Chiral condensate and short-time evolution of (1+1)-dimensional QCD on the light cone

Matthias Burkardt
Department of Physics, New Mexico State University, Las Cruces, New Mexico 88003-0001

Frieder Lenz and Michael Thies
Institut für Theoretische Physik III, Universität Erlangen-Nürnberg, Staudtstraße 7, D-91058 Erlangen, Germany
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Chiral condensates in the trivial light-cone vacuum emerge if defined as short-time limits of fermion propagators. In gauge theories, the necessary inclusion of a gauge string in combination with the characteristic light-cone infrared singularities contain the relevant nonperturbative ingredients responsible for the formation of the condensate, as demonstrated for the 't Hooft model.

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I. INTRODUCTION

The triviality of the light-cone vacuum is the origin of most of the simple properties of field theories if quantized on the light cone (for a recent review, see Ref. [1]). For purely kinematical reasons there is no distinction between the ground state of free and interacting theories. In view of the many physical consequences usually attributed to non-trivial vacua, this has raised considerable concern about the equivalence of light-cone and equal-time quantization schemes. While for bosonic theories the common belief is that the key lies in the constrained zero-mode dynamics [2], in fermionic theories a way out of this apparent contradiction has been sought in prescriptions for regularization of the divergent condensates such as the "parity invariant" regularization relating IR and UV cutoffs [3,4]. It is doubtful that universal, kinematical prescriptions exist that describe the formation of condensates in theories such as gauge theories where chiral symmetry breaking is not tantamount to fermion mass creation. In the present work, the proposal [5] to define order parameters as vacuum expectation values of products of Heisenberg operators, infinitesimally split in a light-cone time direction (in addition to a space direction) will be shown to yield the correct condensate in (1+1)-dimensional QCD (QCD_{1+1})—the 't Hooft model [6]. Unlike in standard quantization, the short-time limit of Heisenberg operators differs non-perturbatively from the corresponding Schrödinger operators. This connection between condensates and the non-triviality of the singular behavior of correlation functions at short light-cone times will be the focus of this work.

The condensate of the 't Hooft model has been evaluated in [7,8]. These studies make use of general relations between properties of the excited states and the condensate, derived in standard quantization (the Oakes-Renner relation and sum rules). Here we will present a direct calculation of the condensate which makes use explicitly of the triviality of the light-cone vacuum on the one hand and the non-perturbative nature of the short-time limit of light-cone correlation functions on the other.

II. CONDENSATE AND SHORT-TIME EVOLUTION

We define the condensate with respect to that of the noninteracting theory,

$$\langle\bar{\psi}\psi\rangle=\lim_{\varepsilon\to 0}[\langle\bar{\psi}\psi\rangle_{\varepsilon}-\langle\bar{\psi}\psi\rangle^{0}_{\varepsilon}], \tag{2.1}$$

with

$$\langle\bar{\psi}\psi\rangle_{\varepsilon}=\langle 0|\bar{\psi}(\varepsilon)Pe^{ig\int_0^{\varepsilon}dx^{\mu}A_{\mu}}\psi(0)|0\rangle \tag{2.2}$$

regularized in a gauge-invariant way. For evaluating the point-split condensate (2.2), we first consider a generic matrix element of the type

$$M(\varepsilon)=\langle 0|A(\varepsilon)P\mathrm{e}^{\mathrm{i}g\int_0^s ds'(dx^{\mu}/ds')A_{\mu}[x(s')]}B(0)|0\rangle \tag{2.3}$$

where $x(s)$ is a straight path with $x(0)=0$ and $x(s)=\varepsilon$. By shifting the argument of the gauge field A_μ and using translational invariance of the vacuum, we can represent $M(\varepsilon)$ as

$$M(\varepsilon)=\langle 0|A(0)W(s)B(0)|0\rangle \tag{2.4}$$

with the point splitting now specified by the operator $W(s)$ (momentum operator P_μ),

$$W(s)=\mathrm{e}^{-\mathrm{i}\varepsilon^{\mu}P_{\mu}}P\mathrm{e}^{\mathrm{i}g\int_0^s ds'(dx^{\mu}/ds')A_{\mu}[x(s')]}. \tag{2.5}$$

