Duality, monodromy, and integrability of two dimensional string effective action

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The monodromy matrix $\hat{\mathcal{M}}$ is constructed for the two dimensional tree level string effective action. The pole structure of $\hat{\mathcal{M}}$ is derived using its factorizability property. It is found that the monodromy matrix transforms nontrivially under the noncompact T-duality group, which leaves the effective action invariant, and this can be used to construct the monodromy matrix for more complicated backgrounds starting from simpler ones. We construct, explicitly, $\hat{\mathcal{M}}$ for the exactly solvable Nappi-Witten model, both when $B=0$ and $B\neq 0$, where these ideas can be directly checked. We consider well known charged black hole solutions in the heterotic string theory that can be generated by T-duality transformations from a spherically symmetric ''seed'' Schwarzschild solution. We construct the monodromy matrix for the Schwarzschild black hole background of the heterotic string theory.

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I. INTRODUCTION

Field theories in two space-time dimensions have attracted considerable attention over the past few decades. They possess a variety of interesting features. Some of these field theories capture several salient characteristics of four dimensional theories and, therefore, such two dimensional models are used as theoretical laboratories. Moreover, the nonperturbative properties of field theories are much simpler to study in two dimensional models. There are classes of two dimensional theories which are endowed with a rich symmetry structure: integrable models [1] and conformal field theories [2,3] belong to this special category, among others. Under special circumstances, in the presence of isometries, a four dimensional theory may also be described by an effective two dimensional theory.

The string theories are abundantly rich in their symmetry content. The tree level string effective action, dimensionally reduced to lower dimensions, is known to possess enlarged symmetries [4–6]. Let us consider toroidal compactification of a heterotic string on a d-dimensional torus, from 10 dimensional space-time to $10-d$ dimensions. The reduced theory is known to be invariant under the noncompact T-duality group $O(d,d+16)$. For the case $d=6$, namely, in the case of reduction to four space-time dimensions, the field strength of the two-form antisymmetric tensor can be traded for the pseudoscalar axion. Furthermore, the dilaton and axion can be combined to parametrize the coset $SL(2,R)/U(1)$. Thus, the four dimensional theory possesses the T-duality as well as the S-duality group of symmetries. When the string effective action is reduced to two space-time dimensions we encounter an enhancement in symmetry, as has been studied by several authors [7–11]. The string effective action describes supergravity theories and the integrability properties of such theories have been investigated in the recent past [12,13]. It is worthwhile to mention that higher dimensional Einstein theory, dimensionally reduced to effective two dimensional theories, has been studied in the past [14]. One of the approaches is to derive the monodromy matrix which encodes some of the essential features of integrable field theories. An effective two dimensional action naturally appears when one considers some aspects of black hole physics, colliding plane waves as well as special types of cosmological models. Recently, we have shown that the two dimensional string effective action has a connection with integrable systems from a new perspective in the sense that one can construct the monodromy matrix for such theories with well defined prescriptions [15]. It was shown, while investigating the collision of plane fronted stringy waves, that the monodromy matrix can be constructed explicitly for a given set of background configurations [16]. Subsequently, we were able to give the procedures for deriving the monodromy matrix under general settings. It is worth mentioning that some of the interesting aspects of black hole physics can be described by an effective two dimensional theory [17]. Moreover, there is an intimate relation between colliding plane waves and the description of four dimensional space-time with two commuting Killing vectors.

Chandrasekhar and Xanthopoulos [18], in their seminal work, have shown that a violently time dependent space-time with a pair of Killing vectors provides a description of plane colliding gravitational waves. Furthermore, Ferrari, Ibanez, and Bruni [19] have demonstrated that the colliding plane wave metric can be identified, locally, as isometric to the interior of a Schwarzschild metric. In another important step,

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Yurtsever [20] constructed the transformation that provides a connection between a metric describing colliding waves and one corresponding to the Schwarzschild black hole. As is well known from the study of colliding plane gravitational waves [21] corresponding to massless states of strings, there is a curvature singularity in the future. Recently, the appearance of the future singularity has found an alternative description in the context of the pre-big-bang (PBB) scenario [22,23]. The incoming plane waves are to be identified as the initial stringy vacuua of the Universe, which collide and lead to the creation of the Universe. Indeed, the solutions correspond to the Kasner type metric and the exponents fulfill the requirements of the PBB conditions. A very important fact, in this context, is that one starts from a four dimensional effective action; however, the physical process is effectively described by a two dimensional theory.

When we focus our attention on addressing these problems in the framework of string theory, it is essential to keep in mind the special symmetries, such as dualities, which are an integral part of the stringy symmetries. We have investigated [15] the behavior of the monodromy matrix under T-duality transformation of the backgrounds under a general setting when the two dimensional action is derived from a D-dimensional string effective action through compactification on a d-dimensional torus T^d. It was shown that the monodromy matrix transforms nontrivially under the duality group $O(d,d)$. Therefore, it opens up the possibility of studying integrable systems which might appear in the context of string theories. As an example, we considered the Nappi-Witten [24] model, which is a solution to an exact conformal field theory described by a Wess-Zumino-Witten (WZW) model. We first obtained the monodromy matrix for the case when the two-form antisymmetric tensor is set to zero. It is also well known that an antisymmetric tensor background can be generated through an $O(2,2)$ transformation from the initial backgrounds [25]. We can construct the monodromy matrix for the new set of backgrounds using our prescriptions. On the other hand, the new monodromy matrix can be constructed directly by utilizing the transformation rules discovered by us. Indeed, we explicitly demonstrated that the monodromy matrices obtained via the two different routes are identical. It is obvious from the preceding discussion that the symmetries of the effective action play an important role in the construction of the monodromy matrix and its transformation properties under those symmetry transformations. Therefore, it is natural to expect intimate connections between the integrability properties of the two dimensional theory and the full stringy symmetry groups of T and S dualities.

The purpose of this article is to present details of our investigations in the directions alluded to above. We provide prescriptions for the construction of the monodromy matrix $\hat{\mathcal{M}}$ for the string theoretic two dimensional effective action. We present the pole structure of the monodromy matrix from general arguments. The duality transformation properties of $\hat{\mathcal{M}}$ follow from the definition and construction of this matrix. For example, if the action respects T duality, then one can derive how $\hat{\mathcal{M}}$ transforms under the group, whereas if the underlying symmetry corresponds to S duality an appropriate transformation rule for the monodromy matrix can also be derived. Our paper is organized as follows. In Sec. II, we recapitulate the form of the two dimensional action obtained by toroidal compactification from higher dimensions. Then we present the equations of motion. A key ingredient in the derivation of the monodromy matrix is the coset space reformulation of the reduced action. We devote Sec. III to the construction of the monodromy matrix for the problem under consideration. The transformation rules for $\hat{\mathcal{M}}$ under a T-duality transformation are derived once the matrix is constructed. An interesting observation is that the expression for $\hat{\mathcal{M}}$ already captures the stringy symmetry in an elegant manner. In this section, we also present explicit forms of $\hat{\mathcal{M}}$ for simple background configurations which still preserve some of the general features. We present some illustrative examples in Sec. IV. The structure of the Nappi-Witten model in the present context is analyzed in detail. Furthermore, we choose an example from black hole physics to construct the monodromy matrix. In this case, the black hole solution can be thought of as a solution to type IIB string effective action and the theory is endowed with S-duality symmetry. Thus we are able to provide an example of how the monodromy matrix transforms under the S-duality group $SL(2,R)$. We present a brief conclusion in Sec. V and some of the useful relations are collected in the Appendix.

II. TWO DIMENSIONAL EFFECTIVE ACTION

In this section, we will briefly recapitulate the form of the string effective action in two dimensions, which will form the basis for all our subsequent discussions. Let us consider, for simplicity, the tree level string effective action in D dimensions consisting of the graviton, the dilaton, and the antisymmetric tensor field,

$$\hat{S}=\int d^D x\sqrt{-\hat{G}}e^{-\hat{\phi}}\left[R_{\hat{G}}+(\hat{\partial}\hat{\phi})^2-\frac{1}{12}\hat{H}_{\hat{\mu}\hat{\nu}\hat{\rho}}\hat{H}^{\hat{\mu}\hat{\nu}\hat{\rho}}\right]. \quad (1)$$

Here, $\hat{G}_{\hat{\mu}\hat{\nu}}$ is the D-dimensional metric in the string frame with signature $(-,+,\cdots,+)$, and $\hat{G}=\det\hat{G}_{\hat{\mu}\hat{\nu}}$. $\hat{R}_{\hat{G}}$ is the scalar curvature, $\hat{\phi}$ is the dilaton, and $\hat{H}_{\hat{\mu}\hat{\nu}\hat{\rho}}=\partial_{\hat{\mu}}\hat{B}_{\hat{\nu}\hat{\rho}}+\partial_{\hat{\rho}}\hat{B}_{\hat{\mu}\hat{\nu}}+\partial_{\hat{\nu}}\hat{B}_{\hat{\rho}\hat{\mu}}$ is the field strength for the second-rank antisymmetric tensor field $\hat{B}_{\hat{\mu}\hat{\nu}}$.

If we compactify this action on a d-dimensional torus T^d, where $d=D-2$, then the resulting dimensionally reduced action will describe the two dimensional string effective action, which has the form [26,27]

$$S=\int dx^0dx^1\sqrt{-g}e^{-\bar{\phi}}\left[R+(\partial\bar{\phi})^2+\frac{1}{8}\mathrm{Tr}(\partial_\alpha M^{-1}\partial^\alpha M)\right]. \quad (2)$$

Here $\alpha,\beta=0,1$ are the two dimensional space-time indices, and $g_{\alpha\beta}$ is the two dimensional space-time metric with $g=\det g_{\alpha\beta}$. R is the corresponding two dimensional scalar curvature, while the shifted dilaton is defined as

$$\overline{\phi}=\phi-\frac{1}{2}\log\det G_{ij} \tag{3}$$

where G_{ij} is the metric in the internal space corresponding to the toroidally compactified coordinates x^i, $i,j=2,3,\dots,D-1$ $(d=D-2)$. Finally, M is a $2d\times 2d$ symmetric matrix of the form

$$M=\begin{pmatrix} G^{-1} & -G^{-1}B \\ BG^{-1} & G-BG^{-1}B \end{pmatrix} \tag{4}$$

where B represents the moduli coming from the dimensional reduction of the $\hat{B}$ field in D dimensions. In general, there will be additional terms in Eq. (2) associated with d Abelian gauge fields arising from the metric $\hat{G}_{\hat{\mu}\hat{\nu}}$ and another set of d Abelian gauge fields coming from the antisymmetric tensor $\hat{B}_{\hat{\mu}\hat{\nu}}$, as a result of dimensional reduction [26]. Furthermore, there would also have been terms involving the field strength of the two dimensional tensor field, $B_{\alpha\beta}$. Since we are in two space-time dimensions, we have dropped the gauge field terms and, in the same spirit, have not kept the field strength of $B_{\alpha\beta}$, which can always be removed, if it depends only on the coordinates x^0 and x^1. Later, we will comment on the gauge fields, which assume a significant role when Abelian gauge fields are present in the original string effective action (1).

