentarily zero and expansion reverses to contraction. The field then rolls back to $-\infty$ (lower left side of the figure). The expanding and contracting phases are exactly symmetrical.

The time scale for one cycle of this empty-universe solution is easily estimated by noting that most of the time is spent near the zero of the potential, where it is rather shallow, and the scale factor a is nearly constant. Therefore, we can neglect gravity and calculate the period for one cycle in the case of the empty universe:

$$t_{empty}\approx\int_{\phi_{min}}^{0}\frac{d\phi}{\sqrt{V_0(e^{-c\phi}-1)}}=\frac{1}{c\sqrt{V_0}}\int_0^{c|\phi_{min}|}\frac{dy}{\sqrt{e^y-1}}\approx\frac{1}{c\sqrt{V_0}} \qquad (44)$$

for large $c|\phi_{min}|$. For the parameters typical in our examples, this corresponds to roughly one-tenth of the current age of the universe, or a billion years.

In Fig. 5, we consider the case where radiation is produced at the bounce, $\chi > 0$. If $H_5(out)$ is too low compared to the cyclic value, the trajectory encounters the zero density boundary (gray region) and reverses to contraction. Solid curves represent the expanding phase of the trajectory, and dashed lines represent the contracting phase of the trajectory. Let us call this an ''undershoot'' solution. H_5 is only constant if ϕ and a are both increasing, or both decreasing. However, if the universe reverses when H_5 is still positive, then the scalar field kinetic energy is blueshifted and H_5 is rapidly driven to more and more positive values. The trajectory flies off to large positive H_5 and ϕ (the upper boundary).

As one increases $H_5(out)$, the behavior of the system changes. For sufficiently large H_5, the system avoids the zero energy surface entirely during the period when ϕ is increasing (the bold solid and dashed trajectory in Fig. 5). The field ''overshoots'' the negative region of the potential and lands on the positive plateau. Exponential expansion begins, followed by a very slow roll of ϕ back towards the potential zero, This period appears as a long, thin excursion on the right hand side of the figure. $|H_5|$ is small because the field is rolling slowing in the quintessence-dominated phase.

The cyclic attractor solution lies between these undershoot and overshoot trajectories. Figure 6 shows trajectories with initial values of $H_5(out)$ both above and below the cyclic value (the middle curve). Here we can study the stability of the cyclic solution. Let us first consider a trajectory with $H_5(out)$ larger than the value in the cyclic solution. This trajectory is indicated by (1) in the figure. Clearly, it overshoots the cyclic trajectory and undergoes a longer period of exponential expansion (the long excursion to the right). During the slow-roll epoch, the difference between this trajectory and the cyclic one damps away until it is exponentially small. The trajectory encounters the zero density surface very slightly later than the cyclic solution does, and, therefore, reverses and ends up with a very slightly smaller value of $H_5(out)$ than that in the cyclic trajectory. Similarly, one can see that starting the system in state (2) with a smaller value of $H_5(out)$ than that of the cyclic trajectory, the system will inflate less and reverse earlier, ending up with a larger value of $H_5(out)$ than that of the cyclic trajectory. This discussion shows that the trajectory is stable and that memory of the initial conditions decays exponentially after just one cycle.

Here we implicitly assumed that χ is a constant independent of $H_5(in)$, the incoming velocity. In the Appendix, we obtain an expression for χ in Eq. (A5) in terms of the matter-radiation energy densities created on the positive and negative tension branes. Assuming the energy density on the negative tension brane ρ_- is significantly greater than the energy density created on the positive tension brane, we have $\chi \propto \rho_-$. Assuming the collision between branes occurs at a low velocity so that one is not far from the adiabatic limit, ρ_- should decrease with decreasing $H_5(in)$. (Note that the low-velocity assumption has been made throughout since it is required for the moduli approximation.) Hence, we would anticipate that χ rises monotonically as the incoming velocity increases. This effect can alter the trajectories and the precise basin of attraction in detail, but does not alter the conclusion that a large basin of attraction exists. This is assured by having a potential plateau or, more generally, a region of the potential in which ϕ slow-rolls with total energy comparable to the current dark energy density.

