Maximum mass of a spherically symmetric isotropic star

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A well known result of a theorem due to Buchdahl, for a regular fluid sphere with a mass density which does not increase outwards, is that the ratio of its gravitational mass M to the coordinate radius R satisfies the inequality $GM/R \leqslant \frac{4}{9}$. This restriction arises from the condition that the isotropic pressure does not become infinity at the center of the sphere to prevent collapse. Buchdahl has also derived an inequality for the value of the central pressure of the sphere which we use to show that the minimum value for this pressure corresponds to a fluid of constant density. Then, using these results and the energy condition ($|p(r)| \leqslant \beta\rho(r)/3$), we find new bounds for the mass to radius ratio given by $2GM/R \leqslant S(\xi)$, where $S(\xi)$ is a nondecreasing function of its argument $\xi = \beta\rho_c/3\overline{\rho}$, where ρ_c is the central density of the star and $\overline{\rho}$ its mean density. For a constant density star, and $\beta = 3$ (which corresponds to the dominant energy condition), we have $S(1) = 3/4$, which implies an upper limit for the gravitational redshift factor for light coming from the surface of the star given by $z \leqslant 1$. We reobtain, for a general model and the values $\beta = 3$, $\rho_c \to \infty$, Buchdahl's limit; however, a comparison of our results with a previous inequality found by Buchdahl shows that for any values of β and the ratio $\rho_c/\overline{\rho}$ our bound of the mass to radius ratio is more strict.

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I. INTRODUCTION

Buchdahl's theorem [1] states that for a star of fixed radius R and gravitational mass M the ratio $GM/R \leqslant 4/9$, if $\rho \geqslant 0$ and $d\rho/dr \leqslant 0$, with the only requirements that the pressure becomes zero at the border of the star and it does not diverge at its center. The equality holds only if ρ is constant, the metric is degenerate $g_{tt}(r=0)=0$, and the pressure diverges. The value 4/9 is therefore an absolute upper limit for all static fluid spheres whose density does not increase outwards. Buchdahl [1] has also derived an inequality for the value of the central pressure of the sphere which we use to show that the minimum value for this pressure corresponds to a fluid of constant density. Then, using these results and the energy condition (Ref. [2]) [$|p(r)| \leqslant \beta\rho(r)/3$], we find new bounds for the mass to radius ratio given by $2GM/R \leqslant S(\xi)$, where $S(\xi)$ is a nondecreasing function of its argument $\xi = \beta\rho_c/3\overline{\rho}$, where ρ_c is the central density of the star and $\overline{\rho}$ its mean density. For a constant density star, and $\beta = 3$ (which corresponds to the dominant energy condition), we have $S(1) = 3/4$, which implies an upper limit for the gravitational redshift factor for light coming from the surface of the star given by $z \leqslant 1$. We reobtain, for a general model and the values $\beta = 3$, $\rho_c \to \infty$, Buchdahl's limit; however, a comparison of our results with a previous inequality found by Buchdahl shows that for any values of β and the ratio $\rho_c/\overline{\rho}$ our bound of the mass to radius ratio is more strict, in the sense that GM/R may be smaller than the limit obtained by Buchdahl [1].

For a static spherically symmetric spacetime the metric takes the Schwarzschild form

$$ds^2 = -B(r)dt^2 + A(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (1)$$

The energy-momentum tensor is assumed to be that for a perfect fluid

$$T_{\mu\nu} = p\, g_{\mu\nu} + (p+\rho)U_\mu U_\nu \quad (2)$$

with p the proper isotropic pressure, ρ the proper total energy density, and U^ν the velocity four-vector, defined so that $g^{\mu\mu}U_\mu U_\nu = -1$.

