Pair production in a rotating strong magnetic field

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We calculate the production probability of an e^--e^+ pair in a strong rotating magnetic field. After deriving some selection rules concerning the states in which the pair can be created and their connection with the time variation of the magnetic field, we conclude that for pair production the change of direction of the magnetic field is a much more efficient mechanism than the change of its strength.

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I. INTRODUCTION

One of the most interesting physical phenomena that still needs a complete explanation is gamma-ray bursts [1,2]. They are pulses of soft gamma rays that satellites register now more than once a day. Many models have been proposed to explain the general features of gamma-ray bursts such as their duration ($\sim$10–100 s), their central frequency ω_c($\hbar\omega_c\sim$10 keV–1 MeV) or the total energy carried ($\sim 10^{50}$–10^{54} ergs) [3]. There are also many hypotheses about their origin but various recent models connect the production of gamma-ray bursts with the formation of neutron stars or black holes [4–9].

Barbiellini *et al.* have proposed a model in which gamma-ray bursts are the synchrotron radiation emitted by electrons and positrons moving in relativistic regimes in the magnetic field around neutron stars or black holes [10]. It is interesting now to understand how electrons and positrons can be created around these massive objects.

It is believed that the huge gravitational fields produced by black holes can prime a mechanism of pair production [11,12] even if there are situations in which the approximation of flat spacetime can be safely used [13]. In the case of neutron stars it seems reasonable to neglect the effects of the gravitational field with respect to that of the magnetic fields produced by these stars.

In this paper we connect the creation of e^--e^+ pairs with the very strong time dependent magnetic fields which are present around these stellar objects ($\gg B_{cr}=m^2c^3/\hbar e\simeq 4.4\times 10^{13}$ G) [14,15] neglecting the effects of the gravitational field. From this point of view we are dealing with a typical problem of pair creation in an external not quantized electromagnetic field. Of course, the peculiar features of our model depend on the particular physical system we are studying and they will be stated in the following paragraph. Similar processes, with different types of magnetic-field variations, have been studied both in the case of a pure strength variation of $\mathbf{B}(t)$ [16] and for some particular variation of strength and direction [17].

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II. THEORETICAL MODEL

In what follows we describe the theoretical model we will use to calculate the production probability of e^--e^+ pairs in a strong magnetic field. Our starting point is a second quantized Dirac field $\Psi(\mathbf{r},t)$ in the presence of a classical four-potential $A^\mu(\mathbf{r},t)=[V(\mathbf{r},t),\mathbf{A}(\mathbf{r},t)]$.

The dynamics of the Dirac field $\Psi(\mathbf{r},t)$ is described by the second quantized Hamiltonian

$$H(t)=\int d\mathbf{r}\Psi^\dagger(\mathbf{r},t)\mathcal{H}(\mathbf{r},-i\boldsymbol{\nabla},t)\Psi(\mathbf{r},t) \tag{1}$$

with

$$\mathcal{H}(\mathbf{r},-i\boldsymbol{\nabla},t)=\boldsymbol{\alpha}\cdot[-i\boldsymbol{\nabla}+e\mathbf{A}(\mathbf{r},t)]+\beta m-eV(\mathbf{r},t), \tag{2}$$

where $-e(e>0)$ is the charge of the electron.

Concerning the magnetic field, we note that the physical system we are studying has two scales of length and time: one related to the elementary particles (Compton wavelength of the electron $\lambdabar=\hbar/mc$ and $\lambdabar/c$) and the other related to the macroscopic source of the magnetic field itself (typical dimension and typical evolution time of a neutron star). The order of magnitude of the macroscopic scale is much larger than that of the microscopic one, and this allows us to consider the magnetic field as uniform in space and slowly varying in time. In this way we can calculate the probability production by using the adiabatic perturbation theory up to the first order [18]. The time variation of $\mathbf{B}(t)$ implies that there are varying currents that give rise to this effect. The calculation we want to present here is intended to apply to the regions where conduction currents are not present; in these regions there are, in general, displacement currents due to the time variation of the induced electric field [see Eq. (5), below]. In turn these currents will yield a correction to $\mathbf{B}(t)$ proportional to $\ddot{\mathbf{B}}(t)$ and not uniform in space. The calculations are carried out at the first order in $\dot{\mathbf{B}}(t)$, so, consistently, in the unperturbed Hamiltonian we include neither the field $\mathbf{E}(t)$, nor any contribution proportional to $\ddot{\mathbf{B}}(t)$.

A particular time evolution for the magnetic field will be considered:

$$\mathbf{B}(t)=\begin{pmatrix} \mathrm{B}_x(t) \\ \mathrm{B}_y(t) \\ \mathrm{B}_z(t) \end{pmatrix}=\mathrm{B}(t)\begin{pmatrix} 0 \\ \cos\vartheta(t) \\ \sin\vartheta(t) \end{pmatrix}, \tag{3}$$

i.e., the field changes both in strength and in direction but it always remains in the y-z plane. We assume $t=0$ as our initial time and $\vartheta(0)=0$ and $\mathrm{B}(0)=\mathrm{B}_0$ as the initial conditions for the magnetic field. The approximation of a strong magnetic field is always valid if we also assume that $\mathrm{B}(t)\geqslant \mathrm{B}_0$ for $t>0$ and that $\mathrm{B}_0 \gg \mathrm{B}_{cr}$.

A convenient choice of the four-potential $\mathrm{A}^\mu(\mathbf{r},t)$ is $V(\mathbf{r},t)=0$ and

$$\mathbf{A}(\mathbf{r},t)=-\frac{1}{2}[\mathbf{r}\times\mathbf{B}(t)]. \tag{4}$$

This is the so-called symmetric gauge and with this choice of the potentials the electric field is given by ($\hbar=c=1$ units are used throughout)

$$\mathbf{E}(\mathbf{r},t)=-\frac{\partial\mathbf{A}(\mathbf{r},t)}{\partial t}=\frac{1}{2}[\mathbf{r}\times\dot{\mathbf{B}}(t)]. \tag{5}$$

We shall see that the electric field plays a fundamental role in the interpretation of our results. We remark, however, that the validity of the adiabatic treatment is ensured in the limited spacetime region where $|\mathbf{E}|<|\mathbf{B}|$ [18].

Finally, in this gauge the one-particle Hamiltonian (2) becomes

$$\mathcal{H}(\mathbf{r},-i\boldsymbol{\nabla},t)=\boldsymbol{\alpha}\cdot[-i\boldsymbol{\nabla}+e\mathbf{A}(\mathbf{r},t)]+\beta m \tag{6}$$

with $\mathbf{A}(\mathbf{r},t)$ given by Eq. (4), while the second quantized Hamiltonian is

$$H(t)=\int d\mathbf{r}\Psi^\dagger(\mathbf{r},t)\mathcal{H}(\mathbf{r},-i\boldsymbol{\nabla},t)\Psi(\mathbf{r},t) \tag{7}$$

with $\mathcal{H}(\mathbf{r},-i\boldsymbol{\nabla},t)$ given by Eq. (6).