The matrix M corresponds to a symmetric representation of the group $O(d,d)$ and the dimensionally reduced action in Eq. (2) is invariant under the global $O(d,d)$ transformations

$$g_{\alpha\beta}\to g_{\alpha\beta}, \quad \overline{\phi}\to\overline{\phi}, \tag{5}$$

$$M\to\Omega^T M\Omega, \tag{6}$$

where $\Omega\in O(d,d)$ is the global transformation matrix, which preserves the $O(d,d)$ metric

$$\eta=\begin{pmatrix} 0 & \mathbf{1}_d \\ \mathbf{1}_d & 0 \end{pmatrix} \tag{7}$$

with $\mathbf{1}_d$ representing the identity matrix in d dimensions; namely, Ω satisfies $\Omega^T\eta\Omega=\eta$.

The equations of motion for the different fields follow from the dimensionally reduced effective action (2). For example, varying the effective action (2) with respect to the shifted dilaton $\overline{\phi}$ and the metric $g_{\alpha\beta}$ leads, respectively, to

$$R+2g^{\alpha\beta}D_\alpha D_\beta\overline{\phi}-g^{\alpha\beta}\partial_\alpha\overline{\phi}\partial_\beta\overline{\phi} + \frac{1}{8}g^{\alpha\beta}\,\mathrm{Tr}(\partial_\alpha M^{-1}\partial_\beta M)=0, \tag{8}$$

$$R_{\alpha\beta}+D_\alpha D_\beta\overline{\phi}+\frac{1}{8}\mathrm{Tr}(\partial_\alpha M^{-1}\partial_\beta M)=0. \tag{9}$$

It follows from these equations that

$$D_\alpha D^\alpha e^{-\overline{\phi}}=0. \tag{10}$$

The variation of the effective action with respect to M needs some care since M is a symmetric $O(d,d)$ matrix satisfying $M\eta M=\eta$. A simple method, for example, would involve adding the constraint to the effective action through a Lagrange multiplier. In any case, since there is no potential term in the effective action involving the matrix M, the Euler-Lagrange equation of motion following from the variation of the action with respect to M has the form of a conservation law:

$$\partial_\alpha(e^{-\overline{\phi}}\sqrt{-g}g^{\alpha\beta}M^{-1}\partial_\beta M)=0. \tag{11}$$

We can further simplify these equations by working in the light-cone coordinates

$$x^+=\frac{1}{\sqrt{2}}(x^0+x^1), \quad x^-=\frac{1}{\sqrt{2}}(x^0-x^1) \tag{12}$$

and choosing the conformal gauge for the two dimensional metric, namely,

$$g_{\alpha\beta}=e^{F(x^+,x^-)}\eta_{\alpha\beta}. \tag{13}$$

In this case, Eqs. (10) and (11), respectively, take the forms

$$\partial_+\partial_- e^{-\overline{\phi}}=0, \tag{14}$$

$$\partial_+(e^{-\overline{\phi}}M^{-1}\partial_- M)+\partial_-(e^{-\overline{\phi}}M^{-1}\partial_+ M)=0, \tag{15}$$

while Eqs. (8) and (9) can be written explicitly as

$$\begin{aligned} &\partial_+^2\overline{\phi}-\partial_+F\partial_+\overline{\phi}+\frac{1}{8}\mathrm{Tr}(\partial_+M^{-1}\partial_+M)=0,\\ &\partial_+\partial_-\overline{\phi}-\partial_+\partial_-F+\frac{1}{8}\mathrm{Tr}(\partial_+M^{-1}\partial_-M)=0,\\ &\partial_-^2\overline{\phi}-\partial_-F\partial_-\overline{\phi}+\frac{1}{8}\mathrm{Tr}(\partial_-M^{-1}\partial_-M)=0. \end{aligned} \tag{16}$$

It is well known [26] that the moduli appearing in the definition of the M matrix [see Eq. (4)] parametrize the coset $O(d,d)/[O(d)\times O(d)]$. Correspondingly, it is convenient to introduce a triangular matrix $V\in O(d,d)/[O(d)\times O(d)]$ of the form

$$V=\begin{pmatrix} E^{-1} & 0 \\ BE^{-1} & E^T \end{pmatrix} \tag{17}$$

such that $M=VV^T$. Here, E is the vielbein in the internal space so that $(E^TE)_{ij}=G_{ij}$. Under a combined global $O(d,d)$ and a local $O(d)\times O(d)$ transformation

$$V\to\Omega^T Vh(x) \tag{18}$$

where $\Omega\in O(d,d)$ and $h(x)\in O(d)\times O(d)$,

$$M=VV^T\to\Omega^T M\Omega. \tag{19}$$

That is, the M matrix is sensitive only to a global $O(d,d)$ rotation.

From the matrix V, we can construct the current $V^{-1}\partial_\alpha V$, which belongs to the Lie algebra of $O(d,d)$ and can be decomposed as

$$V^{-1}\partial_\alpha V = P_\alpha + Q_\alpha . \tag{20}$$

Here, Q_α belongs to the Lie algebra of the maximally compact subgroup $O(d)\times O(d)$ and P_α belongs to the complement. Furthermore, it follows from the symmetric space automorphism property of the coset $O(d,d)/[O(d)\times O(d)]$ that $P_\alpha^T = P_\alpha, Q_\alpha^T = -Q_\alpha$ so that we can identify

$$P_\alpha = \frac{1}{2}[V^{-1}\partial_\alpha V + (V^{-1}\partial_\alpha V)^T],$$

$$Q_\alpha = \frac{1}{2}[V^{-1}\partial_\alpha V - (V^{-1}\partial_\alpha V)^T]. \tag{21}$$

It is now straightforward to check that

$$\mathrm{Tr}(\partial_\alpha M^{-1}\partial_\beta M) = -4\ \mathrm{Tr}(P_\alpha P_\beta). \tag{22}$$

Furthermore, under a global $O(d,d)$ rotation, the currents in Eq. (21) are invariant, while under a local $O(d)\times O(d)$ transformation, $V \rightarrow Vh(x)$,

$$P_\alpha \rightarrow h^{-1}(x) P_\alpha h(x),$$

$$Q_\alpha \rightarrow h^{-1}(x) Q_\alpha h(x) + h^{-1}(x)\partial_\alpha h(x). \tag{23}$$

That is, under a local $O(d)\times O(d)$ transformation, Q_α transforms like a gauge field, while P_α transforms as belonging to the adjoint representation. It is clear, therefore, that Eq. (22) is invariant under the global $O(d,d)$ as well as the local $O(d)\times O(d)$ transformations. Consequently, the action in Eq. (2) is also invariant under local $O(d)\times O(d)$ transformations.

This brings out, naturally, the connection between the system under study and two dimensional integrable systems. First, let us note that, in the absence of gravity and the dilaton (namely, if $g_{\alpha\beta} = \eta_{\alpha\beta}, \bar{\phi} = 0$), the action in Eq. (2) simply corresponds to a flat space sigma model defined over the coset $O(d,d)/[O(d)\times O(d)]$, which can be analyzed through a zero curvature condition with a constant spectral parameter (to be discussed in more detail in the next section). In the presence of gravity as well as the dilaton, we can eliminate the dilaton from the action (2) by choosing the particular conformal gauge $g_{\alpha\beta} = e^{\bar{\phi}}\eta_{\alpha\beta}$. In this case, the action will describe a sigma model, defined over the coset $O(d,d)/[O(d)\times O(d)]$, coupled to gravity. As we will show in the next section, this system can also be analyzed through a zero curvature condition much like the flat space case, although consistency requires the spectral parameter, in this case, to be space-time dependent.

So far, we have discussed only the two dimensional string effective action starting from the D-dimensional action in Eq. (1) involving the graviton, the dilaton, and the antisymmetric tensor field. However, the action in Eq. (1) can be generalized by adding n Abelian gauge fields, with the additional action of the form (such terms naturally arise in heterotic string theory)

$$\hat{S}_{\hat{A}} = -\frac{1}{4}\int d^D x \sqrt{-\hat{G}}\, e^{-\hat{\phi}} (\hat{g}^{\hat{\mu}\hat{\rho}}\hat{g}^{\hat{\nu}\hat{\lambda}}\delta_{IJ}\hat{F}^I_{\hat{\mu}\hat{\nu}}\hat{F}^J_{\hat{\rho}\hat{\lambda}}) \tag{24}$$

where $I,J = 1,2,\ldots,n$ and

$$\hat{F}^I_{\hat{\mu}\hat{\nu}} = \partial_{\hat{\mu}}\hat{A}^I_{\hat{\nu}} - \partial_{\hat{\nu}}\hat{A}^I_{\hat{\mu}} . \tag{25}$$

This action can also be dimensionally reduced [27] to two dimensions and the resulting effective action takes the form

$$S_A = -\frac{1}{4}\int dx^0 dx^1 \sqrt{-g}\, e^{-\bar{\phi}} (F^I_{\alpha\beta}F^{I\alpha\beta} + 2F^I_{\alpha j}F^{I\alpha j}) \tag{26}$$

where we have defined

$$\begin{aligned}
a^I_i &= \hat{A}^I_i ,\\
A^{(1)I}_\alpha &= \hat{G}^I_\alpha ,\\
A^{(3)I}_\alpha &= \hat{A}^I_\alpha - a^I_j A^{(1)j}_\alpha ,\\
F^{(1)i}_{\alpha\beta} &= \partial_\alpha A^{(1)i}_\beta - \partial_\beta A^{(1)i}_\alpha ,\\
F^{(3)I}_{\alpha\beta} &= \partial_\alpha A^{(3)I}_\beta - \partial_\beta A^{(3)I}_\alpha ,\\
F^I_{\alpha\beta} &= F^{(3)I}_{\alpha\beta} + F^{(1)i}_{\alpha\beta} a^I_i ,\\
F^I_{\alpha i} &= \partial_\alpha a^I_i .
\end{aligned} \tag{27}$$