The identity $T^{\mu\nu}{}_{;\nu} = 0$ gives

$$\frac{B_{,r}}{B} = -\frac{2p_{,r}}{p+\rho}. \quad (3)$$

From Einstein equations we can derive an equation for $A(r)$ alone:

$$\frac{R_{rr}}{2A} + \frac{R_{\theta\theta}}{r^2} + \frac{R_{tt}}{2B} = \frac{A_{,r}}{rA} - \frac{1}{r^2} + \frac{1}{Ar^2} = -8\pi G\rho. \quad (4)$$

The solution with $A(0)$ finite is

$$A(r) = \left[1 - \frac{2Gm(r)}{r}\right]^{-1}, \quad (5)$$

where

$$m(r) = \int_0^r 4\pi \acute{r}^2 \rho(\acute{r}) d\acute{r}. \quad (6)$$

Also, using the field equations, we obtain the Tolman-Oppenheimer-Volkoff equation:

$$\frac{dp}{dr} = -(p+\rho)\frac{m(r) + 4\pi r^3 p}{r[r - 2m(r)]}. \quad (7)$$

Thus, for fluid matter with a given equation of state, $p = p(\rho)$, an equilibrium configuration can be determined as follows: We arbitrarily prescribe a central density ρ_c, and hence a central pressure $p_c = p(\rho_c)$. Then we integrate Eqs. (5) and (7) outward until $p(\rho(r))$ drops to zero at some point

$r=R$, which we then interpret as the radius of the particular star where we join the solution onto the vacuum Schwarzschild solution, i.e. $p=0$ and $\rho=0$ for $r>R$. Finally, we solve for $B(r)$ using Eqs. (3) and (7):

$$B(r)=\exp\left[-\int_r^\infty \frac{2G}{\acute{r}^2}[m(\acute{r})+4\pi\acute{r}^3 p(\acute{r})] \times\left(1-\frac{2Gm(\acute{r})}{\acute{r}}\right)^{-1} d\acute{r}\right]. \tag{8}$$

For the particular case of the state equation

$$\rho=\text{const} \tag{9}$$

the solution for $p(r)$,$A(r)$,$B(r)$ is

$$p(r)=\frac{3M}{4\pi R^3}\left[\frac{\sqrt{1-2MGr^2/R^3}-X}{3X-\sqrt{1-2MGr^2/R^3}}\right] \tag{10}$$

where

$$X=\sqrt{1-(2MG/R)} \tag{11}$$

and $0<X<1$,

$$A(r)=\left[1-\frac{2MGr^2}{R^3}\right]^{-1} \tag{12}$$

$$B(r)=\frac{1}{4}\left[3X-\sqrt{1-\frac{2MGr^2}{R^3}}\right]^2. \tag{13}$$

Therefore, the central pressure required for equilibrium of a uniform density star is

$$P_c=\frac{3M}{4\pi R^3}\left[\frac{1-X}{2\sqrt{B(0)}}\right] =\frac{3M}{4\pi R^3}\left[\frac{1-X}{3X-1}\right] \tag{14}$$

and P_c is positive. From Eq. (10) it is easy to see that the pressure is a monotone decreasing function of r, then the maximum for p is at the origin. Usually [3,4] it is required that the pressure does not become infinite anywhere, in particular it is not infinite at the origin, to prevent collapse; therefore $B(0)>0$. The above condition is fulfilled when

$$\frac{GM}{R}<\frac{4}{9}. \tag{15}$$

The existence of an upper mass limit in general relativity, for a given radius R, is not just a consequence of having restricted consideration to stars of uniform density; if we subject ρ only to general requirements, and impose the condition that Eq. (7) must yield a finite solution for $p(r)$, there is an absolute upper limit to MG/R imposed by the structure of the Einstein equations, irrespective of the equation of state [4,3,1]. This limit is just the above limit, calculated for the $\rho=$const model.

On the other hand, for all future directed timelike vector ξ^a, the quantity $-T^a_b\xi^b$ should be a future directed timelike or null vector. Since for an observer with 4-velocity ξ^a the quantity $-T^a_b\xi^b$ is the energy-momentum 4-current density of matter as seen by him; this condition is the dominant energy condition, and it can be interpreted as saying that the speed of energy flow of matter is always less than the speed of light [4,5]. Then, in any orthonormal basis the energy dominates the other components:

$$T^{00}\geqslant|T^{ab}|. \tag{16}$$

In our case the dominant energy condition is equivalent to

$$\rho(r)\geqslant|p(r)|. \tag{17}$$

This last condition is more restrictive than requiring a finite value for the pressure everywhere. Imposing this condition upon Eq. (14), it will be satisfied for all values of r if

$$\left[\frac{1-X}{2\sqrt{B(0)}}\right]\leqslant 1. \tag{18}$$

Therefore, we have

$$\frac{MG}{R}\leqslant\frac{3}{8}, \tag{19}$$

and this is a more restrictive bound than the usual one. In this example the use of the dominant energy condition has resulted in a more restrictive bound for the mass to radius ratio. We wonder if this is a general result, which we analyze in the next section.