In order to apply the adiabatic perturbation theory we have to determine the instantaneous eigenvalues and eigenstates of the Hamiltonian (7) [18]. It is useful to this end to recall some well known results about the motion of a relativistic electron in a constant and uniform magnetic field.

A. A relativistic electron in a constant and uniform magnetic field

In this paragraph we want to collect some results about the motion of a relativistic electron in a constant and uniform magnetic field [19,20]. We will work in the symmetric gauge and, for later convenience, we will put the magnetic field on the y-z plane:

$$\mathbf{B}=\begin{pmatrix} 0 \\ \mathrm{B}_y \\ \mathrm{B}_z \end{pmatrix}, \quad \mathbf{A}(\mathbf{r})=-\frac{1}{2}[\mathbf{r}\times\mathbf{B}]. \tag{8}$$

In classical mechanics the electron performs a motion along a helix whose axis is parallel to $\mathbf{B}$. We will call $R_\perp^2$ the square of the distance of the axis of the helix from the origin. Since we will need it in the following, we define the quantity R_{xy}^2 which is $R_\perp^2$ in the case in which the magnetic field lies along the z axis.

To study the analogous problem in relativistic quantum mechanics we need to solve the eigenvalue equation

$$\mathcal{H}\psi=[\boldsymbol{\alpha}\cdot[-i\boldsymbol{\nabla}+e\mathbf{A}(\mathbf{r})]+\beta m]\psi=\varepsilon\psi. \tag{9}$$

If we define the rotation operator along the x axis

$$\mathcal{R}_x(\vartheta)=e^{-i\vartheta\mathcal{J}_x}, \quad \tan\vartheta=\frac{\mathrm{B}_y}{\mathrm{B}_z}, \tag{10}$$

where $\mathcal{J}_x=\mathcal{L}_x+\mathcal{S}_x$ is the x component of the one-particle total angular momentum operator, then

$$\mathcal{R}_x(\vartheta)\mathcal{H}\mathcal{R}_x^\dagger(\vartheta)=\mathcal{H}'=\boldsymbol{\alpha}\cdot[-i\boldsymbol{\nabla}+e\mathbf{A}'(\mathbf{r})]+\beta m, \tag{11}$$

where $\mathbf{A}'(\mathbf{r})$ is the vector potential in the symmetric gauge corresponding to a magnetic field $\mathbf{B}'$ directed along the z axis and with strength $\mathrm{B}=\sqrt{\mathrm{B}_y^2+\mathrm{B}_z^2}$. Now, we solve the equation

$$\mathcal{H}'\psi'=\varepsilon\psi'. \tag{12}$$

It can easily be shown [20] that the eigenvectors ψ' of $\mathcal{H}'$ can be indicated as

$$\psi'_{\pm,j}(\mathbf{r}), \quad j=\{n_d,k,\sigma,n_g\} \tag{13}$$

and the quantum numbers correspond to the fact that these functions are the common basis of the complete set of commuting observables built up by the Hamiltonian $\mathcal{H}'$, the linear momentum along z $\mathcal{P}_z$, the total angular momentum along z $\mathcal{J}_z$, and R_{xy}^2:

$$\mathcal{H}'\psi'_{\pm,j}(\mathbf{r})=\pm w_j\psi'_{\pm,j}(\mathbf{r}), \tag{14a}$$

$$\mathcal{P}_z\psi'_{\pm,j}(\mathbf{r})=k\psi'_{\pm,j}(\mathbf{r}), \tag{14b}$$

$$\mathcal{J}_z\psi'_{\pm,j}(\mathbf{r})=\left(n_d-n_g+\frac{\sigma}{2}\right)\psi'_{\pm,j}(\mathbf{r}), \tag{14c}$$

$$R_{xy}^2\psi'_{\pm,j}(\mathbf{r})=\frac{2n_g+1}{e\mathrm{B}}\psi'_{\pm,j}(\mathbf{r}), \tag{14d}$$

where

$$w_j=\sqrt{m^2+k^2+e\mathrm{B}(2n_d+1+\sigma)} \tag{15}$$

are the Landau levels for a particle with charge $-e$ and where the quantum numbers can assume the following values:

$$n_d = 0,1, \ldots \tag{16}$$

$$k = \pm\frac{2n\pi}{Z}, \quad n = 0,1 \ldots, \tag{17}$$

$$\sigma = \pm 1, \tag{18}$$

$$n_g = 0,1, \ldots . \tag{19}$$

It is worth noting that the energies w_j never depend on the quantum number n_g and that the energies w_j with $n_d = 0$ and $\sigma = -1$ do not depend on B. We will call the corresponding states with $n_d = 0$ and $\sigma = -1$ transverse ground states.

The longitudinal momenta in Eq. (17) have been discretized by limiting our region of integration by means of the two planes $z = \pm Z/2$ and by applying the periodic boundary conditions to the basis $\psi'_{\pm,j}(\mathbf{r})$ [see Eqs. (20a),(20b) and (22) below]. We have to do this because the adiabatic perturbation theory is easier to use for discrete energy spectra [18]. However, at the end of the calculation we will perform the limit $Z \to \infty$.

The bispinors $\psi'_{\pm,j}(\mathbf{r})$ have the following expressions:

$$\psi'_{+,j}(\mathbf{r}) = \sqrt{\frac{w_j + m}{2w_j}} \begin{pmatrix} \varphi'_j(\mathbf{r}) \\ \dfrac{\mathcal{V}'}{w_j + m}\varphi'_j(\mathbf{r}) \end{pmatrix}, \tag{20a}$$

$$\psi'_{-,j}(\mathbf{r}) = \sqrt{\frac{w_j + m}{2w_j}} \begin{pmatrix} -\dfrac{\mathcal{V}'}{w_j + m}\varphi'_j(\mathbf{r}) \\ \varphi'_j(\mathbf{r}) \end{pmatrix}, \tag{20b}$$

where

$$\mathcal{V}' = \boldsymbol{\sigma}\cdot[-i\boldsymbol{\nabla} + e\mathbf{A}'(\mathbf{r})]. \tag{21}$$