Quantum effects are also unlikely to affect the attractor solution. We have shown that the solution is stable under small perturbations and here the perturbations remain small since $|V_{min}|$ and V_0 are small compared to the Planck scale.

What of the trajectories in Fig. 5, for example, that run away to large ϕ? For these, it is important to understand what happens as ϕ grows more positive. One possibility is that the potential $V(\phi)$ diverges as $\phi \to \infty$. Our example for $V(\phi)$ has an infinite plateau, but, as discussed in Sec. II, this is not a general requirement. If $V(\phi)$ grows sufficiently, ϕ will bounce back towards $-\infty$. Alternatively, the same effect can occur if the theory includes massless fields that couple to the scale factor on the negative tension brane, a_1. The Lagrangian density then includes a term $a_1^4 \dot{\psi}^2 \propto 1/a_1^2$. Increasingly positive ϕ corresponds to shrinking a_1. Hence, this dynamical term can also create a force that causes ϕ to bounce back. The net effect is that ϕ rattles back and forth along the potential, possibly following a chaotic orbit [22,23]. These effects could enhance the basin of attraction for the cyclic solution. That is, some of these trajectories which we ignored in our undershoot and overshoot treatment may eventually hit the plateau with low velocity, at which point they would become drawn to the attractor solution.

Finally, let us emphasize that we have only considered the issue of stability in the context of the very simplified model studied here, with a single scalar field ϕ, and the matching conditions discussed in Sec. IV. It would be very interesting to generalize this discussion to include other moduli, matter which couples in a nontrivial way to ϕ as discussed in Sec. IV, and and also discrete degrees of freedom such as a quantized four-form field, which may change from cycle to cycle so that the system really explores moduli space. The existence or otherwise of an attractor could well be relevant to the determination of the relative abundances of dark energy, dark matter, baryons and photons in the universe, and also to the values of the fundamental constants of nature.

VIII. IMPLICATIONS

The strengths of the cyclic model are its simplicity, its efficient use of all of the dominant elements of the universe and the fact that it is a complete description of all phases of cosmic evolution. This can be contrasted with inflationary cosmology, a highly appealing theoretical model in its own right. Inflationary cosmology focuses on a brief epoch when the universe was 10^{-35} s old. The model relies on assumptions about how the universe emerged from the cosmic singularity. One must postulate the existence of a phase of rapid cosmic acceleration at very high energies, for which their exists no direct proof. (In contrast, the cyclic model relies on low energy cosmic acceleration that has been observed.) Subsequent cosmic events, such as the recent transition from matter domination to dark energy domination and cosmic acceleration, appear to have no direct connection to inflationary theory.

Because the cyclic model ties the past, present and future evolution of the universe in a tight, cross-correlated way, it has surprising explanatory and predictive power. In the Introduction to this paper, we noted a number of the most challenging questions of cosmology and fundamental physics. In this section, we consider each of these questions (and more) and briefly describe the insights the cyclic model provides concerning their answers.

A. Why is the universe homogeneous, isotropic and flat?

The universe is made homogeneous and isotropic during the period of the preceding cycle when quintessence dominates and the universe is undergoing slow cosmic acceleration. This ensures that the branes are flat and parallel as they begin to approach, collide, and emerge in a big bang. Inflation also relies on cosmic acceleration, but driven by very high vacuum energy which produces an acceleration that is nearly 10^{100} times faster.

B. How were density inhomogeneities generated?

In the cyclic model, the observed inhomogeneities in the universe are generated during the contracting phase when the scale factor is nearly static and gravitational effects are weak. Consequently, as in the ekpyrotic scenario, a nearly scale invariant spectrum of adiabatic, Gaussian energy density fluctuations is generated. However, because the expansion rate is negligible and gravitational effects are weak, the tensor (metric fluctuation) spectrum is blue with an exponentially tiny amplitude at long wavelengths.