In the presence of the Abelian gauge fields, the field strength H associated with the second-rank antisymmetric tensor field B needs to be redefined for gauge invariance as

$$\begin{aligned}
H_{\alpha ij} &= \partial_\alpha B_{ij} + \frac{1}{2}(a^I_i \partial_\alpha a^I_j - a^I_j \partial_\alpha a^I_i),\\
H_{\alpha\beta i} &= -C_{ij}F^{(1)j}_{\alpha\beta} + F^{(2)}_{\alpha\beta i} - a^I_i F^{(3)I}_{\alpha\beta} ,\\
H_{\alpha\beta\gamma} &= \partial_\alpha B_{\beta\gamma} - \frac{1}{2}\mathcal{A}^r_\alpha \eta_{rs}\mathcal{F}^s_{\beta\gamma} + \text{cyclic permutations},
\end{aligned} \tag{28}$$

where $\mathcal{A}^r_\alpha = (A^{(1)i}_\alpha, A^{(2)}_{\alpha i}, A^{(3)I}_\alpha)$, $\mathcal{F}^r_{\alpha\beta} = \partial_\alpha \mathcal{A}^r_\beta - \partial_\beta \mathcal{A}^r_\alpha$, and

$$\begin{aligned}
A^{(2)}_{\alpha i} &= \hat{B}_{\alpha i} + B_{ij}A^{(1)j}_\alpha + \frac{1}{2}a^I_i A^{(3)I}_\alpha ,\\
F^{(2)}_{\alpha\beta i} &= \partial_\alpha A^{(2)}_{\beta i} - \partial_\beta A^{(2)}_{\alpha i} ,\\
C_{ij} &= \frac{1}{2}a^I_i a^I_j + B_{ij} .
\end{aligned} \tag{29}$$

Once again, it is easy to see that, in two space-time dimensions, the field strength $H_{\alpha\beta\gamma}$ can be set to zero. Furthermore, keeping all other terms, the complete two dimensional string effective action can be shown to have the same form as in Eq. (2) with

$$M=\begin{pmatrix} G^{-1} & -G^{-1}C & -G^{-1}a^T \\ -CG^{-1} & G+C^TG^{-1}C & C^TG^{-1}a^T+a^T \\ -aG^{-1} & aG^{-1}C+a & 1+aG^{-1}a^T \end{pmatrix}. \tag{30}$$

In this case, M is a symmetric $d\times(d+n)$ matrix $(d=D-2)$ belonging to $O(d,d+n)$. Under an $O(d,d+n)$ transformation

$$M\rightarrow\Omega^T M\Omega \tag{31}$$

where the parameter of transformation $\Omega\in O(d,d+n)$ satisfying $\Omega^T\eta\Omega=\eta$, where

$$\eta=\begin{pmatrix} 0 & \mathbf{1}_d & 0 \\ \mathbf{1}_d & 0 & 0 \\ 0 & 0 & \mathbf{1}_n \end{pmatrix} \tag{32}$$

represents the metric for $O(d,d+n)$. As in the earlier case, it is more convenient to introduce a matrix $V\in O(d,d+n)/[O(d)\times O(d+n)]$ of the form

$$V=\begin{pmatrix} E^{-1T} & 0 & 0 \\ -C^TE^{-1T} & E^T & a^T \\ -aE^{-1T} & 0 & 1 \end{pmatrix} \tag{33}$$

such that $M=VV^T$. As before, under a combined global $O(d,d+n)$ and a local $O(d)\times O(d+n)$ transformation

$$V\rightarrow\Omega^T Vh(x) \tag{34}$$

where $\Omega\in O(d,d+n)$ and $h(x)\in O(d)\times O(d+n)$. However, the matrix M is not sensitive to the local $O(d)\times O(d+n)$ transformations. We can now define the current $V^{-1}\partial_\alpha V$ which belongs to the Lie algebra of $O(d,d+n)$ and which can be decomposed as

$$V^{-1}\partial_\alpha V=P_\alpha+Q_\alpha. \tag{35}$$

In the present case, Q_α belongs to the Lie algebra of the maximal compact subgroup $O(d)\times O(d+n)$, while P_α belongs to the complement. Under a global $O(d,d+n)$ transformation, P_α and Q_α are invariant, while under a local $O(d)\times O(d+n)$ transformation $V\rightarrow Vh(x)$

$$\begin{aligned} P_\alpha&\rightarrow h^{-1}(x)P_\alpha h(x),\\ Q_\alpha&\rightarrow h^{-1}(x)Q_\alpha h(x)+h^{-1}(x)\partial_\alpha h(x), \end{aligned} \tag{36}$$

and all the discussion for the earlier case can again be carried through.

III. MONODROMY MATRIX

In the last section, we saw that the two dimensional string effective action, dimensionally reduced from D dimensions, has the natural description of a sigma model defined on a coset, coupled to gravity. In the case when there are no Abelian gauge fields present in the starting string action, the sigma model is defined on the coset $O(d,d)/[O(d)\times O(d)]$ where $d=D-2$. On the other hand, if n Abelian gauge fields are present in the starting string action, the coset can be identified with $O(d,d+n)/[O(d)\times O(d+n)]$. In this section, we will further analyze the integrability properties of such a system and construct the monodromy matrix associated with the system.

Let us consider a general sigma model in flat space-time, defined on the coset G/H. The two cases of interest for us are when $G=O(d,d),H=O(d)\times O(d)$ and $G=O(d,d+n),H=O(d)\times O(d+n)$. Let $V\in G/H$ and $M=VV^T$. Then, as we have noted in the last section, we can decompose the current $V^{-1}\partial_\alpha V$ belonging to the Lie algebra of G as

$$V^{-1}\partial_\alpha V=P_\alpha+Q_\alpha \tag{37}$$

where Q_α belongs to the Lie algebra of H, while P_α belongs to the complement. The integrability condition following from this corresponds to the zero curvature condition

$$\partial_\alpha(V^{-1}\partial_\beta V)-\partial_\beta(V^{-1}\partial_\alpha V)+[(V^{-1}\partial_\alpha V),(V^{-1}\partial_\beta V)]=0. \tag{38}$$

Explicitly, this equation gives

$$\begin{aligned} \partial_\alpha Q_\beta-\partial_\beta Q_\alpha+[Q_\alpha,Q_\beta]+[P_\alpha,P_\beta]&=0,\\ D_\alpha P_\beta-D_\beta P_\alpha&=0, \end{aligned} \tag{39}$$

where we have defined

$$D_\alpha P_\beta=\partial_\alpha P_\beta+[Q_\alpha,P_\beta]. \tag{40}$$

The equations of motion for the flat space sigma model [see Eq. (11)]

$$\eta^{\alpha\beta}\partial_\alpha(M^{-1}\partial_\beta M)=0 \tag{41}$$

can be rewritten in the form

$$\eta^{\alpha\beta}D_\alpha P_\beta=0. \tag{42}$$

Let us next introduce a one parameter family of matrices $\hat{V}(x,t)$ where t is a constant parameter (and not time), also known as the spectral parameter, such that $\hat{V}(x,t=0)=V(x)$ and

$$\hat{V}^{-1}\partial_\alpha\hat{V}=Q_\alpha+\frac{1+t^2}{1-t^2}P_\alpha+\frac{2t}{1-t^2}\epsilon_{\alpha\beta}P^\beta. \tag{43}$$

Then it is straightforward to check that the integrability condition

$$\partial_\alpha(\hat{V}^{-1}\partial_\beta\hat{V})-\partial_\beta(\hat{V}^{-1}\partial_\alpha\hat{V})+[(\hat{V}^{-1}\partial_\alpha\hat{V}),(\hat{V}^{-1}\partial_\beta\hat{V})]=0 \tag{44}$$

leads naturally to Eqs. (39),(42). That is, the integrability condition (39) as well as the equation of motion for the flat space sigma model are obtained from the zero curvature condition associated with a potential which depends on a constant spectral parameter.

In the presence of gravity, however, the equation for the sigma model is modified [12,13]. In the conformal gauge $g_{\alpha\beta}=e^{\bar{\phi}}\eta_{\alpha\beta}$, Eq. (11) takes the form

$$\eta^{\alpha\beta}\partial_\alpha(e^{-\bar{\phi}}M^{-1}\partial_\beta M)=0,$$
$$\text{or}\quad \eta^{\alpha\beta}D_\alpha(e^{-\bar{\phi}}P_\beta)=D_\alpha(e^{-\bar{\phi}}P^\alpha)=0. \tag{45}$$

As before, we can introduce a one parameter family of potentials depending on a spectral parameter and with a decomposition of the form (43). However, in this case, it is easy to check that the zero curvature condition in Eq. (44) leads to the correct dynamical equation as well as the integrability condition, provided the spectral parameter is space-time dependent and satisfies

$$\partial_\alpha t=-\frac{1}{2}\epsilon_{\alpha\beta}\partial^\beta\left[e^{-\bar{\phi}}\left(t+\frac{1}{t}\right)\right]. \tag{46}$$

In the conformal gauge, as we saw earlier in Eq. (14), the shifted dilaton satisfies a simple equation. Therefore, defining

$$\rho(x)=e^{-\bar{\phi}}, \tag{47}$$

we note that the solution following from the equation for the shifted dilaton can be written as

$$\rho(x)=\rho_+(x^+)+\rho_-(x^-). \tag{48}$$

With this, the solution to Eq. (46) can be written as

$$t(x)=\frac{\sqrt{\omega+\rho_+}-\sqrt{\omega-\rho_-}}{\sqrt{\omega+\rho_+}+\sqrt{\omega-\rho_-}} \tag{49}$$

where ω is the constant of integration, which can be thought of as a global spectral parameter. It is clear that the solutions in Eq. (49) are double valued in nature.