In cylindrical coordinate (ρ,ϕ,z) the spinors $\varphi'_j(\mathbf{r})$ are given by

$$\varphi'_j(\mathbf{r}) = \frac{e^{ikz}}{\sqrt{Z}} \frac{e^{i(n_d - n_g)\phi}}{\sqrt{2\pi}} f'_\sigma R'_{n_d,n_g}(\rho), \tag{22}$$

where the radial functions $R'_{n_d,n_g}(\rho)$ are invariant under the exchange of n_d and n_g and

$$f'_{+1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad f'_{-1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{23}$$

It is now easy to see that the functions

$$\psi_{\pm,j}(\mathbf{r}) = \mathcal{R}_x^\dagger(\vartheta)\psi'_{\pm,j}(\mathbf{r}) \tag{24}$$

solve our initial problem; in fact, they solve the equations

$$\mathcal{H}\psi_{\pm,j}(\mathbf{r}) = \pm w_j\psi_{\pm,j}(\mathbf{r}), \tag{25a}$$

$$\mathcal{P}_\parallel\psi_{\pm,j}(\mathbf{r}) = k\psi_{\pm,j}(\mathbf{r}), \tag{25b}$$

$$\mathcal{J}_\parallel\psi_{\pm,j}(\mathbf{r}) = \left(n_d - n_g + \frac{\sigma}{2}\right)\psi_{\pm,j}(\mathbf{r}), \tag{25c}$$

$$R_\perp^2\psi_{\pm,j}(\mathbf{r}) = \frac{2n_g + 1}{e\mathrm{B}}\psi_{\pm,j}(\mathbf{r}), \tag{25d}$$

with

$$\mathcal{P}_\parallel = \mathcal{R}_x^\dagger(\vartheta)\mathcal{P}_z\mathcal{R}_x(\vartheta), \tag{26}$$

$$\mathcal{J}_\parallel = \mathcal{R}_x^\dagger(\vartheta)\mathcal{J}_z\mathcal{R}_x(\vartheta), \tag{27}$$

$$R_\perp^2 = \mathcal{R}_x^\dagger(\vartheta)R_{xy}^2\mathcal{R}_x(\vartheta). \tag{28}$$

In these equations $\mathcal{P}_\parallel$ and $\mathcal{J}_\parallel$ are the linear momentum and the total angular momentum along the direction of the magnetic field while $R_\perp^2$ has been defined at the beginning of this paragraph.

B. Second quantization of a Dirac field in a constant and uniform magnetic field

The next step in determining the instantaneous eigenvalues and eigenvectors of the Hamiltonian (7) is to second quantize the Dirac field $\Psi(\mathbf{r},t)$ in the presence of the static magnetic field given by Eq. (8).

In order to do this we have to expand the operator $\Psi(\mathbf{r},t)$ in the basis $\psi_{\pm,j}(\mathbf{r})$ obtained in the previous paragraph. Actually, to obtain the right interpretation of the Fock states of the positrons we will consider the following orthonormal basis:

$$u_j(\mathbf{r}) = \psi_{+,n_d,k,\sigma,n_g}(\mathbf{r}),$$

$$v_j(\mathbf{r}) = \sigma\psi_{-,n_g,-k,-\sigma,n_d}(\mathbf{r}). \tag{29}$$

This is an orthonormal basis because we only exchanged the bispinors $\psi_{\pm,j}(\mathbf{r})$, which are an orthonormal basis, among them. As it can easily be obtained from Eqs. (25a)–(25d), the one particle states $v_j(\mathbf{r})$ satisfy the eigenvalue equations

$$\mathcal{H}v_j(\mathbf{r}) = -\tilde{w}_j v_j(\mathbf{r}), \tag{30}$$

$$\mathcal{P}_\parallel v_j(\mathbf{r}) = -k v_j(\mathbf{r}), \tag{31}$$

$$\mathcal{J}_\parallel v_j(\mathbf{r}) = -\left(n_d - n_g + \frac{\sigma}{2}\right)v_j(\mathbf{r}), \tag{32}$$

$$R_\perp^2 v_j(\mathbf{r}) = \frac{2n_d + 1}{e\mathrm{B}}v_j(\mathbf{r}), \tag{33}$$

where

$$\tilde{w}_j = \sqrt{m^2 + k^2 + e\mathrm{B}(2n_g + 1 - \sigma)} \tag{34}$$

are the Landau levels for a particle with charge e.

By expanding the field operator $\Psi(\mathbf{r},t)$ in the $\{u_j(\mathbf{r}),v_j(\mathbf{r})\}$ basis

$$\Psi(\mathbf{r},t)=\sum_j c_j(t)u_j(\mathbf{r})+d_j^\dagger(t)v_j(\mathbf{r}), \tag{35}$$

the second quantized relevant operators become

$$H=\int d\mathbf{r}\Psi^\dagger(\mathbf{r},t)\mathcal{H}\Psi(\mathbf{r},t)$$

$$=\sum_j (w_jN_j+\tilde{w}_j\tilde{N}_j)+E_0, \tag{36a}$$

$$P_\parallel=\int d\mathbf{r}\Psi^\dagger(\mathbf{r},t)\mathcal{P}_\parallel\Psi(\mathbf{r},t)$$

$$=\sum_j k(N_j+\tilde{N}_j), \tag{36b}$$

$$J_\parallel=\int d\mathbf{r}\Psi^\dagger(\mathbf{r},t)\mathcal{J}_\parallel\Psi(\mathbf{r},t)$$

$$=\sum_j \left(n_d-n_g+\frac{\sigma}{2}\right)(N_j+\tilde{N}_j), \tag{36c}$$

where $N_j=c_j^\dagger(t)c_j(t)$ and $\tilde{N}_j=d_j^\dagger(t)d_j(t)$ do not depend on time and where the zero-point energy E_0 will be set to zero. From Eq. (34) it is clear that the energies $\tilde{w}_j$ do not depend on n_d and the transverse ground states of the positron are that with $n_g=0$ and $\sigma=+1$.