II. NEW BOUNDS FOR *M/R*

We will show now a generalization of Buchdahl's theorem which proves that the bound (19) depends on the particular model of the mass distribution for the fluid matter. We begin by deriving Buchdahl's theorem in a new way. Let us assume that $\rho(r)\geqslant 0$ and that it is a monotone decreasing function of r; i.e. $d\rho/dr\leqslant 0$. In fact, the assumption that $\rho\geqslant 0$ follows from the monotone decreases assumption, since the interior solution must eventually match onto the exterior Schwarzschild solution. This hypothesis about ρ give that $w_{,r}\leqslant 0$ where $w(r)=Gm(r)/r^3$. The radius R is fixed through $\rho(r)=0$ for $r>R$. Given any function $\rho(r)$, satisfying these conditions, we can calculate $A(r)$ from Eq. (5); we can then determine $p(r)$ by integrating the Tolman-Oppenheimer-Volkoff equation inward from the surface [with the boundary condition $p(R)=0$] and then calculate $B(r)$ from Eq. (8). Assuming that ρ satisfies $m(r)<r/2G$, we may be sure that $A(r)$ is well behaved, and Eq. (8) will give a finite positive definite $B(r)$. Therefore, any absolute limitation on the input function $\rho(r)$ (such as an upper bound on MG/R) can only come from the condition that Eq. (7) must give not only a finite solution for the pressure $p(r)$, but

also the most restrictive requirement of the dominant energy condition.

In what follows we shall exploit this condition rather directly, at a difference with other authors [4,3,1] who attack the problem by concentrating on the metric coefficient $B(r)$ rather than on $p(r)$ itself.

Let ψ be

$$\psi = A^n(r)\frac{p+\zeta}{p+\eta}, \tag{20}$$

where

$$n = \frac{1}{2}\sqrt{4-3\theta}$$
$$\theta = \frac{w_b}{w_c}$$
$$\zeta = \frac{w_c}{2\pi}(1-n) \tag{21}$$
$$\eta = \frac{w_c}{2\pi}(1+n)$$

and the subscripts c and b refer to the center and the boundary of the sphere, respectively. Incidentally, $\theta \leqslant 1$ because $w_{,r} \leqslant 0$ and then $\frac{1}{2} \leqslant n \leqslant 1$, $\zeta \geqslant 0$ and $\eta \geqslant 0$. With these values of the parameters involved, ψ is then a nonincreasing function of r [1]. Accordingly, a comparison of its central and boundary values gives

$$\psi_c \geqslant \psi_b. \tag{22}$$

Considering $p_b = 0$ we have $A^n(r=R) = 1/X^{2n}$ and $A^n(r=0) = 1$, we obtain

$$p_c + \zeta \geqslant \frac{1}{X^{2n}}\frac{\zeta}{\eta}[p_c + \eta]. \tag{23}$$

Let $\Delta = X^2$. Then using Eq. (21) we have

$$4\pi p_c \geqslant \frac{3w_b(1-\Delta^n)}{2[\Delta^n(1+n)-(1-n)]} \quad \forall n/1 \geqslant n \geqslant \frac{1}{2}. \tag{24}$$

Let us call the right-hand side of the last inequality by $g(n)$. Then, it is not difficult to show that

$$\frac{dg(n)}{dn} = \frac{3w_b[\Delta^{2n} - 2n\Delta^n ln\Delta - 1]}{2[(1+n)\Delta^n + n - 1]^2}. \tag{25}$$