There are several things to note from our discussion so far. First of all, the one parameter family of connections (currents) does not determine the potential $\hat{V}(x,t)$ uniquely, namely, $\hat{V}$ and $S(\omega)\hat{V}$, where $S(\omega)$ is a constant matrix, yield the same one parameter family of connections. Second, in the presence of the spectral parameter, the symmetric space automorphism can be generalized as

$$\eta^\infty(\hat{V}(x,t))=\eta\left(\hat{V}\left(x,\frac{1}{t}\right)\right)=\left(\hat{V}^{-1}\left(x,\frac{1}{t}\right)\right)^T. \tag{50}$$

It can be shown, following from this, that

$$\left(\hat{V}^{-1}\left(x,\frac{1}{t}\right)\partial_\alpha\hat{V}\left(x,\frac{1}{t}\right)\right)^T=-\hat{V}^{-1}(x,t)\partial_\alpha\hat{V}(x,t). \tag{51}$$

Given these, let us define

$$\mathcal{M}=\hat{V}(x,t)\hat{V}^T\left(x,\frac{1}{t}\right). \tag{52}$$

It follows now, from Eq. (51), that

$$\partial_\alpha\mathcal{M}=0; \tag{53}$$

namely, $\mathcal{M}=\mathcal{M}(\omega)$ and is independent of the space-time coordinates. $\mathcal{M}(\omega)$ is known as the monodromy matrix for the system under study and encodes properties of integrability such as the conserved quantities associated with the system.

Let us next describe how the monodromy matrix is constructed for such systems. For simplicity, we will consider the action in Eq. (2), which describes a sigma model defined on the coset $O(d,d)/[O(d)\times O(d)]$. The other case can be studied in a completely analogous manner. To start with, let us set the antisymmetric tensor field to zero, namely, $B=0$. In this case, we can write

$$M^{(B=0)}=\begin{pmatrix} G^{-1} & 0\\ 0 & G\end{pmatrix},\quad V^{(B=0)}=\begin{pmatrix} E^{-1} & 0\\ 0 & E\end{pmatrix}. \tag{54}$$

Let us further assume that the matrix E and therefore G are diagonal, as is relevant in the study of colliding plane waves; namely, let us parametrize

$$E=\mathrm{diag}(e^{(\lambda+\psi_1)/2},e^{(\lambda+\psi_2)/2},\dots,e^{(\lambda+\psi_d)/2}),$$
$$G=\mathrm{diag}(e^{\lambda+\psi_1},e^{\lambda+\psi_2},\dots,e^{\lambda+\psi_d}) \tag{55}$$

with $\Sigma_i\psi_i=0$ so that $\lambda=(1/d)\log\det G$, as adopted in [23].

In this case, it follows that [see Eq. (21)]

$$P_\alpha=\frac{1}{2}[(V^{(B=0)})^{-1}\partial_\alpha V^{(B=0)}+((V^{(B=0)})^{-1}\partial_\alpha V^{(B=0)})^T]$$
$$=\begin{pmatrix} -E^{-1}\partial_\alpha E & 0\\ 0 & E^{-1}\partial_\alpha E\end{pmatrix},$$
$$Q_\alpha=\frac{1}{2}[(V^{(B=0)})^{-1}\partial_\alpha V^{(B=0)}-((V^{(B=0)})^{-1}\partial_\alpha V^{(B=0)})^T]$$
$$=0, \tag{56}$$

so that we have

$$(\hat{V}^{(B=0)})^{-1}\partial_+\hat{V}^{(B=0)}=\frac{1-t}{1+t}P_+,$$
$$(\hat{V}^{(B=0)})^{-1}\partial_-\hat{V}^{(B=0)}=\frac{1+t}{1-t}P_-. \tag{57}$$

Since $P_\pm$ are diagonal matrices and $\hat{V}^{(B=0)}(x,t=0)=V^{(B=0)}(x)$ is diagonal, it follows that we can write

$$\hat{V}^{(B=0)}(x,t)=\begin{pmatrix}\bar{V}^{(B=0)}(x,t) & 0\\ 0 & (\bar{V}^{(B=0)})^{-1}(x,t)\end{pmatrix} \tag{58}$$

with $\overline{V}^{(B=0)}(x,t)$ a diagonal matrix of the form $(\overline{V}_1,\overline{V}_2,\ldots,\overline{V}_d)$. Let us assume that

$$\overline{V}_i=\frac{t_{d+i}}{t_i}\frac{t-t_i}{t-t_{d+i}}E_i^{-1},\quad i=1,2,\ldots,d, \tag{59}$$

where t_i is the spectral parameter corresponding to the constant ω_i. Clearly, for $t=0$, this leads to the diagonal elements of $V^{(B=0)}$. Furthermore, noting the form of $P_\pm$ in Eq. (56) and recalling that the spectral parameters satisfy

$$\partial_\pm t=\frac{1\mp t}{1\pm t}\partial_\pm \ln\rho,\quad \partial_\pm t_i=\frac{1\mp t_i}{1\pm t_i}\partial_\pm \ln\rho, \tag{60}$$

it is easy to verify that

$$\begin{aligned}\overline{V}_i^{-1}\partial_\pm\overline{V}_i&=\partial_\pm \ln E_i^{-1}\mp\frac{t}{1\pm t}\partial_\pm \ln\left(-\frac{t_i}{t_{d+i}}\right)\\&=\frac{1\mp t}{1\pm t}\partial_\pm \ln E_i^{-1}\end{aligned} \tag{61}$$

provided we identify [t_i and t_{d+i} have opposite signatures following from the double valued nature of the solutions in Eq. (49)]

$$-\frac{t_i}{t_{d+i}}=E_i^{-2}. \tag{62}$$

In that case, we can write

$$\begin{aligned}(\hat{V}^{(B=0)})^{-1}\partial_\pm\hat{V}^{(B=0)}&=\frac{1\mp t}{1\pm t}\begin{pmatrix}-E^{-1}\partial_\pm E & 0\\ 0 & E^{-1}\partial_\pm E\end{pmatrix}\\&=\frac{1\mp t}{1\pm t}P_\pm.\end{aligned} \tag{63}$$

Thus, we see that, in the present case,

$$\overline{V}_i=\frac{t_{d+i}}{t_i}\frac{t-t_i}{t-t_{d+i}}E_i^{-1}=\sqrt{-\frac{t_{d+i}}{t_i}}\frac{t-t_i}{t-t_{d+i}} \tag{64}$$

and the matrix $\hat{V}^{(B=0)}(x,t)$ has $2d$ simple poles—one pair for every diagonal element E_i. Furthermore, it is simple to check from Eq. (49) that the spectral parameters satisfy

$$\frac{\omega-\omega_i}{\omega-\omega_{d+i}}=\frac{t_{d+i}}{t_i}\frac{t-t_i}{t-t_{d+i}}\frac{1/t-t_i}{1/t-t_{d+i}}, \tag{65}$$

so that we can determine the monodromy matrix to be of the form

$$\begin{aligned}\hat{\mathcal{M}}^{(B=0)}&=\hat{V}^{(B=0)}(x,t)(\hat{V}^{B=0)})^T\left(x,\frac{1}{t}\right)\\&=\begin{pmatrix}\mathcal{M}(\omega) & 0\\ 0 & \mathcal{M}^{-1}(\omega)\end{pmatrix},\end{aligned} \tag{66}$$

where $\mathcal{M}(\omega)$ is diagonal with

$$\mathcal{M}_i(\omega)=\overline{V}_i(x,t)\overline{V}_i\left(x,\frac{1}{t}\right)=-\frac{\omega-\omega_i}{\omega-\omega_{d+i}}. \tag{67}$$

We note that the double valued relation between the global and the local spectral parameters allows us to choose $\omega_{d+i}=-\omega_i$, in which case, we have

$$\mathcal{M}_i(\omega)=\frac{\omega_i-\omega}{\omega_i+\omega}. \tag{68}$$

This determines the monodromy matrix for the case when $B=0$.

Let us note next that we are dealing with a sigma model, obtained through dimensional reduction of a higher dimensional tree level string effective action. Therefore, the symmetries present in the string theory, such as T duality, should be encoded in the monodromy matrix as well. For example, it is known that one can generate new backgrounds (of the string theory) starting from given ones through T-duality transformations. In particular, starting from a background where $B=0$, it is possible in some models (such as the Nappi-Witten model) to generate backgrounds with $B\neq 0$ through a T-duality rotation. It is natural, therefore, to examine how the monodromy matrix transforms under such transformations, for then we can determine the monodromy matrix for more complicated backgrounds starting from simpler ones.

Let us note that the T-duality transformation, within the context of string theory (without Abelian gauge fields), corresponds to a global $O(d,d)$ rotation. Since the one parameter family of matrices $\hat{V}(x,t)\in O(d,d)/[O(d)\times O(d)]$ much like $V(x)=\hat{V}(x,t=0)$, it follows that under a global $O(d,d)$ rotation

$$\begin{aligned}&\hat{V}(x,t)\to\Omega^T\hat{V}(x,t),\\ \hat{\mathcal{M}}(\omega)&=\hat{V}(x,t)\hat{V}^T\left(x,\frac{1}{t}\right)\to\Omega^T\hat{V}(x,t)\hat{V}^T\left(x,\frac{1}{t}\right)\Omega\\&=\Omega^T\hat{\mathcal{M}}(\omega)\Omega.\end{aligned} \tag{69}$$

Let us also recall that, under a local $O(d)\times O(d)$ transformation, $\hat{V}(x,t)\to\hat{V}(x,t)h(x)$. Therefore, the only local transformations that will preserve the global nature of the monodromy matrix are the ones that do not depend on the local spectral parameter explicitly. We have already seen that the matrix $M=VV^T$ is sensitive only to the global $O(d,d)$ transformations even though $V(x)$ transforms nontrivially under a combined global $O(d,d)$ and local $O(d)\times O(d)$ transformation. In a similar manner, $\hat{\mathcal{M}}(\omega)$ is sensitive only to the global $O(d,d)$ rotation. We will check this explicitly in the case of the Nappi-Witten model in the next section. For the moment, let us note that this brings out an interesting connection between the integrability properties of the two dimensional string effective action and its T-duality properties, which can be used as a powerful tool in determining solutions.