Fluctuations are also created during the quintessence dominated phase, just as they are during inflation. However, because the energy density during the accelerating phase is 100 orders of magnitude smaller than in inflation, the resulting fluctuation amplitude is exponentially smaller in the cyclic model. These fluctuations also have wavelengths that exceed the current Hubble horizon. Hence, they are observationally irrelevant.

C. What is the role of dark energy and the current cosmic acceleration?

Clearly, dark energy and the current cosmic acceleration play an essential role in the cyclic model both by reducing the entropy and black hole density of the previous cycle, and triggering the turnaround from an expanding to a contracting phase. (In all other cosmologies to date, including inflation, dark energy has no essential role.)

D. How old is the Universe?

A truly cyclic universe is clearly infinitely old in terms of cosmic time. As we have noted, the exact cyclic solution can also be an attractor. Hence, the cycling is stable. Consequently, one becomes insensitive to the initial conditions for the universe as long as they were within the basin of attraction of the cyclic solution. We believe that within this framework, the problem of the initial conditions for the universe is significantly altered: as long as the universe has some nonzero probability for entering the cyclic solution, large regions of the universe maintain cyclic evolution for arbitrarily long periods of time.

There is a possible objection to this argument, due to the fact that the four dimensional nonsingular brane space-times in our scenario are past geodesically incomplete. As we have explained, for most of cosmic time they are well approximated by de Sitter space-time, with a cosmological constant (or vacuum energy) close to the currently observed value. This nearly de Sitter space-time is foliated by slices of constant scalar field ϕ, which are nearly geometrically flat. Matter is repeatedly generated on the slices with $\phi=-\infty$, in the rest frame defined by those slices.

As one follows cosmic time t backwards, one must pass an infinite number of these big-crunch–big-bang surfaces. However, even though the cosmic time tends to $-\infty$, the proper time as measured along timelike geodesics running into the past generically is finite even as t tends to $-\infty$. This may be seen as follows. Consider a particle with momentum P in the flat slicing. Its momentum blueshifts as a^{-1} as you follow it back in time. The geodesic becomes nearly null and the proper time measured along the geodesic converges even though t tends to $-\infty$ (this is the crux of the recent argument of Borde, Guth and Vilenkin that inflation is past geodesically incomplete [24]).

In our scenario, however, *all physical particles are created with finite momentum in the flat slicing defined by* ϕ. If we follow a particle present in today's universe back in time, most likely it was created on the last $\phi=-\infty$ surface. With an exponentially smaller probability, it could have been created on the penultimate $\phi=-\infty$ surface, and so on into the past. The probability that any observed particle originated on the $t=-\infty$ flat surface, which is the boundary of the flat slicing of de Sitter space-time, is zero. Therefore we do not attribute any physical significance to the past geodesic incompleteness of the space-time metric in our scenario. In contrast, particle production in standard inflationary models occurred on open slices a finite time ago.

Even if there are no particles present which "saw" the past boundary of the cyclic universe, one might object that the scenario implicitly requires a boundary condition in the infinite past. We do not think this is a strong objection. If the cyclic solution were begun within a finite region (for example a torus) of three dimensional space, it would grow exponentially with each cycle to an arbitrarily large size. After an arbitrarily long time, to any real observer the universe would appear to be infinite both in spatial extent and in lifetime to the past.

So, while the cyclic model still requires an initial condition, provided that state is within the basin of attraction of the cyclic solution, we are completely insensitive to its details. Any features of the initial state (the total size of the universe, or any fluctuations about flatness or homogeneity) become exponentially diluted in each cycle, and since the cyclic solution can repeat forever, they are ultimately completely irrelevant to any observation.

E. What is the ultimate fate of the Universe?

The cycles can be continued to the infinite future, as well as the infinite past. Hence, the universe endures forever.

F. How big is the Universe?

From the effective 4D point-of-view, the universe oscillates between periods of expansion to periods of contraction down to a big crunch. However, from the brane world point-of-view, the universe is always infinite in the sense that the branes always have infinite extent. The fact that the branes are spatially infinite means that it is possibly for the total entropy in the universe to increase from cycle to cycle, and, at the same time, have the entropy density (in particular, the total entropy per Hubble horizon) become nearly zero prior to each bounce.