Differentiation with respect to s,

$$\frac{dW(s)}{ds}=-\mathrm{i}\frac{dx^{\mu}}{ds}[P_{\mu}-gA_{\mu}(0)]W(s), \tag{2.6}$$

and integration of this differential equation with the initial condition $W(0)=1$ allows one to simplify this expression, yielding

$$W(s)=\mathrm{e}^{-\mathrm{i}\varepsilon^{\mu}[P_{\mu}-gA_{\mu}(0)]}. \tag{2.7}$$

With this form of the operator W the chiral condensate of Eq. (2.2) is seen to be given by the space-time evolution of a system of light quarks coupled with the current of an infinitely heavy quark. Thus, on the light cone with its kinematical vacuum, the dynamics of a heavy-light quark system determines the chiral condensate. So far, everything is rather general and applies equally well to QCD in four dimensions. We now specialize to the 't Hooft model [6] and write Eq. (2.2) as

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$$\langle\overline{\psi}\psi\rangle_\varepsilon=\langle 0|\overline{\psi}_i(0)(\mathrm{e}^{-\mathrm{i}\varepsilon^+ H_{\mathrm{eff}}})_{ij}\psi_j(0)|0\rangle \tag{2.8}$$

with the effective Hamiltonian in light-cone gauge ($A_-=0$),

$$(H_{\mathrm{eff}})_{ij}=(P_+ + \lambda P_-)\delta_{ij}-g[A_+(0)]_{ij}. \tag{2.9}$$

We have denoted the slope of the path, $\varepsilon^-/\varepsilon^+$, by λ and will take ε^+ and λ as independent parameters from now on; due to the subtraction of the free value, the condensate will turn out to be independent of λ. P_+ and P_- are the Hamiltonian and momentum operator for the 't Hooft model, respectively.

For evaluation of the chiral condensate, we represent the spinor ψ in terms of the unconstrained right-handed component φ,

$$\psi(x)=\frac{1}{2^{1/4}}\begin{pmatrix}1\\ \dfrac{m}{\mathrm{i}\sqrt{2}\partial_-}\end{pmatrix}\varphi, \tag{2.10}$$

and find (after Fourier transforming the fermion fields)

$$\langle\overline{\psi}\psi\rangle_\varepsilon=\int\frac{\mathrm{d}p}{2\pi}\frac{m}{p}C(p,\varepsilon^+) \tag{2.11}$$

with

$$C(p,t)=\int\frac{\mathrm{d}q}{2\pi}\langle 0|\varphi_i^\dagger(p)(\mathrm{e}^{-\mathrm{i}H_{\mathrm{eff}}t})_{ij}\varphi_j(q)|0\rangle. \tag{2.12}$$

In order to compute this correlation function we derive its equation of motion. In

$$\mathrm{i}\dot{C}(p,t)=\int\frac{\mathrm{d}q}{2\pi}\langle 0|\varphi_i^\dagger(p)(H_{\mathrm{eff}}\mathrm{e}^{-\mathrm{i}H_{\mathrm{eff}}t})_{ij}\varphi_j(q)|0\rangle, \tag{2.13}$$

we treat the two terms of H_{eff} [cf. Eq. (2.9)] separately. In the first term, the product of the operators $(P_+ + \lambda P_-)$ and $\varphi_i^\dagger(p)$ can be replaced by their commutator. In the large N limit, this commutator generates a combination of the quark self-energy and momentum where, following 't Hooft's original paper [6],

$$\langle 0|[\varphi_i^\dagger(p),P_+ + \lambda P_-]=\left(\frac{m_r^2}{2p}+\lambda p\right)\langle 0|\varphi_i^\dagger(p), \tag{2.14}$$

with the self-energy given in principal value prescription by

$$m_r^2\equiv m^2-\frac{Ng^2}{2\pi}. \tag{2.15}$$

The right-hand side of Eq. (2.14) combines with the remainder of the matrix element in Eq. (2.13) to yield again $C(p,t)$. The term proportional to $A_+(0)$ in H_{eff} can be expressed via the Poisson equation in terms of the fermion color charge density:

$$[A_+(0)]_{ij}=-\frac{g}{2}⨍\frac{\mathrm{d}p'}{2\pi}\frac{\mathrm{d}p''}{2\pi}\frac{\varphi_j^\dagger(p')\varphi_i(p'')}{(p'-p'')^2}. \tag{2.16}$$

