Self-gravitating domain walls and the thin-wall limit

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We analyze the distributional thin-wall limit of self-gravitating scalar field configurations representing thick domain wall geometries. We show that thick-wall solutions can be generated by appropiate scaling of the thin-wall ones, and obtain an exact solution for a domain wall that interpolates between AdS_4 asymptotic vacua and has a well-defined thin-wall limit. Solutions representing scalar field configurations obtained via the same scaling but that do not have a thin-wall limit are also presented.

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I. INTRODUCTION

The gravitational properties of domain walls have been studied in the past due to their striking implications for cosmology. Recently, however, they have been the object of intense investigation for different reasons. On the one hand, it has been pointed out that four-dimensional gravity can be realized on a thin wall interpolating between AdS spacetimes [1]. In addition, wall configurations are relevant for the study of renormalization group equation (RGE) flows in the context of AdS/CFT (conformal field theory) correspondence [2].

The first attempts to study these gravitational properties were based on the so-called thin wall limit [3,4]. In a four-dimensional spacetime, the wall is treated as an infinitely thin $2+1$ surface. The spacetime outside the wall is given by vacuum solutions to the Einstein field equations with a planar symmetry, and one can use the Darmois-Israel [5] thin wall formalism to match solutions across the surface. Under this approximation, it is possible to find exact solutions representing an infinitely thin wall with an associated surface energy density. Spacetimes containing a thin domain wall have therefore distributional curvatures and energy-momentum tensors, proportional to delta functions supported in the wall's surface.

However, as pointed out in [6], these thin walls may be very artificial constructions in the sense that they do not necessarily correspond to the appropriate limit of a thick domain wall. Thick domain walls are solutions to the coupled Einstein-scalar field equations interpolating between minima of a potential with a spontaneously broken discrete symmetry. In order for these thick walls to have a thin wall limit an appropriate distributional treatment of the curvature and energy-momentum tensors is required. As is well known, computing the curvature tensor from a metric requires nonlinear operations which are not defined within the framework of distribution theory, and this imposes strong constraints on the class of metrics whose curvature tensors make sense as distributions [7].

The first exact thick wall solution in $D=4$ was obtained by Goetz [8]. The only other solution in the literature is that of Refs. [9,10], for a thick wall without reflection symmetry. Numerical solutions were found in [6] (see also references therein). The thin wall limit of these solutions has not been studied. All of these solutions represent walls in spacetimes without a cosmological constant.

The purpose of this paper is twofold. First, we analyze the thin wall limit of thick wall solutions. We find that the solution of [8] has a well-defined thin wall limit. We also show that this solution can be obtained by an appropriate scaling of a solution of Einstein's equations in vacuum with a planar symmetry. Using this fact, we generate a number of exact solutions to the Einstein-scalar field equations using the vacuum solutions. We then show that the only wall that can be considered a thick domain wall and that possesses a thin wall limit is that of Ref. [8].

Second, we look for solutions representing a domain wall embedded in a four-dimensional spacetime with a negative cosmological constant. We show how the scaling procedure can be modified to generate such solutions, and find an exact solution for a thick domain wall interpolating between two AdS_4 vacua. This wall is then shown to have a curvature and energy-momentum tensor well defined as distributions and the corresponding distributional thin-wall limit is obtained. The possibility of obtaining this solution within the supergravity inspired first order formalism of Refs. [11,12] is also investigated, and the scalar field potential is shown to satisfy the requirements for the existence of stable AdS vacua.

The paper is organized as follows. In the next section we study the thin wall limit of thick wall solutions. In Sec. III, we show how exact solutions can be found by scaling thin wall spacetimes, and show that the new solutions found do not represent true domain walls. In Sec. IV, a new solution for a domain wall in an AdS spacetime is found and analyzed. We summarize our results on Sec. V, and include an Appendix for the reader interested in the details of the distributional analysis.