G. What occurs at the big bang singularity?

The cyclic model utilizes the ekpyrotic notion that the singularity corresponds to the collision and bounce of two outer orbifold branes in a manner that is continuous and well-behaved. The singularity is not a place where energy and curvature diverge and time begins. Rather, formulated in appropriate fields and coordinates, the singularity is a smooth, finite transition from a contracting phase heading towards a big crunch and a big bang evolving into an expanding universe.

H. What determines the arrow of time?

Since the universe is cyclic, it may appear that there is no well-defined means of determining the arrow of time. Indeed, for a local observer, there is no clear means of doing so.

From the global perspective, though, there is a clear means of determining forward from backward in time. First, one of the boundary branes is forever expanding in the ''forward'' time direction in the cyclic model. The other brane is expanding except for brief intervals of contraction, but, averaged over a cycle, the net effect is expansion. The rate changes from phase to phase, as well as the separation. In the contraction phase, the branes themselves stretch at a rate that is slow but their separation rapidly decreases. In the radiation, matter, and quintessence dominated phases, the branes stretch significantly, but their separation remains fixed. During this period, the entropy created during the previous cycle is spread out exponentially, reducing the degrees of freedom per horizon to nearly zero.

I. Why is the cosmological constant so small?

The cyclic model provides a fascinating new outlook on this vexing problem. Historically, the problem is assumed to mean that one must explain why the vacuum energy of the ground state is zero.

In the cyclic model, the vacuum energy of the ground state is not zero. It is negative and its magnitude is large, as is obvious from Fig. 1. If the universe begins in the ground state, the negative cosmological constant will cause rapid recollapse, as expected for an anti–de Sitter phase. In the cyclic scenario, though, we have shown how to arrange conditions where the universe avoids the ground state. Instead, the universe hovers from cycle to cycle above the ground state bouncing from one side of the potential well to the other but spending most time on the positive energy side. The branes are moving too rapidly whenever the separation corresponds to the potential minimum.

There remains the important challenge of explaining why the the current potential energy is so small. The value depends on both the shape of the potential curve and the precise transfer of energy and momentum at the bounce. Perhaps explaining the value will be an issue as knotty as the cosmological constant problem, or perhaps the conditions will prove easier to satisfy. What is certain, though, is that the problem is shifted from tuning a vacuum energy, and this provides an opportunity for new kinds of solutions.

J. Equation-of-state of dark energy

The equation-of-state of the dark energy, w, is the ratio of the pressure to the energy density of ϕ, $(\frac{1}{2}\dot{\phi}^2-V)/(\frac{1}{2}\dot{\phi}^2+V)$. In Sec. V, we discussed the evolution of ϕ in the radiation, matter and quintessence dominated epochs. The generic result is that evolution of ϕ in the positive direction halts and the field begins to roll back towards $-\infty$ in the radiation-dominated epoch. At the turn-around, $w=-1$ since the kinetic energy is zero. As the field rolls back and its kinetic energy increases, w increases. Hence, the generic result is that w is close to -1 today and increasing. Conceivably, cosmological observations could detect this prediction. Tracker models of quintessence, some of the best-motivated alternatives, have the opposite trend: w is near -0.8 or so today and decreasing towards -1 [25]. Other models, such as k-essence, have the same trend as found in the cyclic model [26].

K. Implications for supersymmetry and superstrings

The cyclic model imposes different constraints on fundamental physics compared to previous cosmological models. As an example, consider the problem of designing supergravity potentials. The potentials are constructed from a superpotential W according to the prescription:

$$V=e^{K/M_{pl}^2}\left[K^{ij}D_iW\overline{D}_jW-\frac{3}{M_{pl}^2}W\overline{W}\right], \tag{45}$$

where $D_i=\partial/\partial\phi^i+K_i/M_{pl}^2$ is the Kähler covariant derivative, $K_i=\partial K/\partial\phi^i, K_{ij}=\partial^2K/\partial\phi^i\partial\phi^j$ and a sum over each superfield ϕ_i is implicit. If the ground state is supersymmetric, $D_iW=0$, the first term is zero. In general, unless W is zero for precisely the same values for which $D_iW=0$, the minimum has a negative cosmological constant. In the past, this type of model would have been ruled unacceptable. The possibility of a cosmology in which the universe hovers over the ground state in a state of zero or positive energy revives these models and alters constraints on model building.