We conclude this paragraph by noting that the vectors $v_j(\mathbf{r})$ of the basis (29) can be obtained in a more formal way. First, we solve the eigenvalue equation

$$[\boldsymbol{\alpha}\cdot[-i\boldsymbol{\nabla}-e\mathbf{A}(\mathbf{r})]+\beta m]\tilde{\psi}_{\pm,j}(\mathbf{r})=\pm\tilde{w}_j\tilde{\psi}_{\pm,j}(\mathbf{r}) \tag{37}$$

for a particle with charge e in the magnetic field (8). Then the $v_j(\mathbf{r})$ functions are obtained by applying the charge conjugation operator to the positive energy solutions of this equation:

$$v_j(\mathbf{r})=i\gamma^2\tilde{\psi}^*_{+,j}(\mathbf{r})=i\beta\alpha_2\tilde{\psi}^*_{+,j}(\mathbf{r}). \tag{38}$$

III. TRANSITION MATRIX ELEMENTS AND SELECTION RULES

At this point we have all the theoretical tools needed to calculate the pair creation probability in the framework of the adiabatic perturbation theory. In fact, if the magnetic field is given by Eq. (3) then the instantaneous eigenvectors of the Hamiltonian (7) are the states

$$|\{n_j(t)\};\{\tilde{n}_j(t)\}\rangle \tag{39}$$

with the corresponding instantaneous eigenvalues

$$E(t)=\sum_j w_j(t)n_j(t)+\tilde{w}_j(t)\tilde{n}_j(t). \tag{40}$$

The first order amplitude for creating an e^--e^+ pair in the state $|n_j(t)=1;\tilde{n}_{j'}(t)=1\rangle\equiv|jj'(t)\rangle$ from the vacuum state $|0(t)\rangle$ at a time t is given by [18]

$$\gamma_{jj'}(t)=\int_0^t d\tau\frac{\langle jj'(\tau)|\dot{H}(\tau)|0(\tau)\rangle}{w_j(\tau)+\tilde{w}_{j'}(\tau)}\times\exp\left\{i\int_0^\tau d\tau'[w_j(\tau')+\tilde{w}_{j'}(\tau')]\right\}. \tag{41}$$

In this formula it is tacitly assumed that the depletion of the vacuum due to the transition to the $|jj'(t)\rangle$ states is negligible [18].

The time derivative of the Hamiltonian can be written in the following useful form:

$$\begin{aligned}\dot{H}(t)&=\boldsymbol{\nabla}_\mathbf{B}H(t)\cdot\dot{\mathbf{B}}(t)\\&=\int d\mathbf{r}\Psi^\dagger(\mathbf{r},t)\boldsymbol{\nabla}_\mathbf{B}[\mathcal{H}(\mathbf{r},-i\boldsymbol{\nabla},t)]\Psi(\mathbf{r},t)\cdot\dot{\mathbf{B}}(t)\\&=-\frac{e}{2}\int d\mathbf{r}\Psi^\dagger(\mathbf{r},t)(\mathbf{r}\times\boldsymbol{\alpha})\Psi(\mathbf{r},t)\cdot\dot{\mathbf{B}}(t)\\&=-\frac{e}{2}\int d\mathbf{r}\Psi'^\dagger(\mathbf{r},t)\mathcal{R}_x[\vartheta(t)](\mathbf{r}\times\boldsymbol{\alpha})\\&\quad\times\mathcal{R}_x^\dagger[\vartheta(t)]\Psi'(\mathbf{r},t)\cdot\dot{\mathbf{B}}(t)\\&=-\frac{e}{2}\int d\mathbf{r}\;\;\Psi'^\dagger(\mathbf{r},t)(\mathbf{r}\times\boldsymbol{\alpha})\Psi'(\mathbf{r},t)\cdot\dot{\mathbf{B}}'(t),\end{aligned} \tag{42}$$

where

$$\dot{\mathbf{B}}'(t)=\begin{pmatrix}0\\ \cos\vartheta(t)\dot{\mathrm{B}}_y(t)-\sin\vartheta(t)\dot{\mathrm{B}}_z(t)\\ \sin\vartheta(t)\dot{\mathrm{B}}_y(t)+\cos\vartheta(t)\dot{\mathrm{B}}_z(t)\end{pmatrix}=\begin{pmatrix}0\\ \mathrm{B}(t)\dot{\vartheta}(t)\\ \dot{\mathrm{B}}(t)\end{pmatrix}, \tag{43}$$

with

$$\tan\vartheta(t)=\frac{\mathrm{B}_y(t)}{\mathrm{B}_z(t)} \tag{44}$$

and where $\Psi'(\mathbf{r},t)$ is the field operator in the case in which the magnetic field is along the z axis and its strength is $\mathrm{B}(t)=\sqrt{\mathrm{B}_y^2(t)+\mathrm{B}_z^2(t)}$.

The form of the vector $\dot{\mathbf{B}}'(t)$ shows that the effects of the change of the direction and of the change of the strength of the magnetic field have been completely disentangled. In fact, we can connect the time variation of the magnetic field to some selection rules concerning the states in which the e^--e^+ pair can be created. To do this we define the operator

$$\mathbf{T}'(t)=-\frac{e}{2}\int d\mathbf{r}\Psi'^\dagger(\mathbf{r},t)(\mathbf{r}\times\boldsymbol{\alpha})\Psi'(\mathbf{r},t). \tag{45}$$

TABLE I. Selection rules.

	Transition operator	Selection rule
if $\dot{\vartheta}(t)=0$	$\mathrm{T}'_z(t)$	$j_\parallel(e^-)+j_\parallel(e^+)=0$
if $\dot{\mathrm{B}}(t)=0$	$\mathrm{T}'_y(t)$	$j_\parallel(e^-)+j_\parallel(e^+)=\pm 1$

Since $\mathbf{T}'(t)$ is a vector operator and the states $|0(t)\rangle$ and $|jj'(t)\rangle$ are instantaneous eigenstates of the total angular momentum $J_\parallel(t)$ given in Eq. (36) then see Table I [also see Eq. (43) and the last line of Eq. (42)], where

$$j_\parallel = n_d - n_g + \frac{\sigma}{2}. \tag{46}$$

The first of these selection rules has an interesting classical counterpart. In fact, following the interpretation of the pair creation in the Dirac hole theory, this selection rule means that an electron in a slowly varying magnetic field with constant direction going from a negative energy level to a positive one conserves its total angular momentum. Analogously, in classical mechanics the angular momentum of the electron is, in the same physical situation, an adiabatic invariant [22].