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II. FROM THICK TO THIN DOMAIN WALLS

Consider the spacetime $(R^4, \mathbf{g})$, where the metric in a particular coordinate system is given by

$$g_{ab}=\cosh(\beta x/\delta)^{-2\delta}[-dt_a dt_b+dx_a dx_b +e^{2\beta t}(dy_a dy_b+dz_a dz_b)] \quad (1)$$

where β and δ are constants. In Refs. [8–10], this spacetime has been shown to be the one generated by a ''thick'' domain wall, i.e. it is a solution to the coupled Einstein-scalar field equations:

$$R_{ab}-\frac{1}{2}g_{ab}R=8\pi T_{ab}, \quad (2)$$

$$T_{ab}=\nabla_a\phi\nabla_b\phi-g_{ab}\left(\frac{1}{2}\nabla_c\phi\nabla^c\phi+V(\phi)\right) \quad (3)$$

and

$$\nabla_a\nabla^a\phi-\frac{dV(\phi)}{d\phi}=0, \quad (4)$$

with

$$\phi=\phi_0\tan^{-1}(\sinh(\beta x/\delta)), \quad \phi_0\equiv\sqrt{\frac{\delta(1-\delta)}{4\pi}} \quad (5)$$

and

$$V(\phi)=\frac{1+2\delta}{8\pi\delta}\beta^2[\cos(\phi/\phi_0)]^{2(1-\delta)} \quad (6)$$

where $0<\delta<1$. These solutions represent a domain wall of (finite) thickness δ. The scalar field takes values $\pm\phi_0\pi/2$ at $x=\pm\infty$, corresponding to two consecutive minima of the potential, and interpolates smoothly between these values at the origin.

The metric (1) is only one of many possible thick domain wall solutions which can be obtained under the requirements: (i) $V(\phi)\geqslant 0$, (ii) $g^{ab}\partial_a\phi\partial_b\phi>0$, and (iii) reflection symmetry. However, Eqs. (1), (5), (6) are the only analytic solutions encountered in the literature so far [8] (for a study of their properties, see [10,13]). Numerical treatments can be found in [6] and references therein.

Next, consider the $\delta\to 1$ and $\delta\to 0$ limits of this spacetime:

(1) For $\delta\to 1$ we have $\phi=0, V(\phi)=3\beta^2$, and the metric

$$g_{ab}=(\cosh(\beta x))^{-2}[-dt_a dt_b+dx_a dx_b +e^{2\beta t}(dy_a dy_b+dz_a dz_b)] \quad (7)$$

turns out to be a solution to the Einstein field equations for the vacuum with a cosmological constant term

$$R_{ab}-\frac{1}{2}g_{ab}R+g_{ab}\Lambda=0, \quad (8)$$

where $\Lambda=V(\phi)|_{\delta=1}=3\beta^2$. Under the assumption that the energy-momentum tensor of a thin wall can be approximated by a cosmological constant outside the wall (where the nearly-constant potential term is dominating), the solution (7) of Eq. (8) can be interpreted as representing the spacetime at some distance from the wall [14]. However, notice that there is no thin wall at the origin or elsewhere, the metric being well-defined in all spacetime and having a nonsingular curvature tensor. In fact, the limit $\delta\to 1$ can be considered as representing a wall of infinite thickness.

(2) We turn now to the $\delta\to 0$ limit. In order to compute this limit we consider the curvature tensor and the Einstein tensor as distributional tensor fields. As is well known, the curvature tensor being nonlinear does not make sense in general as a distribution. However, the metric (1) is a smooth metric that belongs to the class of regular metrics [7], and for a regular metric the curvature tensor makes sense as a distribution. Since any contraction of a distribution is also a distribution, for a regular metric the Einstein tensor is well defined as a distribution.

Taking the $\delta\to 0$ limit, we find

$$\lim_{\delta\to 0} g_{ab}=e^{-2\beta|x|}(-dt_a dt_b+dx_a dx_b +e^{2\beta t}(dy_a dy_b+dz_a dz_b)), \quad (9)$$

which for $x<0$ and $x>0$ is a vacuum solution of the Einstein field equations [14], and

$$\lim_{\delta\to 0} G^a_b=-4\beta\delta(x)[\partial t^a dt_b+\partial y^a dy_b+\partial z^a dz_b]. \quad (10)$$

This means that the spacetime (R^4, g), with the metric given by Eq. (1), can be identified in the limit $\delta\to 0$ with the spacetime (R^4, g), with g given by Eq. (9), generated by a thin domain wall with energy-momentum tensor given by

$$8\pi T^a_b=-4\beta\delta(x)[\partial t^a dt_b+\partial y^a dy_b+\partial z^a dz_b]. \quad (11)$$

As expected, Eq. (11) can be obtained from Eq. (9) by using the formalism of Israel [5] to treat surface layers.