An obvious but important implication is that supersymmetry breaking can be achieved without having spontaneous symmetry breaking in the ground state. In this scenario, it suffices if the universe hovers in the radiation, matter and quintessence dominated epochs at some state far above the ground state in energy and the supersymmetry is broken by the appropriate amount in the hovering state, where the radiation, matter and quintessence dominated phases occur. These considerations have a significant impact on the design of phenomenological supersymmetric models.

One other requirement/prediction of the cyclic scenario (and the ekpyrotic models in general) is that the branes move in a space-time with codimension 1. The constraint derives from having a bounce that produces a smooth transition from contraction to expansion. As argued by Khoury *et al.*, the geometry is flat arbitrarily close to the bounce provided there is one extra dimension only. Hence, brane world scenarios based on theories like that of Hořava and Witten are acceptable, but large extra-dimensional models relying on having codimension 2 or greater are problematic.

L. Hoyle's revenge?

Within each cycle, there is a sequence of kinetic energy, radiation, matter and quintessence dominated phases of evolution that are in accord with the standard big bang cosmology. However, averaged over many cycles, the model can be viewed as a remarkable reincarnation of Fred Hoyle's steady state model of the universe. Most of the cycle is spent in a phase with nearly constant energy density, as in the steady state picture. Indeed Hoyle's *C*-field that was introduced to provide a constant supply of matter (and a preferred rest frame) is replaced by our scalar field ϕ, which defines a preferred time slicing and generates matter repeatedly at each bounce, restoring the universe to a state of high temperature and matter density. In Hoyle's steady state model, every flat spatial slice was statistically identical. Here the slices are identical only when separated by one period, so we have a discrete rather than continuous time translation symmetry. Nevertheless when coarse grained over large time spans, the structure is similar to that proposed in the steady state universe. Global properties of the cyclic cosmology will be discussed elsewhere [27].

ACKNOWLEDGMENTS

We thank M. Bucher, S. Gratton, D. Gross, A. Guth, J. Khoury, B. A. Ovrut, J. Ostriker, P.J.E. Peebles, A. Polyakov, M. Rees, N. Seiberg, D. Spergel, A. Tolley, A. Vilenkin, T. Wiseman, and E. Witten for useful conversations. We thank L. Rocher for pointing out Ref. [2] and other historical references. This work was supported in part by U.S. Department of Energy Grant DE-FG02-91ER40671 (P.J.S.) and by PPARC-UK (N.T.).

APPENDIX: MATCHING H_5 ACROSS THE BOUNCE

In this appendix we discuss the matching condition needed to determine $H_5(out)$ in terms of $H_5(in)$. We shall assume that all other extra dimensions and moduli are fixed, and the bulk space-time between the branes settles down to a static state after the collision. (In the simplest brane world models, there is a Birkhoff theorem which ensures that there is a coordinate system in which the bulk metric is static in between the branes.) We shall take the densities of radiation on the branes after collision as being given. By imposing Israel matching in both initial and final states, as well as conservation of total energy and momentum, we shall be able to completely fix the state of the outgoing branes and in particular the expansion rate of the extra dimension $H_5(out)$, in terms of $H_5(in)$. A more complete discussion of this method will be presented in Ref. [28].

The idea is to treat the brane collision as a short-distance phenomenon. The warp factor may be treated as linear between the branes as they approach or recede. Linearity plus Z_2 symmetry ensures that the kinks in the warp factors are equal in magnitude and opposite in sign. Israel matching relates the kink magnitudes to the densities and speeds of the branes, yielding the relations we use below.