With this result, Eq. (2.13) can be simplified by replacing in the large N limit the operator

$$\varphi_i^\dagger(p)\varphi_i(p')\to 2\pi\delta(p-p')N\theta(-p) \tag{2.17}$$

by its expectation value. Here the triviality of the light-cone vacuum is explicitly used. Choosing units such that

$$\frac{Ng^2}{2\pi}=1, \tag{2.18}$$

factoring out a step function $\theta(-p)$ from $C(p,t)$ and changing p into $-p$, the time evolution of C can finally be cast into the form of a typical light-cone Schrödinger equation:

$$\mathrm{i}\dot{C}(p,t)=\left(\frac{m^2-1}{2p}+\lambda p\right)C(p,t)+\frac{1}{2p}C(0,t)-\frac{1}{2}⨍_0^\infty\mathrm{d}p'\frac{1}{p'-p}\frac{\partial C(p',t)}{\partial p'}. \tag{2.19}$$

This evolution equation for $C(p,t)$ at short times together with the initial condition [cf. Eq. (2.12)]

$$C(p,0)=N \tag{2.20}$$

determines the condensate. We note that

$$\mathrm{i}\dot{C}(p,t=0)=N\left(\frac{m^2}{2p}+\lambda p\right).$$

For non-interacting fermions,

$$C_0(p,t)=N\exp\left\{-\mathrm{i}t\left(\frac{m^2}{2p}+\lambda p\right)\right\} \tag{2.21}$$

solves the evolution equation with the correct initial condition. Due to the presence of singularities, $C(p,t)$ deviates significantly for arbitrarily small times from its initial value N. Characteristic for the short-time light-cone dynamics is the infrared singularity which implies

$$\lim_{t\to 0}C(0,t)=0\neq\lim_{p\to 0}C(p,0). \tag{2.22}$$

The short-time behavior for large momenta is the same in the interacting and in the free theory. It is therefore irrelevant for the evaluation of the condensate and we drop in the following the ultraviolet regulator,

$$\lambda=0. \tag{2.23}$$

In this case the evolution equation simplifies significantly due to the underlying covariance. We note that with $C(p,t)$ also $C(\lambda p,\lambda t)$ satisfies the evolution equation (2.19). Furthermore, if $C(p,t)$ satisfies the initial condition (2.20), so does $C(\lambda p,\lambda t)$. Assuming that the initial value problem (2.19), (2.20) defines a unique solution, we conclude that

$$C(p,t)=C(\lambda p,\lambda t), \tag{2.24}$$

which implies

$$C_{(0)}(p,-\mathrm{i}\tau)=NK_{(0)}(p/\tau), \tag{2.25}$$

where we have switched to imaginary time. [In the original units before implementing Eq. (2.18), the scaling variable corresponds to the dimensionless quantity $2\pi p/Ng^2\tau$.] With Eq. (2.25) the time evolution is converted into the integro-differential equation

$$q\frac{\mathrm{d}K(q)}{\mathrm{d}q}=\frac{m^2-1}{2q}K(q)-\frac{1}{2}⨍_0^\infty \mathrm{d}q'\frac{1}{q'-q}\frac{\mathrm{d}K(q')}{\mathrm{d}q'} \tag{2.26}$$

and the asymptotic behavior of $K(q)$ is determined by the initial condition for $C(p,t)$,

$$\lim_{q\to\infty}K(q)=1. \tag{2.27}$$

We can also determine the infrared behavior of $K(q)$. Using results for Hilbert transforms of powers (cf. [9]) it is seen that for small q

$$K(q)\sim q^{\beta_0} \tag{2.28}$$

where β_n denote the solutions of the 't Hooft boundary condition

$$\pi q \cot \pi q=1-m^2 \tag{2.29}$$

ordered according to

$$q=\pm\beta_n, \quad n=0,1,2\ \ldots \quad \text{with} \quad \beta_n\in[n,n+1]. \tag{2.30}$$

Thus the (confining) interaction in the 't Hooft model changes the essential singularity of $K_0(q)$ of the non-interacting theory

$$K_0(q)=\mathrm{e}^{-m^2/2q}$$

to a branch point. This remarkable phenomenon is due to the presence of the gauge string in the correlation function $C(p,t)$.