Another important selection rule can be obtained. In fact, it can easily be shown that

$$\begin{aligned}\langle jj'(t)|\nabla_{\mathbf{B}}H(t)|0(t)\rangle\cdot\dot{\mathbf{B}}(t)\\ &=\langle jj'(t)|\mathbf{T}'(t)|0(t)\rangle\cdot\dot{\mathbf{B}}'(t)\\ &=-\frac{e}{2}\int d\mathbf{r}u_j'^{\dagger}(\mathbf{r},t)(\mathbf{r}\times\boldsymbol{\alpha})v'_{j'}(\mathbf{r},t)\cdot\dot{\mathbf{B}}'(t)\\ &=\mathbf{T}'_{jj'}(t)\cdot\dot{\mathbf{B}}'(t),\end{aligned} \tag{47}$$

where we have defined the one-particle transition matrix elements

$$\mathbf{T}'_{jj'}(t)=-\frac{e}{2}\int d\mathbf{r}u_j'^{\dagger}(\mathbf{r},t)(\mathbf{r}\times\boldsymbol{\alpha})v'_{j'}(\mathbf{r},t). \tag{48}$$

If the magnetic field has a constant direction $[\dot{\vartheta}(t)=0]$ only the matrix elements $\mathrm{T}'_{jj'z}(t)$ play a role (see Table I). It can also be seen that $(\mathbf{r}\times\boldsymbol{\alpha})_z$ anticommutes with the one-particle spin operator $\mathcal{S}_z$:

$$[(\mathbf{r}\times\boldsymbol{\alpha})_z,\mathcal{S}_z]_+=0 \tag{49}$$

and then if $u'_j(\mathbf{r},t)$ and $v'_{j'}(\mathbf{r},t)$ are eigenstates of $\mathcal{S}_z$ the selection rule

$$\sigma(u'_j)+\sigma(v'_{j'})=0 \tag{50}$$

holds. Now, from Eqs. (20a),(20b) and (29) it can be shown that the transverse ground states have the following spinor structure

$$u'_g(\mathbf{r},t)\propto\begin{pmatrix}0\\1\\0\\-\dfrac{k}{w_g(t)+m}\end{pmatrix},$$

$$v'_g(\mathbf{r},t)\propto\begin{pmatrix}0\\-\dfrac{k}{\tilde{w}_g(t)+m}\\0\\1\end{pmatrix}, \tag{51}$$

with

$$w_g(t)=w_{n_d=0,k,\sigma=-1,n_g}(t)=\sqrt{m^2+k^2}, \tag{52a}$$

$$\tilde{w}_g(t)=\tilde{w}_{n_d,k,\sigma=+1,n_g=0}(t)=\sqrt{m^2+k^2}, \tag{52b}$$

and then they are all eigenstates of $\mathcal{S}_z$ with the same eigenvalue $-\frac{1}{2}$ and for every t. This implies that if $\mathbf{B}(t)$ changes only in strength the creation of a pair in which the electron and the positron are both in a transverse ground state is forbidden. In other words, only the rotation of $\mathbf{B}(t)$ allows the creation of a pair with both the electron and the positron in a transverse ground state. In the next section we will see why this selection rule is so important.

IV. ROTATING MAGNETIC FIELD

In this section we want to investigate in detail the effects of the rotation of $\mathbf{B}(t)$; then we will consider the case in which

$$\mathbf{B}(t)=\mathrm{B}_0\begin{pmatrix}0\\ \sin\omega t\\ \cos\omega t\end{pmatrix} \tag{53}$$

and then [see Eq. (43)]

$$\dot{\mathbf{B}}'(t)=\begin{pmatrix}0\\ \omega\mathrm{B}_0\\ 0\end{pmatrix}. \tag{54}$$

We have checked that if $P_{gg'}(t)=P(|0(t)\rangle\to|gg'(t)\rangle)$ is the probability transition to a state in which the electron and the positron are both in a transverse ground state and $P_{jj'}(t)$ is the probability transition to another state then

$$\frac{P_{jj'}(t)}{P_{gg'}(t)}\lesssim\left(\frac{m^2}{e\mathrm{B}_0}\right)^{3/2}\ll 1. \tag{55}$$

In this case we are allowed in first approximation to neglect all the transitions to states in which at least the electron or the positron is not in a transverse ground state. However, in the Appendix we will study the general features of the amplitudes of these remaining transitions.

There are only two transition amplitudes different from zero which contribute to the creation of a pair in which the electron and the positron are both in a transverse ground state. We recall that the transverse ground states are that with $n_d=0$ and $\sigma=-1$ for the electron and that with $n_g=0$ and $\sigma=+1$ for the positron. The only final pair states which are allowed are the states

$$|n_d=0,k,\sigma=-1,n_g;\quad n'_d=n_g+1,k'=-k,\sigma'=+1,n'_g=0\rangle, \tag{56a}$$

$$|n_d=0,k,\sigma=-1,n_g;\quad n'_d=n_g-1,k'=-k,\sigma'=+1,n'_g=0\rangle, \tag{56b}$$

where for simplicity we have omitted their time dependence. The transition amplitudes corresponding to these states are respectively

$$\gamma^{(+)}(t)=\sqrt{\frac{e\mathrm{B}_0(n_g+1)}{32}}\frac{m\omega}{\varepsilon_k^3}e^{i\varepsilon_k t}\sin\varepsilon_k t, \tag{57a}$$

$$\gamma^{(-)}(t)=\sqrt{\frac{e\mathrm{B}_0 n_g}{32}}\frac{m\omega}{\varepsilon_k^3}e^{i\varepsilon_k t}\sin\varepsilon_k t, \tag{57b}$$

where we have set [see Eqs. (52a),(52b)]

$$\varepsilon_k=w_g(t)=\tilde{w}_g(t)=\sqrt{m^2+k^2}. \tag{58}$$

Actually, we want to consider the probability that the electron and the positron are created in wave packets which are linear superpositions of transverse ground states. We will consider the following pair state:

$$|e^-e^+\rangle=|e^-\rangle|e^+\rangle \tag{59}$$

with

$$|e^-\rangle=\sum_{n_g} b_{n_g}^{(-)}|0,k,-1,n_g\rangle, \tag{60a}$$

$$|e^+\rangle=\sum_{n'_d} b_{n'_d}^{(+)}|n'_d,-k,+1,0\rangle, \tag{60b}$$

where

$$b_{n_g}^{(-)}=\frac{1}{\sqrt{n_g!}}e^{-(1/2)|\alpha_-|^2}\alpha_-^{n_g}, \tag{61a}$$

$$b_{n'_d}^{(+)}=\frac{1}{\sqrt{n'_d!}}e^{-(1/2)|\alpha_+|^2}\alpha_+^{n'_d}. \tag{61b}$$

As is evident from these equations we chose a double plane wave for the longitudinal motion and a double coherent state (in n_g for the electron and in n'_d for the positron) for the transverse motion. The coherent states for the transverse motion have a good spatial localization in the plane orthogonal to the magnetic field and this will give us the possibility to understand the role of the electric field which is not uniform in the process of pair creation.