The initial state of empty branes with tensions T and $-T$, and with corresponding velocities $v_+<0$ and $v_->0$ (measured in the frame in which the bulk is static) obeys

$$T\sqrt{1-v_+^2}=T\sqrt{1-v_-^2}$$

$$E_{tot}=\frac{T}{\sqrt{1-v_+^2}}-\frac{T}{\sqrt{1-v_-^2}}$$

$$P_{tot}=\frac{Tv_+}{\sqrt{1-v_+^2}}-\frac{Tv_-}{\sqrt{1-v_-^2}}. \tag{A1}$$

The first equation follows from Israel matching on the two branes as the approach, and equating the kinks in the brane scale factors. The second and third equations are the definitions of the total energy and momentum. The three equations (A1) imply that the incoming, empty state has $v_+=-v_-, E_{tot}=0$ and that the total momentum is

$$P_{tot}=\frac{TLH_5(in)}{\sqrt{1-\frac{1}{4}[LH_5(in)]^2}}<0, \tag{A2}$$

where we identify v_+-v_- with the contraction speed of the fifth dimension, $|LH_5(in)|$.

The corresponding equations for the outgoing state are easily obtained, by replacing T with $T+\rho_+\equiv T_+$ for the positive tension brane, and $-T$ with $-T+\rho_-\equiv -T_-$ for the negative tension brane, assuming the densities of radiation produced at the collision on each brane, ρ_+ and ρ_- respectively, are given from a microphysical calculation, and are both positive.

We now wish to apply energy and momentum conservation, and Israel matching to the final state. The only subtlety is that the (t,y) frame in which the bulk is static is not necessarily the same frame in the final state as it was in the initial state, so one should boost the initial two-momentum (E_{tot},P_{tot}) with a velocity V and then apply the Israel constraints and energy-momentum conservation equations in the new boosted frame. The latter provide three equations for the three unknowns in the final state, namely $v_+(out)$, $v_-(out)$ and V. Writing $v_\pm(out)=\tanh(\theta_\pm)$, where $\theta_\pm$ are the associated rapidities, one obtains two solutions

$$\sinh\theta_+ = -\frac{1}{2T_-}[|P_{tot}|+|P_{tot}|^{-1}(T_+^2-T_-^2)]$$

$$\sinh\theta_- = \pm\frac{1}{2T_+}[|P_{tot}|-|P_{tot}|^{-1}(T_+^2-T_-^2)], \quad \text{(A3)}$$

where $T_+\equiv T+\rho_+$, $T_-\equiv T-\rho_-$ with ρ_+ and ρ_- the densities of radiation on the positive and negative tension branes respectively, after collision. Both ρ_+ and ρ_- are assumed to be positive. In the first solution, with signs $(-+)$, the velocities of the positive and negative tension branes are the same after the collision as they were before it. In the second, with signs $(--)$, the positive tension brane continues in the negative y direction but the negative tension brane is *also* moving in the negative y direction.

The corresponding values for $v_\pm(out)$ and V are

$$v_+(out) = -\frac{|P_{tot}|+|P_{tot}|^{-1}(T_+^2-T_-^2)}{\sqrt{P_{tot}^2+2(T_+^2+T_-^2)+P_{tot}^{-2}(T_+^2-T_-^2)^2}},$$

$$v_-(out) = \pm\frac{|P_{tot}|-|P_{tot}|^{-1}(T_+^2-T_-^2)}{\sqrt{P_{tot}^2+2(T_+^2+T_-^2)+P_{tot}^{-2}(T_+^2-T_-^2)^2}},$$

$$V = -\frac{\sqrt{P_{tot}^2+2(T_+^2+T_-^2)+P_{tot}^{-2}(T_+^2-T_-^2)^2}}{|P_{tot}|(T_+^2+T_-^2)/(T_+^2-T_-^2)+|P_{tot}|^{-1}(T_+^2-T_-^2)},$$

$$\text{or} \quad = -\frac{\sqrt{P_{tot}^2+2(T_+^2+T_-^2)+P_{tot}^{-2}(T_+^2-T_-^2)^2}}{|P_{tot}|+|P_{tot}|^{-1}(T_+^2+T_-^2)}, \quad \text{(A4)}$$

where the first solution for V holds for the $(-+)$ case, and the second for the $(--)$ case.