For the general case, we have not been able to express the solution of Eq. (2.26) via the Mellin transform (see below) in simple terms. In the chiral limit ($\beta_0\approx\sqrt{3}m/\pi\to 0$),

$$C(p,-\mathrm{i}\tau)\approx N(p/\tau)^{\beta_0\theta[1-(p/\tau)]}.$$

It can be verified that Eq. (2.26) is satisfied up to terms of $O(\beta_0)$. This expression displays the subtleties of the $p,t\to 0$ limit. It reproduces in the chiral limit the exact value for the condensate (see below), i.e., the condensate is directly connected to the change in the infrared singularity of the quark propagator.

As a consequence of the light-cone singularity in the infrared, determination of the short-time behavior of $C(p,t)$ requires a non-perturbative calculation of the function $K(q)$. In terms of $K_{(0)}(q)$, the condensate is written [cf. Eqs. (2.1), (2.11)] as

$$\langle\bar{\psi}\psi\rangle=-N\frac{m}{2\pi}\int_0^\infty\frac{\mathrm{d}q}{q}[K(q)-K_0(q)]. \tag{2.31}$$

Due to the scaling property (2.25) the dependence on the regulator ε^+ has disappeared entirely from the expression of the condensate. Since the scaling variable q and the function K are boost invariant, Lorentz invariance of $\langle\bar{\psi}\psi\rangle$ is manifest.

III. CALCULATION OF THE CONDENSATE

We now briefly sketch the evaluation of the condensate by the Mellin transformation of the equation for K. The techniques developed in [10] will be used. We define

$$\gamma(\kappa)=\int_0^\infty \mathrm{d}q\, q^{\kappa-1}K(q). \tag{3.1}$$

The Mellin transform is only well defined if

$$-\beta_0<\kappa<0. \tag{3.2}$$

For non-negative κ the integral (3.1) diverges at large values of q, while the infrared behavior (2.28) entails the lower limit. The Mellin transform converts Eq. (2.26) into the recursion relation (cf. [11])

$$\gamma(\kappa)=-\frac{1}{2\kappa}[m^2-1+\pi(\kappa-1)\cot\pi\kappa]\gamma(\kappa-1). \tag{3.3}$$

As can be easily verified,

$$\gamma(\kappa)=\mathcal{N}\left(\frac{\pi}{2}\right)^{\kappa}\frac{\exp\left\{-2\pi\int_0^\kappa \mathrm{d}u\left(\dfrac{u+\frac{1}{2}\sin^2\pi u}{\sin 2\pi u}\right)\right\}}{\beta_0\cos\pi(\kappa+\beta_0+1/2)}\prod_{n=1}^{\infty}\left(\frac{1+\dfrac{(m^2-1)\tan\pi\kappa}{\pi\beta_{n-1}}}{1+\dfrac{(m^2-1)\tan\pi\kappa}{\pi(\kappa+n)}}\right)\frac{\pi\beta_0-(m^2-1)\tan\pi\kappa}{\pi\kappa+(m^2-1)\tan\pi\kappa} \tag{3.4}$$

solves the recursion relation (3.3). Condition (2.27) determines the normalization $\mathcal{N}$. The solution has the expected poles at the end points of the regularity interval (3.2) and is free of singularities within this interval.

According to Eq. (2.31) the condensate is directly given by the Mellin transform

$$\langle\bar{\psi}\psi\rangle = -N\frac{m}{2\pi}\lim_{\kappa\to 0}[\gamma(\kappa)-\gamma_0(\kappa)] \tag{3.5}$$

where

$$\gamma_0(\kappa) = \left(\frac{m^2}{2}\right)^{\kappa}\Gamma(-\kappa) \tag{3.6}$$

is the Mellin transform of $K_0(q)$ for non-interacting fermions. Expansion of the Mellin transforms (3.4), (3.6) around $\kappa=0$, with the normalization factor $\mathcal{N}$ chosen such that the singular pieces $(\sim 1/\kappa)$ cancel, yields

$$\langle\bar{\psi}\psi\rangle = -N\frac{m}{2\pi}\Bigg\{1+\gamma+\ln\left(\frac{m^2}{\pi}\right) + (1-m^2)\left[\frac{1}{\beta_0}+\sum_{n=1}^{\infty}\left(\frac{1}{\beta_n}-\frac{1}{n}\right)\right]\Bigg\}. \tag{3.7}$$