Actually, to be rigorous, one should prove that the metric (1) provides a sequence of metrics that satisfies the required convergence condition of [7]. Then the limit of the curvature tensor exists and is the curvature tensor of the limit metric. We leave this rather technical proof for the Appendix.

Remarkably enough, the metrics (1), solution to the coupled Einstein-scalar field equations, and Eq. (7), vacuum solution, can be rewritten simply as

$$_{\delta}g_{ab}=f^{2\delta}(x/\delta)[-dt_a dt_b+dx_a dx_b +e^{2\beta t}(dy_a dy_b+dz_a dz_b)] \quad (12)$$

$$g_{ab}=f^2(x)[-dt_a dt_b+dx_a dx_b +e^{2\beta t}(dy_a dy_b+dz_a dz_b)] \quad (13)$$

respectively, where $f(x)=\cosh(\beta x)^{-1}$. This opens up the interesting possibility of generating ''thick wall'' solutions from the ''thin wall'' ones, i.e. by scaling vacuum solutions with positive, negative or null cosmological constant. In the next section, we will explore this possibility.

III. FROM THIN TO THICK WALLS

We wish to find solutions to the coupled Einstein-scalar field equations (2)–(4) with a planar symmetry, where the scalar field ϕ is static. In its most general form, the metric can be written

$$g_{ab}=f(x)^2[-dt_a dt_b+dx_a dx_b]+B(x,t)^2[(1-\kappa r^2)^{-1}dr^2 + r^2 d\varphi^2] \tag{14}$$

where κ is the curvature of the $x,t=\text{const}$ surfaces. Requiring that ϕ be a function of the coordinate perpendicular to the wall only, and that the spacetime be boost-invariant in directions parallel to the wall, it can be shown [14] that

$$B(x,t)=f(x)C(t) \tag{15}$$

where the function $C(t)$ satisfies (dot indicates differentiation with respect to t)

$$\frac{\ddot{C}(t)}{C(t)}=\frac{\dot{C}(t)^2}{C(t)^2}+\frac{\kappa}{C(t)^2}=\beta^2. \tag{16}$$

The time dependence of the solutions depends on the sign of the constant β^2. For positively curved ($\kappa>0$) or planar ($\kappa=0$) walls, β^2 is positive and we have

$$C(t)=e^{\beta t}, \quad \text{when} \quad \kappa=0 \tag{17}$$

$$C(t)=\cosh(\beta t), \ \ \beta^2=\kappa, \quad \text{when} \quad \kappa>0 \tag{18}$$

whereas for negatively curved ($\kappa<0$) walls the sign of β^2 is arbitrary, and three solutions are possible:

$$C(t)=\sinh(\beta t), \quad \beta^2=-\kappa \tag{19}$$

$$=\sqrt{-\kappa}t, \quad \beta^2=0 \tag{20}$$

$$=\sin(\beta t), \quad \beta^2=\kappa. \tag{21}$$

Then the system (2)–(4) reduces to just two equations for the potential and the scalar field. Define (prime denotes derivative respect to x)

$$u(x)\equiv\frac{f(x)'}{f(x)}; \tag{22}$$

then

$$\phi'^2=\frac{1}{4\pi}[-u(x)'+u(x)^2-\beta^2] \tag{23}$$

$$V(\phi)=\frac{1}{8\pi f^2}[-u(x)'-2(u(x)^2+\beta^2)]. \tag{24}$$

The solution of Refs. [8,10] is found with $f(x)=[\cosh(\beta x/\delta)]^{-\delta}$, with $\kappa=0$ and $C(t)$ given by Eq. (17), while as noted in the previous section, a vacuum solution with a cosmological constant is obtained with $f(x)=[\cosh(\beta x)]^{-1}$ and the same curvature and time dependence.

We have found that this is a general result: *the system* (23), (24) *can be integrated with the scaled function*

$$f(x)=f_0(x/\delta)^{\delta}, \tag{25}$$

where $f_0(x)$ is a solution to the Einstein field equations in vacuum with a cosmological constant for the metric (14).