We are interested in the relative speed of the branes in the outgoing state, since that gives the expansion rate of the extra dimension, $-v_+(out)+v_-(out)=LH_5(out)$, compared to their relative speed $-2v_+=-LH_5(in)$ in the incoming state. We find in the $(-+)$ solution,

$$\left|\frac{H_5(out)}{H_5(in)}\right| = \frac{v_+(out)-v_-(out)}{2v_+} = \sqrt{\frac{P_{tot}^2+4T^2}{P_{tot}^2+2(T_+^2+T_-^2)+P_{tot}^{-2}(T_+^2-T_-^2)^2}}, \quad \text{(A5)}$$

and in the $(--)$ solution

$$\left|\frac{H_5(out)}{H_5(in)}\right| = \frac{(T_+^2-T_-^2)}{P_{tot}^2}\sqrt{\frac{P_{tot}^2+4T^2}{P_{tot}^2+2(T_+^2+T_-^2)+P_{tot}^{-2}(T_+^2-T_-^2)^2}} \quad \text{(A6)}$$

with P_{tot} given by Eq. (A2) in both cases.

At this point we need to consider how the densities of radiation ρ_+ and ρ_- depend on the relative speed of approach of the branes. At very low speeds, $|LH_5(in)|\ll 1$, one expects the outer brane collision to be nearly adiabatic and an exponentially small amount of radiation to be produced. The $(-+)$ solution has the speeds of both branes nearly equal before and after collision: we assume that it is this solution, rather than the $(--)$ solution which is realized in this low velocity limit.

As $|LH_5(in)|$ is increased, we expect ρ_+ and ρ_- to grow. Now, if we consider ρ_+ and ρ_- to be both $\ll P_{tot}\ll T$, then the second term in the denominator dominates. If more radiation is produced on the negative tension brane, $\rho_->\rho_+$, then $|H_5(out)/H_5(in)|\equiv(1+\chi)\approx[1+(\rho_- -\rho_+)/2T]$ and so χ is small and positive. This is the condition noted in the text, necessary to obtain cyclic behavior. Conceivably, the brane tension can change from T to $T' = T-t$ at collision. Then, we obtain $(1+\chi)\approx[1+(\rho_- -\rho_+ +2t)/2T]$.

For the $(-+)$ solution, we can straightforwardly determine an upper limit for $|H_5(out)/H_5(in)|\equiv(1+\chi)$. Consider, for example, the case in which the brane tension is unchanged at collision, $t=0$. The expression in Eq. (A5) gives $|H_5(out)/H_5(in)|$ as a function of T_+, T_- and P_{tot}. It is greatest, at fixed T_- and P_{tot}, when $T_+=T$, its smallest value. For $P_{tot}^2<T^2$, it is maximized for $T_-^2=T^2-P_{tot}^2$, and equal to $\sqrt{1+P_{tot}^2/(4T^2)}$ when equality holds. For P_{tot}

$\geqslant T^2$, it is maximized when $T_- = 0$, its smallest value, and $P_{tot}^2 = 2T^2$, when it is equal to $\sqrt{\frac{4}{3}}$. This is more than enough for us to obtain the small values of χ needed to make the cyclic scenario work. A reduction in brane tension at collisions $t > 0$ further increases the maximal value of the ratio. To obtain cyclic behavior, we need χ to be constant from bounce to bounce. That is, compared to the tension before collision, the fractional change in tension and the fractional production of radiation must be constant.

We shall not consider the $(--)$ solution in detail, except to note that in the small P_{tot} limit it allows an arbitrarily large value for $|H_5(out)/H_5(in)|$, which seems unphysical.