It can easily be shown from Eqs. (57a),(57b) that the transition amplitude to the state $|e^-e^+\rangle$ is

$$\begin{aligned}\gamma_{e^-e^+}(t)&=\sum_n b_n^{(-)*}b_{n+1}^{(+)*}\sqrt{\frac{e\mathrm{B}_0(n+1)}{32}}\frac{m\omega}{\varepsilon_k^3}\\&\quad\times e^{i\varepsilon_k t}\sin\varepsilon_k t+\sum_n b_n^{(-)*}b_{n-1}^{(+)*}\\&\quad\times\sqrt{\frac{e\mathrm{B}_0 n}{32}}\frac{m\omega}{\varepsilon_k^3}e^{i\varepsilon_k t}\sin\varepsilon_k t\\&=\mathcal{L}(k,t)\mathcal{T}(\alpha_\pm)\end{aligned} \tag{62}$$

where

$$\mathcal{L}(k,t)=\sqrt{\frac{e\mathrm{B}_0}{32}}\frac{m\omega}{\varepsilon_k^3}e^{i\varepsilon_k t}\sin\varepsilon_k t \tag{63}$$

$$\begin{aligned}\mathcal{T}(\alpha_\pm)=\sum_{n=0}^{\infty}&[b_n^{(-)*}b_{n+1}^{(+)*}\sqrt{n+1}\\&+b_n^{(-)*}b_{n-1}^{(+)*}\sqrt{n}].\end{aligned} \tag{64}$$

In this way the longitudinal and the transverse part of the transition amplitude have been disentangled and only the longitudinal part depends on *t*. Obviously, the corresponding probability transition which is the square modulus of the amplitude (62) can also be divided into a longitudinal part and a transverse part.

We will first calculate the longitudinal part of this probability. The probability that the electron (positron) will be created with a longitudinal momentum between $k(-k)$ and $k+dk(-k-dk)$ is

$$\begin{aligned}dP_L(k,t)&=|\mathcal{L}(k,t)|^2\frac{Z}{2\pi}dk\\&=\frac{e\mathrm{B}_0}{32}(m\omega)^2\frac{\sin^2\varepsilon(k)t}{\varepsilon^6(k)}\frac{Z}{2\pi}dk,\end{aligned} \tag{65}$$

where $\varepsilon_k\to\varepsilon(k)$ for continuous *k* and the corresponding total probability per unit time is

$$\frac{dP_L(t)}{dt}=\frac{e\mathrm{B}_0 Z}{64\pi}(m\omega)^2\int_{-\infty}^{\infty}dk\frac{\sin 2\varepsilon(k)t}{\varepsilon^5(k)}. \tag{66}$$

We can give an asymptotic estimate of this integral by assuming to be interested only in times *t* such that $mt\gg1$. By considering the astrophysical system sketched in the Introduction to which we imagine applying our theory, this assumption is very realistic. The asymptotic estimate can be performed by using the method proposed in [23] and the result is

$$\frac{dP_L(t)}{dt}\sim\frac{e\mathrm{B}_0 Z}{64\pi}\left(\frac{\omega}{m}\right)^2\sqrt{\frac{\pi}{mt}}\sin\left(2mt+\frac{\pi}{4}\right). \tag{67}$$

Now we will calculate the transverse part of Eq. (62) or, equivalently, its complex conjugate

$$\mathcal{T}^*(\alpha_\pm)=\sum_{n=0}^{\infty}[b_n^{(-)}b_{n+1}^{(+)}\sqrt{n+1}+b_n^{(-)}b_{n-1}^{(+)}\sqrt{n}]. \tag{68}$$

From Eqs. (61a),(61b) we have

$$\begin{aligned}\mathcal{T}^*(\alpha_\pm)&=e^{-(|\alpha_-|^2+|\alpha_+|^2)/2}\\&\quad\times\sum_{n=0}^{\infty}\left[\alpha_-\frac{(\alpha_-\alpha_+)^n}{n!}+\alpha_+\frac{(\alpha_-\alpha_+)^n}{n!}\right]\\&=(\alpha_-+\alpha_+)e^{-(|\alpha_-|^2+|\alpha_+|^2)/2}e^{\alpha_-\alpha_+}.\end{aligned} \tag{69}$$

Then the probability to create a pair with the electron in a coherent state between $|\alpha_-\rangle$ and $|\alpha_-+d\alpha_-\rangle$ and the positron in a coherent state between $|\alpha_+\rangle$ and $|\alpha_++d\alpha_+\rangle$ is

$$\begin{aligned}dP(\alpha_\pm)&=|\alpha_-+\alpha_+|^2e^{-(|\alpha_-|^2+|\alpha_+|^2)}e^{2\Re(\alpha_-\alpha_+)}\\&\quad\times\frac{d\alpha_-}{\pi}\frac{d\alpha_+}{\pi}.\end{aligned} \tag{70}$$

It can be shown [19] that the phase of α_- (α_+) is connected to the azimuth of the mean position of the electron (positron) in the plane orthogonal to $\mathbf{B}(t)$ while the modulus $|\alpha_-|$ $(|\alpha_+|)$ is connected to the mean distance of the electron (positron) in the same plane from the origin. Then, since we are not interested in the exact position of the pair we set

$$\alpha_\pm=|\alpha_\pm|e^{i\phi_\pm} \tag{71}$$

and integrate on the angles ϕ_- and ϕ_+:

$$\begin{aligned}dP(|\alpha_\pm|)&=4|\alpha_-||\alpha_+|(|\alpha_-|^2+|\alpha_+|^2)e^{-(|\alpha_-|^2+|\alpha_+|^2)}\\&\quad\times I_0(2|\alpha_-||\alpha_+|)d|\alpha_-|d|\alpha_+|,\end{aligned} \tag{72}$$

where $I_0(x)$ is the modified Bessel function of zero order [21]. In order to obtain a more transparent formula we define the variables α and ψ by means of the equations

$$\begin{aligned}|\alpha_-|&=\alpha\cos\psi,\\|\alpha_+|&=\alpha\sin\psi,\end{aligned} \tag{73}$$

and we integrate on the phase ψ. Since $|\alpha_-|$ and $|\alpha_+|$ vary from 0 to ∞, α and ψ vary from 0 to ∞ and from 0 to $\pi/2$, respectively. After performing the integral on ψ we obtain the differential probability

$$dP(\alpha)=\alpha^3(1-e^{-2\alpha^2})d\alpha. \tag{74}$$

Now, there is an interesting relation between α and the mean value of the operator $R_\perp^2$ in the states $|\alpha_-\rangle$ and $|\alpha_+\rangle$. In fact,

$$R_\perp^2(e^\pm)=\langle\alpha_\pm|R_\perp^2|\alpha_\pm\rangle=\frac{2|\alpha_\pm|^2+1}{e\mathrm{B}_0}\simeq\frac{2}{e\mathrm{B}_0}|\alpha_\pm|^2, \tag{75}$$

where we have assumed that $R_\perp^2(e^\pm)$ are *macroscopic* quantities and then

$$\alpha=\sqrt{|\alpha_-|^2+|\alpha_+|^2}\simeq\sqrt{\frac{e\mathrm{B}_0}{2}[R_\perp^2(e^-)+R_\perp^2(e^+)]}. \tag{76}$$

If we define the "mean" quantity

$$R_{\perp m}=\sqrt{\frac{R_\perp^2(e^-)+R_\perp^2(e^+)}{2}} \tag{77}$$

we can write the transverse probability per unit area as

$$\frac{dP_T}{dA_\perp}\simeq\frac{(e\mathrm{B}_0)^2}{2\pi}R_{\perp m}^2, \tag{78}$$

where $dA_\perp=\pi dR_{\perp m}^2$.