This is easily shown. Substituting Eq. (25) in Eq. (23)

$$\phi'^2=\frac{1}{4\pi}\left[-\frac{1}{\delta}u_0(x/\delta)'+u_0(x/\delta)^2-\beta^2\right] \tag{26}$$

where now prime denotes derivative with respect to the argument. Since f_0 satisfies the Einstein equations (8), we have

$$u_0(x/\delta)'+2(u_0(x/\delta)^2-\beta^2)=-\Lambda f_0(x/\delta)^2 \tag{27}$$

$$u_0(x/\delta)^2-\beta^2-u_0(x/\delta)'=0, \tag{28}$$

and substituting in Eqs. (26), (24)

$$\phi=\sqrt{\frac{\delta(1-\delta)}{4\pi}\frac{\Lambda}{3}}\int_{x_0}^{x/\delta}f_0(\xi)d\xi, \tag{29}$$

$$V(x)=\frac{1}{8\pi}\frac{\Lambda}{3}\left(\frac{1+2\delta}{\delta}\right)f_0(x/\delta)^{2(1-\delta)}. \tag{30}$$

It is then possible to generate solutions representing a self-gravitating scalar field wall using the vacuum solutions of Ref. [14]. The time dependence of the metric and the curvature of the wall will be preserved.

For vacuum solutions with Eqs. (14), (15) and negative cosmological constant we get:

(1) with $f_0(x)=[\sinh(\beta x)]^{-1}$, corresponding to a vacuum solution with $\Lambda=-3\beta^2$, a solution of Eqs. (2)–(6) with

$$f(x)=[\sinh(\beta x/\delta)]^{-\delta}$$

$$\phi(x)=\phi_0\coth^{-1}[\cosh(\beta x/\delta)],$$

$$\phi_0\equiv-\sqrt{\frac{\delta(\delta-1)}{4\pi}}$$

$$V(\phi)=\frac{\beta^2}{8\pi}\frac{2\delta+1}{\delta}[\sinh(\phi/\phi_0)]^{2(1-\delta)}. \tag{31}$$

As in the vacuum case, $C(t)$ is given by either Eqs. (17), (18), or (19) when the curvature of the wall is zero, positive or negative respectively;

(2) with $f_0(x)=[1+\alpha x]^{-1}$, corresponding to a vacuum solution with $\Lambda=-3\alpha^2$, a solution of Eqs. (2)–(6) with

$$f(x)=[1+\alpha x/\delta]^{-\delta}$$

$$\phi(x)=\phi_0\ln(1+\alpha x/\delta),$$

$$\phi_0 \equiv \sqrt{\frac{\delta(\delta-1)}{4\pi}}$$

$$V(\phi) = \frac{\alpha^2}{8\pi}\frac{2\delta+1}{\delta}\exp[2(\delta-1)\phi/\phi_0]. \tag{32}$$

In this case the plane wall corresponds to a static solution $C(t)=1$, but a nonstatic wall is possible with negative curvature and $C(t)$ given by Eq. (20);

(3) with Eqs. (14), (15) and $f_0(x)=[\cos(\beta x)]^{-1}$, vacuum solution with $\Lambda=-3\beta^2$, a solution of Eqs. (2)–(6) with

$$f(x)=[\cos(\beta x/\delta)]^{-\delta}$$

$$\phi(x)=\phi_0 \tanh^{-1}[\sin(\beta x/\delta)],$$

$$\phi_0 \equiv \sqrt{\frac{\delta(\delta+1)}{4\pi}}$$

$$V(\phi) = \frac{\beta^2}{8\pi}\frac{2\delta-1}{\delta}[\cosh(\phi/\phi_0)]^{2(1-\delta)}. \tag{33}$$

The wall must have in this case negative curvature and $C(t)$ given by Eq. (21). A fourth nontrivial vacuum solution exists with $\Lambda=0$, namely

$$f(x)=e^{\pm\beta x} \tag{34}$$

but obviously it cannot generate a thick wall solution by the same scaling procedure.