For completeness, we record here the transformation properties of $\hat{\mathcal{M}}$ under an infinitesimal $O(d,d)$ transformation. Let us denote

$$\hat{\mathcal{M}} = \begin{pmatrix} \hat{\mathcal{M}}_{11} & \hat{\mathcal{M}}_{12} \\ \hat{\mathcal{M}}_{21} & \hat{\mathcal{M}}_{22} \end{pmatrix}, \tag{70}$$

where each element represents a $d\times d$ matrix. Infinitesimally, we can write [26]

$$\Omega = \begin{pmatrix} 1+X & Y \\ Z & 1+W \end{pmatrix}, \tag{71}$$

where the infinitesimal parameters of the transformation satisfy $Y^T=-Y, Z^T=-Z$ and $W=-X^T$. Under such an infinitesimal transformation, it follows from Eq. (71) that

$$\begin{aligned}
\delta\hat{\mathcal{M}}_{11} &= \hat{\mathcal{M}}_{11}X + X^T\hat{\mathcal{M}}_{11} - Z\hat{\mathcal{M}}_{12} + \hat{\mathcal{M}}_{12}Z, \\
\delta\hat{\mathcal{M}}_{12} &= \hat{\mathcal{M}}_{11}Y + X^T\hat{\mathcal{M}}_{12} - Z\hat{\mathcal{M}}_{22} - \hat{\mathcal{M}}_{12}X^T, \\
\delta\hat{\mathcal{M}}_{21} &= -Y\hat{\mathcal{M}}_{11} - X\hat{\mathcal{M}}_{21} + \hat{\mathcal{M}}_{21}X + \hat{\mathcal{M}}_{22}Z, \\
\delta\hat{\mathcal{M}}_{22} &= \hat{\mathcal{M}}_{21}Y - Y\hat{\mathcal{M}}_{12} - X\hat{\mathcal{M}}_{22} - \hat{\mathcal{M}}_{22}X^T.
\end{aligned} \tag{72}$$

IV. APPLICATIONS

The ideas presented in the earlier section can be applied to various physical systems. For example, if we are considering collision of plane fronted waves, which correspond to massless states of closed strings, because of the isometries in the problem, this can be described effectively by a two dimensional theory and all our earlier discussions can be carried over [16]. In this section, we will discuss two other classes of physical phenomena where our results can be explicitly verified and prove quite useful.

A. The Nappi-Witten model

The Nappi-Witten model [24] is an example of a cosmological solution following from the string theory. Let us note that, to leading order in α', the string tension, there are several solutions to the string equations following from Eq. (1) that constitute exact conformal field theory backgrounds. One of these solutions, studied by Nappi and Witten, corresponds to a gauged $\mathrm{SL}(2,R)/\mathrm{SO}(1,1)\times\mathrm{SU}(2)/\mathrm{U}(1)$ Wess-Zumino-Witten model and describes a closed expanding universe in $3+1$ dimensions. The backgrounds consist of the metric, the dilaton, and the antisymmetric tensor fields of the form (here, we identify $x^0=\tau$)

$$\begin{aligned}
fs^2 &= -d\tau^2 + dx^2 + \frac{1}{1-\cos 2\tau \cos 2x} \\
&\quad \times (4\cos^2\tau\cos^2 x\, dy^2 + 4\sin^2\tau \sin^2 x\, dz^2), \\
\phi &= -\frac{1}{2}\log(1-\cos 2\tau\cos 2x), \\
B_{12} &= -B_{21} = b = \frac{(\cos 2\tau - \cos 2x)}{(1-\cos 2\tau \cos 2x)}.
\end{aligned} \tag{73}$$

Here, we have set an arbitrary constant parameter appearing in the Nappi-Witten solution to zero for simplicity.

Note that the backgrounds do not depend on two of the coordinates, namely, (y,z), and, consequently, following from our earlier discussions, the system has an $O(2,2)$ symmetry. It is known that these backgrounds can be obtained from a much simpler background, with a vanishing B field, of the form

$$\begin{aligned}
ds^2 &= -d\tau^2 + dx^2 + \frac{1}{\tan^2\tau}dy^2 + \tan^2 x\, dz^2, \\
\overline{\phi} &= -\log(\sin 2\tau \sin 2x)
\end{aligned} \tag{74}$$

through an $O(2,2)$ rotation. In the language of our earlier discussion, we note that we can write

$$G^{B=0} = \begin{pmatrix} e^{\lambda^{(0)}+\psi^{(0)}} & 0 \\ 0 & e^{\lambda^{(0)}-\psi^{(0)}} \end{pmatrix} = \begin{pmatrix} \xi_1 & 0 \\ 0 & \xi_2 \end{pmatrix} \tag{75}$$

with

$$\begin{aligned}
\xi_1 &= \exp(\lambda^0 + \psi^0) = \frac{1}{\tan^2\tau}, \\
\xi_2 &= \exp(\lambda^0 - \psi^0) = \frac{1}{\tan^2 x},
\end{aligned} \tag{76}$$

where the superscript "0" denotes the vanishing B field. In this case, therefore, we have

$$M^{(B=0)} = \begin{pmatrix} (G^{(B=0)})^{-1} & 0 \\ 0 & G^{(B=0)} \end{pmatrix}. \tag{77}$$

On the other hand, it is easy to check that we can write

$$\begin{aligned}
G^{(B)} &= \begin{pmatrix} \dfrac{2\xi}{1+\xi_1\xi_2} & 0 \\ 0 & \dfrac{2\xi_2}{1+\xi_1\xi_2} \end{pmatrix}, \\
B &= -\frac{1-\xi_1\xi_2}{1+\xi_1\xi_2}\epsilon = b\epsilon,
\end{aligned} \tag{78}$$

where ϵ is the 2×2 antisymmetric matrix

$$\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{79}$$

so that

$$M^{(B)} = \begin{pmatrix} (G^{(B)})^{-1} & -(G^{(B)})^{-1}B \\ B(G^{(B)})^{-1} & G^{(B)} - B(G^{(B)})^{-1}B \end{pmatrix}. \tag{80}$$

It is now a simple matter to check that

$$M^{(B)} = \Omega^T M^{(B=0)} \Omega \tag{81}$$

where [25]

$$\Omega = \frac{1}{\sqrt{2}} \begin{pmatrix} I & \epsilon \\ \epsilon & I \end{pmatrix} \tag{82}$$

belongs to $O(2,2)$. Here, I represents the 2×2 identity matrix while ϵ is the 2×2 anti-symmetric matrix defined in Eq. (79). This shows that the backgrounds with a nontrivial antisymmetric tensor field can be generated from a much simpler background with a vanishing B field through a global $O(2,2)$ rotation.

It follows from Eq. (75) that we can write

$$E^{(B=0)} = \begin{pmatrix} \sqrt{\xi_1} & 0 \\ 0 & \sqrt{\xi_2} \end{pmatrix} \tag{83}$$

so that we have

$$V^{(B=0)} = \begin{pmatrix} (E^{(B=0)})^{-1} & 0 \\ 0 & E^{(B=0)} \end{pmatrix}, \tag{84}$$

and it follows that [see Eq. (56)]

$$P_{\pm}^{(B=0)} = \begin{pmatrix} -(E^{(B=0)})^{-1}\partial_{\pm}E^{(B=0)} & 0 \\ 0 & (E^{(B=0)})^{-1}\partial_{\pm}E^{(B=0)} \end{pmatrix} = \begin{pmatrix} -\frac{(\partial_{\pm}\xi_1)}{\xi_1} & 0 & 0 & 0 \\ 0 & -\frac{(\partial_{\pm}\xi_2)}{\xi_2} & 0 & 0 \\ 0 & 0 & \frac{(\partial_{\pm}\xi_1)}{\xi_1} & 0 \\ 0 & 0 & 0 & \frac{(\partial_{\pm}\xi_2)}{\xi_2} \end{pmatrix},$$

$$Q_{\pm}^{(B=0)} = 0. \tag{85}$$

In this case, therefore, the one parameter family of potentials $\hat{V}^{(B=0)}(x,t)$ have to satisfy (since $Q_{\pm}^{(B=0)}=0$)

$$(\hat{V}^{(B=0)})^{-1}(x,t)\partial_{\pm}\hat{V}^{(B=0)}(x,t) = \frac{1\mp t}{1\pm t} P_{\pm}^{(B=0)} \tag{86}$$

where t is the space-time dependent spectral parameter.

Following our earlier construction [see Eq. (64)], we can determine

$$\hat{V}^{(B=0)}(x,t) = \mathrm{diag}(\bar{V}_1, \ldots, \bar{V}_4) = \mathrm{diag}\left(\sqrt{-\frac{t_3}{t_1}\frac{t-t_1}{t-t_3}}, \sqrt{-\frac{t_4}{t_2}\frac{t-t_2}{t-t_4}}, \sqrt{-\frac{t_1}{t_3}\frac{t-t_3}{t-t_1}}, \sqrt{-\frac{t_2}{t_4}\frac{t-t_4}{t-t_2}} \right) \tag{87}$$

where

$$-\frac{t_1}{t_3} = (E_1^{(B=0)})^{-2} = \frac{1}{\xi_1}, \quad -\frac{t_2}{t_4} = (E_2^{(B=0)})^{-2} = \frac{1}{\xi_2}. \tag{88}$$

It can be checked explicitly, using the equation satisfied by the spectral parameter, Eq. (60), that $\hat{V}^{(B=0)}(x,t)$ in Eq. (87) does indeed satisfy Eq. (86). The monodromy matrix, in this case, follows as

$$\hat{\mathcal{M}}^{(B=0)} = \begin{pmatrix} \mathcal{M}(\omega) & 0 \\ 0 & \mathcal{M}^{-1}(\omega) \end{pmatrix} = \begin{pmatrix} \frac{\omega-\omega_1}{\omega-\omega_3} & 0 & 0 & 0 \\ 0 & \frac{\omega-\omega_2}{\omega-\omega_4} & 0 & 0 \\ 0 & 0 & \frac{\omega-\omega_3}{\omega-\omega_1} & 0 \\ 0 & 0 & 0 & \frac{\omega-\omega_4}{\omega-\omega_2} \end{pmatrix} = \begin{pmatrix} \frac{\omega_1-\omega}{\omega_1+\omega} & 0 & 0 & 0 \\ 0 & \frac{\omega_2-\omega}{\omega_2+\omega} & 0 & 0 \\ 0 & 0 & \frac{\omega_1+\omega}{\omega_1-\omega} & 0 \\ 0 & 0 & 0 & \frac{\omega_2+\omega}{\omega_2-\omega} \end{pmatrix}, \tag{89}$$

where we have identified $\omega_3 = -\omega_1, \omega_4 = -\omega_2$.