The denominator of Eq. (25) and w_b are positive, then the derivative sign depends on the bracket in the numerator: $f(\Delta) = [\Delta^{2n} - 2n\Delta^n ln\Delta - 1]$. Also

$$\frac{df}{d\Delta} = 2n\Delta^{n-1}U(\Delta) \tag{26}$$

where $U(\Delta) = (\Delta^n - 1 - n\, ln\Delta)$,

$$\frac{dU}{d\Delta} = \frac{n(\Delta^n - 1)}{\Delta}. \tag{27}$$

Using $1 > \Delta > 0$, we have $dU/d\Delta < 0$; then $U(\Delta) > U(1) = 0$. Thus, $0 = f(1) > f(\Delta)$ because $df/d\Delta > 0$. This gives $dg(n)/dn < 0$. Then

$$4\pi p_c \geqslant g(1/2) \geqslant g(n) \tag{28}$$

and substituting the value of $n = 1/2$, we obtain

$$p_c \geqslant \left(\frac{3M}{4\pi R^3}\right)\frac{1-X}{3X-1}. \tag{29}$$

According to Eq. (14) the right-hand side of Eq. (29) is the pressure of a star with mass M, radius R and mass distribution $\rho = \bar{\rho} \equiv (3M/4\pi R^3)$; we have called it P_c. Then, for any equation of state P_c is the lowest possible value for the pressure at the center of the star; i.e. given M and R the minimal central pressure for any equation of state is the corresponding $\rho =$ const. Any physical solution will have a finite pressure everywhere. Then, the necessary condition for a physical solution will be that P_c is also bounded and then we have the well known inequality $X > \frac{1}{3}$. The last result gives the bound $GM/R < \frac{4}{9}$, i.e. Buchdahl's theorem. The way we have followed to prove Buchdahl's theorem allows us to generalize it when we have matter which satisfies $p \leqslant \beta/3\rho$ for any β. Now, if this last condition (in the form $p_c \leqslant \beta/3\rho_c$) is not to be violated, it is necessary that $P_c \leqslant \beta/3\rho_c$. If we impose this condition, using Eq. (14) we obtain

$$\frac{1-X}{3X-1} \leqslant \frac{\beta\rho_c}{3\bar{\rho}}. \tag{30}$$

A straightforward calculation shows

$$\frac{2MG}{R} \leqslant S(\xi) \tag{31}$$

for an all spherical star with isotropic pressure and matter which satisfies the energy condition $p \leqslant (\beta/3)\rho$. The function $S(\xi)$, $\xi = \beta\rho_c/3\bar{\rho}$, is a non decreasing function expressed by

$$S(\xi) = 1 - \left(\frac{1+\xi}{1+3\xi}\right)^2. \tag{32}$$

This function satisfies $S(1) = 3/4$, for $\beta = 3$ and a constant density star, and $S(\infty) = 8/9$ if the density goes to infinity at the origin. We can express the inequality (30) in a more convenient way in the form

$$X \geqslant \frac{1}{3}\left(1 + \frac{2\delta}{\beta+\delta}\right) \tag{33}$$

where $\delta = \bar{\rho}/\rho_c$ may take any value in [0,1]. The value $\delta = 0$ corresponds to a central density going to ∞, and $\delta = 1$ to a constant density model.

Buchdahl [1] has found an inequality that in our notation reads

$$X \geqslant \frac{1}{3}\left(1+\frac{2\delta}{\beta+1}\right). \tag{34}$$

It is apparent that the bounds (33) and (34) agree for any value of β when $\delta=0$ or 1. However, for any other values of δ and β, we obtain a smaller limit for the values of GM/R from Eq. (33) than from Eq. (34) which corresponds to the limit obtained by Buchdahl [1]. In general, to have a value of δ we need a detailed static spherically symmetric model for an interior solution. However, an interesting estimate of its value can be achieved for a mass distribution which satisfies the condition that it is always greater or equal than the values represented by a straight line which takes the value ρ_c at the origin, $r=0$, and ρ_R at the border of the star, $r=R$ [for example, when $\rho(r)$ does not have an inflexion point]. A simple calculation shows that, under these conditions, $\delta=1/4$ and, if we consider $\beta=3$, the limit obtained from Eq. (33) is $2MG/R \leqslant 0.8520\ldots$, while the one obtained from Eq. (34) is $2MG/R \leqslant 0.8593\ldots$. In the next section we consider the generalized Buchdahl $n=5$ polytrope (GB5) family of exact interior solutions [6,7] to test the inequality (31).