All of these solutions have an energy-momentum tensor of the form

$$T^a_b = -\rho(x)[\partial t^a dt_b + \partial y^a dy_b + \partial z^a dz_b] + p(x)\partial x^a dx_b \tag{35}$$

compatible with a static scalar field wall. In the first solution, the parameter δ can be interpreted as the wall's thickness, just as in the solution of the previous section. However, notice that this spacetime contains a singularity which seems to be much worse than the one produced by a source concentrated on a thin wall. In this example the metric is not regular, and we cannot assign to it a distributional source following the approach of [7].

In fact, none of these solutions represents a *domain* wall. Namely, none of the potentials above has minima or is even bounded from below. These wall solutions are not topologically protected, and their stability is thus not guaranteed. For example, take case (1) above. Although ϕ takes constant values at infinity, it does not interpolate smoothly between them.

Notice also that far from the walls one recovers the vacuum solutions. Again, take case (1): the metric has the same asymptotic behavior as the domain wall solution of Sec. II,

$$g_{ab} \to e^{-\beta|x|}[-dt_a dt_b + dx_a dx_b + e^{2\beta t}(dy_a dy_b + dz_a dz_b)]$$
$$\text{when} \quad \beta|x| \to \infty. \tag{36}$$

This is an important point: finding a vacuum solution to the Einstein equations with planar symmetry and then using the thin shell formalism of Israel may produce a thin wall which is not the thin limit of a scalar thick domain wall. In this sense, the thin wall solution is less artificial if it can be obtained as the limit of a smooth configuration.

It is possible however to find other solutions to Eqs. (23), (24), not generated by thin wall ones, that do represent a thick domain wall with a well defined thin wall limit, as we do in the next section.

IV. A THICK DOMAIN WALL WITH COSMOLOGICAL CONSTANT

In this section we consider thick domain walls embedded in a spacetime with a negative cosmological constant Λ. The case $\Lambda<0$ is particularly interesting because the positive effective gravitational mass density of AdS_4 spacetime may counteract the negative effective gravitational mass density of the domain wall. On the other hand, a certain type of these solutions may be realized as supersymmetric bosonic field configuration [15]. Clearly, for $\Lambda<0$ we are looking for domain wall solutions where the effective potential $V_{\text{eff}} \equiv V + \Lambda/8\pi$ is not necessarily positive-definite, requiring only that it is bounded from below.

Assuming a conformally flat symmetric *ansatz* for the metric

$$g_{ab} = f(x)^2[-dt_a dt_b + dx_a dx_b + dy_a dy_b + dz_a dz_b] \tag{37}$$

and also that the scalar field depends only on x, the equations of motion are Eqs. (23), (24) where $\beta=0$, and with the addition of a cosmological constant term $-\Lambda/8\pi$ to the right-hand side of Eq. (24).

We look for reflection symmetric domain wall solutions for $\Lambda<0$. Under the requirements

(i) $V(\phi)>0$

(ii) $\lim_{|x|\to 0}|\phi(x)'|=0$

(iii) $\phi'(0)=0$ (nonsingular solution)

(iv) $(f^2)'|_{x=0}=0$ (reflection symmetry)

we find the solution

$$f(x)=(1+\alpha^2 x^2)^{(-1/2)}, \tag{38}$$

$$\phi(x)=\phi_0 \tan^{-1}(\alpha x), \tag{39}$$

$$V(\phi)=\frac{1}{2\pi}\alpha^2\cos^2(\phi/\phi_0), \tag{40}$$

$$\Lambda=-3\alpha^2, \tag{41}$$

where $\phi_0=\sqrt{1/4\pi}$. This solution represents a class of gravitating domain walls that interpolate smoothly between two minima of the potential $V(\phi)$, the spacetime being asymptotically AdS_4. With the coordinate change

$$\alpha\xi = \sinh^{-1}(\alpha x), \tag{42}$$

the line element takes the form

$$g_{ab}=\cosh^{-2}(\alpha\xi)[-dt_a dt_b+dy_a dy_b+dz_a dz_b]+d\xi^2, \tag{43}$$

which asymptotically behaves as AdS_4

$$g_{ab}\to 4e^{-2\alpha|\xi|}[-dt_a dt_b+dy_a dy_b+dz_a dz_b]+d\xi^2 \quad \text{when} \quad \alpha|\xi|\to\infty. \tag{44}$$

We now wish to consider the thin wall limit. However, AdS_4 domain wall solutions generically have two free parameters: one for the asymptotic AdS curvature and one for the wall's width. Thus, in order to introduce a second parameter in the solution found, we make a scaling of the metric tensor as in the previous section.