When $B\neq 0$, we can similarly obtain

$$E^{(B)}=\begin{pmatrix} \sqrt{\dfrac{2\xi_1}{1+\xi_1\xi_2}} & 0 \\ 0 & \sqrt{\dfrac{2\xi_2}{1+\xi_1\xi_2}} \end{pmatrix}, \quad V^{(B)}=\begin{pmatrix} (E^{(B)})^{-1} & 0 \\ B(E^{(B)})^{-1} & E^{(B)} \end{pmatrix}. \tag{90}$$

It follows now that

$$P_{\pm}^{(B)}=\begin{pmatrix} -(E^{(B)})^{-1}\partial_{\pm}E^{(B)} & -\dfrac{1}{2}(E^{(B)})^{-1}(\partial_{\pm}B)(E^{(B)})^{-1} \\ \dfrac{1}{2}(E^{(B)})^{-1}(\partial_{\pm}B)(E^{(B)})^{-1} & (E^{(B)})^{-1}\partial_{\pm}E^{(B)} \end{pmatrix},$$

$$Q_{\pm}=\begin{pmatrix} 0 & \dfrac{1}{2}(E^{(B)})^{-1}(\partial_{\pm}B)(E^{(B)})^{-1} \\ \dfrac{1}{2}(E^{(B)})^{-1}(\partial_{\pm}B)(E^{(B)})^{-1} & 0 \end{pmatrix}. \tag{91}$$

In this case, therefore, $Q_{\pm}\neq 0$ and the one parameter family of potentials has to satisfy

$$(\hat{V}^{(B)})^{-1}(x,t)\partial_{\pm}\hat{V}^{(B)}(x,t)=Q_{\pm}^{(B)}+\frac{1\mp t}{1\pm t}P_{\pm}^{(B)}. \tag{92}$$

It is straightforward to check that

$$V^{(B)}(x)=\Omega^{T}V^{(B=0)}(x)h(x) \tag{93}$$

where Ω is the $O(2,2)$ matrix defined in Eq. (82), and $h(x)\in O(2)\times O(2)$ and is of the form

$$h(x)=\frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{1-b}I & \sqrt{1+b}\epsilon \\ \sqrt{1+b}\epsilon & \sqrt{1-b}I \end{pmatrix}. \tag{94}$$

It can also be checked that

$$\hat{V}^{(B)}(x,t)=\Omega^{T}\hat{V}^{(B=0)}(x,t)h(x) \tag{95}$$

$$=\frac{1}{2}\begin{pmatrix} \sqrt{1-b}\overline{V}_1+\sqrt{1+b}\overline{V}_4 & 0 & 0 & \sqrt{1+b}\overline{V}_1-\sqrt{1-b}\overline{V}_4 \\ 0 & \sqrt{1-b}\overline{V}_2+\sqrt{1+b}\overline{V}_3 & -\sqrt{1+b}\overline{V}_2+\sqrt{1-b}\overline{V}_3 & 0 \\ 0 & -\sqrt{1-b}\overline{V}_2+\sqrt{1+b}\overline{V}_3 & \sqrt{1+b}\overline{V}_2+\sqrt{1-b}\overline{V}_3 & 0 \\ \sqrt{1-b}\overline{V}_1-\sqrt{1+b}\overline{V}_4 & 0 & 0 & \sqrt{1+b}\overline{V}_1+\sqrt{1-b}\overline{V}_4 \end{pmatrix} \tag{96}$$

satisfies the defining relation in Eq. (92). We note here that, when $B\neq 0$, while $V^{(B)}(x)$ is triangular, $\hat{V}^{(B)}(x,t)$ is not in general, and that both $V^{(B)}$ and $\hat{V}^{(B)}$ are related to their counterparts with $B=0$ through a combined global $O(2,2)$ and a local $O(2)\times O(2)$ transformation. Furthermore, as was pointed out earlier, the local transformation does not depend explicitly on the spectral parameter. This is quite crucial, for it immediately leads to

$$\hat{\mathcal{M}}^{(B)}=\hat{V}^{(B)}(x,t)(\hat{V}^{(B)})^T\left(x,\frac{1}{t}\right)=\Omega^T\hat{V}^{(B=0)}(x,t)h(x)h^T(x)(\hat{V}^{(B=0)})^T\left(x,\frac{1}{t}\right)\Omega=\Omega^T\hat{\mathcal{M}}^{(B=0)}\Omega$$
$$=\frac{1}{2}\begin{pmatrix} \mathcal{M}_1+\mathcal{M}_2^{-1} & 0 & 0 & \mathcal{M}_1-\mathcal{M}_2^{-1} \\ 0 & \mathcal{M}_2+\mathcal{M}_1^{-1} & -\mathcal{M}_2+\mathcal{M}_1^{-1} & 0 \\ 0 & \mathcal{M}_2+\mathcal{M}_1^{-1} & \mathcal{M}_2+\mathcal{M}_1^{-1} & 0 \\ \mathcal{M}_1-\mathcal{M}_2^{-1} & 0 & 0 & \mathcal{M}_1+\mathcal{M}_2^{-1}\end{pmatrix} \tag{97}$$

where $\mathcal{M}_1=(\omega_1-\omega)/(\omega_1+\omega)$ and $\mathcal{M}_2=(\omega_2-\omega)/(\omega_2+\omega)$ are the two diagonal elements of $\mathcal{M}(\omega)$ in Eq. (89). This shows explicitly that, for backgrounds related by a duality transformation [in the present example, an $O(2,2)$ rotation], the corresponding monodromy matrices are also related in a simple manner, as was pointed out in the last section.

B. Black holes

As a second application, we will discuss the black hole solutions [28,29] in string theory within the context of our analysis. We are interested in studying systems with charged black hole solutions (electric and magnetic). In this case, in heterotic string theory, one starts from the 10 dimensional string effective action (the bosonic sector)

$$\hat{S}=\int d^{10}x\sqrt{-\hat{G}}e^{-\hat{\phi}}\bigg[R_{\hat{G}}+(\hat{\partial}\hat{\phi})^2-\frac{1}{12}\hat{H}_{\hat{\mu}\hat{\nu}\hat{\lambda}}\hat{H}^{\hat{\mu}\hat{\nu}\hat{\lambda}}-\frac{1}{4}\delta^{IJ}\hat{F}^{I}_{\hat{\mu}\hat{\nu}}\hat{F}^{\hat{\mu}\hat{\nu}I}\bigg] \tag{98}$$

where $\hat{F}^{I}_{\hat{\mu}\hat{\nu}}$ represent Abelian field strengths [see Eqs. (25)–(27)] with $I=1,2,\ldots,16$ for the heterotic string. Commonly, the black hole solutions, in four dimensions, are described in terms of the Einstein metric. The reduction of Eq. (98) to four dimensions, in the Einstein frame, is carried out by identifying [26]

$$\hat{G}_{\hat{\mu}\hat{\nu}}=\begin{pmatrix} e^{2\phi}g_{\mu\nu}+G_{ij}A^{(1)i}_{\mu}A^{(1)j}_{\nu} & A^{(1)i}_{\mu}G_{ij} \\ A^{(1)j}_{\nu}G_{ij} & G_{ij}\end{pmatrix} \tag{99}$$

where $\mu=\tau,r,\theta,\phi$ and $i,j=4,5,\ldots,10$. This leads to a four dimensional action of the form

$$S_4=\int d\tau d^3x\sqrt{-g}\bigg(R_g-\frac{1}{2}(\partial\phi)^2-\frac{1}{12}e^{-2\phi}H_{\mu\nu\lambda}H^{\mu\nu\lambda}-e^{-\phi}F^{i}_{\mu\nu}M^{-1}F^{\mu\nu i}+\frac{1}{8}\mathrm{Tr}(\partial_\mu M^{-1}\partial^\mu M)\bigg), \tag{100}$$

where M is defined in Sec. II [Eq. (30)] along with other relevant parameters. In this case, $M\in O(6,22)$ and the moduli parametrize the coset, $O(6,22)/[O(6)\times O(22)]$.

We are interested in charged, nonrotating, spherically symmetric black hole solutions which are described by a general metric of the form

$$ds^2=g_{\mu\nu}dx^\mu dx^\nu=-\lambda(r)d\tau^2+\lambda^{-1}(r)dr^2+R^2(r)(d\theta^2+\sin^2\theta d\phi^2). \tag{101}$$

Furthermore, the Maxwell equation, together with the Bianchi identity, determines that the only nonzero components of the field strengths have the forms (as 28 dimensional column matrices)

$$F_{\tau r}=\frac{\lambda(r)}{r^2}e^{\phi}M\alpha,\quad F_{\theta\phi}=\sin\theta\,\eta\beta, \tag{102}$$

where α,β are 28 component column vectors representing the electric and the magnetic charges and η is the metric of $O(6,22)$. (The 28 gauge fields correspond to the sum of the original 16 gauge fields and six each coming from the dimensional reduction of the metric and the antisymmetric tensor field.)