Let us reiterate that there are many caveats attached to this calculation. We have not calculated ρ_+ and ρ_- and have left these as parameters. We have set the brane tensions T to be equal before and after the collision, and have neglected possible bulk excitations, treating the bulk space-time as static both before and after collision. We have ignored matter couplings to bulk scalars, and ignored the possible dynamical evolution of additional extra dimensions. Nevertheless we think it encouraging that the unusual behavior of matter bound to a negative tension brane allows $H_5(out)/H_5(in)$ to be slightly greater than unity, which is what we need for cyclic behavior.

[1] P.J. Steinhardt and N. Turok, hep-th/0111030.

[2] See, for example, J. Hastings, *Encyclopaedia of Religion and Ethics* (C. Scribner & Sons, New York, 1927).

[3] R. C. Tolman, *Relativity, Thermodynamics and Cosmology* (Oxford U. Press, Oxford, UK, 1934).

[4] See, for example, R. H. Dicke and J. P. E. Peebles, in *General Relativity: An Einstein Centenary Survey*, edited by S. H. Hawking and W. Israel (Cambridge University Press, Cambridge, England, 1979).

[5] G. Veneziano, Phys. Lett. B **265**, 287 (1991); M. Gasperini and G. Veneziano, Astropart. Phys. **1**, 317 (1993).

[6] For a review, see N. Bahcall, J.P. Ostriker, S. Perlmutter, and P.J. Steinhardt, Science **284**, 1481 (1999).

[7] J. Khoury, B.A. Ovrut, P.J. Steinhardt, and N. Turok, Phys. Rev. D **64**, 123522 (2001).

[8] J. Khoury, B.A. Ovrut, N. Seiberg, P.J. Steinhardt, and N. Turok, Phys. Rev. D **65**, 086007 (2002).

[9] H. Liu, G. Moore, and N. Seiberg, hep-th/0204168; A.J. Tolley and N. Turok, hep-th/0204091.

[10] J. Garriga, E. Pujolas, and T. Tanaka, hep-th/0111277.

[11] N. Seiberg, P. J. Steinhardt, and N. Turok (in preparation).

[12] J. Khoury, B.A. Ovrut, P.J. Steinhardt, and N. Turok, hep-th/0109050.

[13] J. Feng, J. March-Russell, S. Sethi, and F. Wilczek, Nucl. Phys. **B602**, 307 (2001).

[14] R. Bousso and J. Polchinski, J. High Energy Phys. **06**, 006 (2000).

[15] R.R. Caldwell, R. Dave, and P.J. Steinhardt, Phys. Rev. Lett. **80**, 1582 (1998); see also J. Ostriker and P.J. Steinhardt, Sci. Am. January issue (2001).

[16] J. Khoury, B. Ovrut, P.J. Steinhardt, and N. Turok (in preparation).

[17] V. Fock, *Theory of Space, Time and Gravitation* (Pergamon, London, 1959).

[18] N. Turok and P. J. Steinhardt, Science (to be published), hep-th/0111030.

[19] T. Damour and A. Polyakov, Nucl. Phys. **B423**, 532 (1994).

[20] T. Damour, gr-qc/0109063.

[21] R. Brustein and G. Veneziano, Phys. Lett. B **329**, 429 (1994); R. Brustein and R. Madden, *ibid.* **410**, 110 (1997).

[22] T. Damour and M. Henneaux, Phys. Rev. Lett. **85**, 920 (2000).

[23] J.H. Horne and G. Moore, Nucl. Phys. **B432**, 109 (1994).

[24] A. Borde, A. Guth, and A. Vilenkin, gr-qc/0110012.

[25] I. Zlatev and P.J. Steinhardt, Phys. Lett. B **459**, 570 (1999); P.J Steinhardt, L. Wang, and I. Zlatev, Phys. Rev. D **59**, 123504 (1999).

[26] C. Armendariz-Picon, V. Mukhanov, and P.J. Steinhardt, Phys. Rev. Lett. **85**, 4438 (2000).

[27] A. Aguirre, S. Gratton, P. J. Steinhardt, and N. Turok, astro-ph/0111012.

[28] M. Bucher and N. Turok (in preparation).