Consider the scaled metric

$$g_{ab}dx^a dx^b=\cosh^{-2\delta}(\alpha\xi/\delta)[-dt_a dt_b+dy_a dy_b+dz_a dz_b]+d\xi^2. \tag{45}$$

We find this time

$$\phi(\xi)=\phi_0\tan^{-1}\sinh(\alpha\xi/\delta), \tag{46}$$

$$V(\phi)=\frac{1}{8\pi}\alpha^2\left(3+\frac{1}{\delta}\right)\cos^2(\phi/\phi_0), \tag{47}$$

$$\Lambda=-3\alpha^2 \tag{48}$$

with $\phi_0=\sqrt{\delta/4\pi}$. It is easy to see that the metric (45) is a regular metric in the differentiable structure provided by the coordinate chart $\{t,\xi,y,z\}$. It follows that the curvature and Einstein tensor fields are well defined as distributions.

Computations analogous to the ones in the Appendix show that

$$\lim_{\delta\to 0} g_{ab}=4e^{-2\alpha|\xi|}(-dt_a dt_b+dy_a dy_b+dz_a dz_b)+d\xi_a d\xi_b, \tag{49}$$

which is also a regular metric and

$$\lim_{\delta\to 0}(G^a_b+\Lambda g^a_b)=-4\alpha\delta(\xi)(\partial^a_t dt_b+\partial^a_y dy_b+\partial^a_z dz_b) \tag{50}$$

where $\xi=0$ is the codimension one hypersurface where the thin wall is located and Λ is given by Eq. (48).

Thus, we have found a two-parameter family of self-gravitating scalar fields with a thick domain wall profile interpolating between two AdS_4 vacua. Furthermore, these solutions have a distributional curvature tensor with a well-defined thin wall limit in the sense of Ref. [7].

As stated above, the domain wall spacetime considered in this section is asymptotically AdS_4. It is known that restrictions on the potential are to be imposed from the requirement that there exist a stable AdS vacuum [16,17]. In D dimensions, for a model with Lagrangian density

$$\mathcal{L}=\sqrt{-g}\left[\frac{1}{2}R-\frac{1}{2}(\partial\phi)^2-\mathcal{V}(\phi)\right] \tag{51}$$

vacuum stability requires $\mathcal{V}$ to take the form

$$\mathcal{V}=2(D-2)\left[(D-2)\left(\frac{dW}{d\phi}\right)^2-(D-1)W^2\right] \tag{52}$$

where $W(\phi)$ is any function with at least one critical point [17]. Critical points of W are also critical points of $\mathcal{V}$ and in the context of supergravity theories the critical points of W yield stable AdS vacua [11].

For the domain wall solution (45)–(48), this is equivalent to requiring that

$$\mathcal{V}=V_{\rm eff}=V+\Lambda=4\left[2\left(\frac{dW}{d\phi}\right)^2-3W^2\right] \tag{53}$$

with V and Λ given by Eqs. (47) and (48), respectively. It follows that in this case

$$W(\phi)=\frac{1}{2}\beta\sin(\phi/\phi_0) \tag{54}$$

whose critical points are $\phi=\pm\pi\phi_0/2$. Whether a supergravity theory can be constructed so that the supersymmetry conditions lead to Eq. (54) is a question beyond the scope of this paper. But since the critical points of Eq. (54) are the asymptotic values of ϕ, as given by Eq. (46), this suggests that these asymptotic AdS vacua are stable.

It should be noted that Eqs. (45)–(48) can be parametrized by Eq. (54), so we could have found it using the first order formalism of [11,12]. However, Eq. (46) is not the familiar kink which one usually encounters in the literature, whether supersymmetric or not [18,12]. Finally, in Ref. [19] an example with a superpotential similar to Eq. (54) has been considered in $D=5$, in the study of how four-dimensional gravity arises on a thick wall interpolating between two AdS_5 spacetimes.