As is clear from this, in the case of black holes, there are two Abelian isometries since the variables are independent of time as well as the azimuthal angle. Therefore, the proper way to analyze this problem would be to dimensionally reduce the effective action to two dimensions, as has been done in the earlier sections. This, however, leads to some technical issues and, therefore, to keep our discussion simple, we will dimensionally reduce the effective action to three dimensions first. Since the black hole solutions are independent of time, we dimensionally reduce time as well as six spatial dimensions and, keeping in mind the Einstein frame, we parametrize the metric as

$$\hat{G}_{\hat{\mu}\hat{\nu}}=\begin{pmatrix} e^{2\bar{\phi}}h_{\alpha\beta}+G_{mn}A^{(1)m}_{\alpha}A^{(1)n}_{\beta} & A^{(1)m}_{\alpha}G_{mn} \\ A^{(1)n}_{\beta}G_{mn} & G_{mn}\end{pmatrix} \tag{103}$$

where $\alpha,\beta=1,2,3$ and $m,n=0,4,5,\ldots,10$. Here $\bar{\phi}=\hat{\phi}-\frac{1}{2}\log\det G_{mn}$ is the shifted dilaton and the metric $h_{\alpha\beta}$ is in the Einstein frame (since the dilaton term has been factored out explicitly) with Euclidean signature. The dimensionally reduced effective action can be determined following the discussion in Sec. II and has the form [30–32]

$$S_3=\int d^3x\sqrt{h}\bigg[R_h-(\partial\bar{\phi})^2-\frac{1}{12}e^{-4\bar{\phi}}H_{\alpha\beta\gamma}H^{\alpha\beta\gamma}-e^{-2\bar{\phi}}F^T_{\alpha\beta}(\eta M\eta)F^{\alpha\beta}+\frac{1}{8}\mathrm{Tr}(\partial_\alpha M^{-1}\partial^\alpha M)\bigg], \tag{104}$$

where η is the metric of $O(7,23)$

$$\eta=\begin{pmatrix}0 & 1_7 & 0\\ 1_7 & 0 & 0\\ 0 & 0 & 1_{16}\end{pmatrix} \tag{105}$$

and the matrix $M\in O(7,23)$ has the form given in Eq. (30). We can set the field strength $H_{\alpha\beta\gamma}$ to zero since, in three dimensions, $B_{\alpha\beta}$ carries no physical degree of freedom. Furthermore, now we have 30 gauge fields—16 from the starting action and seven each coming from the dimensional reduction of the metric and the antisymmetric tensor field. $F_{\alpha\beta}$ correspondingly represents a 30 component column matrix.

The equations of motion for the gauge fields, following from the action in Eq. (104), are (in matrix notation)

$$\partial_\alpha(e^{-2\bar{\phi}}\sqrt{h}(\eta M\eta)F^{\alpha\beta})=0. \tag{106}$$

In three dimensions, the solution of this can be represented through a duality relation as

$$e^{-2\bar{\phi}}\sqrt{h}(\eta M\eta)F^{\alpha\beta}=\frac{1}{2}\epsilon^{\alpha\beta\gamma}\partial_\gamma\chi, \tag{107}$$

where χ represents 30 scalar fields (in a column matrix representation). Furthermore, the Bianchi identity

$$\epsilon^{\alpha\beta\gamma}\partial_\alpha F_{\beta\gamma}=0 \tag{108}$$

can now be written in terms of the 30 scalar fields as

$$D_\alpha(e^{2\bar{\phi}}(\eta M\eta)\partial^\alpha\chi)=0, \tag{109}$$

where D_α represents the gravitational covariant derivative. The important point of this analysis is that, in three-dimensions, the gauge fields can be traded in for scalars, which can, in principle, enlarge the coset parametrized by the moduli.

In fact, let us define a 32×32 matrix as

$$\bar{M}=\begin{pmatrix} M-e^{-2\bar{\phi}}\chi\chi^T & e^{2\bar{\phi}}\chi & M\eta\chi-\frac{1}{2}e^{2\bar{\phi}}(\chi^T\eta\chi)\chi\\ e^{2\bar{\phi}}\chi^T & -e^{2\bar{\phi}} & \frac{1}{2}e^{2\bar{\phi}}\chi^T\eta\chi\\ \chi^T\eta M-\frac{1}{2}e^{2\bar{\phi}}(\chi^T\eta\chi)\chi^T & \frac{1}{2}e^{2\bar{\phi}}\chi^T\eta\chi & -e^{-2\bar{\phi}}+\chi^T(\eta M\eta)\chi-\frac{1}{4}e^{2\bar{\phi}}(\chi^T\eta\chi)^2\end{pmatrix}. \tag{110}$$

This is manifestly symmetric and satisfies

$$\bar{M}\bar{\eta}\bar{M}=\bar{\eta} \tag{111}$$

where

$$\bar{\eta}=\begin{pmatrix}\eta & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix} \tag{112}$$

corresponds to the metric for $O(8,24)$. Therefore, the symmetric matrix $\bar{M}\in O(8,24)$. It is straightforward to verify that the action in Eq. (104) can be rewritten as

$$S=\int d^3x\sqrt{h}\left(R_h+\frac{1}{8}\mathrm{Tr}(\partial_\alpha\bar{M}^{-1}\partial^\alpha\bar{M})\right) \tag{113}$$

and is invariant under the $O(8,24)$ transformations

$$h_{\alpha\beta}\rightarrow h_{\alpha\beta},\quad \bar{M}\rightarrow\Omega^T\bar{M}\Omega, \tag{114}$$

where Ω is a global $O(8,24)$ matrix satisfying $\Omega^T\bar{\eta}\Omega=\bar{\eta}$. Thus, we note that, in three dimensions, the action is a sum of the Einstein Hilbert action and a nonlinear sigma model coupled to gravity defined over $O(8,24)/[O(8)\times O(24)]$. We note here that this is, in fact, the symmetry content we would have obtained had we dimensionally reduced to two dimensions directly.

The three dimensional metric corresponding to the black hole solution of Eq. (101) has the form

$$ds^2=h_{\alpha\beta}dx^\alpha dx^\beta=dr^2+\tilde{R}^2(r)(d\theta^2+\sin^2\theta d\phi^2) \tag{115}$$

where $\tilde{R}(r)=\lambda(r)R(r)$. Furthermore, the relations between the three-dimensional fields and the four-dimensional ones are given by T-duality relations [33] which we give in the Appendix. For the present, let us note that there is no dependence on the azimuthal angle in any of the variables. Consequently, we can integrate out ϕ in Eq. (113) to obtain

$$S=\int d^2\xi\sqrt{\gamma^{(2)}}\left(R_\gamma+\frac{1}{8}\gamma^{ab}\mathrm{Tr}(\partial_a\bar{M}^{-1}\partial_b\bar{M})\right), \tag{116}$$

where ξ^1,ξ^2 denote, respectively, r,θ, and the two dimensional metric has the form

$$\gamma_{ab}=\begin{pmatrix} \tilde{R}(r) & 0 \\ 0 & \tilde{R}(r)\sin\theta \end{pmatrix}. \tag{117}$$

This gives the effective two dimensional action in the context of black hole solutions and our general analysis of Sec. III can now be applied.

In the preceding discussion, we considered a four dimensional action (100) with metric and matter fields such as the shifted dilaton, the two-form Neveu-Schwarz–Neveu-Schwarz (NS-NS) potential, the gauge fields, and the moduli matrix M. We have also considered a three dimensional reduced effective action (104) with corresponding fields and an $\bar{M}$ matrix which parametrizes the coset $O(8,24)/[O(8)\times(24)]$. The field configurations such as the gauge potentials, shifted dilatons and the moduli appearing in the two action are related by T duality and these relations are given in the Appendix. The charged black hole solutions of heterotic string theory are described by the moduli and the gauge field configurations, which are also presented in the Appendix. One demands that the M matrix or $\bar{M}$ matrix tend to a constant as $r\rightarrow\infty$ and similarly the gauge potentials have appropriate asymptotic behavior in order to define the associated charges. As is well known, the charged black hole solutions can be obtained by applying the solution generating techniques [28,29,33,34] (see [35] for generating black hole solutions in type IIB theory). The starting point is the spherically symmetric Schwarzschild black hole solution that appears as a solution of the heterotic string effective action. Subsequently, a series of T-duality transformations, often called "boosts," are implemented in order to obtain charged black hole solutions with 28 charges (we are thinking of electrically charged black holes; there will be another 28 magnetic charges too). Next, one can obtain the extremal black hole solution by tuning an appropriate parameter to zero.

We have derived the transformation properties of the monodromy matrix under the noncompact T-duality group in Sec. III. Therefore, it will suffice to construct the monodromy matrix for the Schwarzschild black hole solution in the heterotic string theory. One can derive the monodromy matrix for the general charged black hole from the monodromy matrix associated with the Schwarzschild black hole solution, since the T-duality transformations are well known (see [33], for example).

In what follows, we focus our attention on explicit construction of the monodromy matrix, for the simplest of black holes, namely, the Schwarzschild black hole, following from our general analysis. In this case, $B=0$ and we can write, inside the trapped region,

$$V(x)=\mathrm{diag}(\lambda_1^{-1},\lambda_2^{-1},\ldots,\lambda_{32}^{-1})=\left(\sqrt{-\frac{r}{r-m}},1,\ldots,1,\sqrt{-\frac{r-m}{r}},1,\ldots,1,\sqrt{-\frac{r}{r-m}},\sqrt{-\frac{r-m}{r}}\right) \tag{118}$$

where r denotes the radial coordinate and $\lambda_1=\lambda_{31},\lambda_8=\lambda_{32}$. Correspondingly, the M matrix has the form

$$\bar{M}=VV^T=\mathrm{diag}\left(-\frac{r}{r-m},1,\ldots,1,-\frac{r-m}{r},1,\ldots,1,-\frac{r}{r-m},-\frac{r-m}{r}\right). \tag{119}$$

Here we follow the notation of [33] and choose the moduli such that they go over to the $O(8,24)$ metric in the asymptotic limit. Note that usually in the Schwarzschild metric the term appears as $r-2M$; here we have $r-m$. This is just for notational convenience. In this case, we can obtain, in a straightforward manner,

$$Q_\alpha=0,$$
$$P_\alpha=\mathrm{diag}(-\lambda_1^{-1}\partial_\alpha\lambda_1,0,\ldots,0,-\lambda_8^{-1}\partial_\alpha\lambda_8,0,\ldots,0,-\lambda_{31}^{-1}\partial_\alpha\lambda_{31},-\lambda_{32}\partial_\alpha\lambda_{32}). \tag{120}$$

The one parameter family of potentials, in this case, satisfy

$$\hat{V}^{-1}(x,t)\partial_\pm\hat{V}(x,t)=\frac{1\mp t}{1\pm t}P_\pm \tag{121}$$

and can be determined to have the form

$$\hat{V}(x,t)=\mathrm{diag}(\bar{V}_1,1,\ldots,1,\bar{V}_8,1,\ldots,1,\bar{V}_{31},\bar{V}_{32}), \tag{122}$$

where $(i=1,8,31,32)$

$$\bar{V}_i=\frac{t_{d+i}}{t_i}\frac{t-t_i}{t-t_{d+i}}\lambda_i=\sqrt{-\frac{t_{d+i}}{t_i}\frac{t-t_i}{t-t_{d+i}}}. \tag{123}$$

Here, we have made the identification, following our discussion in Sec. III,

$$-\frac{t_i}{t_{d+i}}=\lambda_i^{-2}. \tag{124}$$

The monodromy matrix, in this case, follows as

$$\hat{\mathcal{M}}(\omega)=\mathrm{diag}(\mathcal{M}_1(\omega),1,\ldots,1,\mathcal{M}_8(\omega),1,\ldots,1,\mathcal{M}_{31}(\omega),\mathcal{M}_{32}(\omega)) \tag{125}$$

with $(i=1,8,31,32)$

$$\mathcal{M}_i(\omega)=\frac{\omega_i-\omega}{\omega_i+\omega}. \tag{126}$$

Since other black hole solutions can be obtained from the Schwarzschild one by *T*-duality transformations, the corresponding monodromy matrices can also be obtained from the one constructed above following the procedure described in Sec. III.