Let now ω_e be the frequency of emission of light at the surface of the star and ω_o the frequency measured by an observer at a given position. Then, the redshift factor is, as usual, $z=\omega_e/\omega_o-1$; if we use explicitly the values of g_{tt} at both positions we would have

$$z=\frac{\left(1-\dfrac{2MG}{r_o}\right)^{1/2}}{\left(1-\dfrac{2MG}{r_e}\right)^{1/2}}-1 \tag{35}$$

where r_e is the emission radius and r_o is the observation radius.

Using Eq. (30) we obtain that the maximum redshift factor of light emitted from the surface of a constant density static star is for $r_e=\frac{8}{3}GM$ and an observation radius $r_o=\infty$. Then

$$z_{max}=\left(\frac{\omega_e}{\omega_o}\right)_{max}-1=1. \tag{36}$$

On the other hand, for a mass distribution which satisfies the condition that it is always greater or equal than the values represented by a straight line which takes the value ρ_c at the origin, $r=0$, and ρ_R at the border of the star, $r=R$, the maximum redshift factor is $z=1.6$.

III. THE GB5 INTERIOR SOLUTIONS

In this section we put the gravitational constant $G=1$. We just introduce the GB5 family of interior solutions as a testing ground for the inequality (31). We write the metric in a proper time general radial gauge as

$$ds^2=-Y^2(x)dt^2+N^2(x)dx^2+S^2(x)(d\theta^2+\sin^2\theta d\phi^2). \tag{37}$$

In terms of the general radial variable x, the Schwarzschild radial variable is given by $r=S(x)$. The GB5 family of interior solutions is given by ("s" means the value at the surface of the star)

$$Y=(1-2M/R)^{1/2}\left(\frac{T_s+X_s}{T_s-X_s}\right)\frac{T-X}{T+X},$$
$$N=S=(T+X)^2 \tag{38}$$

where

$$T(x)=\sqrt{\frac{b\lambda^{-3}}{\cosh(x-\Delta)}},\quad X(x)=\sqrt{\frac{b}{\cosh x}}. \tag{39}$$

The constant Δ is a nontrivial integration constant, and the constants b and λ characterize the equation of state given by

$$p=\frac{a(u^6-\lambda^6)}{(1+u)^5(1-u)},\quad \rho=\frac{3a(u^5+\lambda^6)}{(1+u)^5} \tag{40}$$

where $a=1/(8\pi b^2)$, $u=X/T$ and $u_s=\lambda$, $u_c=\lambda^{3/2}e^{-\Delta/2}$; ("$c$" means the value at the center of the star, $S=0$). The parameters λ and u_c characterize a given member of the GB5 family $(0<\lambda\leqslant u\leqslant u_c<1)$. It is convenient to identify a given solution by λ and a new parameter, μ, given by $\mu=(u_c-\lambda)/(1-\lambda)$. The GB5 family in the (λ,μ) parameter space corresponds to the unit square, $0<\lambda<1$, $0<\mu<1$ [7]. For our purposes we need to complete our set of equations by expressing M and R in terms of λ, u_c, and b. The desired relations are [7]

$$M=\frac{4bu_c(u_c^2-\lambda^2)^{3/2}(u_c^2-\lambda^4)^{3/2}}{(u_c^2+\lambda^3)(u_c^2-\lambda^3)^3},$$
$$R=\frac{2bu_c(u_c^2-\lambda^2)^{1/2}(u_c^2-\lambda^4)^{1/2}(1+\lambda)^2}{\lambda(u_c^2+\lambda^3)(u_c^2-\lambda^3)}. \tag{41}$$

We are now ready to compute all the necessary ingredients to construct the inequality (31). We begin by finding a value of β for each solution. To this end from Eq. (40), we obtain

$$\frac{3p}{\rho}\leqslant\frac{(u_c^6-\lambda^6)}{(\lambda^5+\lambda^6)(1-u_c)}:=\beta. \tag{42}$$