V. CONCLUDING REMARKS

We have studied the thin-wall limit of thick domain wall solutions in a (3+1)-dimensional spacetime. We have shown that the solution (1) of Ref. [8] represents a spacetime with a regular metric in the sense of Ref. [7], and that the thin wall limit can be taken rigorously in distribution theory. Not surprisingly, in the thin-wall limit solution (1) becomes the well-known thin wall solution of [3]. We have also demonstrated that this thick solution can be obtained by appropriately scaling thin, i.e. vacuum, ones. However, although other solutions to the Einstein-scalar field coupled equations can be systematically obtained by the same procedure, it does not follow that these new solutions are thick walls, even if they have the appropriate asymptotic behavior far from the origin and the stress-energy tensor of a wall. The scalar field potential is in general not bounded from below and the scalar field configurations are not topologically protected, thus probably unstable.

Using a similar scaling procedure, we have obtained a solution representing a thick domain wall embedded in an AdS_4 spacetime. The cosmological constant, as expected, is related to the wall's surface energy density. This solution was shown to have a thin-wall limit, with a stress-energy tensor which is well defined as a distribution. The potential is positive definite and the scalar field smoothly interpolates between two AdS vacua. Moreover, the scalar field potential for this solution has been shown to satisfy the requirements for the existence of stable AdS vacua, being derivable from a superpotential function. The possible connection with supergravity theories, and the stability of the solution under perturbations are currently under investigation.

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APPENDIX

In this paper we use the definition of tensor distribution given by Geroch and Traschen. The reader is referred to [7] for details.

Definition. A symmetric tensor field g_{ab} will be called a regular metric provided that (i) g_{ab} and $(g^{-1})^{ab}$ exist everywhere and are locally bounded and (ii) the weak derivative of g_{ab} in some smooth metric η_{ab} exists and is locally square integrable.

The curvature tensor and the Einstein tensor of a regular metric make sense as distributions, therefore it makes sense to write Einstein's equations with a distributional energy-momentum tensor. Furthermore, these idealized matter sources are necessarily concentrated on submanifolds of codimension of at most one.

First define conveniently the smooth tensor fields

$$S_{ab}\equiv -dt_a dt_b + dx_a dx_b + e^{2\beta t}(dy_a dy_b + dz_a dz_b) \tag{A1}$$

$$(S^{-1})^{ab}\equiv -\partial_t^a\partial_t^b + \partial_x^a\partial_x^b + e^{-2\beta t}(\partial_y^a\partial_y^b + \partial_z^a\partial_z^b). \tag{A2}$$

Now let

$${}_n g_{ab} = \cosh(\beta n x)^{-2/n} S_{ab}, \quad g_{ab} = e^{-2\beta|x|} S_{ab} \tag{A3}$$

and

$$({}_n g^{-1})^{ab} = \cosh(\beta n x)^{2/n} (S^{-1})^{ab},$$
$$(g^{-1})^{ab} = e^{2\beta|x|}(S^{-1})^{ab}. \tag{A4}$$

Let U^{ab} be a test tensor field defined on R^4. We have

$${}_n g_{ab} U^{ab} = \cosh(n\beta x)^{-2/n}[-U^{tt} + U^{xx} + e^{2\beta t}(U^{yy} + U^{zz})] \tag{A5}$$

and

$$g_{ab}U^{ab} = e^{-2\beta|x|}[-U^{tt} + U^{xx} + e^{2\beta t}(U^{yy}+U^{zz})]. \tag{A6}$$

Clearly, ${}_n g_{ab}$ and g_{ab} are locally bounded. Let U_{ab} be a test tensor field defined on R^4. We have

$$({}_n g^{-1})^{ab} U_{ab} = \cosh(n\beta x)^{2/n}[-U_{tt} + U_{xx} + e^{-2\beta t}(U_{yy} + U_{zz})] \tag{A7}$$

and

$$(g^{-1})^{ab}U_{ab} = e^{-2\beta|x|}[-U_{tt} + U_{xx} + e^{-2\beta t}(U_{yy}+U_{zz})]. \tag{A8}$$

Hence $({}_n g^{-1})^{ab}$ and $(g^{-1})_{ab}$ are locally bounded also.