Since Eq. (78) does not depend on t, if we put together Eqs. (67) and (78) and divide by Z we obtain the probability per unit time and unit volume $dV=ZdA_\perp$

$$\frac{dP(t)}{dVdt}\sim\frac{m^4}{2}\left(\frac{e\mathrm{B}_0}{4\sqrt{\pi}m^2}\right)^3(R_{\perp m}\omega)^2\frac{\sin(2mt+\pi/4)}{\sqrt{mt}}, \tag{79}$$

where the $\sim$ reminds us that this is an asymptotic formula valid for $mt\gg1$. The fact that this probability grows with $R_{\perp m}$ can be understood in terms of the electric field $\mathbf{E}(\mathbf{r},t)$ [see Eq. (5)]. In fact, for a purely rotating magnetic field

$$\mathrm{E}_y^2(\mathbf{r},t)+\mathrm{E}_z^2(\mathbf{r},t)=\frac{\omega^2\mathrm{B}_0^2}{4}x^2. \tag{80}$$

In order to connect this quantity with $R_{\perp m}$ we observe that

$$x=x_\perp=R_\perp(t)\cos\phi(t), \tag{81}$$

where $R_\perp(t)$ and $\phi(t)$ are the polar coordinates in the plane orthogonal to $\mathbf{B}(t)$. If we define the function $\mathcal{E}_{yz}^2(R_\perp)$ as the average of $\mathrm{E}_y^2(\mathbf{r},t)+\mathrm{E}_z^2(\mathbf{r},t)$ on the angle $\phi(t)$ we obtain from Eqs. (80) and (81)

$$\mathcal{E}_{yz}^2(R_\perp)\equiv\langle\mathrm{E}_y^2+\mathrm{E}_z^2\rangle_\phi=\frac{\omega^2\mathrm{B}_0^2}{8}R_\perp^2. \tag{82}$$

With this formula the probability (79) can be written in c.g.s. units as

$$\frac{dP(t)}{dVdt}\sim\frac{c}{\lambda\!\!\!^{-4}16\pi^{3/2}}\frac{\mathrm{B}_0\mathcal{E}_{yz}^2(R_{\perp m})}{B_{cr}^3}\frac{\sin(2ct/\lambda\!\!\!^{-}+\pi/4)}{\sqrt{ct/\lambda\!\!\!^{-}}}, \tag{83}$$

where $\lambda\!\!\!^{-}=\hbar/mc$ is the Compton wavelength of the electron and where $B_{cr}=m^2c^3/e\hbar$. The fact that the probability depends only on $\mathrm{E}_y(\mathbf{r},t)$ and $\mathrm{E}_z(\mathbf{r},t)$ can be explained by rewriting the transition matrix elements (47) as

$$\langle jj'(t)|\nabla_{\mathbf{B}}H(t)|0(t)\rangle\cdot\dot{\mathbf{B}}(t)=-e\int d\mathbf{r}\mathbf{E}'(\mathbf{r},t)\cdot[u_j'^{\dagger}(\mathbf{r},t)\boldsymbol{\alpha}v_{j'}'(\mathbf{r},t)], \tag{84}$$

where

$$\mathbf{E}'(\mathbf{r},t)=\frac{1}{2}[\mathbf{r}\times\dot{\mathbf{B}}'(t)] \tag{85}$$

is the electric field seen from the frame which rotates around the x axis and whose z axis is instantaneously parallel to $\mathbf{B}(t)$. Now, the particular structure of the functions $u_j'(\mathbf{r},t)$ and $v_j'(\mathbf{r},t)$ for transverse ground states [see Eq. (51)] makes it so that only the α_z term contributes to the transition. This term contains only $\mathrm{E}_z'(\mathbf{r},t)$ which is a linear superposition of $\mathrm{E}_y(\mathbf{r},t)$ and $\mathrm{E}_z(\mathbf{r},t)$; in fact

$$\mathrm{E}_z'(\mathbf{r},t)=\cos\omega t\mathrm{E}_z(\mathbf{r},t)+\sin\omega t\mathrm{E}_y(\mathbf{r},t). \tag{86}$$

V. CONCLUSIONS

In this paper we have continued the study of the electron-positron production in a slowly varying strong magnetic field $\mathbf{B}(t)$. We have seen that the time variation of the field is connected to selection rules on the state in which the pair can be created. In particular, only if the direction of $\mathbf{B}(t)$ changes with time is it possible to create a pair with the electron and the positron both in a transverse ground state. Since the energies of these states do not depend on the magnetic field and since $\mathbf{B}(t)$ is very strong these energies are much lower than that of the other levels. This feature makes the probability of creating a pair in these states much larger than the other probabilities and we can conclude that the change of direction of the magnetic field is a pair creation mechanism much more efficient than the change of its strength.

Finally, by choosing a well spatially localized state in the plane orthogonal to $\mathbf{B}(t)$ as the pair state, we have shown how the pair production probability can be interpreted in terms of the nonuniform electric field present in consequence of the time variation of $\mathbf{B}(t)$. Our results indicate that the probability per unit volume and unit time depends on the square of the electric field.

As we recalled in the Introduction, the interest for the e^--e^+ production in these physical situations arises from the fact that these particles could be an intermediate step in the final generation of gamma rays. In view of this fact we intend to explore also the related process where the variation of a strong magnetic field gives rise to the direct production of photons.