This agrees with the result obtained in [8]. In particular, in the chiral limit where the $1/\beta_0$ term dominates, the result

$$\langle\bar{\psi}\psi\rangle = -\frac{N}{\sqrt{12}}, \tag{3.8}$$

first derived in [7], is reproduced. The calculations [7,8], are based on low energy theorems and connect the condensate with properties of the mesons of the 't Hooft model (in particular the "Goldstone boson" in the chiral limit). In our approach with the quark propagator as the essential ingredient, the condensate is determined by the short-time behavior of a heavy-light quark system. The equivalence of these quite different approaches suggests a relation in light-cone quantization similar to the relation in ordinary coordinates based on chiral Ward identities which connects quark propagators and the Goldstone boson Bethe-Salpeter wave function (cf. [12]).

A comparison between the evaluation of the condensate in the 't Hooft and the Gross-Neveu model is instructive. Our technique of evaluating the condensate via fermionic two-point functions applies as well to non-gauged theories like the Gross-Neveu model. In this model, Eq. (2.19) becomes simply

$$i\dot{C}_{\mathrm{GN}}(p,t) = \left(\frac{\hat{m}^2}{2p}+\lambda p\right)C_{\mathrm{GN}}(p,t), \tag{3.9}$$

with $\hat{m} = -g^2\langle\bar{\psi}\psi\rangle_\epsilon$. The solution of Eq. (3.9) together with Eq. (2.11) yields the well-known "gap equation" (cf. [5])

$$\hat{m}\left(1+\frac{Ng^2}{2\pi}\ln\frac{\hat{m}^2}{\Lambda^2}\right)=0, \quad \Lambda = \frac{\sqrt{2}}{\varepsilon^+\sqrt{\lambda}}e^{-\gamma}, \tag{3.10}$$

describing either the chirally symmetric $(\hat{m}=0)$ or the broken phase. Unlike in the 't Hooft model the procedure to regularize the condensate by subtracting the condensate of the non-interacting theory is not available in the Gross-Neveu model. This is why we have to keep the ultraviolet regulator λp in the evolution equation.

Strictly speaking, also in the 't Hooft model there is the possibility of a chirally symmetric solution. Indeed at $m=0$ (and $\lambda=0$), Eq. (2.19) can be solved by $C(p,t)=1$. By approaching the chiral limit from finite fermion mass this trivial solution cannot be reached. One might expect that as in standard coordinates the difference in energy density between the chirally symmetric and broken phases is infinite.

IV. CONCLUSIONS

In quantum field theory, condensates are calculated in general as expectation values of Schrödinger operators in the corresponding vacua. Non-vanishing order parameters reflect the non-triviality of the vacuum. In light-cone quantization, the kinematical structure of the light-cone vacuum gives rise to trivial vacuum expectation values of Schrödinger operators which therefore cannot serve as order parameters. On the light-cone, condensates have to be defined as vacuum expectation values of the limits of Heisenberg operators [5]. In this way, the condensate is obtained as the short light-cone time limit of an appropriate correlation function. In light-cone quantization, non-vanishing order parameters reflect the non-triviality of the short-time limit of the relevant Heisenberg operators.

We have carried out a study of the chiral condensate of two-dimensional QCD within light-cone quantization. A direct and explicit calculation of the condensate within the 't Hooft model has been presented. The kinematical structure of the light-cone vacuum has been an essential ingredient in this calculation. Our calculation shows that the short-time limit of the quark propagator is afflicted by non-perturbative physics. On the light cone this correlation function is singular in the infrared; the singularity depends on the dynamics. The essential singularity of the non-interacting theory is converted by the interactions in QCD_{1+1} to a branch point—a phenomenon which defies a perturbative description and is responsible for a generation of the condensate in the chiral limit. The resulting fermion correlation function differs significantly at short light-cone times from the correlation function of the Gross-Neveu model. In this two dimensional model a chiral condensate emerges in the process of mass generation [5]. Here the essential singularity in the infrared persists, its parameters are modified by interactions. The successful description of the condensates in these dynamically very different models strongly supports the idea of reconciling the non-trivial vacuum properties with the kinematical nature of the light-cone vacuum by the dynamical, nonperturbative short light-cone time limit of Heisenberg operators.

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