Then, from Eqs. (40), (41), (42), and the definitions of δ and ξ, we obtain

$$\xi=\frac{u_c^2(u_c^6-\lambda^6)(u_c^5+\lambda^6)(1+\lambda)^5}{3\lambda^8(1-u_c)(1+\lambda)^5(u_c^2+\lambda^3)^2}. \tag{43}$$

In the calculations that we have performed we have used $\alpha:=R/M$ as an input parameter and λ as an independent variable to cover the whole family of solutions. The relations that yield u_c in terms of α and λ are [7]

$$u_c=\lambda^{3/2}\left(\frac{1+\zeta}{1-\zeta}\right)^{1/4},\quad \zeta:=\frac{(1-\lambda^2)\sqrt{\alpha(\alpha-2)}}{\alpha(1+\lambda^2)-(1+\lambda)^2}. \tag{44}$$

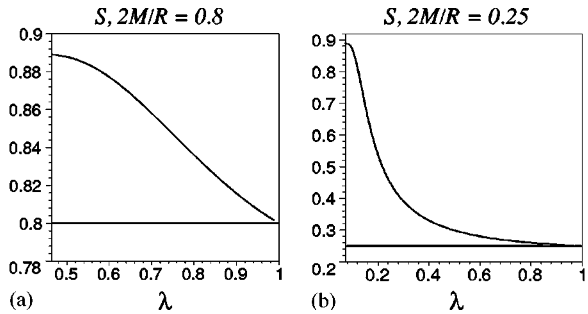

FIG. 1. Two typical plots of the bound $S(\xi)$ and $2M/R$ vs the parameter λ for different values of α showing that $2M/R \leqslant S(\xi)$. The lower limit for the values of λ, for each α, is obtained from the condition that the parameter μ should not exceed the value one in the parameter space of the GB5 family. The curve on the left corresponds to $\alpha = 2.5$, while that on the right corresponds to $\alpha = 8$.

The computational procedure is as follows: First, we choose a value of the parameter α. Second, we compute the range of λ by finding its lower limit as the largest root of Eq. (44) for $u_c = 1$ (this value corresponds to $\mu = 1$). And finally, we compute the function $S(\xi)$ and check its value against $(2M/R) = 2/\alpha$ which should be larger than this in the whole range of λ. Then, we repeat the procedure for a new value of α. Although we have carried out the calculations for a wide range of values of α between 2 and ∞, we reproduce in Fig. 1 only two of the curves $S(\xi)$ vs λ, which show that the inequality (31) is verified, as expected.

IV. CONCLUSIONS

In this paper we have followed a different procedure from Buchdahl [1] to obtain new bounds for the mass to radius ratio for a spherically symmetric isotropic star. We have analyzed the properties of the mass distribution rather than the metric of the spacetime. We have obtained, using Buchdahl's results and the energy condition, $(|p(r)| \leqslant \beta\rho(r)/3)$, an inequality given by $2GM/R \leqslant S(\xi)$, where $S(\xi)$ is a nondecreasing function of the ratio of the density of the star at the center to the mean density, $\xi = \beta\rho_c/3\bar{\rho}$. For a wide class of models of mass distribution we have been able to improve the maximum limit of GM/R. In particular, for a constant density star, and $\beta = 3$, we have $GM/R < 3/8$, which implies an upper limit for the gravitational redshift factor for light coming from the surface of the star given by $z \leqslant 1$; then, the use of the dominant energy condition gives a more strict bound. However, for a general model the use of the dominant energy condition does not modify Buchdahl's limit, $GM/R \leqslant 4/9$, which arises from the requirement that the central pressure must not diverge. On the other hand, a comparison of our results with a previous inequality found by Buchdahl shows that for any values of the parameters β and δ our bound of the mass to radius ratio is more strict, in the sense that GM/R may be smaller than the limit obtained by Buchdahl [1].

In Sec. III we have used the GB5 family of interior solutions as a testing ground for the inequality (31) and have obtained that it is verified, as expected. Given a model for the mass distribution of the star, we have definite values of ρ_c and $\bar{\rho}$; with these values we get a more strict bound for the gravitational redshift factor than the value of z obtained by Buchdahl. If the kinematical effects on the redshift may be taken into account, we hope that this difference may become observable.

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