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APPENDIX

In this appendix we study the general features of the transition amplitudes to those states in which the electron and/or the positron are not in a transverse ground state. First of all we write the corresponding transition matrix elements $\langle jj'(t)|\dot{H}(t)|0(t)\rangle$. In fact, since in the case of a purely rotating magnetic field these transition matrix elements and the energies $w_j(t)$ and $\tilde{w}_j(t)$ are actually time independent the corresponding amplitudes are given by [see Eq. (41)]

$$\gamma_{jj'}(t)=2\frac{\langle jj'(t)|\dot{H}(t)|0(t)\rangle}{(w_j+\tilde{w}_{j'})^2}e^{(i/2)(w_j+\tilde{w}_{j'})t}\times\sin\frac{1}{2}(w_j+\tilde{w}_{j'})t. \tag{A1}$$

The transition matrix elements different from zero are

$$\begin{aligned}\langle n_d,k,\sigma,n_g;n_g+1,-k,-\sigma,n_d|\dot{H}(t)|0\rangle&=\langle n_d,k,\sigma,n_g+1;n_g,-k,-\sigma,n_d|\dot{H}(t)|0\rangle\\&=\sigma\mathcal{N}_{jj'}\sqrt{\frac{e\mathrm{B}_0}{2}(n_g+1)}\left\{\frac{1}{E_{jj'}^2}+\sigma\left[\frac{1}{e\mathrm{B}_0}\left(1-\frac{k^2}{E_{jj'}^2}\right)+\frac{2n_d+1}{E_{jj'}^2}\right]\right\},\end{aligned} \tag{A2}$$

$$\begin{aligned}\langle n_d+1,k,\sigma,n_g;n_g,-k,-\sigma,n_d|\dot{H}(t)|0\rangle&=\langle n_d,k,\sigma,n_g;n_g,-k,-\sigma,n_d+1|\dot{H}(t)|0\rangle\\&=\sigma\mathcal{N}_{jj'}\sqrt{\frac{e\mathrm{B}_0}{2}(n_d+1)}\left\{\frac{3}{E_{jj'}^2}+\sigma\left[\frac{1}{e\mathrm{B}_0}\left(1-\frac{k^2}{E_{jj'}^2}\right)+\frac{2n_d+2}{E_{jj'}^2}\right]\right\},\end{aligned} \tag{A3}$$

$$\begin{aligned}\langle n_d,k,\sigma,n_g;n_g,k',-\sigma,n_d+1|\dot{H}(t)|0\rangle&=\langle n_d+1,k,\sigma,n_g;n_g,k',-\sigma,n_d|\dot{H}(t)|0\rangle\\&=\mathcal{N}_{jj'}\sqrt{\frac{e\mathrm{B}_0}{2}(n_d+1)}\frac{i\Delta k}{E_{jj'}^2}\mathcal{I}(K),\end{aligned} \tag{A4}$$

$$\langle n_d,k,\sigma,n_g;n_g,k',\sigma,n_d|\dot{H}(t)|0\rangle$$

$$=\mathcal{N}_{jj'}\left[\sigma\left(\frac{kk'}{E^2_{jj'}}-1\right)\mathcal{I}(K)-\frac{ik}{E^2_{jj'}}(2n_d+1)\delta_{k,-k'}\right], \tag{A5}$$

$$\langle n_d,k,-1,n_g;n_g+1,-k,-1,n_d-1|\dot{H}(t)|0\rangle$$

$$=\langle n_d,k,-1,n_g+1;n_g,-k,-1,n_d-1|\dot{H}(t)|0\rangle$$

$$=-\mathcal{N}_{jj'}\frac{2ik}{E^2_{jj'}}\sqrt{n_d(n_g+1)}, \tag{A6}$$

$$\langle n_d,k,+1,n_g;n_g+1,-k,+1,n_d+1|\dot{H}(t)|0\rangle$$

$$=\langle n_d,k,+1,n_g+1;n_g,-k,+1,n_d+1|\dot{H}(t)|0\rangle$$

$$=\mathcal{N}_{jj'}\frac{2ik}{E^2_{jj'}}\sqrt{(n_d+1)(n_g+1)}, \tag{A7}$$

$$\langle n_d,k,+1,n_g;n_g,k',+1,n_d+2|\dot{H}(t)|0\rangle$$

$$=\langle n_d+2,k,-1,n_g;n_g,k',-1,n_d|\dot{H}(t)|0\rangle$$

$$=-\frac{2\mathcal{N}_{jj'}}{E^2_{jj'}}[ik\delta_{k,-k'}+e\mathrm{B}_0\mathcal{I}(K)]\times\sqrt{(n_d+2)(n_d+1)}. \tag{A8}$$

In order to simplify the previous formulas we omitted the time dependence of the states and we used a compact notation to label the pair states [see Eqs. (56a),(56b)]. We also defined the quantities

$$\Delta k=k'-k, \tag{A9}$$

$$K=k'+k, \tag{A10}$$

$$E^2_{jj'}=(w_j+m)(\tilde{w}_{j'}+m), \tag{A11}$$

$$\mathcal{N}_{jj'}=\frac{e\mathrm{B}_0}{2}\,\omega\sqrt{\frac{w_j+m}{2w_j}\,\frac{\tilde{w}_{j'}+m}{2\tilde{w}_{j'}}}. \tag{A12}$$

Finally, we note that the function

$$\mathcal{I}(k)=\int_{-Z/2}^{Z/2}dz\,z\frac{e^{-ikz}}{Z}=\begin{cases}0 & \text{if } k=0,\\ i\dfrac{(-1)^n}{k} & \text{if } k=\dfrac{2\pi n}{Z}\neq 0\end{cases} \tag{A13}$$

becomes proportional to the derivative of the δ function in the limit $Z\to\infty$.

All the previous matrix elements can be divided into two groups: the ones related to transitions in which the longitudinal linear momentum conserves and the others characterized by the presence of the function $\mathcal{I}(K)$ in which it does not. Of course, this fact is due to the dependence of the transition operators on z [see Eqs. (47) and (48)]. As can be seen from Eqs. (84) and (85) it is the x component of the "rotated" electric field $\mathbf{E}'(\mathbf{r},t)$ which depends on z.

Finally, we observe that if we sum the probabilities corresponding to the previous matrix elements by means of Eq. (A1) with respect to the quantum number n_d all the series converge. Only the series corresponding to the matrix elements (A3) diverge logarithmically and it is not so obvious how to give a physical interpretation of such a kind of divergence. However, we can understand qualitatively why the probability of creating a pair with larger and larger n_d and then with larger and larger energy decreases so slowly. In fact, the quantum number n_d is connected to the radius $\rho_\perp$ of the helix along which a classical electron performs its motion. In particular, it can be shown that $\rho_\perp^2\sim n_d$ [19]. From this point of view while creating a pair with larger and larger n_d needs an amount of energy proportional to $\sqrt{n_d}$ [see Eqs. (15) and (34)], the pair itself extends over a volume increasing with n_d, so one can think that the energy of the magnetic field available for the pair creation also increases with n_d.

We conclude by noting that all the amplitudes have been calculated up to the first order in the derivative of the magnetic field. It is possible to estimate the second order contributions and, in the case of a purely rotating magnetic field, this task is not particularly difficult. One can compute, e.g., the depletion of the vacuum state and the scattering of the produced pair in the nonstationary field. In view of the present level of knowledge of the phenomenological aspects these refinements have not been worked out completely and are not presented.

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