V. SUMMARY AND DISCUSSION

We have described the prescriptions for the construction of the monodromy matrix for two dimensional string effective action. We adopted the procedure commonly followed in the construction of the monodromy matrix for a class of two dimensional σ models in curved space. As mentioned earlier, in most of the cases, the σ model arises from the dimensional reduction of higher dimensional Einstein-Hilbert action to two dimensional space-time due to the presence of isometries. In the context of string theory, a similar approach was adopted in the past to construct the monodromy matrix, as was the case with dimensionally reduced models in gravity.

One of our principal objectives was to take into account the symmetries associated with the string effective action and construct the monodromy matrix that contains information about these symmetries. We have succeeded in introducing a procedure for the construction of the monodromy matrix under general grounds with some mild requirements such as factorizability and the presence of isolated poles. Furthermore, we have demonstrated that the monodromy matrix transforms nontrivially under the noncompact *T*-duality group when the two dimensional string effective action respects that symmetry. We feel that this is an interesting and important result. The procedure adopted by us allows us to construct the monodromy matrix, once a set of string background configurations are known. As a result, if we know the monodromy matrix for a given set of simple string vacuum backgrounds, we can directly obtain the corresponding monodromy matrix for another set of more complicated backgrounds, if the latter can be derived by duality transformations from the simpler backgrounds.

We have discussed two illustrative examples in Sec. IV as applications of our methods. First, we considered the Nappi-Witten model which is exactly solvable for both vanishing and nonvanishing two-form potential *B*. This is a good testing ground for the duality transformation properties of the monodromy matrix. We have constructed this matrix for the case $B=0$. Subsequently, we have also constructed it for the case $B\neq 0$. Then, as a consistency check, we have derived the $\hat{\mathcal{M}}^B$ from $\hat{\mathcal{M}}^{B=0}$ following our rules of the transformations of $\hat{\mathcal{M}}$ under duality. Indeed, it is found that the monodromy matrix computed using the two different ways mentioned above coincide. Our second example is that of black hole solutions in heterotic string theory. After recapitulating the charged black hole solutions, we construct the monodromy matrix for the "seed" Schwarzschild black hole in heterotic string theory. One can construct the monodromy matrix for charged black hole solutions since the *T*-duality transformations that generate charged black hole solutions are already known. For the sake of completeness, we have given the corresponding metric for plane waves in the trapped region for the charged black holes [36,37]. The isometries are quite transparent and the monodromy matrix for the colliding wave case can be constructed by the techniques used by us [16].

It is worthwhile to mention that all our results are derived for the case of classical two dimensional effective theory as is the case for effective two dimensional theories derived from higher dimensional Einstein-Hilbert action. It might be interesting to explore systematically the construction of the monodromy matrix and its properties in quantum theory. We hope the work presented here will find applications in diverse directions where one encounters effective two dimensional models in the context of string theory.

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APPENDIX: SOME USEFUL RELATIONS

In this appendix, we collect some relations that are useful in understanding the details of various issues, but are not essential to the logic presented in the text. As is mentioned in the section on black holes, the fields in three and four dimensions are related by duality transformations of the form (tilde quantities are three dimensional while the ones without tildes are four dimensional)

$$G_{ij}=\tilde{G}_{1+i,1+j}, \quad B_{ij}=\tilde{B}_{1+i,1+j}, \quad a^I_j=\tilde{a}^I_{1+j},$$

$$\phi=\tilde{\phi}+\frac{1}{2}[\log\det\tilde{G}_{mn}-\log\det G_{ij}],$$

$$\lambda=e^{-\phi}\left[\frac{\det\tilde{G}_{mn}}{\det G_{ij}}\right]^{1/2}, \quad R=\frac{\tilde{R}}{\lambda},$$

$$A_t^{(1)i}=G^{ij}\tilde{G}_{1,1+j}, \quad A_\phi^{(1)i}=G^{ij}\tilde{A}^m_\phi\tilde{G}_{m,1+j}, \tag{A1}$$

$$A_t^{(3)I}=\tilde{a}^I_t-a^I_jA_t^{(1)j},$$

$$A_\phi^{(3)I}=\tilde{A}_\phi^{14+I}+a^I_m\tilde{A}^n_\phi,$$

$$A^{(2)}_{ti}=\tilde{B}_{1,1+i}+B_{ij}A_t^{(1)j}+\frac{1}{2}a^I_iA_t^{(3)I},$$

$$A^{(2)}_{\phi i}=\tilde{A}_\phi^{8+i}-\tilde{B}_{1+i,n}\tilde{A}_\phi^{14+I}+B_{ij}A_\phi^{(1)j}+\frac{1}{2}a^I_iA_\phi^{(3)I}.$$

We will now give the explicit forms of some of the black hole solutions as well as discuss briefly the connection between the black holes and the colliding waves. We know that in four dimensions, inside the Schwarzschild horizon, $r\leqslant 2M$, the black hole (BH) metric has the following form:

$$ds^2=\left(\frac{2M-r}{r}\right)dt^2-\left(\frac{r}{2M-r}\right)dr^2+r^2(d\theta^2+\sin^2\theta d\varphi^2). \tag{A2}$$

On the other hand, for colliding plane waves, the metric, in general, can be represented as

$$ds^2=-e^{-M(u,v)}du dv+e^{-U(u,v)}(e^{V(u,v)}dx^2+e^{-V(u,v)}dy^2) \tag{A3}$$

where u,v are light-cone coordinates. Let us consider the region $u\geqslant 0, v\geqslant 0, u+v\leqslant \pi/2$. In this region, if we make the transformations

$$r\to M[1-\sin(u+v)],$$

$$\theta\to\frac{\pi}{2}+(v-u), \tag{A4}$$

$$t\to x, \varphi\to 1+\frac{y}{M}$$

and analytically continue y beyond the cyclic boundary condition on the angle φ, then the metric for the black hole becomes

$$ds^2=-4M^2[1-\sin(u+v)]^2du dv+\frac{\cos^2(u+v)}{[1-\sin(u+v)]^2}dx^2 + \cos^2(u-v)[1-\sin(u+v)]^2dy^2, \tag{A5}$$

which has the form of that for colliding waves.

A Reissner-Nordstrom BH will have the form of the four dimensional metric, given by

$$ds^2=\lambda(r)dt^2-\lambda^{-1}(s)dr^2+R^2(r)(d\theta^2+\sin^2\theta d\varphi^2), \tag{A6}$$

$$\lambda=\frac{(r+\beta)(r-\beta)}{(XY-Z^2)^{1/2}},$$

$$R=(XY-Z^2)^{1/2},$$

$$e^{2\phi}=\frac{W^2}{XY-Z^2},$$

$$X=r^2+\overline{Q_2}\cos^2 h\,\delta_1+\overline{Q_1}\sin^2 h\,\delta_1,$$

$$Y=r^2+\overline{Q_1}r,$$

$$Z=Q_1\sin h\,\delta_1 r,$$

$$W=r^2,$$

$$\overline{Q_2}=\pm\sqrt{Q_2+\beta^2}, \tag{A7}$$

$$\overline{Q_1}=\pm\sqrt{Q_1+\beta^2}.$$

Here β is the nonextremality parameter, where the extremal limit corresponds to $\beta\to 0$.

The nonextremal BH metric has the generic form

$$ds^2=\frac{(r_+-r)(r-r_-)}{(r^2-R_0^2)}dt^2-\frac{(r^2-R_0^2)}{(r_+-r)(r-r_-)}dr^2 +(r^2-R_0^2)(d\theta^2+\sin^2\theta d\varphi^2) \tag{A8}$$

where R_0^2 is expressed in terms of charges and boost parameters of the $O(d,d)$ transformation; $r_\pm=M\pm r_0$ and again r_0 is expressed in terms of charges as well as the $O(d,d)$ boost parameters.

We can go from this black hole metric to that of colliding waves through the transformation [36]

$$r\to M\pm r_0\left(\frac{u}{a}+\frac{v}{b}\right),$$

$$\theta\to\frac{\pi}{2}\pm\left(\frac{u}{a}-\frac{v}{b}\right), \tag{A9}$$

$$t\to\frac{xr_0}{(M^2-R_0^2)^{1/2}},$$

$$\varphi\to 1+\frac{y}{(M^2-R_0^2)^{1/2}}.$$

Therefore, we see again that the trapped region of the BH is locally isometric to the interaction region of the colliding plane waves. The periodic coordinate φ goes to y (which is nonperiodic) to represent a plane wave in the x-y plane, so that, in the trapped region,

$$g_{\mu\nu}=\frac{-2\{[M\pm r_0\sin(u/a+v/b)]^2-R_0^2\}}{ab},$$

$$g_{xx}=\frac{(M^2-R_0^2)\cos(u/a+v/b)^2}{\{[M\pm r_0\sin(u/a+v/b)]^2-R_0^2\}}, \tag{A10}$$

$$g_{yy}=\cos^2\left(\frac{u}{a}-\frac{v}{b}\right)\frac{\{[M\pm r_0\sin(u/a+v/b)]^2-R_0^2\}}{(M^2-R_0^2)}.$$

In the asymptotic limit, the Einstein metric $g_{\mu\nu}=\eta_{\mu\nu}$ for $u=v=0$. The incoming parameters a and b are required to satisfy the following relations:

$$ab=(M^2-R_0^2)=\frac{4S_{ext}}{\pi}, \tag{A11}$$

where S_{ext} is the entropy of the extremal BH.

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