Now choose as a smooth derivative operator ∇_a, the one compatible with the Minkowski metric η_{ab} and let U^{cab} be a test tensor field on R^4. The weak derivative in η_{ab} of ${}_n g_{ab}$ and g_{ab} exist everywhere and are given by

$$\nabla_c({}_n g_{ab})[U^{cab}] \equiv -{}_n g_{ab}[\nabla_c U^{cab}] = \int_{R^4} {}_n W_{cab} U^{cab}\omega_\eta \tag{A9}$$

and the equivalent expression for g_{ab}, where

$${}_n W_{cab} = 2\beta\cosh(n\beta x)^{-2/n} \times\{\tanh(n\beta x)dx_c[-dt_a dt_b dx_a dx_b + e^{2\beta t}(dy_a dy_b + dz_a dz_b)] - e^{2\beta t}dt_c(dy_a dy_b + dz_a dz_b)\} \tag{A10}$$

and

$$W_{cab} = \begin{cases} 2\beta e^{2\beta x}[dx_c(-dt_a dt_b + dx_a dx_b) + e^{2\beta t}(dt_c + dx_c)(dy_a dy_b + dz_a dz_b)], & x<0 \\ 2\beta e^{-2\beta x}[-dx_c(-dt_a dt_b + dx_a dx_b) + e^{2\beta t}(dt_c - dx_c)(dy_a dy_b + dz_a dz_b)], & x>0, \end{cases} \tag{A11}$$

with ω_η the volume element in η_{ab} and where it is understood that η_{ab} and its inverse are used to raise and lower tensor indices. It then follows that $\nabla_c({}_n g_{ab}) \equiv {}_n W_{cab}$ and $\nabla_c g_{ab} \equiv W_{cab}$ are locally square integrable. Therefore both ${}_n g_{ab}$ and g_{ab} are regular metrics.

Now we can consider the limit $n\to\infty$.

Theorem. Let ${}_n g_{ab}$ and g_{ab} be regular metrics. Let (i) ${}_n g_{ab}$ and $({}_n g^{-1})^{ab}$ be locally uniformly bounded and (ii) ${}_n g_{ab}$, $({}_n g^{-1})^{ab}$ and the weak derivative $\nabla_c({}_n g_{ab})$ converge locally in square integral to g_{ab}, $(g^{-1})^{ab}$ and $\nabla_c g_{ab}$ respectively. Then the corresponding curvature distributions ${}_nR^d_{abc}$ converge to R^d_{abc} in the following sense: for any test field U^{abc}_d,

$$\lim_{n\to\infty} {}_nR^d_{abc}[U^{abc}_d]=R^d_{abc}[U^{abc}_d]. \tag{A12}$$

(See [7] for the proof.)

It is straightforward to prove that Eqs. (A3)–(A8) and Eqs. (A10), (A11) satisfy the conditions of the above theorem. We have

$$|({}_ng_{ab}|U^{ab})|\leqslant|(S_{ab}|U^{ab})| \tag{A13}$$

and

$$|(({}_ng^{-1})^{ab}|U_{ab})|\leqslant[2\cosh(\beta x)]^2|((S^{-1})^{ab}|U_{ab})|. \tag{A14}$$

It follows that ${}_ng_{ab}U^{ab}$ and $({}_ng^{-1})^{ab}U_{ab}$ are bounded by smooth tensor fields with compact support, i.e. test fields. Therefore ${}_ng_{ab}$ and $({}_ng^{-1})^{ab}$ are locally uniformly bounded.

Let U^{abcd} be a test tensor field on R^4. Define

$$\rho_n({}_ng,g)\equiv\int_{R^4}({}_ng_{ab}-g_{ab})({}_ng_{cd}-g_{cd})U^{abcd}\omega_\eta. \tag{A15}$$

It is easy to see that

$$\lim_{n\to\infty}\rho_n({}_ng,g)=0. \tag{A16}$$

Then ${}_ng_{ab}$ converges locally in square integral to g_{ab}. The equivalent relation holds true for $({}_ng^{-1})^{ab}$. Finally, let U^{abcdef} be a test tensor field on R^4. We have

$$\lim_{n\to\infty}\int_{R^4}({}_nW_{abc}-W_{abc})({}_nW_{cdf}-W_{cdf})U^{abcdef}\omega_\eta=0. \tag{A17}$$

Therefore ${}_nW_{abc}$ converges locally in square integral to W_{abc}.

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